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Multipliers of de Branges-Rovnyak spaces in H^2

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1. Introduction.

In 1966 de Branges and Rovnyak introduced a concept of complementation associated to a contraction between Hilbert spaces that generalizes the classical concept of orthogonal complement. When applied to Toeplitz operators on the Hardy space of the disc, H^2, this notion turned out to be the starting point of a beautiful subject, with many applications to function theory. The work has been in constant progress for the last few years. We study here the multipliers of some de Branges-Rovnyak spaces contained in H^2.

This introductory section is devoted mainly to general background on Hilbert spaces contained contractively in H^2; all its material can be found in [15], and especially in [13]. Also, at the end of the section we give an account of the main results obtained in this paper.

Let H, H_1 be Hilbert spaces, and $A : H_1 \to H$ be a contraction. We denote by $M(A)$ the space formed by the range of A with the Hilbert space structure that makes A a coisometry from H_1 onto $M(A)$. With this structure the inclusion of $M(A)$ in H is a contraction, so we say that $M(A)$ is contained contractively in H. The space $\mathcal{H}(A) = M[(1 - AA^*)^{1/2}]$ is called the complementary space of $M(A)$. The overlapping space $M(A) \cap \mathcal{H}(A)$ equals $A\,\mathcal{H}(A^*)$, and it is not difficult to prove that if $a \in H$, then $a \in \mathcal{H}(A)$ if and only if $A^*a \in \mathcal{H}(A^*)$. If A is a partial isometry (and only in this case), $M(A)$ and $\mathcal{H}(A)$ are closed subspaces of H, orthogonal complements of each other; otherwise the overlapping space $A\mathcal{H}(A^*)$ is always nontrivial.

Let b be an element of the unit ball $B(H^\infty)$ in H^∞, and let T_b and $T_{\overline{b}}$ be the Toeplitz operators associated to b and $\overline{b}$ acting on H^2. Since these operators are contractions, we can consider the spaces $\mathcal{H}(T_b)$ and $\mathcal{H}(T_{\overline{b}})$, which from now on will be denoted by $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$, respectively. Using a classical criterion of Douglas to factorize contractions, it is easy to show that $\mathcal{H}(\overline{b})$ is contained contractively in $\mathcal{H}(b)$ (see [15, II-2]). Now a simple calculation shows that if $f, g \in \mathcal{H}(b)$, then

$$\langle f, g\rangle_{\mathcal{H}(b)} = \langle f, g\rangle_{H^2} + \langle T_{\overline{b}}(f), T_{\overline{b}}(g)\rangle_{\mathcal{H}(\overline{b})}\,.$$

If $b = b_1 b_2$, with b_1 and b_2 in $B(H^\infty)$, then $\mathcal{H}(b) = \mathcal{H}(b_1) + b_1\mathcal{H}(b_2)$, where $\mathcal{H}(b_1)$ is contained contractively in $\mathcal{H}(b)$ and T_{b_1} implements a contraction from $\mathcal{H}(b_2)$ into $\mathcal{H}(b)$. Besides, this sum is direct (*i.e.* $\mathcal{H}(b_1) \cap b_1\mathcal{H}(b_2) = \{0\}$) if and only if $\mathcal{H}(b_1)$ is the orthogonal complement of $b_1\mathcal{H}(b_2)$ in $\mathcal{H}(b)$. In particular this holds if b_1 is an inner function, because since in this case T_{b_1} is an isometry, so that $(1 - T_{b_1}T_{\overline{b}_1})^{1/2}$ is the projection (in H^2) onto the orthogonal complement of $b_1 H^2$. Moreover, $\mathcal{H}(b_1)$ is an ordinary closed subspace of H^2.

For $\varphi \in H^\infty$, the Toeplitz operator $T_{\overline{\varphi}}$ is a bounded operator on $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ with norm (in both cases) not exceeding $\|\varphi\|_\infty$.

The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ can be represented in terms of Cauchy integrals. Let μ be a Borel finite positive measure on $\partial\mathbb{D}$, the boundary of the unit disc. For $f \in L^2(\mu)$, define the Cauchy transform of f respect to μ as

$$K_\mu(f)(z) = \int_{\partial\mathbb{D}} \frac{1}{1 - e^{-i\theta}z}\, f(e^{i\theta})\, d\mu(e^{i\theta})\,, \qquad z \in \mathbb{C} \setminus \partial\mathbb{D}.$$

It is an analytic function on $\mathbb{C} \setminus \partial\mathbb{D}$. We often (not always) use the restriction of this function to $\mathbb{D}$, its meaning being clear from the context. If the measure μ is given by a weight, $d\mu(e^{i\theta}) = g(e^{i\theta})\, d\theta/2\pi$ with $g \in L^1\,(= L^1(d\theta/2\pi))$, $g \geq 0$, we simply write K_g for K_μ. In particular, if $g \equiv 1$ we write K.

Let $b \in B(H^\infty)$. The real part of the function $(1+b(z))/(1-b(z))$ is $(1-|b(z)|^2)/|1-b(z)|^2 \geq 0$, so it can be represented by the Herglotz formula

$$(1)\quad \frac{1+b(z)}{1-b(z)} = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu_b(e^{i\theta}) + i\,\mathrm{Im}\left(\frac{1+b(0)}{1-b(0)}\right), \qquad z \in \mathbb{D},$$

where

$$d\mu_b(e^{i\theta}) = \frac{1-|b(e^{i\theta})|^2}{|1-b(e^{i\theta})|^2}\frac{d\theta}{2\pi} + d\mu_S(e^{i\theta}),$$

with μ_S a positive finite singular measure and

$$\sigma = \frac{1-|b|^2}{|1-b|^2} \in L^1.$$

First Clark [3] for b inner and then Sarason in general [17] proved that the operator given by $V_b(f)(z) = (1-b(z))K_{\mu_b}(f)(z)$ (for $f \in L^2(\mu_b)$ and $z \in \mathbb{D}$), establishes an isometry from $H^2(\mu_b)$ onto $\mathcal{H}(b)$, where $H^2(\mu_b)$ is the closure in $L^2(\mu_b)$ of the analytic polynomials (see [1] and [2] for vector valued versions). Also, in [13] it is proved that if $\rho(e^{i\theta}) = 1-|b(e^{i\theta})|^2$, then K_ρ is an isometry from $H^2(\rho)$ $(=$ $H^2(\rho(e^{i\theta})\,d\theta/2\pi))$ onto $\mathcal{H}(\overline{b})$. For a given $b \in B(H^\infty)$, ρ, σ and μ_b will always denote the functions and measure associated to b as in the above paragraph.

At this point two very different cases appear in the study of the spaces $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$, according to whether b is or is not an extreme point of $B(H^\infty)$, or equivalently, according to whether ρ is not or is log-integrable on $\partial\mathbb{D}$ (see [11, p. 138]). The reason for this distinction is a famous theorem of Szegö ([11, p. 49]), which asserts that for a positive finite measure μ on $\partial\mathbb{D}$, $H^2(\mu) = L^2(\mu)$ if and only if the Radon-Nikodym derivative of μ with respect to the Lebesgue measure is not log-integrable. Thus, if b is extreme in $B(H^\infty)$ (and only in this case), $H^2(\rho) = L^2(\rho)$ and $H^2(\mu_b) = L^2(\mu_b)$. Notice that $\log\sigma = \log\rho - \log|1-b|^2$, where $\log|1-b|^2$ is integrable because $1-b \in H^1$ ([11, p. 51]).

A multiplier of $\mathcal{H}(b)$ (or of $\mathcal{H}(\overline{b})$) is a function $m \in H^\infty$ such that $\mathcal{H}(b)$ (respectively $\mathcal{H}(\overline{b})$) is invariant by T_m. If $f \in H^2$, then $f \in \mathcal{H}(\overline{b})$ if and only if $bf \in \mathcal{H}(b)$. This immediately implies that every multiplier of $\mathcal{H}(b)$ is also a multiplier of $\mathcal{H}(\overline{b})$. Also, for u an inner function, the decomposition $\mathcal{H}(ub) = u\mathcal{H}(b) + \mathcal{H}(u)$ together with the fact that $uH^2 \cap \mathcal{H}(u) = \{0\}$, implies that every multiplier of $\mathcal{H}(ub)$ is a multiplier of $\mathcal{H}(b)$. It is known that both inclusions of multipliers can be proper. D. Sarason [16] gave an example of a nonextreme outer function b for which the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are different. However, it is unknown if the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ can be different when b is outer and extreme. If b is inner, $\mathcal{H}(\bar{b})$ is trivial, and it is easy to see that only the constant functions are multipliers of $\mathcal{H}(b)$; otherwise there are plenty of nonconstant multipliers (see [13]).

Information about multipliers for the nonextreme case can be found in [5], [12], [14] and [16]. The main source for the extreme case is the paper of Lotto and Sarason [13]. The latter case is the subject of this paper, so we assume from now on that b is an extreme point of $B(H^\infty)$ unless the contrary is stated. Also, we exclude the trivial case b inner.

Since in our case the backward shift S^* is an invertible operator on $\mathcal{H}(\bar{b})$ ([13, Theorem 3.6]), it is easy to prove that every multiplier of $\mathcal{H}(\bar{b})$ is in $\mathcal{H}(\bar{b}) + \mathbb{C}$. Since $\mathcal{H}(\bar{b})$ has no other constants except the zero function, the above space is a one-dimensional linear extension of $\mathcal{H}(\bar{b})$. If $f \in \mathcal{H}(\bar{b}) + \mathbb{C}$, the Cauchy representation of $\mathcal{H}(\bar{b})$ shows that for $z \in \mathbb{D}$, $f(z) = K_\rho(q)(z) + c$ with $q \in L^2(\rho)$ and $c \in \mathbb{C}$. Now define the following conjugation in $\mathcal{H}(\bar{b}) + \mathbb{C}$, $f_*(z) = -K_\rho(\bar{q})(z) + K_\rho(\bar{q})(0) + \bar{c}$. A straightforward calculation shows that if we think of f as defined on $\mathbb{C} \setminus \partial\mathbb{D}$, then

$$f_*(z) = \overline{f(1/\bar{z})}\,.$$

Let us denote by $\mathcal{M}(b)$ and $\mathcal{M}(\bar{b})$ the algebras of multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ respectively. The above conjugation has the important property that if m belongs to any of these algebras, then m_* belongs to the same algebra. In particular, $m(z)$ and $\overline{m(1/\bar{z})}$ are in H^∞ (for $z \in \mathbb{D}$), which implies that $m(z) = K_\rho(q)(z) + c$ must be bounded for all $z \in \mathbb{C} \setminus \partial\mathbb{D}$. In other words, the algebras of multipliers are contained in the space

$$K^\infty(\rho) = \{m = K_\rho(q) + c : \ q \in L^2(\rho)\,,\ c \in \mathbb{C}\,,\ \sup_{z \in \mathbb{C}\setminus\partial\mathbb{D}} |m(z)| < +\infty\}\,.$$

The space $K^\infty(\rho)$ is closed under multiplication, and if $f, g \in K^\infty(\rho)$ then $(fg)_* = f_* g_*$. Moreover, if $m = K_\rho(q) + c \in K^\infty(\rho)$, the norm $\|m\|_{K^\infty(\rho)} = \sup_{z\in\mathbb{C}\setminus\mathbb{D}} |m(z)| + \|q\|_{L^2(\rho)}$ makes $K^\infty(\rho)$ a $*$-Banach algebra. Summing up, we have the following string of inclusions

$$\mathcal{M}_\infty(b) \subset \mathcal{M}(ub) \subset \mathcal{M}(b) \subset \mathcal{M}(\bar{b}) \subset K^\infty(\rho)\,, \tag{2}$$

where u is an inner function and $\mathcal{M}_\infty(b) = \bigcap_{v \text{ inner}} \mathcal{M}(vb)$. If m belongs to any of these algebras, the spectrum of m in the respective algebra is the closure of $m(\mathbb{C}\backslash\partial\mathbb{D})$. Also, the operation $m \to m_*$ is a multiplicative conjugation in all the algebras (see [13]).

The paper is organized as follows. In Section 2 we give a characterization of the group $\Gamma = \{f \in K^\infty(\rho) : \ f_* = f^{-1}\}$ and we show that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are any of the algebras in (2), then $\mathcal{M}_1 = \mathcal{M}_2$ if and only if $\mathcal{M}_1 \cap \Gamma = \mathcal{M}_2 \cap \Gamma$. This observation will be fundamental in the sequel. In Section 3 we establish some known relations between multipliers and weighted norm inequalities. We study these relations in terms of our characterization of Γ. Section 4 answers a question by Lotto and Sarason by giving an example of $b \in B(H^\infty)$ extreme, such that $\mathcal{M}(\overline{b})$ does not coincide with $K^\infty(\rho)$. We obtain a complete characterization of $\mathcal{M}(\overline{b})$ for this example. In Section 5 it is proved that $\mathcal{M}_\infty(b)$ is dense in $\mathcal{M}(b)$ and $\mathcal{M}(\overline{b})$ with the respective strong operator topologies. Section 6 discusses the way in which the singular component of μ_b affects the algebras $\mathcal{M}(b)$ and $K^\infty(\rho)$. In Section 7 we introduce a partial isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(\overline{b})$, which is used to obtain a sufficient condition for a function $m \in K^\infty(\rho)$ to belong to $\mathcal{M}(b)$. It follows as a corollary that $\mathcal{H}(b)$ is imbedded in $L^2(\rho/|1-ub|^2)$ for every inner function u. Also, we show several characterizations of the equality $\mathcal{M}_\infty(b) = K^\infty(\rho)$. In particular, this turns out to be equivalent to $\mathcal{M}_\infty(b) = \mathcal{M}(b)$. In Section 8 we investigate how $\mathcal{H}(b)$, $\mathcal{H}(\overline{b})$ and their multipliers are affected if we replace b by $\tau \circ b$, where τ is an analytic authomorphism of the unit disk. In Section 9 we prove that the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ coincide when b is continuous up to the boundary of the disk. Finally, Section 10 contains some information about the interaction between the conjugation $*$ and the inner factors of functions in any of the algebras $\mathcal{M}(b)$, $\mathcal{M}(\overline{b})$ and $K^\infty(\rho)$.

