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L^p multipliers and their H^1-L^1 estimates on the Heisenberg group

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Abstract. We give a Hörmander-type sufficient condition on an operator-valued function M that implies the L^p-boundedness result for the operator T_M defined by $(T_M f)\hat{} = M\hat{f}$ on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$. Here "$\hat{}$" denotes the Fourier transform on $\mathbb{H}^n$ defined in terms of the Fock representations. We also show the H^1-L^1 boundedness of T_M, $\|T_M f\|_{L^1} \le C\,\|f\|_{H^1}$, for $\mathbb{H}^n$ under the same hypotheses of L^p-boundedness.

1. Introduction.

Let $f \mapsto \hat{f}$ be the Fourier transform, $f \mapsto \check{f}$ the inverse Fourier transform, and m a bounded measurable function on $\mathbb{R}^n$. We say that m is a *multiplier* for $L^p(\mathbb{R}^n)$, $1 \le p \le +\infty$, if $f \in L^2 \cap L^p$ implies $(m\hat{f})\check{}$ is in L^p and satisfies

$$\big\|(m\hat{f})\check{}\big\|_{L^p} \le C_p\,\|f\|_{L^p}\,,$$

with C_p independent of f. The multiplier theorem was originally due to Hörmander [H] on $\mathbb{R}^n$:

Theorem (Multiplier theorem for $L^p(\mathbb{R}^n)$). *Let m be a function in $C^k(\mathbb{R}^n \setminus \{0\})$, $k > n/2$. Assume that $m \in L^\infty(\mathbb{R}^n)$ and*

$$\sup_{R>0} R^{2|\alpha|-n} \int_{R<|x|\le 2R} |D^\alpha m(x)|^2\, dx < +\infty$$

for all differential monomials D^α of order $|\alpha| \le k$. Then the multiplier operator mapping f into $(m\hat{f})\check{}$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

There are two general methods of proving multiplier theorems. The first one follows Hörmander's process [H] and works mostly on the Fourier transform side. The second one applies the theory developed by Coifman and Weiss [CW1] of constructing a well-behaved approximate identity and working mostly on the group. DeMichele and Mauceri [DMM] applied Coifman and Weiss' theory to extend the L^p multiplier theorem to the three-dimensional Heisenberg group $\mathbb{H}$. Here we follow the same machinery as in [DMM], and extend to the more general case of the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$.

Theorem 1 (Multiplier theorem for $L^p(\mathbb{H}^n)$). *Let M be an operator-valued function with each entry in $C^k(\mathbb{R}\setminus\{0\})$, $k \ge 4\left[(n+5)/4\right]$, where $[\,\cdot\,]$ denotes the greatest integer function. Also assume*

$$\sup_{\lambda\neq 0} \|M(\lambda)\| < +\infty\,,$$

$$\sup_{R>0} R^{\deg P-n-1} \int_{-\infty}^{+\infty} \left\|[\Delta_P M(\lambda)]\hat{\Pi}_R(\lambda)\right\|_{HS}^2 |\lambda|^n\, d\lambda < +\infty\,,$$

for every monomial P with $\deg P \le 4\left[(n+5)/4\right]$, where Δ_P is a difference-differential operator, $\hat{\Pi}_R(\lambda)$ a projection operator to a part of main diagonal (both operators will be defined in the next section). Then M is a multiplier of $L^p(\mathbb{H}^n)$, $1 < p < \infty$, and is of weak type $(1,1)$.

We also show the H^1-L^1 boundedness of T_M as follows.

Theorem 2. *Suppose M satisfies the same hypotheses as Theorem 1. Then T_M maps $H^1(\mathbb{H}^n)$ boundedly into $L^1(\mathbb{H}^n)$. Moreover, there exists a constant $C > 0$, independent of f, such that $\|T_M f\|_{L^1} \le C\,\|f\|_{H^1}$ for all $f \in H^1(\mathbb{H}^n)$.*

In Section 2 we review some basic tools of harmonic analysis on $\mathbb{H}^n$. In Section 3 we prove the $L^p(\mathbb{H}^n)$ multiplier theorem, and in Section 4 we show the H^1-L^1 estimate. Finally, we mention that C will be used to denote a constant which may vary from line to line.

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2. Preliminaries.

Most results in this section were given in [DMM] for the three-dimensional Heisenberg group $\mathbb{H}$. We extend those to $\mathbb{H}^n$ and give more detailed proofs here.

$\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{R} \times \mathbb{C}^n$ and multiplication defined by

$$(t, z)\,(t', z') = (t + t' + 2\,\mathrm{Im}\,(z \cdot \bar{z}'), z + z')\,,$$

where $z \cdot \bar{z}' = \sum_{j=1}^{n} z_j \bar{z}_j'$. The Heisenberg Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on $\mathbb{H}^n$ is generated by

$$\begin{aligned} T &= \frac{\partial}{\partial t}\,, \\ Z_j &= \frac{\partial}{\partial z_j} + i\,\bar{z}_j\,\frac{\partial}{\partial t}\,, \\ \overline{Z}_j &= \frac{\partial}{\partial \bar{z}_j} - i\,z_j\,\frac{\partial}{\partial t}\,, \qquad j = 1, 2, \ldots, n\,, \end{aligned}$$

and the only non-zero commutations are

$$[Z_j, \overline{Z}_j] = -2\,i\,T\,, \qquad j = 1, 2, \ldots, n\,.$$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure dV on $\mathbb{R} \times \mathbb{C}^n$. $\mathbb{H}^n$ is endowed with a family of dilations $\{\delta_r : r > 0\}$ defined by $\delta_r(t, z) = (r^2 t, rz)$. We say a function f on $\mathbb{H}^n$ is *homogeneous of degree* d if $f \circ \delta_r = r^d f$ for every $r > 0$. Furthermore, we define the *homogeneous norm* of $(t, z) \in \mathbb{H}^n$, denoted by $|(t, z)|$, to be $(t^2 + |z|^4)^{1/4}$. The norm is homogeneous of degree 1. For simplification of notation, sometimes we use $x = (t, z)$ to denote a point of $\mathbb{H}^n$, $rx = \delta_r(t, z) = (r^2 t, rz)$ a dilation of x, and $|x| = (t^2 + |z|^4)^{1/4}$ a homogeneous norm on $\mathbb{H}^n$. Guivarch [Gu] has shown that the triangle inequality $|xy| \leq |x| + |y|$, $x, y \in \mathbb{H}^n$, holds. Moreover, using polar coordinates, we have (*cf.* [FS, Corollary 1.16])

$$\int\limits_{a \leq |x| \leq b} |x|^\alpha \, dx = \frac{C}{\alpha + 2n + 2} \left(b^{\alpha+2n+2} - a^{\alpha+2n+2} \right), \tag{1}$$

for $\alpha \neq -2n - 2$, $0 \leq a < b \leq +\infty$, where C is an absolute constant.

The convolution of two functions $f, g \in L^1(\mathbb{H}^n)$ is defined as usual,

$$(g * f)(x) = \int_{\mathbb{H}^n} g(xy^{-1}) \, f(y) \, dy = \int_{\mathbb{H}^n} g(y) \, f(y^{-1}x) \, dy \, .$$

Let $\mathcal{S}(\mathbb{H}^n)$ and $\mathcal{S}'(\mathbb{H}^n)$ denote the Schwartz space of rapidly decreasing smooth functions and space of tempered distributions, respectively.

It was observed by Stone, von Neumann, and Weyl in the early 1930's that the irreducible unitary representations of $\mathbb{H}^n$ split into two classes. The representations which are trivial on the center $\mathcal{Z} = \{(t, 0) : t \in \mathbb{R}\}$ of $\mathbb{H}^n$ are just the usual one-dimensional representations of $\mathbb{C}^n \cong \mathbb{H}^n/\mathcal{Z}$ lifted to $\mathbb{H}^n$. Since these representations form a set of measure zero in the decomposition of $L^2(\mathbb{H}^n)$, we will not consider them further. The representations which are nontrivial on $\mathcal{Z}$ are classified by a parameter $\lambda \in \mathbb{R}^*$ $(\equiv \mathbb{R} \setminus \{0\})$ and may be described as follows. For $\lambda > 0$, let $\mathcal{H}_\lambda$ be the Bargmann space

$$\mathcal{H}_\lambda = \Big\{ F \text{ holomorphic on } \mathbb{C}^n : \\ \|F\|^2 = \left(\frac{2\lambda}{\pi}\right)^n \int_{\mathbb{C}^n} |F(\zeta)|^2 \, e^{-2\lambda|\zeta|^2} d\zeta < +\infty \Big\} .$$

Then $\mathcal{H}_\lambda$ is a Hilbert space and the monomials

$$F_{\alpha,\lambda}(\zeta) = \sqrt{\frac{(2\lambda)^{|\alpha|}}{\alpha!}} \, \zeta^\alpha, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \overline{\mathbb{N}}^n$$

$(\overline{\mathbb{N}} \equiv \mathbb{N} \cup \{0\})$, form an orthonormal basis for $\mathcal{H}_\lambda$, where

$$\alpha! = (\alpha_1!) \, (\alpha_2!) \cdots (\alpha_n!) \, , \qquad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \, ,$$

and

$$\zeta^\alpha = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\cdots\zeta_n^{\alpha_n}\,.$$

For $\lambda \in \mathbb{R}^*$, the representation Π_λ acts on $\mathcal{H}_{|\lambda|}$ by

$$\Pi_\lambda(t,z)F(\zeta) = \begin{cases} e^{i\lambda t + 2\lambda(\zeta\cdot z - |z|^2/2)}\,F(\zeta - \bar z)\,, & \text{for } \lambda > 0\,,\\ e^{i\lambda t - 2\lambda(\zeta\cdot \bar z - |z|^2/2)}\,F(\zeta - z)\,, & \text{for } \lambda < 0\,. \end{cases}$$

A straightforward calculation shows that $\Pi_\lambda(t,z)$ is unitary and its adjoint operator $\Pi_\lambda(t,z)^* = \Pi_\lambda(-t,-z)$. For $f \in L^1(\mathbb{H}^n)$, $\lambda \in \mathbb{R}^*$, set

$$\hat f(\lambda) = \int_{\mathbb{H}^n} f(t,z)\,\Pi_\lambda(t,z)\,dV$$

where the integral is defined in the weak sense, and the operator $\hat f(\lambda)$ is called the *Fourier transform* of f. It follows immediately from the definition that for $f, g \in L^1(\mathbb{H}^n)$

$$\begin{aligned} \Pi_\lambda(y)\,\hat f(\lambda)\,F(\zeta) &= \Pi_\lambda(y)\int f(x)\,\Pi_\lambda(x)\,F(\zeta)\,dx\\ &= \int f(x)\,\Pi_\lambda(yx)\,F(\zeta)\,dx\\ &= \int f(y^{-1}x)\,\Pi_\lambda(x)\,F(\zeta)\,dx \end{aligned}$$

and

$$\begin{aligned} \hat g(\lambda)\,\hat f(\lambda) &= \int g(y)\,\Pi_\lambda(y)\,dy\,\hat f(\lambda)\\ &= \iint g(y)\,f(y^{-1}x)\,\Pi_\lambda(x)\,dx dy\\ &= \int (g*f)(x)\,\Pi_\lambda(x)\,dx\\ &= (g*f)\hat{\ }(\lambda)\,. \end{aligned}$$