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2. Some special functions in $K^\infty(\rho)$.

One of the main problems when studying the algebras of multipliers is the lack of examples, in particular, the difficulty to exhibit nonconstant elements of $K^\infty(\rho)$. The next theorem will allow us to construct functions $m = K_\rho(q) + c$ in $K^\infty(\rho)$, where $r = |q\rho|$ has a preestablished behaviour. We need two lemmas.

Lemma 2.1. *Let* $f = K_\rho(q) + c \in \mathcal{H}(\overline{b}) + \mathbb{C}$*. Then the (inner) boundary function of* $f(z) - \overline{f_*(z)}$ *equals* $q\rho$*. Conversely, if* f *and* g *are analytic functions on* $\mathbb{D}$ *such that* $f - \overline{g} = P * q\rho$*, where* P *denotes the Poisson kernel and* $q \in L^2(\rho)$*, then in* $\mathbb{D}$*,*

$$f(z) = K_\rho(q)(z) + \overline{g(0)}$$

and

$$g(z) = \overline{K_\rho(q)(1/\overline{z})} + g(0)\,.$$

The lemma is just a particular case of Lemmas 10.1 and 10.2 in [13].

Lemma 2.2. *Let* s *be a real valued function in* L^∞*. Then*

$$2\,|s| \le |e^s - e^{-s}| \le 2\,e^{|s|}\,|s|\,.$$

Proof. Both inequalities follow from simple calculations with the Taylor series

$$\frac{|e^s - e^{-s}|}{|s|} = 2\sum_{n\ge 0} \frac{s^{2n}}{(2n+1)!}\,.$$

If f and g are functions defined almost everywhere in $\partial\mathbb{D}$, and f takes the the value zero whenever g does (except for a null set), the quotient f/g makes sense and it is finite almost everywhere with the convention $0/0 = 0$.

Theorem 2.3. *Let* s *be a real valued bounded function defined on* ∂D *such that* $s^2/\rho \in L^1$*. Then* $m = e^{s+i\tilde{s}} \in K^\infty(\rho)$*, where* $\tilde{s}$ *is any harmonic conjugate of* s*. Moreover, if* $m = K_\rho(q) + c$ *with* $q \in L^2(\rho)$ *and* $c \in \mathbb{C}$*, then*

1) $q\rho = (|m|^2 - 1)/\overline{m}\,.$
2) *If* $r = |q\rho|$*, then* $2\,|s| \le |r| \le 2\,e^{\|s\|_\infty}\,|s|\,.$
3) $m_* = m^{-1}\,.$

Conversely, every $m \in K^\infty(\rho)$ *such that* $m_* = m^{-1}$ *is of the above form with* $s = \log|m|$*.*

PROOF. The function $m = e^{s+i\tilde{s}}$ is invertible in H^∞. Hence, the bounded harmonic function $m-\overline{m}^{-1}$ is the Poisson integral of its (inner) boundary function $(|m|^2-1)/\overline{m}$. Write $q = (|m|^2-1)/\overline{m}\rho$. Since $|(|m|^2-1)/\overline{m}| = |e^s - e^{-s}|$, Lemma 2.2 asserts that

$$2\,\frac{|s|}{\rho} \le |q| \le C\,\frac{|s|}{\rho}\,, \qquad \text{with } C = 2\,e^{\|s\|_\infty}\,. \tag{3}$$

Therefore $|q|^2\rho \le C^2\,(s^2/\rho) \in L^1$ and consequently $q \in L^2(\rho)$. By Lemma 2.1, $m = K_\rho(q) + \overline{m^{-1}(0)}$ and $m_* = m^{-1}$.

On the other hand, if $m = K_\rho(q) + c$ is any element of $K^\infty(\rho)$ such that $m_* = m^{-1}$, then by Lemma 2.1 the boundary function of $m-\overline{m}_* = (|m|^2-1)/\overline{m}$ equals $q\rho$. Hence $q = (|m|^2-1)/\overline{m}\rho \in L^2(\rho)$. Since m is an invertible function of H^∞ then $m = e^{s+i\tilde{s}}$, where $s = \log|m| \in L^\infty$. A new application of Lemma 2.2 shows that the inequalities (3) hold for these q and s, thus $s^2/\rho \le (1/4)|q|^2\rho \in L^1$.

Definition. *Let $b \in B(H^\infty)$ and $\rho(e^{i\theta}) = 1 - |b(e^{i\theta})|^2$. If s is a real valued, essentially bounded function on $\partial\mathbb{D}$ such that $s^2/\rho \in L^1$, we will say that s is an admissible function for ρ, or simply, that s is admissible.*

Theorem 2.3 implies that for every s admissible there is $m = K_\rho(q) \in K^\infty(\rho)$, where $r = |q\rho|$ behaves like $|s|$. On the other hand, if $m = K_\rho(q) + c$ is any element of $K^\infty(\rho)$, then $r = |q\rho|$ is admissible.

We fix for the rest of the paper the notation E for the set where ρ does not vanish. That is,

$$E = \{e^{i\theta} \in \partial\mathbb{D} :\ \rho(e^{i\theta}) \neq 0\}\,.$$

In Theorem 13.3 of [13] it is proved that $m = K_\rho(q) + c \in K^\infty(\rho)$ is a multiplier of $\mathcal{H}(vb)$ for every inner function v if and only if $q^2\rho \in L^\infty$. If we write $r = |q\rho|$, this condition can be rewritten as $r^2/\rho \in L^\infty$. Since r is bounded, the above condition holds for all $m \in K^\infty(\rho)$ if $\chi_E/\rho \in L^\infty$ (where χ_E denotes the characteristic function of E). Theorem 2.3 immediately implies that the converse also holds, because if $\chi_E/\rho \notin L^\infty$ then there is an admissible function s such that $s^2/\rho \notin L^\infty$.

Theorem 2.3 gives a characterization of the functions in

$$\Gamma = \{f \in K^\infty(\rho) :\ f_* = f^{-1}\}\,.$$

Denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ two different algebras of the string of inclusions (2), with $\mathcal{M}_1 \subset \mathcal{M}_2$.

Proposition 2.4. *$\mathcal{M}_2 \subset \mathcal{M}_1$ if and only if $\mathcal{M}_2 \cap \Gamma \subset \mathcal{M}_1$.*

PROOF. Suppose that there is $m \in \mathcal{M}_2 \setminus \mathcal{M}_1$. Since $m = (m+m_*)/2 + i(m-m_*)/2i$, then $(m+m_*)/2$ or $(m-m_*)/2i$ is not in $\mathcal{M}_1$. Hence there is $f \in \mathcal{M}_2 \setminus \mathcal{M}_1$ such that $f = f_*$. Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$ be a number which does not belong to the spectrum of f. Then

$$\frac{f-\overline{\alpha}}{f-\alpha} = 1 + \frac{\alpha-\overline{\alpha}}{f-\alpha} \in (\mathcal{M}_2 \cap \Gamma) \setminus \mathcal{M}_1 .$$

For $m \in K^\infty(\rho)$ denote by $\mathrm{sp}\,(m)$ the spectrum of m. Let $\mathcal{M}$ be any of the Banach algebras $\mathcal{M}(b)$, $\mathcal{M}(\overline{b})$ or $K^\infty(\rho)$.

Lemma 2.5. *Let $m \in \mathcal{M}$ with $\mathrm{sp}\,(m) \cap \partial\mathbb{D} = \varnothing$. If f is a continuous function on $\partial\mathbb{D}$, then*

$$I_f(m) = \int_0^{2\pi} \frac{m_* - e^{-i\theta}}{m - e^{i\theta}}\, f(e^{i\theta})\, \frac{d\theta}{2\pi}$$

is in $\mathcal{M}$.

PROOF. The continuity of the map $\omega \to (m_* - \overline{\omega})/(m-\omega)$ for $\omega \notin \mathrm{sp}\,(m) = \overline{\mathrm{sp}\,(m_*)}$ assures that $I_f(m)$ is well defined (because $\mathrm{sp}\,(m) \cap \partial\mathbb{D} = \varnothing$), and that it is the limit (in norm) of

$$S_n = \sum_{k=0}^{n-1} \frac{1}{n}\, f(e^{2\pi i k/n})\, \frac{m_* - e^{-2\pi i k/n}}{m - e^{2\pi i k/n}} .$$

Proposition 2.6. *Let $m \in K^\infty(\rho)$ with $\|m\| < 1$. If $f_k(e^{i\theta}) = e^{ik\theta}$ (with k an integer) then*

$$I_{f_k}(m) = \begin{cases} m^{k-2}\,(1 - m\, m_*)\,, & \text{if } k \geq 2\,, \\ -m_*\,, & \text{if } k = 1\,, \\ 0\,, & \text{if } k \leq 0\,. \end{cases}$$

PROOF. It is a straightforward calculation with the power series expansion (in $e^{i\theta}$) of $(m_* - e^{-i\theta})/(m - e^{i\theta})$.

Corollary 2.7. *Let $\mathcal{M}$ be as in the preceding lemma. Then the span of $\Gamma \cap \mathcal{M}$ is dense in $\mathcal{M}$.*

PROOF. The proof of Lemma 2.5 shows that if f is continuous on $\partial\mathbb{D}$ and $m \in \mathcal{M}$ is such that $\operatorname{sp}(m) \cap \partial\mathbb{D} = \varnothing$, then $I_f(m)$ is in the closure of $\operatorname{span}(\Gamma \cap \mathcal{M})$. Given any $m \in \mathcal{M}$, take $m^1 = m_*/2\|m_*\|$, where the norm $\|m_*\|$ is taken in $K^\infty(\rho)$, and $f_1(e^{i\theta}) = e^{i\theta}$. By Proposition 2.6, $-I_{f_1}(m^1) = m^1_* = m/2\|m_*\|$. Hence, $m = -2\,\|m_*\|\, I_{f_1}(m^1)$ is in the closure of $\operatorname{span}(\Gamma \cap \mathcal{M})$.

3. Weights and Multipliers.

In [13] some criteria are given for a function $m \in K^\infty(\rho)$ to belong to $\mathcal{M}(b)$ or $\mathcal{M}(\overline{b})$. Those criteria are the starting point of most of the sequel. The next theorem is a different formulation of Theorem 12.2 and Lemma 13.1 in [13].

Theorem 3.1. *Let $m = K_\rho(q) + c \in K^\infty(\rho)$. If $r = |q\rho|$, then*

1) *$m \in \mathcal{M}(\overline{b})$ if and only if $f \in L^2(r^2/\rho)$ for every $f \in \mathcal{H}(\overline{b})$.*

2) *$m \in \mathcal{M}(b)$ if and only if $f \in L^2(r^2/\rho)$ for every $f \in \mathcal{H}(b)$.*

3) *If $m \in \mathcal{M}(b)$ and u is an inner function, then $m \in \mathcal{M}(ub)$ if and only if $f \in L^2(r^2/\rho)$ for every $f \in \mathcal{H}(u)$.*

The advantage of this point of view for the present paper is that Theorem 3.1 is given in terms of the admissible function r. Theorems 3.1 and 2.3 immediately yield the fact that $\mathcal{M}(b)$ (or $\mathcal{M}(\overline{b})$) coincides with $K^\infty(\rho)$ if and only if for every admissible function r, $f \in L^2(r^2/\rho)$ for all $f \in \mathcal{H}(b)$ (respectively, for all $f \in \mathcal{H}(\overline{b})$).

By a standard argument involving the closed graph theorem, if any of the conditions of Theorem 3.1 holds, then it holds with continuity.