For $(\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \overline{\mathbb{N}}^n$, we use the notations

$$\begin{aligned} m_i^+ &= \max\{m_i, 0\}\,, & m_i^- &= -\min\{m_i, 0\}\,,\\ m^+ &= (m_1^+, m_2^+, \ldots, m_n^+)\,, & m^- &= (m_1^-, m_2^-, \ldots, m_n^-)\,, \end{aligned}$$

and define the partial isometry $W_\alpha^m(\lambda)$ on $\mathcal{H}_{|\lambda|}$ by

$$W_\alpha^m(\lambda)\, F_{\beta,\lambda} = (-1)^{|m^+|}\, \delta_{\alpha+m^+,\beta}\, F_{\alpha+m^-,\lambda}\,, \qquad \text{for } \lambda > 0\,,$$

and for negative λ by

$$W_\alpha^m(\lambda) = [W_\alpha^m(-\lambda)]^*\,.$$

Thus $\{W_\alpha^m(\lambda) : m \in \mathbb{Z}^n, \alpha \in \overline{\mathbb{N}}^n\}$ is an orthonormal basis for the Hilbert-Schmidt operators on $\mathcal{H}_{|\lambda|}$, and W_α^m has the matrix expression $W_{\alpha_1}^{m_1} \otimes W_{\alpha_2}^{m_2} \otimes \cdots \otimes W_{\alpha_n}^{m_n}$ if we view $F_{\alpha,\lambda}(\zeta)$ as $F_{\alpha_1,\lambda}(\zeta_1) \otimes F_{\alpha_2,\lambda}(\zeta_2) \otimes \cdots \otimes F_{\alpha_n,\lambda}(\zeta_n)$, where $\otimes$ means tensor product, $W_{\alpha_i}^{m_i}$ corresponds to the infinite dimensional matrix whose $(\alpha_i, m_i+\alpha_i)$-entry is $(-1)^{m_i}$ and zero everywhere else for $m_i \geq 0$, $(\alpha_i - m_i, \alpha_i)$-entry is 1 and zero everywhere else for $m_i < 0$, and $F_{\alpha_i,\lambda}(\zeta_i)$ corresponds to the infinite dimensional vector whose α_i-th components is 1 and all other components are zero. (Note: all entries and components here are counted from the 0-th position.) For each i, the matrix form of $W_{\alpha_i}^{m_i}$ is

$$(-1)^{m_i} \begin{bmatrix} 0 & & & \\ & \ddots & & 1 \\ & & \ddots \; {\swarrow}{}^{m_i}{\nearrow} & \\ & & & \\ & & & \end{bmatrix} \begin{array}{l} \leftarrow \text{0-th row} \\ \leftarrow \alpha_i\text{-th row} \\ \\ \\ \\ \end{array} \qquad \text{for } m_i \geq 0,$$

$$\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & & \ddots \\ & & {\swarrow}{}^{-m_i}{\nearrow} & \\ & 1 & & \end{bmatrix} \begin{array}{l} \leftarrow \text{0-th row} \\ \\ \\ \\ \leftarrow (\alpha_i - m_i)\text{-th row} \end{array} \qquad \text{for } m_i < 0.$$

Proposition [G]. *If $f \in L^1 \cap L^2(\mathbb{H}^n)$ is of the form*

$$f(t,z) = \sum_{m \in \mathbb{Z}^n} f_m(t, |z_1|, \ldots, |z_n|)\, e^{i(m_1\theta_1 + \cdots + m_n\theta_n)}\,, \qquad z_j = |z_j|\, e^{i\theta_j}\,,$$

then

$$\hat{f}(\lambda) = \sum_{\substack{m \in \mathbb{Z}^n \\ \alpha \in \overline{\mathbb{N}}^n}} \mathcal{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\,,$$

where

$$\mathcal{R}_f(\lambda, m, \alpha) = \int_{\mathbb{H}^n} f_m(t, |z_1|, \ldots, |z_n|)\, e^{i\lambda t} \cdot l_{\alpha_1}^{|m_1|}(2\,|\lambda|\,|z_1|^2) \cdots l_{\alpha_n}^{|m_n|}(2\,|\lambda|\,|z_n|^2)\, dV\,, \tag{2}$$

and $l_{\alpha_j}^{|m_j|}$ *is the Laguerre function.*

NOTE. For a poly-radial function $f(t,z) = f(t, |z_1|, \ldots, |z_n|)$, the summation $\sum_{m\in\mathbb{Z}^n}$ in the above proposition contains only the term with $m = 0$. Hence $\hat{f}(\lambda)$ is poly-diagonal.

Recall $(\delta_r f)\hat{}\,(\xi) = r^{-n}\,\hat{f}(r^{-1}\xi)$ on $\mathbb{R}^n$, where $\delta_r f(x) = f(rx)$. If we define

$$f_r(t,z) = r^{-(n+1)/2}\, f(r^{-1/2}t, r^{-1/4}z)\,, \qquad r > 0\,,$$

on $\mathbb{H}^n$, from identity (2) and a change of variables we have a similar relationship between Fourier coefficients $\mathcal{R}_{f_r}(\lambda, m, \alpha)$ and $\mathcal{R}_f(\lambda, m, \alpha)$ as follows:

$$\mathcal{R}_{f_r}(\lambda, m, \alpha) = \mathcal{R}_f(\sqrt{r}\lambda, m, \alpha)\,. \tag{3}$$

We also have $[-ix_j f(x)]\hat{}\,(\xi) = \partial_j \hat{f}(\xi)$ on $\mathbb{R}^n$. More generally if $P(x) = P(x_1, x_2, \ldots, x_n)$ is a polynomial on $\mathbb{R}^n$ and the differential operator $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ is defined as usual, then $[P(-ix)\, f(x)]\hat{}\,(\xi) = P(D)\hat{f}(\xi)$. Let P be a polynomial in $t, z_j,\ \bar{z}_j$ on $\mathbb{H}^n$. Define the difference-differential operator Δ_P acting on the Fourier transform of $f \in L^1 \cap L^2(\mathbb{H}^n)$ by

$$\Delta_P\Big(\sum_{m,\alpha} \mathcal{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\Big) = \sum_{m,\alpha} \mathcal{R}_{Pf}(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\,.$$

Let $\{e_j : 1 \le j \le n\}$ be the standard basis of $\mathbb{Z}^n$. We have the following explicit expressions for Δ_t, Δ_{z_j}, and $\Delta_{\bar{z}_j}$.

$$\Delta_t \hat{f}(\lambda) = -\, i \sum_{m,\alpha} \big(\frac{\partial}{\partial\lambda}\, \mathcal{R}_f(\lambda, m, \alpha)\big)\, W_\alpha^m(\lambda) - \frac{ni}{2\lambda} \sum_{m,\alpha} \mathcal{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{i}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^{n} \sqrt{\alpha_j(\alpha_j + |m_j|)}\, \mathcal{R}_f(\lambda, m, \alpha - e_j)\, W_\alpha^m(\lambda)$$

$$+ \frac{i}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^{n} \sqrt{(\alpha_j + 1)(\alpha_j + |m_j| + 1)}$$

$$\cdot\, \mathcal{R}_f(\lambda, m, \alpha + e_j)\, W_\alpha^m(\lambda)\ ;$$

$$\Delta_{z_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + m_j}\, \mathcal{R}_f(\lambda, m - e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + 1}\, \mathcal{R}_f(\lambda, m - e_j, \alpha + e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \geq 1$;

$$\Delta_{z_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j - m_j + 1}\, \mathcal{R}_f(\lambda, m - e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j}\, \mathcal{R}_f(\lambda, m - e_j, \alpha - e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \leq 0$;

$$\Delta_{\bar{z}_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + m_j + 1}\, \mathcal{R}_f(\lambda, m + e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j}\, \mathcal{R}_f(\lambda, m + e_j, \alpha - e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \geq 0$;

$$\Delta_{\bar{z}_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j - m_j}\, \mathcal{R}_f(\lambda, m + e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + 1}\, \mathcal{R}_f(\lambda, m + e_j, \alpha + e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \leq -1$.

Using these formulas, we obtain similar results for polynomials and extend the operators Δ_P as formal difference-differential operators acting on operators which are of type

$$M(\lambda) = \sum_{m,\alpha} B(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\,.$$

We establish the formula for Δ_{z_j}. The others are proved similarly by using the recurrence relations and differential properties of $\{l_\alpha^m\}$. We use identity (2) and write

$$\begin{aligned}\Delta_{z_j}\hat{f}(\lambda) &= \sum_{m,\alpha}\mathcal{R}_{z_jf}(\lambda,m,\alpha)\,W_\alpha^m(\lambda)\\ &= \sum_{m,\alpha}\int_{\mathbb{H}^n}|z_j|\,f_{m-e_j}\,e^{i\lambda t}l_{\alpha_1}^{|m_1|}(2\,|\lambda|\,|z_1|^2)\cdots l_{\alpha_j}^{|m_j|}(2\,|\lambda|\,|z_j|^2)\\ &\qquad\qquad \cdots l_{\alpha_n}^{|m_n|}(2\,|\lambda|\,|z_n|^2)\,dV\;W_\alpha^m(\lambda)\,.\end{aligned}$$

The recurrence relations for Laguerre functions tell us

$$\begin{aligned}&\sqrt{2|\lambda|}\;|z_j|\;l_{\alpha_j}^{|m_j|}(2\,|\lambda|\,|z_j|^2)\\ &\quad= \sqrt{\alpha_j+m_j}\;l_{\alpha_j}^{m_j-1}(2\,|\lambda|\,|z_j|^2)-\sqrt{\alpha_j+1}\;l_{\alpha_j+1}^{m_j-1}(2\,|\lambda|\,|z_j|^2)\,,\end{aligned}$$

if $m_j\geq 1$, and

$$\begin{aligned}&\sqrt{2|\lambda|}\;|z_j|\;l_{\alpha_j}^{|m_j|}(2\,|\lambda|\,|z_j|^2)\\ &\quad= \sqrt{\alpha_j-m_j+1}\;l_{\alpha_j}^{-m_j+1}(2\,|\lambda|\,|z_j|^2)-\sqrt{\alpha_j}\;l_{\alpha_j-1}^{-m_j+1}(2\,|\lambda|\,|z_j|^2)\,,\end{aligned}$$

if $m_j\leq 0$.

Thus, we have

$$\begin{aligned}\Delta_{z_j}\hat{f}(\lambda) &= \frac{1}{\sqrt{2|\lambda|}}\sum_{m,\alpha}\sqrt{\alpha_j+m_j}\;\mathcal{R}_f(\lambda,m-e_j,\alpha)\,W_\alpha^m(\lambda)\\ &\quad-\frac{1}{\sqrt{2|\lambda|}}\sum_{m,\alpha}\sqrt{\alpha_j+1}\;\mathcal{R}_f(\lambda,m-e_j,\alpha+e_j)\,W_\alpha^m(\lambda)\,,\end{aligned}$$

for $m_j\geq 1$, and

$$\begin{aligned}\Delta_{z_j}\hat{f}(\lambda) &= \frac{1}{\sqrt{2|\lambda|}}\sum_{m,\alpha}\sqrt{\alpha_j-m_j+1}\;\mathcal{R}_f(\lambda,m-e_j,\alpha)\,W_\alpha^m(\lambda)\\ &\quad-\frac{1}{\sqrt{2|\lambda|}}\sum_{m,\alpha}\sqrt{\alpha_j}\;\mathcal{R}_f(\lambda,m-e_j,\alpha-e_j)\,W_\alpha^m(\lambda)\,,\end{aligned}$$

for $m_j\leq 0$.

This proves the formula for Δ_{z_j}. Similarly, applying the same techniques we can obtain the formulas for $\Delta_{\bar{z}_j}$ and Δ_t.