Let μ be a finite Borel measure on $\partial\mathbb{D}$ and $f \in L^1(\mu)$. Then, as a function on $\mathbb{D}$, $K_\mu(f)$ belongs to H^p for $0 < p < 1$; so it has a finite nontangential limit for almost every $e^{i\theta} \in \partial\mathbb{D}$ (see [8, pages 17 and 39]). Most of the time it will be convenient to think of $K_\mu(f)$ as its (inner) boundary function. Since $K_\rho : L^2(\rho) \to \mathcal{H}(\overline{b})$ is an onto isometry, then for $f = K_\rho(q)$,

$$\|f\|_{\mathcal{H}(\overline{b})} = \|q\|_{L^2(\rho)} = \|q\rho^{1/2}\|_{L^2} .$$

Thus every $f \in \mathcal{H}(\bar{b})$ can be written as $f = K_{\rho^{1/2}}(h)$ with $q\rho^{1/2} = h \in L^2$, $h = 0$ outside of E, and $\|f\|_{\mathcal{H}(\bar{b})} = \|h\|_{L^2} = \|h\|_{L^2(\chi_E)}$. Conversely, if $h \in L^2$ then $h\chi_E = q\rho^{1/2}$ with $q \in L^2(\rho)$ (take $q = h\chi_E/\rho^{1/2}$), and $\|h\chi_E\|_{L^2} = \|q\|_{L^2(\rho)}$. Then $K_{\rho^{1/2}} : L^2(\chi_E) \to \mathcal{H}(\bar{b})$ is an onto isometry. On the other hand, if $d\mu_b = \sigma\, d\theta/2\pi + d\mu_S$ is the measure associated to b by formula (1), then $K_{\mu_b} = K_\sigma + K_{\mu_S}$, and $V_b = (1-b)\, K_{\mu_b}$ is an onto isometry from $L^2(\mu_b)$ onto $\mathcal{H}(b)$. As before, we can replace the operator K_σ on $L^2(\sigma)$ by $K_{\sigma^{1/2}}$ on $L^2(\chi_E)$. We just obtained that $W_b = (1-b)\,(K_{\sigma^{1/2}} + K_{\mu_S})$ is an isometry from $L^2(\chi_E) \oplus L^2(\mu_S)$ onto $\mathcal{H}(b)$. With these facts in mind we can rewrite Theorem 3.1 once more.

Theorem 3.2. *Let $m = K_\rho(q) + c \in K^\infty(\rho)$. If $r = |q\rho|$, then*

1) *$m \in \mathcal{M}(\bar{b})$ if and only if $K_{\rho^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(r^2/\rho)$.*

2) *$m \in \mathcal{M}(b)$ if and only if $K_{\sigma^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(r^2/\sigma)$ and K_{μ_S} maps $L^2(\mu_S)$ into $L^2(r^2/\sigma)$.*

3) *If $m \in \mathcal{M}(b)$ then $m \in \mathcal{M}(ub)$ if and only if $(1-u)\, K_{\mu_u}$ maps $L^2(\mu_u)$ into $L^2(r^2/\rho)$, where μ_u is the measure associated to u in the representation* (1) .

Proof. 1) and 3) are immediate. By Theorem 3.1 and the above comment, $m \in \mathcal{M}(b)$ if and only if for every $q_1 \in L^2(\chi_E)$ and $q_2 \in L^2(\mu_S)$,

$$(1-b)\,(K_{\sigma^{1/2}}(q_1) + K_{\mu_S}(q_2)) \in L^2(r^2/\rho)\,.$$

Since $r^2/\sigma = r^2|1-b|^2/\rho$, this is equivalent to $K_{\sigma^{1/2}}(q_1) + K_{\mu_S}(q_2) \in L^2(r^2/\sigma)$, and clearly this is the same as 2).

Again, if any of the conditions of the theorem holds, it does with continuity. Then, the problem of establishing whether a given $m \in K^\infty(\rho)$ is a multiplier is transformed into a problem of weighted norm inequalities. It is not surprising then that Helson-Szegö weights play an important role in the theory. A Helson-Szegö weight is a function $\gamma = e^{\varphi + \tilde{\psi}}$, where φ and ψ are bounded real valued functions on $\partial\mathbb{D}$ and $\|\psi\|_\infty < \pi/2$. The relevance of these functions is that they are precisely the positive weights γ in L^1 such that the Cauchy transform is a bounded operator from $L^2(\gamma)$ into itself [10].

Theorem 3.3. *Let r be an admissible function. If there is a Helson-Szegö weight γ_r such that $r^2/\rho = \chi_E\,\gamma_r$, then $K_{\rho^{1/2}}$ is a bounded operator from $L^2(\chi_E)$ into $L^2(r^2/\rho)$. The statement also holds replacing ρ by σ everywhere.*

PROOF. Take $f \in L^2(\chi_E)$; then $f\rho^{1/2} \in L^2(\chi_E/\rho) \subset L^2(r^2/\rho)$, and since $f\rho^{1/2} = 0$ outside of E, then $f\rho^{1/2} \in L^2(\gamma_r)$. By the Helson-Szegö theorem $K_{\rho^{1/2}}(f) \in L^2(\gamma_r)$, hence $K_{\rho^{1/2}}(f) \in L^2(\gamma_r\chi_E) = L^2(r^2/\rho)$. The same argument works for σ.

Corollary 3.4. *Let $b \in B(H^\infty)$. If there is a Helson-Szegö weight γ such that $\chi_E/\rho = \chi_E\,\gamma$, then $K_{\rho^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(r^2/\rho)$ for every admissible function r. If $d\mu_b = \sigma\, d\theta/2\pi$, the same holds replacing ρ by σ everywhere.*

PROOF. Since Helson-Szegö weights are in L^1, $\chi_E/\rho \in L^1$ (*i.e.* χ_E is admissible). By Theorem 3.3 $K_{\rho^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(\chi_E/\rho)$, and since r is bounded, $L^2(\chi_E/\rho) \subset L^2(r^2/\rho)$.

The assertion for σ can be similarly deduced from Theorem 3.3 if we show that χ_E is admissible, that is, $\chi_E/\rho \in L^1$. So we assume that $\chi_E/\sigma = \chi_E\gamma$, with γ a Helson-Szegö weight. Clearly γ^{-1} is also a Helson-Szegö weight, thus $\sigma^2\gamma = \chi_E/\gamma \in L^1$, or what is the same, $\sigma \in L^2(\gamma)$. Then, by the Helson-Szegö theorem, $K(\sigma) \in L^2(\gamma) \subset L^2(\chi_E\gamma)$. Since $d\mu_b = \sigma\, d\theta/2\pi$, then by [15, III-7],

$$K(\sigma) = K_\sigma(1) = K_{\mu_b}(1) = (1-b)^{-1}(1-\overline{b(0)})^{-1}(1-\overline{b(0)}b)\,,$$

which implies that $(1-b)^{-1} \in L^2(\chi_E\gamma)$. Thus,

$$|1-b|^{-2}\chi_E\,\gamma = |1-b|^{-2}\,\chi_E/\sigma = \chi_E/\rho$$

is in L^1, as claimed.

The statement for σ in the above corollary already appears in [13, Theorem 14.1] with a different formulation and a similar (slightly different) proof.

4. An example.

It is asked in [13] if for b extreme, not an inner function, the algebras $\mathcal{M}(\bar{b})$ and $K^\infty(\rho)$ coincide. We give here an example for which those algebras do not coincide. We also obtain for this example a complete characterization of the multipliers of $\mathcal{H}(\bar{b})$ among the elements of $K^\infty(\rho)$.

When convenient, we identify a function $f(e^{i\theta})$ defined almost everywhere on $\partial\mathbb{D}$ with a function $f(\theta)$ defined for almost every $\theta \in (-\pi, \pi]$. Let β be a function in L^1 (of $\partial\mathbb{D}$). For $f \in L^1(\beta)$ define the Hilbert transform of f with weight β as

$$H_\beta(f)(\theta) = \lim_{\varepsilon \to 0} \int_{\varepsilon<|\varphi-\theta|<\pi} \frac{f(\varphi)}{\theta - \varphi}\, \beta(\varphi)\, d\varphi\,.$$

We write H if $\beta = 1$.

Proposition 4.1. *In Theorem 3.2 we can replace $K_{\rho^{1/2}}$ and $K_{\sigma^{1/2}}$ by $H_{\rho^{1/2}}$ and $H_{\sigma^{1/2}}$, respectively.*

PROOF. We prove the proposition for $K_{\sigma^{1/2}}$, the proof for $K_{\rho^{1/2}}$ is the same. Let $f \in L^2(\chi_E)$; then for $z \in \mathbb{D}$,

$$K_{\sigma^{1/2}}(f)(z) = \frac{1}{2}\left((P * f\sigma^{1/2})(z) + i(Q * f\sigma^{1/2})(z) + (P * f\sigma^{1/2})(0)\right),$$

where P is the Poisson kernel and Q is its harmonic conjugate. Since f and $\sigma^{1/2}$ are in L^2, $f\sigma^{1/2} \in L^1$; hence the boundary function of $(P * f\sigma^{1/2})(z)$ is $f\sigma^{1/2}$. The fact that $f \in L^2$ and $r \in L^\infty$ now implies that $f\sigma^{1/2} \in L^2(r^2/\sigma)$. Also $L^2(r^2/\sigma)$ contains the constants because $r^2/\sigma \in L^1$. That is, $K_{\sigma^{1/2}}(f) \in L^2(r^2/\sigma)$ if and only if the boundary function of $(Q * f\sigma^{1/2})(z)$ is in $L^2(r^2/\sigma)$. Let us denote this boundary function also by $Q * f\sigma^{1/2}$. A simple computation shows that

$$Q * f\sigma^{1/2} = \frac{1}{\pi}\, H_{\sigma^{1/2}}(f) + d * f\sigma^{1/2}\,,$$

where $d(\theta) = \operatorname{cotg}\theta/2 - 2/\theta$ is a bounded function, $|d(\theta)| \le 2/\pi$ (see [9, p. 105]). Hence $|d * f\sigma^{1/2}| \le C\,\|f\sigma^{1/2}\|_{L^1} < +\infty$, and then $d * f\sigma^{1/2}$ always belongs to $L^2(r^2/\sigma)$.

For $\theta \in (0, 2\pi]$ the function $(1 - e^{-1/\theta})^{1/2}$ is log-integrable, so that there is $b \in H^\infty$ such that $|b(e^{i\theta})| = (1 - e^{-1/\theta})^{1/2}$ almost everywhere with respect to $d\theta$. Furthermore, $\rho(\theta) = 1 - |b(e^{i\theta})|^2 = e^{-1/\theta}$ is not log-integrable; thus b is an extreme point of $B(H^\infty)$. We consider this b for the rest of the section. It will be convenient to think of ρ as defined on $(-\pi, \pi]$,

$$\rho(\theta) = \begin{cases} e^{-1/\theta}\,, & \text{if } \ 0 < \theta \leq \pi\,, \\ e^{-1/(2\pi+\theta)}\,, & \text{if } -\pi < \theta \leq 0\,. \end{cases}$$

Theorem 4.2. *For $m = K_\rho(q) + c \in K^\infty(\rho)$, put $r = |q\rho|$. If $m \in \mathcal{M}(\overline{b})$, there is a constant $C > 0$ such that*

$$\int_0^\varepsilon r^2(\theta)\, e^{1/\theta}\, d\theta \leq C\,\varepsilon\,, \qquad \textit{for all } \varepsilon \in (0,\pi)\,.$$

PROOF. For $\theta \in (0, \pi)$, the function $r^2(\theta)\, e^{1/\theta} = r^2(\theta)/\rho(\theta) \in L^1$, from which it is immediate that the conclusion of the theorem is equivalent to

$$\sup \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon r^2(\theta)\, e^{1/\theta}\, d\theta < +\infty\,. \tag{4}$$

If (4) does not hold, there are γ, $0 < \gamma < 1$, and two sequences (α_k), $(\beta_k) \subset (0, \pi)$ such that $\alpha_k < \gamma \beta_k$ for all k, $\alpha_k \to 0$, $\beta_k \to 0$ and

$$\frac{1}{\beta_k - \alpha_k} \int_{\alpha_k}^{\beta_k} r^2(\theta)\, e^{1/\theta}\, d\theta \longrightarrow +\infty\,.$$

Taking suitable subsequences of (α_k) and (β_k) we can also assume that $\beta_{k+1} < \alpha_k$ for all k. Let (s_k) be a sequence in ℓ^1 (the space of absolutely summable sequences) such that $s_k > 0$ for all k, and

$$\sum_{k \geq 1} s_k \frac{1}{\beta_k - \alpha_k} \int_{\alpha_k}^{\beta_k} r^2(\theta)\, e^{1/\theta}\, d\theta = +\infty\,. \tag{5}$$

Take

$$d_k = \Big(\frac{s_k}{\beta_k - \alpha_k}\Big)^{1/2} \qquad \text{for } k \geq 1\,,$$

and consider the function

$$f(\theta) = \sum_{k\geq 1} d_k \, \chi_{(-\beta_k, -\alpha_k)}(\theta) \,.$$

Then $\rho^{-1/2} f \in L^2$, because

$$\begin{aligned}\int_{-\pi}^{\pi} \rho^{-1} |f|^2 \, d\theta &\leq \sup_{-\pi<\theta<0} |\rho^{-1}(\theta)| \sum_{k\geq 1} \frac{s_k}{\beta_k - \alpha_k} (\beta_k - \alpha_k) \\ &= e^{1/\pi} \, \|(s_k)\|_{l^1} < +\infty \,.\end{aligned}$$

By Proposition 4.1, if we show that $H_{\rho^{1/2}}(\rho^{-1/2} f) = H(f)$ does not belong to $L^2(r^2/\rho)$, then m is not a multiplier of $\mathcal{H}(\overline{b})$.

A simple computation shows that the Hilbert transform of $\chi_{(-\beta_k - \alpha_k)}$ is $\log(|\theta + \beta_k|/|\theta + \alpha_k|)$, and this function is positive for $\theta > 0$. Thus, for $\theta > 0$ we have

$$H(f)(\theta) \geq d_k \log \frac{\theta + \beta_k}{\theta + \alpha_k} \,, \qquad \text{for all } k \geq 1 \,. \tag{6}$$

In particular, (6) holds for $\alpha_k < \theta < \beta_k$. Besides, if $\alpha_k < \theta < \beta_k$, $(2\alpha_k)^{-1} > (\theta + \alpha_k)^{-1} > (\beta_k + \alpha_k)^{-1}$, and consequently

$$\begin{aligned}\frac{\theta + \beta_k}{\theta + \alpha_k} &= 1 + \frac{\beta_k - \alpha_k}{\theta + \alpha_k} > 1 + \frac{\beta_k - \alpha_k}{\beta_k + \alpha_k} = \frac{2\,\beta_k}{\beta_k + \alpha_k} \\ &= \frac{2}{1 + \alpha_k/\beta_k} > \frac{2}{1+\gamma} = c > 1 \,.\end{aligned}$$

Therefore,

$$\log \frac{\theta + \beta_k}{\theta + \alpha_k} \geq \log c \,, \qquad \text{for all } \theta \in (\alpha_k, \beta_k) \,. \tag{7}$$

Now (6) and (7) yield

$$\begin{aligned}\int_{\alpha_k}^{\beta_k} |H(f)|^2 \, r^2 \, e^{1/\theta} \, d\theta &\geq \int_{\alpha_k}^{\beta_k} d_k^2 \, \log^2 \Big(\frac{\theta + \beta_k}{\theta + \alpha_k}\Big) \, r^2 \, e^{1/\theta} \, d\theta \\ &\geq \frac{s_k}{\beta_k - \alpha_k} \, \log^2 c \int_{\alpha_k}^{\beta_k} r^2 \, e^{1/\theta} \, d\theta \,.\end{aligned} \tag{8}$$

Then,

$$\int_0^\pi |H(f)|^2 r^2 e^{1/\theta}\, d\theta \geq \log^2 c \sum_{k\geq 1} \frac{s_k}{\beta_k - \alpha_k} \int_{\alpha_k}^{\beta_k} r^2\, e^{1/\theta}\, d\theta = +\infty$$

by (8) and (5). That is, $H(f) \notin L^2(r^2/\rho)$.

Theorem 4.3. *For $m = K_\rho(q) + c \in K^\infty(\rho)$, put $r = |q\rho|$. If for some constant $C > 0$,*

$$\int_0^\varepsilon r(\theta)^2\, e^{1/\theta}\, d\theta \leq C\,\varepsilon\,, \qquad \text{for } 0 < \varepsilon < \pi\,,$$

then m is a multiplier of $\mathcal{H}(\overline{b})$.

PROOF. By Proposition 4.1, we must show that $H_{\rho^{1/2}}(f) \in L^2(r^2/\rho)$ for every $f \in L^2$. For $f \in L^2$, the function $f\rho^{1/2}$ is in L^2, and the Hilbert transform maps L^2 into itself (see [9, III]), so that $H_{\rho^{1/2}}(f) \in L^2$. Besides, for $-\pi < \theta < 0$, $\rho^{-1}(\theta) = e^{1/(2\pi+\theta)}$ is bounded, and so is r^2/ρ. Thus $H_{\rho^{1/2}}(f)$ is square integrable with respect to the measure $r^2/\rho\, d\theta$ in $(-\pi, 0)$. So, we only have to show the square integrability in $(0,\pi)$. We can assume $f \geq 0$. Write $f = f_1 + f_2$, where $f_1 = f\chi_{(-\pi,0)}$ and $f_2 = f\chi_{(0,\pi)}$. For $0 < \theta < \pi$,

$$\begin{aligned} H_{\rho^{1/2}}(f_1)(\theta) &= \lim_{\varepsilon\to 0} \int\limits_{\varepsilon<|\varphi-\theta|<\pi} \frac{f_1(\varphi)\,\rho^{1/2}(\varphi)}{\theta-\varphi}\, d\varphi \\ &= \int_{-\pi}^0 \frac{f_1(\varphi)\,\rho^{1/2}(\varphi)}{\theta-\varphi}\, d\varphi\,. \end{aligned}$$

Since $f_1 \geq 0$, this equality shows that $H_{\rho^{1/2}}(f_1)(\theta)$ is decreasing for $0 < \theta < \pi$. Then for $\lambda > 0$, the set

$$E_\lambda = \{\theta \in (0,\pi) : \ |H_{\rho^{1/2}}(f_1)(\theta)| > \lambda\}$$

is some interval $(0, a_\lambda)$ with $0 \leq a_\lambda < \pi$ (the possibility $E_\lambda = \varnothing$ is covered by $a_\lambda = 0$). Denote by ν the measure on $(0,\pi)$ defined by $d\nu = r^2(\theta)\, e^{1/\theta}\, d\theta$. For a (Lebesgue) measurable set $F \subset \partial\mathbb{D}$ we write $|F|$ for its Lebesgue measure. By the hypothesis of the theorem,

$$\nu(E_\lambda) = \nu\big((0,a_\lambda)\big) = \int_0^{a_\lambda} r^2(\theta)\, e^{1/\theta}\, d\theta \leq C\, a_\lambda = C\,|E_\lambda|\,.$$

Hence,

$$\int_0^\pi |H_{\rho^{1/2}}(f_1)|^2\,d\nu = \int_0^\infty 2\,\lambda\,\nu(E_\lambda)\,d\lambda$$
$$\leq C\int_0^\infty 2\,\lambda\,|E_\lambda|\,d\lambda = C\int_0^\pi |H_{\rho^{1/2}}(f_1)|^2\,d\theta\,,$$

and the last integral is finite because $f_1\rho^{1/2}\in L^2$. For f_2 and $\theta\in(0,\pi)$ we have

$$\begin{aligned} H_{\rho^{1/2}}(f_2)(\theta) &= H(f_2(\varphi)e^{-1/2\varphi})(\theta) \\ &= H[f_2(\varphi)\,(e^{-1/2\varphi}-e^{-1/2\theta})](\theta) + H(f_2(\varphi)\,e^{-1/2\theta})(\theta) \\ &= I_1(\theta)+I_2(\theta)\,. \end{aligned}$$

The function $I_2(\theta)$ is equal to $e^{-1/2\theta}H(f_2)(\theta)$, hence

$$\begin{aligned} \int_0^\pi |I_2(\theta)|^2\,r^2\,e^{1/\theta}\,d\theta &= \int_0^\pi e^{-1/\theta}\,|H(f_2)(\theta)|^2\,r^2\,e^{1/\theta}\,d\theta \\ &\leq \|r\|_{L^\infty}^2\,\|H(f_2)\|_{L^2}^2 < +\infty\,. \end{aligned}$$

Finally,

$$I_1(\theta) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|\varphi-\theta|<\pi} f_2(\varphi)\,N(\varphi,\theta)\,d\theta\,,$$

where

$$N(\varphi,\theta) = \frac{e^{-1/2\varphi}-e^{-1/2\theta}}{\theta-\varphi}$$

can be continuously extended to $[0,\pi]\times[0,\pi]$, and therefore is bounded. Hence $|I_1(\theta)|\leq C\,\|f_2\|_{L^1}<+\infty$, which implies that $I_1(\theta)$ is square integrable with respect to the (finite) measure $r^2/\rho\;d\theta = r^2\,e^{1/\theta}\,d\theta$ in $(0,\pi)$.

For our example, Theorems 4.2 and 4.3 give a complete characterization of the multipliers of $\mathcal{H}(\bar{b})$ among the elements of $K^\infty(\rho)$. However, it is not clear at this point that there are elements in $K^\infty(\rho)$ which fail to satisfy the condition of the theorems. Theorem 2.3 will be the fundamental tool to construct such an element.

Corollary 4.4. *There are elements in $K^\infty(\rho)$ which are not multipliers of $\mathcal{H}(\bar{b})$.*

PROOF. If s is an admissible function for ρ, then Theorem 2.3 asserts that $m = e^{s+i\tilde{s}}$ is in $K^\infty(\rho)$. Besides, if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon s^2(\theta)\, e^{1/\theta}\, d\theta = +\infty\,,$$

part 2) of Theorem 2.3 together with Theorem 4.2 immediately implies that $m \notin \mathcal{M}(\overline{b})$. A straightforward calculation shows that if $0 < \alpha < 1/2$, then

$$s(\theta) = \begin{cases} \dfrac{e^{-1/2\theta}}{\theta^\alpha}\,, & \text{if } 0 < \theta \le \pi\,, \\ 0\,, & \text{if } -\pi < \theta \le 0\,, \end{cases}$$

does the job.

If we take as b an outer function such that $|b(e^{i\theta})| = (1 - e^{-1/\theta})^{1/2}$ almost everywhere with respect to $d\theta$, then b is invertible in H^∞. Hence by [13, Theorem 7.1], $\mathcal{H}(b)=\mathcal{H}(\overline{b})$.

5. Strong operator topology.

Let f and q be measurable functions on $\partial\mathbb{D}$. Denote

$$J_q(f) = \|qf\|_{L^2(\rho)} = \Big(\frac{1}{2\pi} \int_0^{2\pi} |qf|^2\, \rho\, d\theta\Big)^{1/2}.$$

Notice that if $K_\rho(q) + c$ is in $\mathcal{M}(b)$ (or in $\mathcal{M}(\overline{b})$) then by Theorem 3.1, $J_q(f) < \infty$ for all $f \in \mathcal{H}(b)$ (respectively $f \in \mathcal{H}(\overline{b})$). Actually, the above conditions are equivalent.

Lemma 5.1. *Let $m = K_\rho(q) + c \in \mathcal{M}(\overline{b})$ and $f = K_\rho(g) \in \mathcal{H}(\overline{b})$. Then*

$$\|mf\|_{\mathcal{H}(\overline{b})} \le J_q(f) + J_{\overline{m}_*}(g)\,.$$

PROOF. By Lemma 2.1, $mf = K_\rho(h)$, where h is the boundary function of $mf - \overline{m}_* \overline{f}_*$. This boundary function is

$$mf - \overline{m}_* \overline{f}_* = (m - \overline{m}_*)\, f + \overline{m}_*\, (f - \overline{f}_*) = (qf + \overline{m}_* g)\, \rho\,.$$

Thus $\|mf\|_{\mathcal{H}(\overline{b})} = \|qf + \overline{m}_* g\|_{L^2(\rho)} \le J_q(f) + J_{\overline{m}_*}(g)\,.$

Lemma 5.2. *Let* $m = K_\rho(q) + c \in \mathcal{M}(b)$*,* $f \in \mathcal{H}(b)$ *and* g *be the function in* $L^2(\rho)$ *such that* $T_{\overline{b}} f = K_\rho(g)$*. Then*

$$\|mf\|_{\mathcal{H}(b)} \leq \|mf\|_{H^2} + 2\, J_q(T_{\overline{b}} f) + J_{\overline{m}_*}(g) + J_q(f)\,.$$

Proof. The equality $\|mf\|^2_{\mathcal{H}(b)} = \|mf\|^2_{H^2} + \|T_{\overline{b}}(mf)\|^2_{\mathcal{H}(\overline{b})}$ implies

$$\|mf\|_{\mathcal{H}(b)} \leq \|mf\|_{H^2} + \|T_{\overline{b}}(mf)\|_{\mathcal{H}(\overline{b})}\,. \tag{9}$$

We have

$$T_{\overline{b}}(mf) = m\, T_{\overline{b}} f + P_+\{(\overline{b}f - P_+(\overline{b}f))\, m\}\,, \tag{10}$$

where P_+ is the orthogonal projection from L^2 onto H^2. The function $h = \overline{(1 - P_+)(\overline{b}f)}$ is in H^2_0, so Lemma 12.1 of [13] says that

$$P_+\{(\overline{b}f - P_+(\overline{b}f))\, m\} = P_+(\overline{h}(K_\rho(q) + c)) = K_\rho(\overline{h}q)\,.$$

Thus

$$\begin{aligned} \|P_+(\overline{h}m)\|_{\mathcal{H}(\overline{b})} &= \|\overline{h}q\|_{L^2(\rho)} \\ &= \|(\overline{b}f - P_+(\overline{b}f))\, q\|_{L^2(\rho)} \\ &\leq J_q(f) + J_q(T_{\overline{b}} f)\,. \end{aligned} \tag{11}$$

Besides, $m \in \mathcal{M}(\overline{b})$ (because $\mathcal{M}(b) \subset \mathcal{M}(\overline{b})$), so by Lemma 5.1,

$$\|m\, T_{\overline{b}} f\|_{\mathcal{H}(\overline{b})} \leq J_q(T_{\overline{b}} f) + J_{\overline{m}_*}(g)\,. \tag{12}$$

Therefore (9), (10), (11) and (12) yield the conclusion.

Theorem 5.3. $\mathcal{M}_\infty(b)$ *is dense in* $\mathcal{M}(b)$ *and* $\mathcal{M}(\overline{b})$ *with the respective strong operator topologies.*

Proof. We prove the theorem for $\mathcal{M}(b)$; the same argument works for $\mathcal{M}(\overline{b})$. Let $\Gamma = \{m \in K^\infty(\rho) : \ m_* = m^{-1}\}$. By Corollary 2.7, span$\,(\Gamma \cap \mathcal{M}(b))$ is dense in $\mathcal{M}(b)$ with the operator norm. So, it is enough to prove that every $m \in \Gamma \cap \mathcal{M}(b)$ can be approached (in the strong operator topology of $\mathcal{M}(b)$) by a sequence $(m_n) \subset \Gamma \cap \mathcal{M}_\infty(b)$.