Denoting $\|A\|_{HS}^2 = \operatorname{tr}(A^*A)$, the square of the Hilbert-Schmidt norm of A, we have the following Plancherel formula

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{+\infty} \left\|\hat{f}(\lambda)\right\|_{HS}^2 |\lambda|^n \, d\lambda\,, \qquad f \in L^1 \cap L^2(\mathbb{H}^n)\,.$$

By this we can extend the Fourier transform as an isometry from $L^2(\mathbb{H}^n)$ onto the Hilbert space of the operator-valued functions $\lambda \mapsto A(\lambda)$, $\lambda \in \mathbb{R}^*$, satisfying

i) $A(\lambda)$ is a Hilbert-Schmidt operator on $\mathcal{H}_\lambda$, for almost every $\lambda \in \mathbb{R}^*$,

ii) $(A(\lambda)P, Q)$ is a measurable function of λ, for every polynomial P, Q on $\mathbb{C}^n$,

iii) $\|A\|_2^2 = \dfrac{2^{n-1}}{\pi^{n+1}} \displaystyle\int_{-\infty}^{+\infty} \|A(\lambda)\|_{HS}^2 |\lambda|^n \, d\lambda < +\infty\,.$

Definition. *A left invariant multiplier of $L^p(\mathbb{H}^n)$, $1 \le p \le \infty$, is an operator-valued function $M : \lambda \mapsto M(\lambda)$, $\lambda \in \mathbb{R}^*$, such that*

a) for every $\lambda \in \mathbb{R}^*$, $M(\lambda)$ is a bounded operator on $\mathcal{H}_\lambda$,

b) the operator T_M defined by

$$(T_M f)\hat{}\,(\lambda) = M(\lambda)\,\hat{f}(\lambda)\,, \qquad f \in L^1 \cap L^p(\mathbb{H}^n)\,,$$

extends to a bounded operator on $L^p(\mathbb{H}^n)$.

From the Plancherel formula iii) above it follows immediately that M is a left $L^2(\mathbb{H}^n)$ multiplier if and only if $\sup_{\lambda \neq 0} \|M(\lambda)\| < +\infty$. We also remark that everything we say for left multipliers may be translated for right multipliers similarly defined, because the group $\mathbb{H}^n$ is unimodular.

On $\mathbb{R}^n$ we have $(\partial_j f)\hat{}\,(\xi) = i\,\xi_j \hat{f}(\xi)$. On $\mathbb{H}^n$ we have the following analogues: for $\lambda > 0$,

$$\begin{aligned}
&(Z_j f)\hat{}\,(\lambda)\, F(\zeta) \\
&= -2\,\lambda\, \hat{f}(\lambda)\, \zeta_j\, F(\zeta) \\
&= -\sqrt{2\lambda}\, \hat{f}(\lambda) \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\} F(\zeta)
\end{aligned}$$

and

$$
\begin{aligned}
&(\overline{Z}_j f)\hat{}(\lambda)\, F(\zeta) \\
&= \hat{f}(\lambda)\, \frac{\partial F}{\partial \zeta_j}(\zeta) \\
&= \sqrt{2\lambda}\, \hat{f}(\lambda) \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\} F(\zeta)\,,
\end{aligned}
$$

where I_k, $k = 1, 2, 3, \ldots$, is the infinite dimensional identity matrix; for $\lambda < 0$, we switch the formulas for Z_j and $\overline{Z}_j$. For all $\lambda \in \mathbb{R} \setminus \{0\}$, $(Tf)\hat{}(\lambda) = -i\lambda \hat{f}(\lambda)$. Thus, for $\lambda > 0$, we can consider the corresponding multiplier of the differential operators Z_j, $\overline{Z}_j$, T to be

$$
-\sqrt{2\lambda} \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\},
$$

$$
\sqrt{2\lambda} \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\},
$$

and

$$
-i\,\lambda \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes I_j \otimes I_{j+1} \otimes \cdots \right\},
$$

respectively. To prove these formulas we consider the case $n = 1$, $\lambda > 0$, and the formula for $\hat{Z}$ only; for all other cases the following proof can be easily carried over. By definition

$$
(Zf)\hat{}(\lambda) = (\partial_z f)\hat{}(\lambda) + i\, (\bar{z}\, \partial_t f)\hat{}(\lambda)
$$

and integration by parts yields

$$
(\bar{z}\, \partial_t f)\hat{}(\lambda)\, F(\zeta) = -i\,\lambda\, (\bar{z} f)\hat{}(\lambda)\, F(\zeta)\,.
$$

We have

$$(Zf)\widehat{\ }(\lambda) = (\partial_z f)\widehat{\ }(\lambda) + \lambda\,(\bar{z}f)\widehat{\ }(\lambda)\,.$$

Furthermore,

$$\begin{aligned}\hat{f}(\lambda)\,\big(\zeta\,F(\zeta)\big) &= \int f\,e^{i\lambda t+2\lambda(\zeta\cdot z-|z|^2/2)}(\zeta-\bar{z})\,F(\zeta-\bar{z})\,dV\\ &= \zeta\,\hat{f}(\lambda)\,F(\zeta) - (\bar{z}f)\widehat{\ }(\lambda)\,F(\zeta)\,,\end{aligned}$$

so

$$\begin{aligned}(\partial_z f)\widehat{\ }(\lambda)\,F(\zeta) &= -\int f\,\partial_z(\Pi_\lambda F)\,dV\\ &= -2\,\lambda\,\zeta\,\hat{f}(\lambda)\,F(\zeta) + \lambda\,(\bar{z}f)\widehat{\ }(\lambda)\,F(\zeta)\\ &= -2\,\lambda\,\hat{f}(\lambda)\,\big(\zeta\,F(\zeta)\big) - \lambda\,(\bar{z}f)\widehat{\ }(\lambda)\,F(\zeta)\,.\end{aligned}$$

Hence,

$$(Zf)\widehat{\ }(\lambda)\,F(\zeta) = -2\,\lambda\,\hat{f}(\lambda)\,\big(\zeta\,F(\zeta)\big)\,.$$

As for the matrix form, we recall that

$$F_{\alpha,\lambda}(\zeta) = \frac{(2\lambda)^{\alpha/2}}{\sqrt{\alpha!}}\,\zeta^\alpha$$

is an orthonormal basis for $\mathcal{H}_\lambda$, and write

$$F(\zeta) = \sum_{\alpha=0}^{\infty} a_\alpha\,F_{\alpha,\lambda}(\zeta) \equiv \begin{bmatrix} a_0\\ a_1\\ a_2\\ \vdots\end{bmatrix}\,, \qquad a_\alpha \in \mathbb{C}\,.$$

Then

$$\begin{aligned}\zeta\,F(\zeta) &= \sum_{\alpha=0}^{\infty} a_\alpha\,\zeta\,F_{\alpha,\lambda}(\zeta)\\ &= \frac{1}{\sqrt{2\lambda}}\sum_{\alpha=0}^{\infty}\sqrt{\alpha+1}\,a_\alpha\,F_{\alpha+1,\lambda}(\zeta)\\ &= \frac{1}{\sqrt{2\lambda}}\begin{bmatrix}0\\ a_0\\ \sqrt{2}\,a_1\\ \sqrt{3}\,a_2\\ \vdots\end{bmatrix}\end{aligned}$$

$$= \frac{1}{\sqrt{2\lambda}} \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix} F(\zeta)\,.$$

On $\mathbb{H}^n$ the sub-Laplacian is the differential operator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_jZ_j\right).$$

The above calculations can be used to compute $\hat{\mathcal{L}}_0$. We apply the matrix expressions of Z_j and $\overline{Z}_j$ to get

$$\begin{aligned}
\hat{\mathcal{L}}_0(\lambda) &= -\frac{1}{2}\sum_{j=1}^{n}\left\{\hat{\overline{Z}}_j(\lambda)\hat{Z}_j(\lambda) + \hat{Z}_j(\lambda)\hat{\overline{Z}}_j(\lambda)\right\} \\
&= \sum_{j=1}^{n}\Bigg\{I_1\otimes\cdots\otimes I_{j-1}\otimes \\
&\qquad |\lambda|\left(\begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & \ddots \end{bmatrix} + \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{bmatrix}\right)\otimes I_{j+1}\otimes\cdots\Bigg\} \\
&= \sum_{j=1}^{n}\left\{I_1\otimes\cdots\otimes I_{j-1}\otimes|\lambda|\begin{bmatrix} 1 & & & & \\ & 3 & & & \\ & & 5 & & \\ & & & 7 & \\ & & & & \ddots \end{bmatrix}\otimes I_{j+1}\otimes\cdots\right\} \\
&= \sum_{\alpha\in\overline{\mathbb{N}}^n}(2\,|\alpha| + n)\,|\lambda|\,W_\alpha^0(\lambda)\,.
\end{aligned}$$

We now introduce the partition of the identity $I = \sum_{k=-\infty}^{+\infty}\hat{\Pi}_{2^kR}$, $R > 0$, where $\hat{\Pi}_s$ is the spectral projection of $\mathcal{L}_0$ corresponding to the multiplier

$$\hat{\Pi}_s(\lambda) = \sum_{s<(2|\alpha|+n)\,|\lambda|\le\sqrt{2}s} W_\alpha^0(\lambda)\,.$$

Then the $L^p(\mathbb{H}^n)$ multiplier theorem can be stated in the following way.

Theorem 1 (Multiplier theorem for $L^p(\mathbb{H}^n)$). *Let M be an operator-valued function with each entry in $C^k(\mathbb{R}\setminus\{0\})$, $k \geq 4\,[(n+5)/4]$, where $[\,\cdot\,]$ denotes the greatest integer function. Also assume*

$$\sup_{\lambda\neq 0} \|M(\lambda)\| < +\infty\,, \tag{4}$$

$$\sup_{R>0}\ R^{\deg P-n-1}\int_{-\infty}^{+\infty} \big\|[\Delta_P M(\lambda)]\hat{\Pi}_R(\lambda)\big\|_{HS}^2\,|\lambda|^n\,d\lambda < +\infty\,, \tag{5}$$

for every monomial P with $\deg P \leq 4\,[(n+5)/4]$. Then M is a multiplier of $L^p(\mathbb{H}^n)$, $1<p<\infty$, and is of weak type $(1,1)$.

3. Proof of the $L^p(\mathbb{H}^n)$ multiplier theorem.

We follow [DMM] fairly closely. According to [CW1, Theorem 3.1], to establish the multiplier theorem it suffices to construct a well-behaved approximate identity $\{\phi_r : r>0\}$ satisfying

$$\int_{\mathbb{H}^n} |T_M\psi_r(x)|\left(1+\left(\frac{\rho(x)}{r}\right)^{\varepsilon}\right)dx \leq C\,, \qquad 0<r<+\infty\,, \tag{6}$$

for some $\varepsilon>0$ and $C>0$, where

$$\psi_r = \phi_r - \phi_{r/2} \qquad \text{and} \qquad \rho(x) = |x|^4 = t^2 + \Big(\sum_{j=1}^n |z_j|^2\Big)^2.$$

Note that [CW1, Theorem 3.1] adopts $|x|^{2n+2}$ rather than $|x|^4$. However, if we check their proof carefully, we find that the inequality (6) also implies the $L^p(\mathbb{H}^n)$ boundedness of T_M due to Lemma 3 and Lemma 4 below.