By Theorem 2.3, $m = e^{s+i\tilde{s}}$, with s some admissible function. Consider

$$s_n(e^{i\theta}) = \begin{cases} s(e^{i\theta})\,, & \text{if } |s(e^{i\theta})| \le n\,\rho^{1/2}(e^{i\theta})\,, \\ n\,\rho^{1/2}(e^{i\theta})\,, & \text{if } |s(e^{i\theta})| > n\,\rho^{1/2}(e^{i\theta})\,. \end{cases}$$

Since $s_n^2/\rho \le n^2$, $m_n = e^{s_n+i\tilde{s}_n}$ is in $\mathcal{M}_\infty(b)$. Clearly $s_n \to s$ in L^2, so by the continuity in L^2 of the harmonic conjugation, also $\tilde{s}_n \to \tilde{s}$ in L^2. Taking a suitable subsequence, we can assume that $s_n(e^{i\theta}) \to s(e^{i\theta})$ and $\tilde{s}_n(e^{i\theta}) \to \tilde{s}(e^{i\theta})$ for almost every $e^{i\theta} \in \partial\mathbb{D}$.

By Theorem 2.3, $m = K_\rho(q) + c$ with $q = e^{i\tilde{s}}(e^s - e^{-s})/\rho$ and $c \in \mathbb{C}$; and $m_n = K_\rho(q_n) + c_n$ with $q_n = e^{i\tilde{s}_n}(e^{s_n} - e^{-s_n})/\rho$ and $c_n \in \mathbb{C}$. Hence, $m_n \to m$, $q_n \to q$ and $(m_n)_* = m_n^{-1} \to m^{-1} = m_*$ almost everywhere. Theorem 2.3 also shows that

$$|q_n| \le 2\,e^{\|s_n\|}\,\frac{|s_n|}{\rho} \le 2\,e^{\|s\|}\,\frac{|s|}{\rho} \le e^{\|s\|}\,|q|\,.$$

Thus $|q - q_n| \le C\,|q|$ for all $n \ge 1$, where $C > 0$. Since $m \in \mathcal{M}(b)$, then $hq \in L^2(\rho)$ for any $h \in \mathcal{H}(b)$. Hence, if $f \in \mathcal{H}(b)$ then $J_{q-q_n}(T_{\bar{b}}f)$ and $J_{q-q_n}(f)$ tend to zero when $n \to \infty$ by the dominated convergence theorem. Besides,

$$\max\{\|(m_n)_*\|_\infty\,,\ \|m_n\|_\infty\} \le e^{\|s\|_\infty}\,.$$

So, if $T_{\bar{b}}f = K_\rho(g)$, then $J_{m_*-(m_n)_*}(g)$ and $\|(m - m_n)f\|_{H^2}$ also tend to zero when $n \to \infty$ by the dominated convergence theorem. Thus, Lemma 5.2 shows that $\|(m - m_n)f\|_{\mathcal{H}(b)} \to 0$.

6. The singular component of the measure μ_b.

It is natural to ask how the singular component of the measure μ_b affects the algebras $\mathcal{M}(\bar{b})$, $\mathcal{M}(b)$ and $K^\infty(\rho)$. We address now this problem. Let b, b_1 be extreme points of $B(H^\infty)$, and u be an inner function such that $\mu_b = \mu_{b_1} + \mu_u$. Since u is inner, it is clear from the Herglotz representation (1) that μ_u is a singular measure. Conversely, every Borel positive finite singular measure is associated (via the Herglotz formula) to an inner function. Put $\rho_1 = 1 - |b_1|^2$, $\rho = 1 - |b|^2$ and σ for the Radon-Nikodym derivative of μ_b (and of μ_{b_1}) with respect to the normalized Lebesgue measure. In order to simplify notation, we assume without loss of generality that the respective additive imaginary constant for b_1, b and u in formula (1) is trivial.

Lemma 6.1. *Let $q \in L^2(\rho)$ and $q_1 \in L^2(\rho_1)$. Then $K_\rho(q) = K_{\rho_1}(q_1)$ if and only if $q\rho = q_1\rho_1$.*

PROOF. Suppose that $K(q\rho - q_1\rho_1) = 0$; then $q\rho - q_1\rho_1 \in \overline{H}^2_0$, so it must be trivial if it is not log-integrable. The equality

$$\frac{\rho_1}{|1-b_1|^2} = \sigma = \frac{\rho}{|1-b|^2}$$

implies that the sets $E = \{z \in \partial\mathbb{D} : \rho(z) \neq 0\}$ and $\{z \in \partial\mathbb{D} : \rho_1(z) \neq 0\}$ coincide almost everywhere. Then,

$$\begin{aligned} q\rho - q_1\rho_1 &= \Big(q\rho^{1/2}\Big(\frac{\rho}{\rho_1}\Big)^{1/2} - q_1\rho_1^{1/2}\Big)\,\rho_1^{1/2} \\ &= \Big(q\rho^{1/2}\Big|\frac{1-b}{1-b_1}\Big| - q_1\rho_1^{1/2}\Big)\,\rho_1^{1/2} \\ &= \big(q\rho^{1/2}|1-b| - q_1\rho_1^{1/2}|1-b_1|\big)\,\sigma^{1/2} = h\,\sigma^{1/2}\,, \end{aligned}$$

where the function h is in L^2. Thus, $\log|q\rho - q_1\rho_1| \leq \log^+|h| + (1/2)\log\sigma$ is not integrable and the lemma follows.

Lemma 6.2. *Let b, b_1 and u be as before. Then*

i) $\displaystyle 2\,\frac{1-b}{1-b_1} = 3 - b - 2\,\frac{1-b}{1-u}\,,$

ii) $\displaystyle 2\,\frac{1-b_1}{1-b} = 1 + b_1 + 2\,\frac{1-b_1}{1-u}\,.$

PROOF. Both formulas are straightforward calculations from the identity

$$\frac{1+b}{1-b} = \frac{1+b_1}{1-b_1} + \frac{1+u}{1-u}$$

given by the Herglotz representations associated to b, b_1 and u.

Theorem 6.3. *Let b, b_1 and u be as before.*

1) *Let $m = K_\rho(q) \in K^\infty(\rho)$, $|q\rho| = r$. Then*

$$m \in K^\infty(\rho_1) \ \text{ if and only if } \ (1-u)^{-1} \in L^2(r^2/\sigma)\,.$$

2) *Let $m_1 = K_{\rho_1}(q_1) \in K^\infty(\rho_1)$, $|q_1\rho_1| = r_1$. Then*

$$m_1 \in K^\infty(\rho) \ \text{ if and only if } \ (1-u)^{-1} \in L^2(r_1^2/\sigma)\,.$$

PROOF. 1) Let $m = K_\rho(q) \in K^\infty(\rho)$. If $m(z) = K_\rho(q)(z) = K_{\rho_1}(q_1)(z) + c$, with $q_1 \in L^2(\rho_1)$ and $c \in \mathbb{C}$, then letting $z \to \infty$ we obtain that $c = 0$. Hence $K_\rho(q) = K_{\rho_1}(q_1)$, and Lemma 6.1 says that this happens if and only if $r = |q\rho|$ is admissible for ρ_1 (so $q_1 = q\,\rho/\rho_1$). That is, if and only if $r^2/\rho_1 \in L^1$. Now

$$\frac{r^2}{\rho_1} = \frac{\rho}{\rho_1}\,\frac{r^2}{\rho} = \left|\frac{1-b}{1-b_1}\right|^2 \frac{r^2}{\rho} = \left|\frac{3-b}{2} - \frac{1-b}{1-u}\right|^2 \frac{r^2}{\rho}\,,$$

where the last equality follows from i) of Lemma 6.2. Since $(3-b)/2$ is bounded and r^2/ρ is in L^1, we have that $r^2/\rho_1 \in L^1$ if and only if

$$\left|\frac{1-b}{1-u}\right|^2 \frac{r^2}{\rho} \in L^1\,,$$

or, what is the same, if and only if $(1-u)^{-1} \in L^2(r^2/\sigma)$. Assertion 2) follows in the same way using formula ii) of Lemma 6.2.

Theorem 6.4. *Let b and b_1 be as before. Then $\mathcal{M}(b) \subset \mathcal{M}(b_1)$.*

PROOF. Let $m = K_\rho(q) + c \in \mathcal{M}(b)$. We will show first that m belongs to $K^\infty(\rho_1)$. The measure μ_{b_1} decomposes as $d\mu_{b_1} = \sigma\, d\theta/2\pi + d\mu_{S_1}$, where μ_{S_1} is the singular component of μ_{b_1}. On the other hand, μ_u can be decomposed as $d\mu_u = \alpha\, d\mu_{S_1} + d\mu_0$, where $\alpha \in L^1(\mu_{S_1})$, $\alpha \geq 0$ is the Radon-Nikodym derivative of μ_u with respect to μ_{S_1}, and μ_0 is singular with respect to μ_{S_1}. These decompositions together show that the measure $d\nu = (1+\alpha)\, d\mu_{S_1} + d\mu_0$ is the singular component of $d\mu_b$. Put $r = |q\rho|$; since $m \in \mathcal{M}(b)$, Theorem 3.2.2) asserts that $K_\nu(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\nu)$. Let χ be a function which takes the value 1 almost everywhere with respect to $d\mu_{S_1}$ and the value 0 almost everywhere with respect to $d\mu_0$, and consider $f = \alpha(1+\alpha)^{-1}\chi + 1 - \chi$. Since $\alpha \geq 0$ and μ_{S_1} and μ_0 are finite measures, then $f \in L^2(\nu)$ (f is bounded almost everywhere with respect to $d\nu$). Thus $K_\nu(f)$ is in $L^2(r^2/\sigma)$. But

$$\begin{aligned} K_\nu(f) &= K_{(1+\alpha)\mu_{S_1}}\,(\alpha(1+\alpha)^{-1}\chi) + K_{\mu_0}\,(1-\chi) \\ &= K_{\mu_{S_1}}(\alpha\chi) + K_{\mu_0}(1) \\ &= K_{\alpha\mu_{S_1}+\mu_0}(1) = K_{\mu_u}(1)\,. \end{aligned}$$

Hence $K_{\mu_u}(1) \in L^2(r^2/\sigma)$. It is well known [15, III-7] that

$$(1-u)\,K_{\mu_u}(1) = (1-\overline{u(0)})^{-1}\,(1-\overline{u(0)}u)\,. \tag{13}$$

Since $|(1-\overline{u(0)})^{-1}(1-\overline{u(0)}u)|$ is bounded from below by a positive constant, we obtain that $(1-u)^{-1} \in L^2(r^2/\sigma)$. Now Theorem 6.3.1) says that $m \in K^\infty(\rho_1)$.

The fact that $m \in \mathcal{M}(b)$ implies by Theorem 3.2.2), that $K_{\sigma^{1/2}}(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\chi_E)$. So, by the same theorem, in order to prove that $m \in \mathcal{M}(b_1)$ we must show that if $g \in L^2(\mu_{S_1})$ then $K_{\mu_{S_1}}(g) \in L^2(r^2/\sigma)$. Consider the function $g(1+\alpha)^{-1}\chi$. Since

$$\begin{aligned}|g\,(1+\alpha)^{-1}\chi|^2\,d\nu &= |g|^2\,|1+\alpha|^{-2}\,(1+\alpha)\,d\mu_{S_1}\\ &= |g|^2\,(1+\alpha)^{-1}\,d\mu_{S_1} \le |g|^2\,d\mu_{S_1}\,,\end{aligned}$$

then $g(1+\alpha)^{-1}\chi$ belongs to $L^2(\nu)$. Therefore, since m is a multiplier of $\mathcal{H}(b)$, Theorem 3.2.2) says that $K_\nu\,(g\,(1+\alpha)^{-1}\chi)$ is in $L^2(r^2/\sigma)$; but $K_\nu\,(g\,(1+\alpha)^{-1}\chi) = K_{(1+\alpha)\mu_{S_1}}(g\,(1+\alpha)^{-1}) = K_{\mu_{S_1}}(g)$, and the theorem follows.

Two particular cases are of special interest in Theorem 6.4, when μ_{b_1} is absolutely continuous, and when μ_u is singular with respect to the singular component of μ_{b_1} (*i.e.* $\alpha = 0$ in the proof of the theorem). If b_1 is a nonextreme point of $B(H^\infty)$ and μ_{b_1} is absolutely continuous, Theorem 6.4 was obtained by Davis and McCarthy [5].

Theorem 6.5. *Let b_1 be an extreme point of $B(H^\infty)$ and $\mu_S = \mu_1 + \cdots + \mu_n$ be a purely atomic measure, where each μ_j $(1 \le j \le n)$ is an atom at the point $\omega_j = e^{i\varphi_j} \in \partial\mathbb{D}$ (with $\omega_j \ne \omega_k$ if $j \ne k$). Let $b \in B(H^\infty)$ such that $\mu_b = \mu_{b_1} + \mu_S$. If $m = K_{\rho_1}(q_1) + c$ (with $q \in L^2(\rho_1)$, $c \in \mathbb{C}$ and $r = |q\rho_1|$) is a multiplier of $\mathcal{H}(b_1)$, then the following conditions are equivalent.*

1) $m \in K^\infty(\rho)$.

2) $K_{\mu_S}(1) \in L^2(r^2/\sigma)$.

3) $K_{\mu_j}(1) \in L^2(r^2/\sigma)$ *for every* j.

4) $m \in \mathcal{M}(b)$.

5) $f_j(\theta) = (\theta-\varphi_j)^{-2}\,r^2(e^{i\theta})/\sigma(e^{i\theta}) \in L^1[d\theta,(\varphi_j-\pi,\varphi_j+\pi)]$ *for all* j.