Since we assume $\sup_{\lambda\neq 0}\|M(\lambda)\| < +\infty$, by Plancherel formula, the homogeneity of $\phi_r(x) = r^{-(n+1)/2}\phi_1(r^{-1/4}x)$ (see Lemma 1 below), and changing variables, we have

$$\int_{\mathbb{H}^n} |T_M\psi_r(x)|^2\,dx \leq C\,r^{-(n+1)/2}\,, \qquad 0<r<+\infty\,. \tag{7}$$

If we can also obtain

$$\int_{\mathbb{H}^n} |T_M\psi_r(x)|^2\,\rho(x)^{2[(n+5)/4]}\,dx \leq C\,r^{2[(n+5)/4]-(n+1)/2}\,, \tag{8}$$

for $0 < r < +\infty$, then we claim both inequalities (7) and (8) imply (6), and hence the multiplier theorem for $L^p(\mathbb{H}^n)$ follows. To see this we choose $0 < \varepsilon < [(n+5)/4] - (n+1)/4$. Then by (8) and (1)

$$\begin{aligned}
&\int\limits_{\rho(x)>r} |T_M \psi_r(x)|\, \rho(x)^{\varepsilon}\, dx \\
&\le \Big(\int\limits_{\rho(x)>r} |T_M \psi_r(x)|^2\, \rho(x)^{2[(n+5)/4]}\, dx\Big)^{1/2} \Big(\int\limits_{\rho(x)>r} \rho(x)^{2\varepsilon - 2[(n+5)/4]}\, dx\Big)^{1/2} \\
&\le C\, r^{[(n+5)/4]-(n+1)/4}\, r^{\varepsilon - [(n+5)/4] + (n+1)/4} \\
&= C\, r^{\varepsilon}\,.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int\limits_{\rho(x)>r} |T_M \psi_r(x)|\, dx \\
&\le \Big(\int\limits_{\rho(x)>r} |T_M \psi_r(x)| \Big(\frac{\rho(x)}{r}\Big)^{\varepsilon} dx\Big)^{1/2} \Big(\int\limits_{\rho(x)>r} |T_M \psi_r(x)| \Big(\frac{r}{\rho(x)}\Big)^{\varepsilon} dx\Big)^{1/2} \\
&\le C \Big(\int\limits_{\rho(x)>r} |T_M \psi_r(x)|\, dx\Big)^{1/2}.
\end{aligned}$$

Combining this and the previous inequality, we obtain

$$\int\limits_{\rho(x)>r} |T_M \psi_r(x)| \Big(1 + \Big(\frac{\rho(x)}{r}\Big)^{\varepsilon}\Big)\, dx \le C\,. \tag{9}$$

On the other hand, from (7) we have

$$\int\limits_{\rho(x)\le r} |T_M \psi_r(x)|\, dx \le \Big(\int\limits_{\rho(x)\le r} dx\Big)^{1/2} \Big(\int\limits_{\rho(x)\le r} |T_M \psi_r(x)|^2\, dx\Big)^{1/2} \le C\,.$$

Thus

$$\int\limits_{\rho(x)\le r} |T_M \psi_r(x)| \Big(1 + \Big(\frac{\rho(x)}{r}\Big)^{\varepsilon}\Big)\, dx \le 2 \int\limits_{\rho(x)\le r} |T_M \psi_r(x)|\, dx \le C\,. \tag{10}$$

Combining (9) and (10) establishes the claim. Therefore, we only have to prove the inequality (8) to establish the multiplier theorem.

Compare with the construction in [DMM]. The construction of the approximate identity is contained in the following.

Lemma 1. *Let $\phi_1 \in \mathcal{S}(\mathbb{H}^n)$ be the poly-radial function with Fourier coefficients*

$$\mathcal{R}_{\phi_1}(\lambda, 0, \alpha) = \exp\{-(2\,|\alpha| + n)^4\,\lambda^4\}\,, \qquad \lambda \in \mathbb{R}^*, \quad \alpha \in \overline{\mathbb{N}}^n\,.$$

Define

$$\phi_r(t, z) = r^{-(n+1)/2}\,\phi_1\big(r^{-1/2}t, r^{-1/4}z\big)\,, \qquad r > 0\,.$$

Then $\mathcal{R}_{\phi_r}(\lambda, 0, \alpha) = \exp\{-r^2\,(2\,|\alpha| + n)^4\,\lambda^4\}$ and satisfies, for some $\eta > 0$,

i) $\displaystyle\int_{\mathbb{H}^n} |\phi_r(t, z)|\,\Big(1 + \frac{\rho(t, z)}{r}\Big)^{\eta}\,dV \le C\,,$

ii) $\displaystyle\int_{\mathbb{H}^n} \phi_r(t, z)\,dV = 1\,,$

iii) $\phi_r * \phi_s = \phi_s * \phi_r\,,$

iv) $\displaystyle\int_{\mathbb{H}^n} \big|\phi_r\big((t, z)(t_0, z_0)^{-1}\big) - \phi_r(t, z)\big|\,dV \le C\,\Big(\frac{\rho(t_0, z_0)}{r}\Big)^{\eta}\,,$

v) $\phi_r(t, z) = \phi_r(-t, -z)\,.$

Proof. The identity (3) gives

$$\mathcal{R}_{\phi_r}(\lambda, 0, \alpha) = \mathcal{R}_{\phi_1}(\sqrt{r}\lambda, 0, \alpha) = \exp\{-r^2(2\,|\alpha| + n)^4\,\lambda^4\}\,.$$

By the homogeneity of ϕ_r and a change of variables, we have

$$\int_{\mathbb{H}^n} |\phi_r(t, z)|\,\Big(1 + \frac{\rho(t, z)}{r}\Big)^{\eta}\,dV = \int_{\mathbb{H}^n} |\phi_1(t, z)|\,(1 + \rho(t, z))^{\eta}\,dV < +\infty$$

for all $\eta > 0$, since $\phi_1 \in \mathcal{S}(\mathbb{H}^n)$. This proves i). Since $l^0_\alpha(0) = 1$ and

$$\begin{aligned}\exp\{(-r^2(2\,|\alpha| + n)^4\,\lambda^4\} &= \mathcal{R}_{\phi_r}(\lambda, 0, \alpha)\\ &= \int_{\mathbb{H}^n} \phi_r(t, |z_1|, \dots, |z_n|)\,e^{i\lambda t}\\ &\qquad \cdot l^0_{\alpha_1}(2\,|\lambda|\,|z_1|^2) \cdots l^0_{\alpha_n}(2\,|\lambda|\,|z_n|^2)\,dV\,.\end{aligned}$$

Letting $\lambda \to 0$ and applying the Lebesgue dominated convergence theorem, we have ii). Properties iii) and v) follow from the facts that $\hat{\phi}_r(\lambda)$ is poly-diagonal and $\hat{\phi}_r(\lambda) = \hat{\phi}_r(-\lambda)$. To prove iv) it suffices to prove it for $r = 1$ by the homogeneity of ϕ_r and changing variables. Let $L \in \mathfrak{h}$ be the normalized generator of the one parameter subgroup through $(0, z_0)^{-1}$. Then the fundamental theorem of calculus gives

$$\begin{aligned}
\int_{\mathbb{H}^n} \big|\phi_1((t,z)(0,z_0)^{-1}) - \phi_1(t,z)\big|\, dV \qquad\qquad& \\
&\le \int_{\mathbb{H}^n}\int_0^{|z_0|} \big|L\phi_1((t,z)\,\exp(sL))\big|\, ds\, dV \\
&= |z_0|\,\|L\phi_1\|_1 \\
&= \rho(0,z_0)^{1/4}\,\|L\phi_1\|_1\,,
\end{aligned}$$

which proves iv) for $t_0 = 0$. For the general case, we write

$$(t_0, z_0) = \big(t_0, (z_0)_1, \ldots, (z_0)_n\big) \in \mathbb{H}^n$$

and let

$$\begin{aligned}
h_{0j} &= \big(0, 0, \ldots, 0, (z_0)_j, 0, \ldots, 0\big)\,, \\
h_{1j} &= \big(0, 0, \ldots, 0, \frac{i}{2\sqrt{n}}\sqrt{t_0}, 0, \ldots, 0\big)\,, \\
h_{2j} &= \big(0, 0, \ldots, 0, \frac{1}{2\sqrt{n}}\sqrt{t_0}, 0, \ldots, 0\big)\,,
\end{aligned}$$

where each h_{kj}, $(k = 0, 1, 2)$, $(j = 1, 2, \ldots, 5n)$, has its only non-zero entry in the z_j-component. By a straightforward calculation we have

$$\begin{aligned}
(t_0, z_0) &= \prod_{j=1}^{n}\big(\frac{t_0}{n}, 0, \ldots, 0, (z_0)_j, 0, \ldots, 0\big) \\
&= \begin{cases} \prod\limits_{j=1}^{n} h_{0j}h_{1j}h_{2j}h_{1j}^{-1}h_{2j}^{-1}\,, & \text{if} \quad t_0 \ge 0\,, \\ \prod\limits_{j=1}^{n} h_{0j}h_{2j}h_{1j}h_{2j}^{-1}h_{1j}^{-1}\,, & \text{if} \quad t_0 < 0\,. \end{cases}
\end{aligned}$$

Thus we can express $\phi_1\big((t,z)(t_0,z_0)^{-1}\big) - \phi_1(t,z)$ as a sum of $5n$ differences

$$\begin{aligned}\phi_1\big((t,z)(t_0,z_0)^{-1}\big) - \phi_1(t,z) &\equiv \phi_1(xx_1x_2\cdots x_{5n}) - \phi_1(x) \\ &= \phi_1(xx_1x_2\cdots x_{5n}) - \phi_1(xx_1x_2\cdots x_{5n-1}) \\ &\quad + \phi_1(xx_1x_2\cdots x_{5n-1}) - \phi_1(xx_1x_2\cdots x_{5n-2}) \\ &\quad + \phi_1(xx_1x_2\cdots x_{5n-2}) - \phi_1(xx_1x_2\cdots x_{5n-3}) \\ &\quad\ \vdots \\ &\quad + \phi_1(xx_1) - \phi_1(x)\end{aligned}$$

for which each x_j ($= h_{kj}$ or h_{kj}^{-1}, $k = 0,1,2$), $j = 1,2,\ldots,5n$, has t-component zero, and apply the result just established to complete the proof of iv).

Lemma 2. *For every homogeneous polynomial P in $\mathbb{H}^n$ with $1 \le \deg P \le 4\,[(n+5)/4]$, one has*

$$\begin{aligned}&\sup\Big\{\big|\mathcal{R}_{P\psi_r}(\lambda,m,\alpha)\big|^2 :\ m\in\mathbb{Z}^n,\ R<(2\,|\alpha|+n)\,|\lambda|\le\sqrt{2}\,R\Big\} \\ &\qquad \le C_{P}\, r^{(1-n)/2+2[(n+5)/4]}\, R^{1-n+4\,[(n+5)/4]-\deg P}\, f_{P}(rR^2)\,,\end{aligned}\tag{11}$$

for $0 < r < +\infty$, where $f_{P} \in L^1(\mathbb{R}_+)$. Moreover,

$$\begin{aligned}&\big|\mathcal{R}_{\psi_r}(\lambda,0,\alpha)\big| \\ &\qquad \le \begin{cases} C_0\big(r\,(2\,|\alpha|+n)^2\,\lambda^2\big)^2, & \text{for } r\,(2\,|\alpha|+n)^2\,\lambda^2 \le 1\,, \\ 1\,, & \text{for } r\,(2\,|\alpha|+n)^2\,\lambda^2 > 1\,. \end{cases}\end{aligned}\tag{12}$$

Because the proof of Lemma 2 is messy, it will be postponed to the appendix. That will enable the reader to follow the paper without getting lost. We now let $k = [(n+5)/4]$, $\rho(t,z)^k = (t^2+|z|^4)^{[(n+5)/4]}$. By the Plancherel formula, the inequality (8) is equivalent to the following inequality:

$$\int_{-\infty}^{+\infty}\sum_{m,\alpha}\big|\mathcal{R}_{\rho^k T_M\psi_r}(\lambda,m,\alpha)\big|^2\,|\lambda|^n\,d\lambda \le C\,r^{2k-(n+1)/2}\,,\tag{13}$$

for $0 < r < +\infty$. Assume we can express $\rho(t,z)^k$ as a linear combination of products of powers of z_j', $\bar z_j'$, $z_j - z_j'$, $\bar z_j - \bar z_j'$, $\delta(t',z')$, $\bar\delta(t',z')$,

$\delta((t,z)(t',z')^{-1})$, $\bar{\delta}((t,z)(t',z')^{-1})$, and $\rho(t',z')$, where $\delta(t,z) = t + i|z|^2$ and each term is homogeneous of degree $4k$. (We use $\tilde{\sum}$ to denote this linear combination.) Then we have a Leibniz formula for the operator Δ_{ρ^k}

$$
\begin{aligned}
\Delta_{\rho^k}\left[M(\lambda)\,\hat{\psi}_r(\lambda)\right] = \sum_{\text{finite}} C_j\left[\Delta_{P_j}M(\lambda)\right]\hat{\psi}_r(\lambda) \\
+ \sum_{\substack{\text{finite} \\ \deg R_j \geq 1}} C'_j\left[\Delta_{Q_j}M(\lambda)\right]\left[\Delta_{R_j}\hat{\psi}_r(\lambda)\right]
\end{aligned}
\tag{14}
$$

for some homogeneous polynomials P_j, Q_j, R_j with $\deg P_j = \deg Q_j + \deg R_j = 4k$, and constants C_j, C'_j. To see this we check some of terms in $\tilde{\sum}$, for instance $[\rho(t',z')]^k$ and $\bar{z}'_j(z_j - z'_j)^{4k-3}\delta((t,z)(t',z')^{-1})$. Write $M(\lambda) = \hat{f}(\lambda)$ for some f and $\psi_r = g$. Then

$$\Delta_{\rho^k}[M(\lambda)\,\hat{\psi}_r(\lambda)] = \Delta_{\rho^k}[(f*g)\hat{}\,(\lambda)] = [\rho^k(f*g)]\hat{}\,(\lambda)$$

and

$$
\begin{aligned}
&\rho^k(f*g)(t,z) \\
&\quad = \int_{\mathbb{H}^n} \rho(t,z)^k\, f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z') \\
&\quad = \int_{\mathbb{H}^n} \left\{C_1\,\rho(t',z')^k + C_2\,\bar{z}'_j(z_j - z'_j)^{4k-3}\delta((t,z)(t',z')^{-1}) + \cdots\right\} \\
&\qquad\qquad \cdot f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z') \\
&\quad \equiv C_1\,A_1(t,z) + C_2\,A_2(t,z) + \cdots,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(t,z) &= \int_{\mathbb{H}^n} \rho(t',z')^k\, f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z') \\
&= (f * \rho^k g)(t,z)\,.
\end{aligned}
$$

Thus

$$\hat{A}_1(\lambda) = \hat{f}(\lambda)\,(\rho^k g)\hat{}\,(\lambda) = M(\lambda)[\Delta_{\rho^k}\hat{\psi}_r(\lambda)]\,.$$

Also

$$
\begin{aligned}
&A_2(t,z) \\
&= \int_{\mathbb{H}^n} \bar{z}'_j(z_j - z'_j)^{4k-3}\delta((t,z)(t',z')^{-1})\, f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z') \\
&= (Qf * Rg)(t,z)\,,
\end{aligned}
$$

where $Q(t,z)=(z_j)^{4k-3}\delta(t,z)$ and $R(t,z)=\bar{z}_j$. Thus,

$$\hat{A}_2(\lambda)=(Qf)\hat{}(\lambda)\,(Rg)\hat{}(\lambda)=[\Delta_{Q_j}M(\lambda)]\,[\Delta_{R_j}\hat{\psi}_r(\lambda)]\,,$$

and the same process can be carried over to the other terms of $\tilde{\sum}$. We now show that $\rho(t,z)^k$ can be expressed as the linear combination $\tilde{\sum}$. We note that

$$\begin{aligned}\rho(t,z)-\rho(t',z')=(t-t')^2&+2\,t'(t-t')\\&+\Bigl(\sum|z_j|^2-\sum|z_j'|^2\Bigr)^2\\&+2\Bigl(\sum|z_j'|^2\Bigr)\Bigl(\sum|z_j|^2-\sum|z_j'|^2\Bigr).\end{aligned}$$

Since

$$\begin{aligned}\sum|z_j|^2&=\sum|z_j'|^2\\&\quad+\sum\bigl(|z_j-z_j'|^2-2|z_j'|^2+(z_j-z_j')\bar{z}_j'+(\bar{z}_j-\bar{z}_j')z_j'+2\,|z_j'|^2\bigr)\end{aligned}$$

is a linear combination of products of powers of z_j', $\bar{z}_j'$, z_j-z_j', $\bar{z}_j-\bar{z}_j'$. Thus $\rho(t,z)$ is a linear combination of products of powers of t', $t-t'$, z_j', $\bar{z}_j'$, z_j-z_j', $\bar{z}_j-\bar{z}_j'$, and $\rho(t',z')$ with homogeneous degree 4 in each term. Also

$$t'=\frac{1}{2}\bigl(\delta(t',z')+\bar{\delta}(t',z')\bigr)\,,$$

$$\begin{aligned}t-t'=\frac{1}{2}&\bigl(\delta((t,z)(t',z')^{-1})+\bar{\delta}((t,z)(t',z')^{-1})\bigr)\\&-i\sum_{j=1}^{n}\bigl((z_j-z_j')\bar{z}_j'-(\bar{z}_j-\bar{z}_j')\bar{z}_j'\bigr)\,.\end{aligned}$$

This gives $\rho(t,z)^k=\tilde{\sum}$ as a linear combination of products of powers of z_j', $\bar{z}_j'$, z_j-z_j', $\bar{z}_j-\bar{z}_j'$, $\delta(t',z')$, $\bar{\delta}(t',z')$, $\delta((t,z)(t',z')^{-1})$, $\bar{\delta}((t,z)(t',z')^{-1})$ and $\rho(t',z')$, in which each term is homogeneous of degree $4k$.

By the Leibniz formula (14) we write

$$\int_{-\infty}^{+\infty}\sum_{m,\alpha}\bigl|\mathcal{R}_{\rho^k T_M\psi_r}(\lambda,m,\alpha)\bigr|^2\,|\lambda|^n\,d\lambda$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \left\| \Delta_{\rho^k} [M(\lambda)\hat{\psi}_r(\lambda)] \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&\le C \Bigg(\sum_{\text{finite}} \int_{-\infty}^{+\infty} \left\| [\Delta_{P_j} M(\lambda)] \hat{\psi}_r(\lambda) \right\|_{HS}^2 |\lambda|^n \, d\lambda \qquad (15) \\
&\qquad + \sum_{\substack{\text{finite} \\ \deg R_j \ge 1}} \int_{-\infty}^{+\infty} \left\| [\Delta_{Q_j} M(\lambda)] \, [\Delta_{R_j} \hat{\psi}_r(\lambda)] \right\|_{HS}^2 |\lambda|^n \, d\lambda \Bigg) \\
&\equiv C \Big(\sum_{\text{finite}} I_j + \sum_{\substack{\text{finite} \\ \deg R_j \ge 1}} J_j \Big) .
\end{aligned}
$$

Recall that $\{W_\alpha^m(\lambda) : m \in \mathbb{Z}^n, \ \alpha \in \overline{\mathbb{N}}^n\}$ is an orthonormal basis for the Hilbert-Schmidt operators on $\mathcal{H}_{|\lambda|}$, and

$$
\hat{\Pi}_s(\lambda) = \sum_{s < (2|\alpha|+n)|\lambda| \le \sqrt{2} s} W_\alpha^0(\lambda) .
$$

If P is a homogeneous polynomial with degree $4k$, then

$$
\begin{aligned}
\mathrm{I} &\equiv \int_{-\infty}^{+\infty} \left\| [\Delta_P M(\lambda)] \hat{\psi}_r(\lambda) \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&= \int_{-\infty}^{+\infty} \left\| [\Delta_P M(\lambda)] \Big[\sum_{j \in \mathbb{Z}} \mathcal{R}_{\psi_r}(\lambda, 0, \alpha) \, \hat{\Pi}_{2^{j/2} r^{-1/2}}(\lambda) \Big] \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&= \sum_{j \in \mathbb{Z}} \int_{-\infty}^{+\infty} \left\| [\Delta_P M(\lambda)] \, \hat{\Pi}_{2^{j/2} r^{-1/2}}(\lambda) \right\|_{HS}^2 \left| \mathcal{R}_{\psi_r}(\lambda, 0, \alpha) \right|^2 |\lambda|^n \, d\lambda \\
&\equiv \sum_{j<0} + \sum_{j \ge 0} \\
&\equiv \mathrm{I}' + \mathrm{I}'' ,
\end{aligned}
$$

where the coefficients $\mathcal{R}_{\psi_r}(\lambda, 0, \alpha)$ satisfy

$$
2^{j/2} \, r^{-1/2} < (2\,|\alpha| + n)\,|\lambda| \le \sqrt{2} \; 2^{j/2} \, r^{-1/2} .
$$

For $j < 0$, we have $r^{1/2}(2\,|\alpha| + n)\,|\lambda| \le \sqrt{2}\; 2^{j/2} \le 1$, so

$$
\left| \mathcal{R}_{\psi_r}(\lambda, 0, \alpha) \right| \le C \left(r \, (2|\,\alpha| + n)^2 \, \lambda^2 \right)^2 \le C \, 2^{2j+2}
$$

by (12). For $j \geq 0$, we get $r^{1/2}(2\,|\alpha| + n)\,|\lambda| > 2^{j/2} \geq 1$, and hence from (12)

$$\left|\mathcal{R}_{\psi_r}(\lambda, 0, \alpha)\right| \leq 1\,.$$

The basic assumption (5) on the multiplier now implies

$$\begin{aligned}
\mathrm{I}' &\leq C \sum_{j<0} 2^{4j+4} \int_{-\infty}^{+\infty} \left\|[\Delta_P M(\lambda)]\,\hat{\Pi}_{2^{j/2}r^{-1/2}}(\lambda)\right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&\leq C \sum_{j<0} 2^{4j+4} \left(2^{j/2}\, r^{-1/2}\right)^{1+n-4k} \\
&= C\, r^{2k-(n+1)/2}\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}'' &\leq \sum_{j\geq 0} \int_{-\infty}^{+\infty} \left\|[\Delta_P M(\lambda)]\,\hat{\Pi}_{2^{j/2}r^{-1/2}}(\lambda)\right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&\leq C \sum_{j\geq 0} \left(2^{j/2}\, r^{-1/2}\right)^{1+n-4k} \\
&= C\, r^{2k-(n+1)/2}\,.
\end{aligned}$$

For n fixed, there are at most a finite number (depending only on n) of terms of the form I. This proves (13) for the first sum of (15). Next consider two homogeneous polynomials Q, R with $\deg Q + \deg R = 4k$, $\deg R \geq 1$, and

$$\begin{aligned}
\mathrm{J} &\equiv \int_{-\infty}^{+\infty} \left\|[\Delta_Q M(\lambda)]\,[\Delta_R \hat{\psi}_r(\lambda)]\right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&= \int_{-\infty}^{+\infty} \sum_{\alpha\in\overline{\mathbb{N}}^n} \left\|[\Delta_Q M(\lambda)]\, W_\alpha^{m_R}(\lambda)\right\|_{HS}^2 \left|\mathcal{R}_{R\psi_r}(\lambda, m_R, \alpha)\right|^2 |\lambda|^n \, d\lambda
\end{aligned}$$

since $\hat{\psi}_r(\lambda)$ is a poly-diagonal matrix and each of $\{\Delta_t, \Delta_{z_j}, \Delta_{\bar{z}_j}\}_{j=1}^n$ maps a poly-diagonal matrix into a pseudo-poly-diagonal matrix (*i.e.* the one in which each factor has its only non-zero entries on one sub or super diagonal), $\Delta_R\hat{\psi}_r(\lambda)$ is pseudo-poly-diagonal and hence

$$\Delta_R \mathcal{R}_{\psi_r}(\lambda, m, \alpha) = 0 \qquad \text{except for some } m_R \in \mathbb{Z}^n\,.$$

Using (11), (5), and the orthonormality of $\{W_\alpha^m(\lambda)\}$, we have

$$\mathrm{J} = \sum_{j\in\mathbb{Z}} \int_{-\infty}^{+\infty} \left\|[\Delta_Q M(\lambda)]\,\hat{\Pi}_{2^{j/2}}(\lambda)\right\|_{HS}^2 \left|\mathcal{R}_{R\psi_r}(\lambda, m_R, \alpha)\right|^2 |\lambda|^n \, d\lambda$$

$$
\begin{aligned}
&\le \sum_{j\in\mathbb{Z}} C_R\, r^{(1-n)/2+2k} \left(2^{j/2}\right)^{1-n+4k-\deg R} f_R(2^j r) \\
&\qquad \cdot \int_{-\infty}^{+\infty} \left\| [\Delta_Q M(\lambda)]\, \hat{\Pi}_{2^{j/2}}(\lambda) \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&\le \sum_{j\in\mathbb{Z}} C\, r^{(1-n)/2+2k} \left(2^{j/2}\right)^{1-n+4k-\deg R} f_R(2^j r) \left(2^{j/2}\right)^{1+n-\deg Q} \\
&= C\, r^{(1-n)/2+2k-1} \sum_{j\in\mathbb{Z}} 2^j\, r\, f_R(2^j\, r) \\
&\approx C\, r^{(1-n)/2+2k-1} \|f_R\|_1 \\
&= C\, r^{2k-(n+1)/2} \,.
\end{aligned}
$$

There are only finitely many terms of the form J, so the inequality (13) for the second sum in (15) is proved. This establishes the multiplier theorem for $L^p(\mathbb{H}^n)$.