Proof. 1) if and only if 2) is in Theorem 6.3.2), using again that if u is the inner function associated to μ_S, then $(1-u)^{-1}$ behaves like $K_{\mu_S}(1)$ (formula (13)).

2) implies 3). Let $V \subset \partial\mathbb{D}$ be an open neighborhood of ω_1 such that the closure of V does not contain any of the ω_j, $2 \le j \le n$. Then $K_{\mu_1}(1)$ is continuous on $\partial\mathbb{D} \setminus V$ and therefore it is square integrable with respect to the measure $r^2/\sigma\, d\theta$ there. On the other hand,

$$K_{\mu_1}(1) = K_{\mu_S}(1) - \sum_{j=2}^{n} K_{\mu_j}(1)\,,$$

and since $\sum_{j=2}^{n} K_{\mu_j}(1)$ is continuous on V and by hypothesis $K_{\mu_S}(1) \in L^2(r^2/\sigma)$, then $K_{\mu_1}(1)$ is also square integrable with respect to $r^2/\sigma\, d\theta$ in V. Analogously, $K_{\mu_j}(1) \in L^2(r^2/\sigma)$ for all $2 \le j \le n$.

3) implies 4). Hypothesis 3) clearly implies that $K_{\mu_S}(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\mu_S)$. In particular 2) holds, and since 2) implies 1), $m \in K^\infty(\rho)$. Since $m \in \mathcal{M}(b_1)$, by Theorem 3.2.2) and the comments preceding it, $K_{\mu_{b_1}}(h) \in L^2(r^2/\sigma)$ for all $h \in L^2(\mu_{b_1})$. The decomposition $\mu_b = \mu_{b_1} + \mu_S$ now clearly implies that $K_{\mu_b}(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\mu_b)$. Hence by Theorem 3.2 again, $m \in \mathcal{M}(b)$.

Obviously 4) implies 1). To prove the equivalence between 3) and 5), write $\alpha_j = \|\mu_j\|$. Then $K_{\mu_j}(1)(e^{i\theta}) = \alpha_j(1 - \overline{\omega}_j e^{i\theta})^{-1}$. Therefore,

$$|K_{\mu_j}(1)(e^{i\theta})|^2 = |\alpha_j|^2 |e^{i\varphi_j} - e^{i\theta}|^{-2} = |\alpha_j|^2\, 2^{-1} \left(1 - \cos(\theta - \varphi_j)\right)^{-1}.$$

The equivalence now follows from the fact that $1 - \cos(\theta - \varphi_j)$ behaves like $(\theta - \varphi_j)^2$ when $|\theta - \varphi_j| < \pi$.

7. A partial isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(\overline{b})$.

If φ and f are measurable functions on $\partial\mathbb{D}$ such that $\varphi f \in L^2$, we define $T_\varphi(f) = P_+(\varphi f)$, where P_+ is the orthogonal projection from L^2 onto H^2. Hence T_φ is an operator defined on the space $\{f$ measurable: $\varphi f \in L^2\}$. If $\psi \in L^\infty$, M_ψ will denote the operator on L^2 of multilplication by ψ.

Lemma 7.1. *The operators $T_{1-\overline{b}} K_{\sigma^{1/2}}$ and $K_{\sigma^{1/2}} M_{1-\overline{b}}$ are contractions from $L^2(\chi_E)$ into L^2 and coincide.*

PROOF. Notice that since $(1-b)\, K_{\sigma^{1/2}}(f) \in \mathcal{H}(b) \subset H^2$ for $f \in L^2(\chi_E)$, then $(1 - \overline{b})\, K_{\sigma^{1/2}}(f) \in L^2$, so $T_{1-\overline{b}} K_{\sigma^{1/2}}$ is well defined on $L^2(\chi_E)$.

Let $f = (1-\overline{b})\, g$, with $g \in L^2(\chi_E)$, then

$$\begin{aligned} T_{1-\overline{b}}\, K_{\sigma^{1/2}}\big((1-\overline{b})\, g\big) &= T_{1-\overline{b}}\, K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, g\Big) \\ &= K_{\rho^{1/2}}\Big(\frac{(1-\overline{b})^2}{|1-b|}\, g\Big) \\ &= K_{\sigma^{1/2}}\big((1-\overline{b})\, f\big) \\ &= K_{\sigma^{1/2}}\, M_{1-\overline{b}} f\,, \end{aligned}$$

where the second equality follows from [13, Corollary 3.5]. Hence both operators coincide on $(1-\overline{b})L^2(\chi_E)$. This is a dense subspace of $L^2(\chi_E)$, because if h is orthogonal to this subspace, then for all $g \in L^2(\chi_E)$,

$$0 = \langle h, (1-\overline{b})\, g\rangle = \langle (1-b)\, h, g\rangle\,,$$

which implies $(1-b)\, h\chi_E = 0$, so $h = 0$ almost everywhere with respect to $d\theta$ on E. Therefore, we only have to show that both operators are contractions. Let $f \in L^2(\chi_E)$; then

$$\begin{aligned} \|T_{1-\overline{b}}\, K_{\sigma^{1/2}}(f)\|_{L^2} &= \|P_+[(1-\overline{b})\, K_{\sigma^{1/2}}(f)]\|_{L^2} \\ &\le \|(1-\overline{b})\, K_{\sigma^{1/2}}(f)\|_{L^2} \\ &= \|(1-b)\, K_{\sigma^{1/2}}(f)\|_{H^2} \\ &\le \|(1-b)\, K_{\sigma^{1/2}}(f)\|_{\mathcal{H}(b)} \\ &= \|f\|_{L^2(\chi_E)}\,. \end{aligned}$$

Also,

$$\begin{aligned} \|K_{\sigma^{1/2}}\big((1-\overline{b})\, f\big)\|_{L^2} &= \Big\|K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, f\Big)\Big\|_{L^2} \\ &= \Big\|K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, f\Big)\Big\|_{H^2} \\ &\le \Big\|K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, f\Big)\Big\|_{\mathcal{H}(\overline{b})} \\ &= \Big\|\frac{1-\overline{b}}{|1-b|}\, f\Big\|_{L^2(\chi_E)} = \|f\|_{L^2(\chi_E)}\,. \end{aligned}$$

The decomposition of the measure $\mu_b = \sigma\, d\theta/2\pi + d\mu_S$ induces an orthogonal decomposition $L^2(\mu_b) = L^2(\sigma) \oplus L^2(\mu_S)$, which according to our treatment we identify with $L^2(\chi_E) \oplus L^2(\mu_S)$ (via the onto isometry $(f,g) \mapsto (\sigma^{1/2} f, g)$). This decomposition translates into an orthogonal decomposition for $\mathcal{H}(b)$ as $\mathcal{H}(b) = \mathcal{H}(b)^\sigma \oplus \mathcal{H}(b)^S$, where

$$\mathcal{H}(b)^\sigma = (1-b)\, K_{\sigma^{1/2}}(L^2(\chi_E))$$

and

$$\mathcal{H}(b)^S = (1-b)\, K_{\mu_S}(L^2(\mu_S))\,.$$

Theorem 7.2. *$T_{(1-\overline{b})/(1-b)}$ is a partial isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(\overline{b})$ with initial space $\mathcal{H}(b)^\sigma$. Further, if $g \in L^2(\chi_E)$,*

$$T^*_{(1-\overline{b})/(1-b)}(K_{\rho^{1/2}}(g)) = (1-b)\, K_{\sigma^{1/2}}\Big(\frac{1-b}{|1-b|}\, g\Big)\,.$$

PROOF. First we show that $\mathcal{H}(b)^S$ is contained in the kernel of $T_{(1-\overline{b})/(1-b)}$. Denote by u the inner function associated to μ_S in (1). Let $f \in \mathcal{H}(b)^S$; then there is $g \in L^2(\mu_S)$ such that

$$f = (1-b)\, K_{\mu_S}(g) = \frac{1-b}{1-u}\,(1-u)\, K_{\mu_S}(g) \in \frac{1-b}{1-u}\, \mathcal{H}(u)\,.$$

Besides, $\|f\|_{\mathcal{H}(b)} = \|g\|_{L^2(\mu_S)} = \|(1-u)\, K_{\mu_S}(g)\|_{\mathcal{H}(u)}$. We can now begin with $g \in L^2(\mu_S)$, obtaining that

$$\mathcal{H}(b)^S = \frac{1-b}{1-u}\, \mathcal{H}(u)\,.$$

It is well known that the span of the functions

$$k^u_\omega(e^{i\theta}) = \frac{1-\overline{u(\omega)}\, u(e^{i\theta})}{1-\overline{\omega}\, e^{i\theta}}\,, \qquad \omega \in \mathbb{D}\,,$$

is dense in $\mathcal{H}(u)$. Thus the span of the functions $(1-b)\,(1-u)^{-1} k^u_\omega$ ($\omega \in \mathbb{D}$) is dense in $\mathcal{H}(b)^S$. Hence, it is enough to prove that these functions belong to the kernel of $T_{(1-\overline{b})/(1-b)}$. Let us denote by z the function $z(e^{i\theta})=e^{i\theta}$. Then

$$T_{(1-\overline{b})/(1-b)}\Big(\frac{1-b}{1-u}\,k_\omega^u\Big)=P_+\Big(\frac{(1-\overline{b})\,(1-\overline{u(\omega)})\,u}{(1-u)\,(1-\overline{\omega}\,z)}\Big)$$
$$=P_+\Big(\frac{(1-\overline{b})\,(\overline{u}-\overline{u(\omega)})\,\overline{z}}{(\overline{u}-1)\,(\overline{z}-\overline{\omega})}\Big)=P_+(\overline{g})\,,$$

where
$$g=-\frac{(1-b)\,(u-u(\omega))\,z}{(1-u)\,(z-\omega)}\,.$$

In [15, III-11] it is proved that $(1-b)\,(1-u)^{-1}$ belongs to H^2; therefore $g\in H_0^2$ and consequently $P_+(\overline{g})=0$. Now let $f\in L^2(\chi_E)$. By Lemma 7.1,

$$T_{(1-\overline{b})/(1-b)}\big((1-b)\,K_{\sigma^{1/2}}(f)\big)=T_{1-\overline{b}}\,K_{\sigma^{1/2}}(f)$$
$$=K_{\sigma^{1/2}}\big((1-\overline{b})\,f\big)=K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\,f\Big)\,,$$

and clearly
$$\Big\|\frac{1-\overline{b}}{|1-b|}\,f\Big\|_{L^2(\chi_E)}=\|f\|_{L^2(\chi_E)}\,.$$

That is, $T_{(1-\overline{b})/(1-b)}$ maps $\mathcal{H}(b)^\sigma$ isometrically into $\mathcal{H}(\overline{b})$. To see that this map is onto, let $g\in L^2(\chi_E)$ and take $f=(1-b)\,g/|1-b|$. By Lemma 7.1,

$$T_{(1-\overline{b})/(1-b)}\big((1-b)\,K_{\sigma^{1/2}}(f)\big)=K_{\sigma^{1/2}}\Big(\frac{|1-b|^2}{|1-b|}\,g\Big)=K_{\rho^{1/2}}(g)\,.$$

This also proves the formula for $T^*_{(1-\overline{b})/(1-b)}$.

Corollary 7.3. *The measure μ_b is absolutely continuous if and only if*

$$T_{(1-\overline{b})/(1-b)}\,(1-T_b\,T_{\overline{b}})^{1/2}$$

is one-to-one (from H^2 into H^2).

PROOF. By Theorem 7.2, μ_b is absolutely continuous if and only if $T_{(1-\overline{b})/(1-b)}|_{\mathcal{H}(b)}$ is one-to-one. Hence, the corollary will follow if we show that $(1-T_b T_{\overline{b}})^{1/2}$ is one-to-one. Since b is not an inner function, $\|T_{\overline{b}}f\|_{H^2} \le \|\overline{b}\, f\|_{L^2} < \|f\|_{H^2}$ unless $f=0$. Hence, $f \ne T_b T_{\overline{b}} f$ if $f \ne 0$.

Theorem 7.4. *K_{μ_b} maps $L^2(\mu_b)$ into $L^2(\rho)$.*

PROOF. Let $h \in L^2(\mu_b)$, and consider $f=(1-b)\,K_{\mu_b}(h) \in \mathcal{H}(b)$. Then $T_{\overline{b}}f$ is in $\mathcal{H}(\overline{b})$, and

$$\begin{aligned} T_{\overline{b}}f &= P_+\big((\overline{b}-1+1-|b|^2)\,K_{\mu_b}(h)\big) \\ &= -P_+\big((1-\overline{b})\,K_{\mu_b}(h)\big) + P_+\big(\rho\,K_{\mu_b}(h)\big) \\ &= -T_{(1-\overline{b})/(1-b)}f + K\big(\rho\,K_{\mu_b}(h)\big)\,. \end{aligned}$$

Notice that $\rho\,|K_{\mu_b}(h)| \le 2\,|(1-b)\,K_{\mu_b}(h)| \in L^2$. By Theorem 7.2 the first summand is in $\mathcal{H}(\overline{b})$, therefore $K(\rho\,K_{\mu_b}(h))$ belongs to $\mathcal{H}(\overline{b})$, too. Then there is $q \in L^2(\rho)$ such that $K(\rho\,K_{\mu_b}(h)-\rho\,q)=0$, or equivalently, $\rho\,K_{\mu_b}(h)-\rho\,q \in \overline{H}_0^2$. Now,

$$\log|\rho\,K_{\mu_b}(h)-\rho\,q| \le \log^+|\rho^{1/2}\,K_{\mu_b}(h)-\rho^{1/2}\,q| + \frac{1}{2}\log\rho\,,$$

and since ρ is not log-integrable, $\rho\,K_{\mu_b}(h)-\rho\,q$ cannot be log-integrable if we prove that $\rho^{1/2}\,K_{\mu_b}(h)-\rho^{1/2}\,q$ is in L^1. The function $\rho^{1/2}\,q$ is in L^2. Besides

$$\rho^{1/2}\,|K_{\mu_b}(h)| = \frac{\rho^{1/2}}{|1-b|}\,|(1-b)\,K_{\mu_b}(h)| = \sigma^{1/2}\,|f|\,,$$

which is in L^1 because it is the product of two functions of L^2. Hence $K_{\mu_b}(h)(e^{i\theta}) = q(e^{i\theta})$ almost everywhere with respect to the measure $\rho(e^{i\theta})\,d\theta$, so $K_{\mu_b}(h) \in L^2(\rho)$.