4. H^1-L^1 estimate.

In Theorem 1, the multiplier theorem is valid for L^p, $p > 1$. For $p = 1$ we only have a weak-type estimate, so in this section we are trying to extend to another sense of strong type $\|T_M f\|_{L^1} \le C\|f\|_{H^1}$. Here H^1 is the Hardy space on $\mathbb{H}^n$ defined either in terms of maximal functions or in terms of an atomic decomposition [FS]. When $p > 1$, L^p and H^p are essentially the same.

REMARK. We have in fact proved that $\|T_M f\|_{H^p} \le C\|f\|_{H^p}$ for $0 < p \le 1$. The proof is more complicated than the one here and requires the theory of molecules (*cf.* [TW]). The details of this proof will appear elsewhere.

Specifically, we define a $(1,2,0)$-*atom* as an L^2-function f having support in a ball $B_R = \{x \in \mathbb{H}^n \,:\, |x| \le R\}$ and satisfying

$$
\|f\|_2 \le \left|B_R\right|^{-1/2} \qquad \text{and} \qquad \int_{\mathbb{H}^n} f(x)\, dx = 0\,.
$$

It is obvious that $\|f\|_1 \le 1$ for any $(1,2,0)$-atom f.

Theorem (Atomic decomposition of H^1) [FS, Chapter 3]. *Any f in H^1 can be represented as a linear combination of $(1,2,0)$-atoms $f=\sum_{i=1}^{\infty}\lambda_i f_i$, $\lambda_i\in\mathbb{C}$, where the f_i are $(1,2,0)$-atoms and the sum converges in H^1. Moreover,*

$$\|f\|_{H^1}\approx\inf\Big\{\sum_{i=1}^{\infty}|\lambda_i| : \sum_{i=1}^{\infty}\lambda_i f_i \textit{ is a decomposition of } f \textit{ into } (1,2,0)\textit{-atoms}\Big\}.$$

Let $\{\phi_r : r>0\}$ be the approximate identity in Section 3. It is easy to check that $\{\phi_r*\phi_r : r>0\}$ is also an approximate identity and satisfies the same properties i)-v) of Lemma 1. Therefore,

$$\phi_{2^{-i}}*\phi_{2^{-i}}*f\to f \qquad \text{in } L^p$$

and

$$\begin{aligned} f&=\lim_{m\to\infty}\sum_{i=0}^{m}\big(\phi_{2^{-i-1}}*\phi_{2^{-i-1}}-\phi_{2^{-i}}*\phi_{2^{-i}}\big)*f+\phi_1*\phi_1*f\\ &=\lim_{m\to\infty}\sum_{i=0}^{m}-\psi_{2^{-i}}*\big(\phi_{2^{-i}}+\phi_{2^{-i-1}}\big)*f+\phi_1*\phi_1*f \qquad \text{in } L^p. \end{aligned}$$

Since we only concern the tail terms in the approach $\phi_{2^{-i}}*\phi_{2^{-i}}*f\to f$, we may assume $\phi_1\equiv 1$. Thus if $\int f(x)\,dx=0$, $\phi_1*\phi_1*f=0$.

In the proof of Theorem 1, we have shown

$$\int_{\mathbb{H}^n}|T_{\scriptscriptstyle M}\psi_r(x)|\Big(1+\Big(\frac{\rho(x)}{r}\Big)^{\varepsilon}\Big)\,dx\le C, \qquad \text{for all } r>0. \tag{6}$$

Let η and ε be the constants in Lemma 1 and (6), respectively. Setting $\widetilde{\phi}_r=\phi_r+\phi_{r/2}$, $a_r=T_{\scriptscriptstyle M}\psi_r*\widetilde{\phi}_r$, $K_m=-\sum_{i=0}^{m}a_{2^{-i}}$, and $\delta=\min\{\eta,\varepsilon\}$, we now have the following two lemmas.

Lemma 3.

a) $\displaystyle\int_{\mathbb{H}^n}|a_r(x)|\,\rho(x)^{\delta}\,dx\le C\,r^{\delta}$,

b) $\displaystyle\int_{\mathbb{H}^n}|a_r(xy^{-1})-a_r(x)|\,dx\le C\Big(\frac{\rho(y)}{r}\Big)^{\delta}$.

PROOF. Lemma 1.i) and inequality (6) give the uniform boundedness of $\|\widetilde{\phi}_r\|_1$ and $\|T_M \psi_r\|_1$. Applying the triangle inequality, we have

$$\rho(x)^\delta \leq C_\delta \left(\rho(y)^\delta + \rho(y^{-1}x)^\delta\right)$$

and then

$$\begin{aligned}\int_{\mathbb{H}^n} |a_r(x)|\, \rho(x)^\delta\, dx &\leq \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} |T_M \psi_r(y)|\, |\widetilde{\phi}_r(y^{-1}x)|\, \rho(x)^\delta\, dxdy \\ &\leq C_\delta \left\|\widetilde{\phi}_r\right\|_1 \int_{\mathbb{H}^n} |T_M \psi_r(y)|\, \rho(y)^\delta\, dy \\ &\qquad + C_\delta \|T_M \psi_r\|_1 \int_{\mathbb{H}^n} \left|\widetilde{\phi}_r(x)\right| \rho(x)^\delta\, dx \\ &\leq C\, r^\delta .\end{aligned}$$

The last inequality is given by Lemma 1.i) and (6) again. The inequality b) is an easy consequence of Lemma 1.iv).

Lemma 4. *Suppose M satisfies the same hypotheses as Theorem 1. Then there exist constants C_1 and C_2, independent of m and y, such that*

$$\int_{\rho(x)>C_1\rho(y)} |K_m(xy^{-1}) - K_m(x)|\, dx \leq C_2 ,$$

for all $y \in \mathbb{H}^n$ and for all $m \geq 0$.

PROOF. For $i \in \mathbb{Z}^+$, Lemma 3.a) shows

$$\int_{\rho(x)>\lambda} \left|a_{2^{-i}}(x)\right| dx \leq \frac{1}{\lambda^\delta} \int_{\rho(x)>\lambda} \left|a_{2^{-i}}(x)\right| \rho(x)^\delta\, dx \leq \frac{C}{(2^i\lambda)^\delta} .$$

Therefore, choosing $C_1 > 16$, we have

$$\begin{aligned}&\int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1}) - a_{2^{-i}}(x)\right| dx \\ &\qquad \leq \int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1})\right| dx + \int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx\end{aligned}$$

$$\leq \int\limits_{\rho(xy)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx + \int\limits_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx$$
$$\leq \int\limits_{\rho(x)>C_3\rho(y)} \left|a_{2^{-i}}(x)\right| dx + \int\limits_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx$$
$$\leq \frac{C}{(2^i\,\rho(y))^\delta}$$

since $16\,(\rho(x)+\rho(y)) > \rho(xy) > C_1\,\rho(y)$ implies

$$\rho(x) > \frac{C_1-16}{16}\,\rho(y) \equiv C_3\,\rho(y)\,.$$

The above inequality and Lemma 3.b) get

$$\int\limits_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1}) - a_{2^{-i}}(x)\right| dx \leq C\,\min\left\{\left(2^i\,\rho(y)\right)^\delta, \frac{1}{(2^i\,\rho(y))^\delta}\right\}.$$

Taking the summation of these inequalities, we obtain

$$\int\limits_{\rho(x)>C_1\rho(y)} |K_m(xy^{-1}) - K_m(x)|\,dx$$
$$\leq \sum_{i=0}^{m} \int\limits_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1}) - a_{2^{-i}}(x)\right| dx$$
$$\leq C \sum_{i<-\log_2\rho(y)} \left(2^i\,\rho(y)\right)^\delta + C \sum_{i\geq -\log_2\rho(y)} \frac{1}{(2^i\,\rho(y))^\delta}$$
$$\leq \frac{C}{2^\delta-1} + \frac{C}{1-2^{-\delta}}\,.$$

The proof is thus complete.

Now we are ready to prove the H^1-L^1 estimate of $T_{_M}$.

Theorem 2. *Suppose M satisfies the same hypotheses as Theorem 1. Then $T_{_M}$ maps $H^1(\mathbb{H}^n)$ boundedly into $L^1(\mathbb{H}^n)$. Moreover, there exists a constant $C>0$, independent of f, such that $\|T_{_M} f\|_{L^1} \leq C\,\|f\|_{H^1}$ for all $f \in H^1(\mathbb{H}^n)$.*

PROOF. By the atomic decomposition of H^1, it suffices to show

$$\|T_{_M} f\|_{L^1} \leq C\,, \qquad \text{for any } (1,2,0)\text{-atom } f\,.$$

Given a $(1,2,0)$-atom f with $\operatorname{supp} f \subseteq \{x \in \mathbb{H}^n : |x| \leq R\}$, then $\|f\|_2 \leq C\,R^{-n-1}$ and $\int f(x)\,dx = 0$. The L^2-boundedness of $T_{_M}$ implies

$$T_{_M} f = \lim_{m\to\infty} \sum_{i=0}^{m} -T_{_M} \psi_{2^{-i}} * \widetilde{\phi}_{2^{-i}} * f = \lim_{m\to\infty} K_m * f \qquad \text{in } L^2\,.$$

Thus there exists a subsequence $\{m_j\}$, such that

$$T_{_M} f = \lim_{j\to\infty} K_{m_j} * f \qquad \text{almost everywhere.}$$

Let C_1, C_2 be the constants in Lemma 4. Then

$$\begin{aligned}
&\int\limits_{|x|>C_1^{1/4}R} |T_{_M} f(x)|\,dx \\
&\qquad \leq \liminf_{j\to\infty} \int\limits_{|x|>C_1^{1/4}R} |K_{m_j} * f(x)|\,dx \\
&\qquad \leq \liminf_{j\to\infty} \int\limits_{|x|>C_1^{1/4}R} \left| \int\limits_{|y|\leq R} \big(K_{m_j}(xy^{-1}) - K_{m_j}(x)\big)\, f(y)\,dy \right| dx \\
&\qquad \leq \liminf_{j\to\infty} \int\limits_{|y|\leq R} |f(y)|\,dy \int\limits_{|x|>C_1^{1/4}|y|} \big|K_{m_j}(xy^{-1}) - K_{m_j}(x)\big|\,dx \\
&\qquad \leq C_2\,\|f\|_{L^1} \\
&\qquad \leq C_2\,.
\end{aligned}$$

On the other hand the Schwartz inequality gives

$$\int\limits_{|x|\leq C_1^{1/4}R} |T_{_M} f(x)|\,dx \leq C\,R^{n+1}\,\|T_{_M} f\|_2 \leq C\,R^{n+1}\,\|f\|_2 \leq C\,.$$

This completes the proof.

Appendix. The proof of Lemma 2.

The purpose of this appendix is to prove the lemma that occurs in Section 3.