A direct consequence of the above theorem is that $V_b=(1-b)\,K_{\mu_b}$ maps $L^2(\mu_b)$ into $L^2(\sigma)$, in other words $\mathcal{H}(b) \subset L^2(\sigma)$. Let us return to the multipliers.

Corollary 7.5. *Let $m = K_\rho(q)+c \in K^\infty(\rho)$, and put $r=|q\,\rho|$. A sufficient condition for m to be a multiplier of $\mathcal{H}(b)$ is that there exists a constant $C>0$ such that $r^2/\sigma \le C\rho$ (or what is equivalent, $|q|\chi_E \le C^{1/2}|1-b|^{-1}\chi_E$, where $E=\{z \in \partial\mathbb{D}:\ \rho(z)\ne 0\}$).*

PROOF. By Theorem 3.2, $m \in \mathcal{M}(b)$ if and only if $K_{\mu_b}(h) \in L^2(r^2/\sigma)$ for all $h \in L^2(\mu_b)$. By Theorem 7.4 this holds if $L^2(\rho) \subset L^2(r^2/\sigma)$, and this is clearly equivalent to $r^2/\sigma \leq C\rho$ for some constant $C > 0$. Besides,

$$\frac{r^2}{\sigma} \leq C\,\rho \;\text{ if and only if }\; |q|^2\,\rho^2 = r^2 \leq C\,\rho\sigma = C\,\frac{\rho^2}{|1-b|^2}\,,$$

which is equivalent to

$$|q|^2\,\chi_E \leq C\,\frac{\chi_E}{|1-b|^2}$$

and

$$|q|\,\chi_E \leq C^{1/2}\,\frac{\chi_E}{|1-b|}\,.$$

REMARK 7.6. If s is any bounded real valued function which satisfies $s^2/\sigma \leq C\,\rho$ for some constant $C > 0$, then $s^2/\rho \leq C\,\sigma \in L^1$, that is, s is admissible for ρ. Hence $m = e^{s+i\tilde{s}} \in K^\infty(\rho)$, and if $m = K_\rho(q) + c$, then $r = |q\,\rho|$ behaves like s. Therefore the corollary asserts that $m \in \mathcal{M}(b)$.

The unexpected condition for multipliers given by Corollary 7.5 is not always necessary. For instance, let b be an outer function such that $\rho(e^{i\theta}) = e^{-1/|\theta|}$ for $\theta \in [-\pi, \pi)$. Then b is continuous on $\partial\mathbb{D}$ because b is outer and $|b|$ is continuously differentiable on $\partial\mathbb{D}$. Moreover, $|b(1)| = 1$, so we can assume multiplying by $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ if need be, that $b(1) = -1$. The function $\rho^{1/2}$ is admissible, even more, $m = e^{\rho^{1/2}+\tilde{\rho}^{1/2}} \in \mathcal{M}_\infty(b)$ because $\left(\rho^{1/2}\right)^2/\rho = \chi_E$ is bounded (see Section 2). If $m = K_\rho(q) + c$, $r = |q\,\rho|$, and r satisfies the condition of Corollary 7.5, then also $\rho^{1/2}$ satisfies this condition, that is, $\rho/\sigma \leq C\,\rho$. This is equivalent to $|1 - b(z)|^2 \leq C\,(1 - |b(z)|^2)$ for all $z \in \partial\mathbb{D}$. And this inequality obviously does not hold for z close to 1.

Corollary 7.7. *Let b be an extreme point and u be an inner function. If $\sigma_{ub} = \rho/|1-ub|^2$, then $\mathcal{H}(b) \subset L^2(\sigma_{ub})$.*

PROOF. If $s = \rho^{1/2}\,\sigma_{ub}^{1/2}$ then $s^2/\sigma_{ub} = \rho$, so by Remark 7.6, $m = e^{s+i\tilde{s}}$ belongs to $\mathcal{M}(ub)$. In particular, m is in $\mathcal{M}(b)$, thus

$$(1-b)\,K_{\mu_b}(f) \in L^2(s^2/\rho) = L^2(\sigma_{ub})\,, \qquad \text{for all } \; f \in L^2(\mu_b)\,.$$

That is, $\mathcal{H}(b) \subset L^2(\sigma_{ub})$.

The idea of the example in Remark 7.6 will be exploited more in the sequel. For expository reasons, it will be convenient to prove the next lemma in $\mathbb{C}_+ = \{z = x + iy \in \mathbb{C} : \ y > 0\}$. Of course, the result also holds in the disc (with obvious translation).

Lemma 7.8. *Let (α_k) be a sequence of real numbers such that $\alpha_k \neq \alpha_j$ if $k \neq j$ and $\lim \alpha_k = \alpha$, with $\alpha \neq \alpha_k$ for all k. Let (ω_k) be a sequence in $\partial\mathbb{D}$ and (ε_k) be a decreasing sequence of positive numbers that tends to zero. Then there exists an interpolating Blaschke product B, continuous on the closure of $\mathbb{C}_+$ except in $z = \alpha$, such that $|B(\alpha_k) - \omega_k| < \varepsilon_k$ for all k .*

Proof. We can assume $\varepsilon_k < 1$ for all k. Take $d_1 = (1/4)\inf_{j\neq 1}|\alpha_1 - \alpha_j|$ and $r_1 = \varepsilon_1 d_1/2^2$. Consider the half circle $S_1 = \{z \in \mathbb{C}_+ : \ |z-\alpha_1| = r_1\}$. There is $z_1 \in S_1$ such that

$$\left|\operatorname{Arg}\left(\frac{\alpha_1 - z_1}{\alpha_1 - \overline{z}_1}\right) - \operatorname{Arg}\omega_1\right| < \frac{\varepsilon_1}{2} \ ,$$

where Arg is the argument taken in $[0, 2\pi)$. Hence, if $b_1(z) = (z - z_1)/(z-\overline{z}_1)$ then $|b_1(\alpha_1)-\omega_1| < \varepsilon_1/2$. If $x \in \mathbb{R}$ is such that $|x-\alpha_1| > d_1$, then $\operatorname{Arg}((x - z_1)/(x - \overline{z}_1))$ belongs to the union of the intervals $(0, a_1)$ and $(2\pi - a_1, 2\pi)$, where $a_1 = 2\arctan(r_1/d_1) \leq 2\,r_1/d_1 = \varepsilon_1/2$. We can repeat the process with α_2, taking $d_2 = (1/4)\inf_{j\neq 2}|\alpha_2 - \alpha_j|$, $r_2 = \varepsilon_2 d_2/2^3$ and $\overline{b_1(\alpha_2)}\,\omega_2$ instead of ω_2. So, we obtain a point $z_2 \in S_2 = \{z \in \mathbb{C}_+ : \ |z_2 - \alpha_2| = r_2\}$ such that if $b_2(z) = (z - z_2)/(z - \overline{z}_2)$, then

$$|b_2(\alpha_2) - \overline{b_1(\alpha_2)}\,\omega_2| < \frac{\varepsilon_2}{2} \ ,$$

and for $x \in \mathbb{R}$ with $|x - \alpha_2| > d_2$, $\operatorname{Arg} b_2(x) \in (0, a_2) \cup (2\pi - a_2, 2\pi)$, where $a_2 < 2\,r_2/d_2 = \varepsilon_2/2^2$. Consider the Blaschke product $B_2 = b_2\,b_1$. Then,

$$\begin{aligned} |B_2(\alpha_2) - \omega_2| &= |b_2(\alpha_2)\,b_1(\alpha_2) - \omega_2| \\ &= |b_2(\alpha_2) - \omega_2\,\overline{b_1(\alpha_2)}| < \frac{\varepsilon_2}{2} \end{aligned} \tag{1}$$

and

$$\begin{aligned} |B_2(\alpha_1) - \omega_1| &\leq |b_2(\alpha_1)\,b_1(\alpha_1) - b_2(\alpha_1)\,\omega_1| + |b_2(\alpha_1)\,\omega_1 - \omega_1| \\ &= |b_1(\alpha_1) - \omega_1| + |b_2(\alpha_1) - 1| \\ &< \frac{\varepsilon_1}{2} + a_2 < \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} \ , \end{aligned} \tag{2}$$

where (2) holds because $|\alpha_1 - \alpha_2| > d_2$. Repeating this process k times, where $d_k = (1/4)\inf_{j\neq k}|\alpha_k - \alpha_j|$, $r_k = \varepsilon_k d_k/2^{k+1}$ and ω_k is replaced by $\overline{B_{k-1}(\alpha_k)}\,\omega_k$, we obtain a point $z_k \in S_k = \{z \in \mathbb{C}_+ : |z - \alpha_k| = r_k\}$ such that if $B_k = b_k B_{k-1}$, then

$$(1') \qquad |B_k(\alpha_k) - \omega_k| < \frac{\varepsilon_k}{2}$$

and

$$(2') \qquad |B_k(\alpha_j) - \omega_j| < \frac{\varepsilon_j}{2^j} + \frac{\varepsilon_{j+1}}{2^{j+1}} + \cdots + \frac{\varepsilon_k}{2^k}\,, \qquad \text{for all } j < k\,.$$

For $j < k$ the fact that (ε_n) is a decreasing sequence implies

$$(14) \qquad |B_k(\alpha_j) - \omega_j| < \sum_{n=j}^{k} \frac{\varepsilon_n}{2^n} < \varepsilon_j \sum_{n=j}^{k} \frac{1}{2^n} < \varepsilon_j\,.$$

The sequence (B_k) obtained in this process is the sequence of partial products of $B(z) = \prod_{k=1}^{\infty}(z - z_k)/(z - \overline{z}_k)$, where the points z_k are as above. The usual factors used to make the arguments convergent are not required because $\{z_k : k \geq 1\}$ is bounded.

Simple estimations show that $|z_k - z_j|/|z_k - \overline{z}_j| > 1/3$ for $k \neq j$. Since $\operatorname{Im} z_k \leq r_k < C\,2^{-k}$ for some $C > 0$, it is clear that $B(z)$ is an interpolating Blaschke product (see [9, VII]). It is well known that the set of continuity on $\mathbb{C}_+$ of a Blaschke product coincides with the complement of the limit set of its zeros in $\mathbb{R}$. Then B is continuous on $\mathbb{C}_+ \setminus \{\alpha\}$ and by (14), $|B(\alpha_k) - \omega_k| < \varepsilon_k$ for all $k \geq 1$.

Theorem 7.9. *The following conditions are equivalent.*

1) $\mathcal{M}_\infty(b) = K^\infty(\rho)$.

2) $\mathcal{M}_\infty(b) = \mathcal{M}(b)$.

3) *There is a constant* $\delta > 0$ *such that* $\rho(e^{i\theta}) \geq \delta\,\chi_E(e^{i\theta})$ *almost everywhere with respect to* $d\theta$.

4) *For every inner function* u *there is a constant* $C = C(u) > 0$ *such that*

$$\frac{1 - |b(e^{i\theta})|^2}{|1 - u(e^{i\theta})\,b(e^{i\theta})|^2} \leq C$$

almost everywhere with respect to $d\theta$.

5) *For every inner function* u *there is a constant* $\varepsilon = \varepsilon(u) > 0$ *such that*

$$\varepsilon\, \chi_E(e^{i\theta}) \leq \frac{1 - |b(e^{i\theta})|^2}{|1 - u(e^{i\theta})\, b(e^{i\theta})|^2}$$

almost everywhere with respect to $d\theta$.

6) *Condition* 4) *holds with* C *independent of* u.

7) *Condition* 5) *holds with* ε *independent of* u.

PROOF. The equivalence of 1) and 3) is in the comments following the definition of admissible function (Section 2). The string of inclusions (2) in Section 1 clearly shows that 1) implies 2).