Lemma 2. *For every homogeneous polynomial P in $\mathbb{H}^n$ with $1 \leq \deg P \leq 4\,[(n+5)/4]$, one has*

$$\begin{aligned}\sup\Big\{\big|\mathcal{R}_{P\psi_r}(\lambda,m,\alpha)\big|^2 :\ m\in\mathbb{Z}^n,\ R<(2\,|\alpha|+n)\,|\lambda|\leq\sqrt{2}\,R\Big\}\\ \leq C_{P}\, r^{(1-n)/2+2[(n+5)/4]}\,R^{1-n+4\,[(n+5)/4]-\deg P}\, f_{P}(rR^2)\,,\end{aligned}\tag{11}$$

for $0<r<+\infty$, where $f_P\in L^1(\mathbb{R}_+)$. Moreover,

$$\begin{aligned}&\big|\mathcal{R}_{\psi_r}(\lambda,0,\alpha)\big|\\ &\qquad\leq\begin{cases} C_0\big(r\,(2\,|\alpha|+n)^2\,\lambda^2\big)^2, & \text{for}\ \ r\,(2\,|\alpha|+n)^2\,\lambda^2\leq 1\,,\\ 1\,, & \text{for}\ \ r\,(2\,|\alpha|+n)^2\,\lambda^2> 1\,.\end{cases}\end{aligned}\tag{12}$$

PROOF. The mean value theorem gives $|e^{-x}-e^{-x/4}|\leq C_0\,x$ for $0\leq x\leq 1$. Then,

$$\begin{aligned}\big|\mathcal{R}_{\psi_r}(\lambda,0,\alpha)\big| &= \big|\mathcal{R}_{\phi_r}(\lambda,0,\alpha)-\mathcal{R}_{\phi_{r/2}}(\lambda,0,\alpha)\big|\\ &= \big|e^{-r^2(2|\alpha|+n)^4\lambda^4}-e^{-r^2(2|\alpha|+n)^4\lambda^4/4}\big|\\ &\leq C_0\,r^2\,(2\,|\alpha|+n)^4\,\lambda^4\end{aligned}$$

for $0\leq r\,(2\,|\alpha|+n)^2\,\lambda^2\leq 1$. For the second estimate of (12), we note that $|e^{-x}-e^{-x/4}|\leq 1$ for $x>1$. Thus (12) is proved. As for (11) we let $\sigma\equiv(2\,|\alpha|+n)\,|\lambda|$ and claim

$$\begin{aligned}\big|\mathcal{R}_{P\phi_r}(\lambda,m,\alpha)\big| \leq C_{P}\,e^{-r^2\sigma^4}\,\sigma^{-\deg P/2}\\ \cdot\Big(\sum_{k=1}^{\deg P}(r\sigma^2)^{2k}+(r\sigma^2)^{2\deg P}\,e^{80\,r^2\sigma^4/81}\Big)\,,\end{aligned}\tag{16}$$

for $R<(2\,|\alpha|+n)\,|\lambda|\leq\sqrt{2}\,R$. Assuming the claim for a moment, we have

$$\begin{aligned}&\big|\mathcal{R}_{P\phi_r}(\lambda,m,\alpha)\big|\\ &\qquad\leq C_{P}\,(r\sigma^2)^{(1-n)/4+[(n+5)/4]}\,\sigma^{-\deg P/2}\,(r\sigma^2)^{2-(1-n)/4-[(n+5)/4]}\end{aligned}$$

$$\cdot\Big(e^{-r^2\sigma^4}\sum_{k=0}^{\deg P-1}(r\sigma^2)^{2k}+(r\sigma^2)^{2\deg P-2}\,e^{-r^2\sigma^4/81}\Big)$$
$$= C_{P}\,(r\sigma^2)^{(1-n)/4+[(n+5)/4]}\,\sigma^{-\deg P/2}\,g_{P}(r\sigma^2)\,,$$

where

$$g_{P}(x) = x^{2-(1-n)/4-[(n+5)/4]}\Big(e^{-x^2}\sum_{k=0}^{\deg P-1}x^{2k}+e^{-x^2/81}\,x^{2\deg P-2}\Big)\,.$$

We note that $\sigma \approx R$ and the exponent $2-(1-n)/4-[(n+5)/4] \geq 1/2$ for $n \in \mathbb{N}$. Hence, we set $f_{P} \equiv g_{P}^2 \in L^1(\mathbb{R}_+)$.

To prove the claim, we need the following well-known summation formula

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\,i^k = 0\,, \qquad \text{for}\ \ 0 \leq k \leq m-1\,,\ m \in \mathbb{N}. \tag{17}$$

The idea of the proof of (16) is quite simple, but calculation is messy. We use binomial expansion and apply (17) again and again to establish the inequality (16). We show detailedly the inequality only for $P(t,z) = z_1^a\,\bar{z}_1^b$ with $a \geq b$ and $P(t,z) = z_1^a\,\bar{z}_2^b$; the proof can be carried over to the other cases with minor modifications. It follows from (2) and the recurrence relations and differential properties of $\{l_\alpha^m\}$ that, for $a \geq b$,

$$\begin{aligned}
&\mathcal{R}_{z_1^a\,\bar{z}_1^b\phi_r}(\lambda,(a-b)e_1,\alpha)\\
&\quad= (2|\lambda|)^{-(a+b)/2}\sqrt{\frac{(\alpha_1+a-b)!}{\alpha_1!}}\\
&\qquad\cdot\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\,\mathcal{R}_{\phi_r}(\lambda,0,\alpha+(i-j)e_1)\,,
\end{aligned}$$

and $\mathcal{R}_{z_1^a\,\bar{z}_1^b\phi_r}(\lambda,m,\alpha) = 0$ for $m \neq (a-b)e_1$. Then

$$\begin{aligned}
&\big|\mathcal{R}_{z_1^a\,\bar{z}_1^b\phi_r}(\lambda,(a-b)e_1,\alpha)\big|\\
&\quad\leq C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\\
&\qquad\cdot\Big|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\,e^{-r^2(\sigma+2(i-j)|\lambda|)^4}\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\left|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\right| \\
&\quad+C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4} \\
&\qquad\cdot\left|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=1}^{a+b-1}\frac{A_{i,j}^k}{k!}\right| \qquad (18)\\
&\quad+C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4} \\
&\qquad\cdot\left|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=a+b}^{\infty}\frac{A_{i,j}^k}{k!}\right| \\
&\equiv S_1+S_2+S_3\;,
\end{aligned}
$$

where

$$
\begin{aligned}
A_{i,j} &= -\,r^2\big(\{\sigma+2\,(i-j)\,|\lambda|\}^4-\sigma^4\big) \\
&= -\,8\,r^2\,|\lambda|\,(i-j) \\
&\qquad\cdot\big(\sigma^3+3\,\sigma^2\,|\lambda|\,(i-j)+4\,\sigma\,|\lambda|^2\,(i-j)^2+2\,|\lambda|^3\,(i-j)^3\big)\,.
\end{aligned}
$$

We immediately obtain $S_1=0$ since the summation $\sum_{j=0}^{b}(-1)^j\binom{b}{j}=0$. To estimate S_3, we have

$$
\begin{aligned}
S_3 &\leq C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4} \\
&\qquad\cdot\left|\sum_{\substack{0\leq i\leq a\\ 0\leq j\leq b\\ i\geq j}}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=a+b}^{\infty}\frac{A_{i,j}^k}{k!}\right| \\
&\quad+C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4} \\
&\qquad\cdot\left|\sum_{\substack{0\leq i\leq a\\ 0\leq j\leq b\\ i< j}}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=a+b}^{\infty}\frac{A_{i,j}^k}{k!}\right| \\
&\leq C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\sum_{\substack{0\leq i\leq a\\ 0\leq j\leq b\\ i\geq j}}\binom{a}{i}\binom{b}{j}\frac{\sigma^b}{|\lambda|^b}\,(r^2\,|\lambda|\,\sigma^3)^{a+b} \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\; e^{-r^2\sigma^4} \\
&\quad \cdot \sum_{\substack{0\le i\le a\\ 0\le j\le b\\ i<j}} \binom{a}{i}\binom{b}{j}\frac{\sigma^b}{|\lambda|^b}\,(r^2|\,\lambda|\,\sigma^3)^{a+b}\, e^{80\,r^2\sigma^4/81} \\
&\le C\,\sigma^{-(a+b)/2}\, e^{-r^2\sigma^4}\,(r^2\,\sigma^4)^{a+b} \\
&\quad + C\,\sigma^{-(a+b)/2}\, e^{-r^2\sigma^4}\,(r^2\,\sigma^4)^{a+b}\, e^{80\,r^2\sigma^4/81}
\end{aligned}
$$

since we apply the property of alternating series and the following estimate to the last two inequalities

$$
\begin{aligned}
\sum_{k=a+b}^{\infty}\frac{A_{i,j}^k}{k!} &= C\,A_{i,j}^{a+b}\sum_{k=a+b}^{\infty}\frac{A_{i,j}^{k-(a+b)}}{k!} \\
&\le C\,A_{i,j}^{a+b}\,e^{A_{i,j}} \\
&\le C\,(r^2\,|\lambda|\,\sigma^3)^{a+b}\, e^{r^2(\sigma^4-\{\sigma-|2\lambda(i-j)|\}^4)} \\
&\le C\,(r^2\,|\lambda|\,\sigma^3)^{a+b}\, e^{80\,r^2\sigma^4/81},
\end{aligned}
$$

for $i<j$ and $2\,|\alpha|+n\neq|2\,(i-j)|$.

For $i<j$ and $2\,|\alpha|+n=|2\,(i-j)|$, the term $e^{-r^2(\sigma+2(i-j)|\lambda|)^4}$ in (18) disappears. Thus

$$
\begin{aligned}
\left|\mathcal{R}_{z_1^a\bar z_1^b\phi_r}(\lambda,(a-b)e_1,\alpha)\right| &\le C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a} \\
&\quad\cdot\left|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\right| \\
&= C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a} \\
&\quad\cdot\left|\sum_{i=0}^{a}(-1)^i\binom{a}{i}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{j=0}^{b}(-1)^j\binom{b}{j}\right| \\
&= 0\,.
\end{aligned}
$$

Hence S_3 fits the format of (16). As to S_2, we write

$$
\begin{aligned}
S_2 = C\,&\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2\sigma^4} \\
&\cdot \Big|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\Big(\sum_{1\le u+v\le b} c_{u,v}\,\alpha_1^u\, i^v\Big) \\
&\quad\cdot \sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\frac{\big(-8\,r^2|\lambda|\,(i-j)\big)^k c_{k,m}\,\sigma^{3k-m}|\lambda|^m(i-j)^m}{k!}\Big| \\
= C\,&\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2\sigma^4} \\
&\cdot \Big|\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\sum_{1\le u+v\le b}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\quad\cdot c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k(i-j)^{k+m}c_{k,m}\,\sigma^{3k-m}|\lambda|^{k+m}\Big| \\
= C\,&\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2\sigma^4} \\
&\cdot \Big|\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\sum_{1\le u+v\le b}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\quad\cdot c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k\sum_{l=0}^{k+m}c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}|\lambda|^{k+m}\Big| \\
= C\,&\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2\sigma^4}\,|S_4+S_5|\,,
\end{aligned}
$$

where $c_{u,v}$ and $c_{k,m}$ denote constants dependent on their indexes,

$$
\begin{aligned}
S_4 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a-1}\sum_{1\le u+v\le b}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j}& \\
\cdot\, c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}&\,,
\end{aligned}
$$

and

$$
S_5 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\ge a}\sum_{1\le u+v\le b}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j}
$$

$$\cdot\, c_{u,v}\, \alpha_1^u\, i^v (-8\, r^2)^k c_{k,m,l}\, i^{k+m-l} (-j)^l \sigma^{3k-m}\, |\lambda|^{k+m}\,.$$

It follows from (17) that

$$\begin{aligned} S_4 = & \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{1\le k+m\le a-1} \frac{1}{k!}\, (-8\, r^2)^k \sigma^{3k-m}\, |\lambda|^{k+m} \\ & \cdot \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{1\le u+v\le b} \Big(\sum_{l=0}^{b-1} + \sum_{l=b}^{k+m} \Big)\, c_{u,v,k,m,l}\, (-1)^{i+j} \\ & \qquad\qquad\qquad \cdot \binom{a}{i} \binom{b}{j} \alpha_1^u\, i^{k+m+v-l} (-j)^l \\ = & \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{1\le k+m\le a-1} \frac{1}{k!}\, (-8\, r^2)^k \sigma^{3k-m}\, |\lambda|^{k+m} \\ & \cdot \sum_{i=0}^{a} \sum_{1\le u+v\le b} \sum_{l=0}^{b-1} c_{u,v,k,m,l}\, (-1)^{i+l} \binom{a}{i} \alpha_1^u\, i^{k+m+v-l} \\ & \qquad\qquad \cdot \sum_{j=0}^{b} (-1)^j \binom{b}{j} j^l \\ & + \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{1\le k+m\le a-1} \frac{1}{k!}\, (-8\, r^2)^k \sigma^{3k-m}\, |\lambda|^{k+m} \\ & \cdot \sum_{j=0}^{b} \sum_{1\le u+v\le b} \sum_{l=b}^{k+m} c_{u,v,k,m,l}\, (-1)^{j+l} \binom{b}{j} \alpha_1^u\, j^l \\ & \qquad\qquad \cdot \sum_{i=0}^{a} (-1)^i \binom{a}{i} i^{k+m+v-l} \\ = & \; 0\,, \end{aligned}$$

since $l \geq b$ implies $k+m+v-l \leq k+m+b-l \leq k+m \leq a-1$. Thus we infer

$$S_2 = C\, \frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2 \sigma^4}\, |S_5|\,.$$

Similarly we estimate S_5 as follows.

$$S_5 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\sum_{l=0}^{k+m}\frac{1}{k!}(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$
$$\cdot\, c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$+\sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\sum_{\substack{1\leq u+v\leq b\\ u\leq k+m-a}}\sum_{l=0}^{k+m}\frac{1}{k!}(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$
$$\cdot\, c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$\equiv S_6 + S_7\ .$$

Using (17) again, we obtain

$$S_6 = \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\left(\sum_{l=0}^{b-1}+\sum_{l=b}^{k+m}\right)\frac{1}{k!}(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$
$$\cdot\, c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$= \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\frac{1}{k!}(-8\,r^2)^k\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$\cdot\sum_{i=0}^{a}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\sum_{l=0}^{b-1}c_{u,v,k,m,l}\,(-1)^{i+l}\binom{a}{i}$$
$$\cdot\,\alpha_1^u\, i^{k+m+v-l}\sum_{j=0}^{b}(-1)^j\binom{b}{j}j^l$$
$$+\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\leq k+m\leq a-1}\frac{1}{k!}(-8\,r^2)^k\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$\cdot\sum_{j=0}^{b}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\sum_{l=b}^{k+m}c_{u,v,k,m,l}\,(-1)^{j+l}\binom{b}{j}$$

$$\cdot\, \alpha_1^u\, j^l \sum_{i=0}^{a} (-1)^i \binom{a}{i} i^{k+m+v-l}$$

$$= 0\,,$$

since $k+m+v-l \le b-l+a-1 \le a-1$ for $1 \le u+v \le b$, $u \ge k+m-a+1$, and $l \ge b$.

$$|S_7| = \Bigg| \sum_{i=0}^{a} \sum_{j=0}^{b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\ge a} \sum_{\substack{1\le u+v\le b\\ u\le k+m-a}} \sum_{l=0}^{k+m} \frac{1}{k!}\, (-1)^{i+j} \binom{a}{i} \binom{b}{j}$$

$$\cdot\, c_{u,v}\, \alpha_1^u\, i^v (-8\, r^2)^k c_{k,m,l}\, i^{k+m-l} (-j)^l \sigma^{3k-m}\, |\lambda|^{k+m} \Bigg|$$

$$\le C_{a,b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\ge a} |\alpha|^{k+m-a}\, r^{2k}\, \sigma^{3k-m}\, |\lambda|^{k+m}$$

$$= C_{a,b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\ge a} \sigma^{k+m-a}\, r^{2k}\, \sigma^{3k-m}\, |\lambda|^{a}$$

$$\le C_{a,b} \sum_{k=1}^{a+b-1} r^{2k}\, |\lambda|^a\, \sigma^{4k-a}\,.$$

Hence,

$$S_2 \le C\, \frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2\sigma^4} \sum_{k=1}^{a+b-1} r^{2k}\, |\lambda|^a\, \sigma^{4k-a}$$

$$= C\, e^{-r^2\sigma^4}\, \sigma^{-(a+b)/2} \sum_{k=1}^{a+b-1} (r\,\sigma^2)^{2k}\,,$$

which combined with (18), (19), and $S_1 = 0$ proves (16) for $P(t,z) = z_1^a\, \bar z_1^b$, $a \ge b$. For $P(t,z) = z_1^a\, \bar z_2^b$, we use (2) and the recurrence relations and differential properties of Laguerre functions again to obtain

$$\mathcal{R}_{z_1^a \bar z_2^b \phi_r}(\lambda, ae_1 - be_2, \alpha) = (2\,|\lambda|)^{-(a+b)/2} \sqrt{\frac{(\alpha_1+a)!}{\alpha_1!}} \sqrt{\frac{(\alpha_2+b)!}{\alpha_2!}}$$

$$\cdot \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \mathcal{R}_{\phi_r}(\lambda, 0, \alpha + ie_1 + je_2)$$

and $\mathcal{R}_{z_1^a \bar{z}_2^b \phi_r}(\lambda, m, \alpha) = 0$ for $m \neq ae_1 - be_2$. Then

$$\begin{aligned}
&\left|\mathcal{R}_{z_1^a \bar{z}_2^b \phi_r}(\lambda, ae_1 - be_2, \alpha)\right| \\
&\quad \leq C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} e^{-r^2(\sigma + 2(i+j)|\lambda|)^4} \right| \\
&\quad \leq C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2\sigma^4} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \right| \\
&\qquad + C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2\sigma^4} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \sum_{k=1}^{a+b-1} \frac{B_{i,j}^k}{k!} \right| \qquad (20) \\
&\qquad + C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2\sigma^4} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \sum_{k=a+b}^{\infty} \frac{B_{i,j}^k}{k!} \right| \\
&\quad \equiv T_1 + T_2 + T_3\,,
\end{aligned}$$

where

$$\begin{aligned}
B_{i,j} &= -\, r^2 \big((\sigma + 2\,(i+j)|\,\lambda|)^4 - \sigma^4 \big) \\
&= -\, 8\, r^2\, |\lambda|\, (i+j) \\
&\qquad \cdot \big(\sigma^3 + 3\,\sigma^2\, |\lambda|\, (i+j) + 4\,\sigma\, |\lambda|^2\, (i+j)^2 + 2\, |\lambda|^3\, (i+j)^3 \big)\,.
\end{aligned}$$

We immediately obtain $T_1 = 0$ since the summation

$$\sum_{j=0}^{b} (-1)^j \binom{b}{j} = 0\,.$$

To estimate T_3, we use $|\lambda| \leq \sigma$ and the property of alternating series to get

$$\begin{aligned}
T_3 &\leq C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2\sigma^4} \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} (r^2\, |\lambda|\, \sigma^3)^{a+b} \qquad (21) \\
&\leq C\, e^{-r^2\sigma^4}\, \sigma^{-(a+b)/2}\, (r^2\, \sigma^4)^{a+b}\,.
\end{aligned}$$

As to T_2, we write

$$
\begin{aligned}
T_2 = C\, & \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4} \\
& \cdot \Bigg| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \\
& \qquad \cdot \sum_{k=1}^{a+b-1} \sum_{m=0}^{3k} \frac{\big(-8\, r^2\, |\lambda|\, (i+j)\big)^k c_{k,m}\, \sigma^{3k-m}\, |\lambda|^m (i+j)^m}{k!} \Bigg| \\
= C\, & \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4} \\
& \cdot \Bigg| \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{k=1}^{a+b-1} \sum_{m=0}^{3k} \frac{1}{k!}\, (-1)^{i+j} \binom{a}{i} \binom{b}{j} \\
& \qquad \cdot (-8\, r^2)^k\, (i+j)^{k+m}\, c_{k,m}\, \sigma^{3k-m}\, |\lambda|^{k+m} \Bigg| \\
= C\, & \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4} \\
& \cdot \Bigg| \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{k=1}^{a+b-1} \sum_{m=0}^{3k} \frac{1}{k!}\, (-1)^{i+j} \binom{a}{i} \binom{b}{j} \\
& \qquad \cdot (-8\, r^2)^k \sum_{l=0}^{k+m} c_{k,m,l}\, i^{k+m-l}\, j^l\, \sigma^{3k-m}\, |\lambda|^{k+m} \Bigg| \\
= C\, & \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4}\, |T_4 + T_5|\,,
\end{aligned}
$$

where $c_{k,m}$ and $c_{k,m,l}$ denote constants dependent on their indexes,

$$
\begin{aligned}
T_4 = \sum_{i=0}^{a} \sum_{j=0}^{b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{1\le k+m\le a+b-1} \sum_{l=0}^{k+m} \frac{1}{k!}\, (-1)^{i+j} \binom{a}{i} \binom{b}{j} & \\
\cdot (-8\, r^2)^k\, c_{k,m,l}\, i^{k+m-l}\, j^l\, \sigma^{3k-m}\, |\lambda|^{k+m}\,, &
\end{aligned}
$$

and

$$
T_5 = \sum_{i=0}^{a} \sum_{j=0}^{b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\ge a+b} \sum_{l=0}^{k+m} \frac{1}{k!}\, (-1)^{i+j} \binom{a}{i} \binom{b}{j}
$$

$$\cdot\,(-8\,r^2)^k\,c_{k,m,l}\,i^{k+m-l}\,j^l\,\sigma^{3k-m}\,|\lambda|^{k+m}\,.$$

It follows from (17) that

$$\begin{aligned}
T_4 &= \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a+b-1} \frac{1}{k!}\,(-8\,r^2)^k\,\sigma^{3k-m}\,|\lambda|^{k+m} \\
&\qquad \cdot\sum_{i=0}^{a}\sum_{j=0}^{b}\left(\sum_{l=0}^{b-1}+\sum_{l=b}^{k+m}\right) c_{k,m,l}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j}\,i^{k+m-l}\,j^l \\
&= \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a+b-1} \frac{1}{k!}\,(-8\,r^2)^k\,\sigma^{3k-m}\,|\lambda|^{k+m} \\
&\qquad \cdot\sum_{i=0}^{a}\sum_{l=0}^{b-1} c_{k,m,l}\,(-1)^i\binom{a}{i}\,i^{k+m-l}\sum_{j=0}^{b}(-1)^j\binom{b}{j}\,j^l \\
&\qquad + \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a+b-1} \frac{1}{k!}\,(-8\,r^2)^k\,\sigma^{3k-m}\,|\lambda|^{k+m} \\
&\qquad\quad \cdot\sum_{j=0}^{b}\sum_{l=b}^{k+m} c_{k,m,l}\,(-1)^j\binom{b}{j}\,j^l\sum_{i=0}^{a}(-1)^i\binom{a}{i}\,i^{k+m-l} \\
&= 0\,,
\end{aligned}$$

since $l\ge b$ implies $k+m-l\le a+b-1-l\le a-1$. Thus

$$\begin{aligned}
T_2 &= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4}\,|T_5| \\
&= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4} \\
&\qquad \cdot\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\ge a+b}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\qquad\quad \cdot\,(-8\,r^2)^k\,c_{k,m,l}\,i^{k+m-l}\,j^l\,\sigma^{3k-m}\,|\lambda|^{k+m}\Bigg|
\end{aligned}$$

$$\leq C_{a,b}\, \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\geq a+b} r^{2k}\, \sigma^{3k-m}\, |\lambda|^{k+m}$$

$$\leq C_{a,b}\, \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\geq a+b} r^{2k}\, \sigma^{3k-m}\, |\lambda|^{a+b}\, \sigma^{k+m-a-b}$$

$$= C_{a,b}\, e^{-r^2\sigma^4}\, \sigma^{-(a+b)/2} \sum_{k=1}^{a+b-1} (r\,\sigma^2)^{2k}\,,$$

which combined with (20), (21), and $T_1 = 0$ proves (16) for

$$P(t,z) = z_1^a\, \bar{z}_2^b\,,$$

and ends the proof of Lemma 2.

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