2) implies 4). Take $s : \partial\mathbb{D} \to \mathbb{R}$ bounded such that $s^2/\rho \leq C\,\sigma$ (where C is some positive constant). As we pointed out in Remark 7.6, s is admissible and $m = e^{s + i\tilde{s}} = K_\rho(q) + c$ belongs to $\mathcal{M}(b)$, where $r = |q\,\rho|$ behaves like s. Hypothesis 2) says that $m \in \mathcal{M}_\infty(b)$. This is equivalent to the boundedness of s^2/ρ (Section 2). So, $s^2/\rho \leq C\,\sigma$ implies that s^2/ρ is bounded. Take $s = \rho^{1/2}\sigma^{1/2} = (1 - |b|^2)\,|1 - b|^{-1} \leq 2$. Then $s^2/\rho = \rho\,\sigma/\rho = \sigma$, and consequently $s^2/\rho = \sigma$ must be bounded. We arrived to this conclusion only assuming $\mathcal{M}_\infty(b) = \mathcal{M}(b)$, and if this happens, then $\mathcal{M}_\infty(b) = \mathcal{M}(ub)$ for every inner function u. Besides, the characterization of $\mathcal{M}_\infty(b)$ given in Section 2 is not sensitive to the inner factor u, thus $\mathcal{M}_\infty(ub) = \mathcal{M}_\infty(b) = \mathcal{M}(ub)$. Therefore $\sigma_{ub} = (1 - |b|^2)/|1 - ub|^2$ must be bounded for every inner function u.

4) implies 3). If 3) does not hold, there is a positive decreasing sequence (ε_k) which tends to zero, such that the sets

$$T_k = \{z \in \partial\mathbb{D} :\ \varepsilon_k < \rho \leq \varepsilon_{k-1}\}\,, \qquad k \geq 2\,,$$

all have positive measure. Then there are points $\omega_k \in \partial\mathbb{D}$ such that

$$E_k = \Big\{ z \in T_k :\ \Big| \frac{\overline{b(z)}}{|b(z)|} - \omega_k \Big| < \varepsilon_k \Big\}$$

also have positive measure. For each $k \geq 2$ let α_k be a density point of E_k. By compactness we can extract a convergent subsequence of (α_k), we also denote this sequence by (α_k). Even more, we can assume that $\alpha_k \neq \alpha_j$ for $k \neq j$ and $\lim \alpha_k \neq \alpha_j$ for all j. By Lemma 7.8 there is an interpolating Blaschke product B continuous on $\{\alpha_k : k \geq 2\}$ such that

$$|B(\alpha_k) - \omega_k| < \varepsilon_k\,, \qquad \text{for all } k \geq 2\,.$$

Since α_k is a density point of E_k, any open arc-interval centered at α_k small enough satisfies $|E_k \cap I_k| > |I_k|/2$. Furthermore, by the continuity of B in α_k we can assume (shrinking I_k if necessary) that

$$|B(z) - \omega_k| < \varepsilon_k \,, \qquad \text{for all } z \in I_k \text{ and all } k \geq 2\,.$$

Hence, for almost every $z \in E_k \cap I_k$,

$$\begin{aligned} |B(z)\, b(z) - |b(z)|| &\leq |B(z)\, b(z) - \omega_k\, b(z)| + \left|\omega_k\, b(z) - \frac{\overline{b(z)}}{|b(z)|}\, b(z)\right| \\ &\leq |B(z) - \omega_k| + \left|\omega_k - \frac{\overline{b(z)}}{|b(z)|}\right| < 2\,\varepsilon_k\,. \end{aligned} \tag{15}$$

The first summand is smaller than ε_k because $z \in I_k$ and the second because $z \in E_k$. Then, for almost every $z \in E_k \cap I_k$,

$$\begin{aligned} |1 - B(z)\, b(z)| &\leq |1 - |b(z)|| + |\,|b(z)| - B(z)\, b(z)| \\ &< \rho(z) + 2\,\varepsilon_k < 3\,\rho(z)\,, \end{aligned}$$

because since $z \in E_k \cap I_k \subset T_k$ then $\varepsilon_k < \rho(z)$.

Hypothesis 4) says that there is a constant $C = C(B) > 0$ such that for almost every $z \in E$,

$$C^{-1}\,\rho(z) \leq |1 - B(z)\, b(z)|^2\,,$$

and since $T_k \subset E$ this equality holds in $E_k \cap I_k$. Therefore, for almost every $z \in E_k \cap I_k$,

$$C^{-1}\,\rho(z) \leq |1 - B(z)\, b(z)|^2 < 3^2\,\rho^2(z)\,.$$

Then $(9C)^{-1} \leq \rho$ in $E_k \cap I_k$, and since $E_k \cap I_k \subset T_k$, also $(9C)^{-1} \leq \rho \leq \varepsilon_{k-1}$, which contradicts the fact that (ε_k) tends to zero.

5) implies 3). We assume that 3) does not hold and retain the notations of the above proof. Consider the Blaschke product $-B$. For almost every $z \in E_k \cap I_k$,

$$\begin{aligned} |1 + B(z)\, b(z)| &\geq \left|\,\left|1 + b(z)\,\frac{\overline{b(z)}}{|b(z)|}\right| - \left|b(z)\,\frac{\overline{b(z)}}{|b(z)|} - b(z)\, B(z)\right|\,\right| \\ &= |1 + |b(z)| - |\,|b(z)| - b(z)\, B(z)|\,| \\ &> 1 + |b(z)| - 2\,\varepsilon_k > \frac{1}{2} \end{aligned} \tag{16}$$

if $\varepsilon_k < 1/4$ (*i.e.* for k big enough), by (15). By hypothesis there is $\varepsilon = \varepsilon(B) > 0$ such that

$$|1 + B(z)\, b(z)|^2 < \varepsilon^{-1} \rho(z)\,, \qquad \text{for almost every } z \in E\,.$$

In particular this holds for almost every $z \in E_k \cap I_k$, and since $\rho(z) \leq \varepsilon_{k-1}$ in this set, (16) implies

$$\frac{1}{4} \leq |1 + B(z)\, b(z)|^2 < \varepsilon^{-1} \rho(z) \leq \varepsilon^{-1} \varepsilon_{k-1}$$

for almost every $z \in E_k \cap I_k$. Again, this contradicts $\varepsilon_k \to 0$.

Clearly 6) implies 4) and 7) implies 5), so the theorem will follow if we show that 3) implies 6) and 7). If $\rho \geq \delta\, \chi_E$, then $|1 - ub|\, \chi_E \geq (\delta/2)\, \chi_E$ for every inner function u. Then,

$$\frac{\delta}{4}\, \chi_E \leq \frac{\rho}{4} \leq \frac{1 - |b|^2}{|1 - ub|^2} \leq 4\, \frac{\rho}{\delta^2} \leq \frac{4}{\delta^2}\, \chi_E\,.$$

8. Almost conformal invariance.

Lemma 8.1. *Let b be extreme and $\rho = 1 - |b|^2$. For $z_0 \in \mathbb{D}$ put $b_0 = (b - z_0)/(1 - \overline{z}_0 b)$, $\rho_0 = 1 - |b_0|^2$, $\sigma_{b_0} = \rho_0/|1 - b_0|^2$ and $\lambda = (1 + z_0)/(1 + \overline{z}_0)$. Then*

1) $\displaystyle \rho_0 = \rho\, \frac{1 - |z_0|^2}{|1 - \overline{z}_0 b|^2}\,,$

2) $\displaystyle 1 - b_0 = (1 + z_0)\, \frac{1 - \overline{\lambda}\, b}{1 - \overline{z}_0\, b}\,,$

3) $\displaystyle \sigma_{b_0} = \frac{\rho_0}{|1 - b_0|^2} = \frac{1 - |z_0|^2}{|1 + z_0|^2}\, \frac{\rho}{|\lambda - b|^2}\,.$

PROOF. The above formulas follow from straightforward calculations with the following two identities (for $z \in \mathbb{C}$)

(i) $$1 - \Big| \frac{z - z_0}{1 - \overline{z}_0\, z} \Big|^2 = (1 - |z|^2)\, \frac{1 - |z_0|^2}{|1 - \overline{z}_0\, z|^2}\,,$$

(ii) $$1 - \frac{z - z_0}{1 - \overline{z}_0\, z} = \Big(\frac{1 + z_0}{1 + \overline{z}_0} - z \Big)\, \frac{1 + \overline{z}_0}{1 - \overline{z}_0\, z}\,.$$

Theorem 8.2. *Let b be extreme, $z_0 \in \mathbb{D}$ and $b_0 = (b - z_0)(1 - \overline{z}_0 b)^{-1}$. Then $\mathcal{H}(\overline{b}) = \mathcal{H}(\overline{b}_0)$ and $(1 - \overline{z}_0 b)\,\mathcal{H}(b_0) = \mathcal{H}(b)$.*

Proof. The easy estimate

$$\frac{1-|z_0|^2}{4} \le \frac{1-|z_0|^2}{|1-\overline{z}_0\, b|^2} \le \frac{1+|z_0|}{1-|z_0|}$$

together with Lemma 8.1.1) shows that b_0 is also an extreme point of $B(H^\infty)$, and that

$$E = \{e^{i\theta} \in \partial\mathbb{D} :\ \rho(e^{i\theta}) \neq 0\} = \{e^{i\theta} \in \partial\mathbb{D} :\ \rho_0(e^{i\theta}) \neq 0\}$$

almost everywhere. Also, if $f \in L^2(\chi_E)$, Lemma 8.1.1) implies

$$K_{\rho_0^{1/2}}(f) = K_{\rho^{1/2}}\left(f\ \frac{(1-|z_0|^2)^{1/2}}{|1-\overline{z}_0\, b|}\right),$$

and consequently $\mathcal{H}(\overline{b}) = \mathcal{H}(\overline{b}_0)$. Write $c = (1-|z_0|^2)/|1+\overline{z}_0|^2$ and $\lambda = (1+z_0)/(1+\overline{z}_0)$. By formula 3) of Lemma 8.1, for $z \in \mathbb{D}$,

$$\sigma_{b_0}(z) = c\,\frac{\rho(z)}{|1-\overline{\lambda}\, b(z)|^2} = c\,\sigma_{\overline{\lambda} b}(z)\,.$$

Hence,

$$\operatorname{Re}\left(\frac{1+b_0(z)}{1-b_0(z)}\right) = \sigma_{b_0}(z) = c\,\sigma_{\overline{\lambda} b}(z) = c\,\operatorname{Re}\left(\frac{1+\overline{\lambda}\, b(z)}{1-\overline{\lambda}\, b(z)}\right).$$

Two analytic functions with the same real part must differ in an imaginary constant. Thus, there are $\gamma,\ \delta \in \mathbb{R}$ such that for $z \in \mathbb{D}$,

$$i\,\gamma = \frac{1+b_0(z)}{1-b_0(z)} - c\,\frac{1+\overline{\lambda}\, b(z)}{1-\overline{\lambda}\, b(z)} = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(e^{i\theta}) + i\,\delta\,,$$

where $d\mu(e^{i\theta}) = d\mu_{b_0}(e^{i\theta}) - c\, d\mu_{\overline{\lambda} b}(e^{i\theta})$. Since μ is a real measure, evaluating at $z = 0$ we obtain $\gamma = \delta$. The identity

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} = 1 + 2\sum_{n\ge 1} z^n\, e^{-in\theta}\,,$$

with uniform convergence of the series in $|z| \leq r < 1$, now shows that $\int e^{-in\theta}\, d\mu(e^{i\theta}) = 0$ for all $n \geq 0$. Since μ is a real measure, taking complex conjugation we also obtain that $\int e^{in\theta}\, d\mu(e^{i\theta}) = 0$ for all $n \geq 1$. Then $\mu \equiv 0$ and therefore $\mu_{b_0} = c\, \mu_{\overline{\lambda} b}$. Thus, for $f \in L^2(\mu_{b_0}) = L^2(\mu_{\overline{\lambda} b})$,

$$\begin{aligned} V_{b_0}(f) &= (1 - b_0)\, K_{\mu_{b_0}}(f) \\ &= (1 + z_0) \frac{1 - \overline{\lambda}\, b}{1 - \overline{z}_0\, b}\, K_{c\mu_{\overline{\lambda} b}}(f) \\ &= \frac{1 + z_0}{1 - \overline{z}_0\, b}\, c\,(1 - \overline{\lambda}\, b)\, K_{\mu_{\overline{\lambda} b}}(f) \\ &= \frac{1}{1 - \overline{z}_0\, b}\, \frac{1 - |z_0|^2}{1 + \overline{z}_0}\, V_{\overline{\lambda} b}(f) \end{aligned}$$

by Lemma 8.1.2) and the equality of the measures. Thus

$$(1 - \overline{z}_0\, b)\, V_{b_0}(f) = \frac{1 - |z_0|^2}{(1 + \overline{z}_0)}\, V_{\overline{\lambda} b}(f)\,,$$

which clearly implies that $(1 - \overline{z}_0 b)\, \mathcal{H}(b_0) = \mathcal{H}(\overline{\lambda} b)$. Since $\mathcal{H}(\overline{\lambda}\, b) = \mathcal{H}(b)$, the theorem follows.

Corollary 8.3. *Let b be extreme, and denote by $\operatorname{sp}(b)$ the spectrum of b in H^∞. Then for $z_0 \neq 0$ the following conditions are equivalent.*

1) $z_0 \in \mathbb{D} \setminus \operatorname{sp}(b)$.
2) $(1 - \overline{z}_0\, b)\, \mathcal{H}(\overline{b}) = \mathcal{H}(b)$.
3) $(1 - \overline{z}_0\, b)^{-1} \in \mathcal{M}(b)$.

PROOF. 1) if and only if 2). $z_0 \in \mathbb{D} \setminus \operatorname{sp}(b)$ if and only if $b_0 = (b - z_0)/(1 - \overline{z}_0 b)$ is invertible. Since b_0 is extreme, Theorem 7.1 of [13] says that b_0 is invertible if and only if $\mathcal{H}(b_0) = \mathcal{H}(\overline{b}_0)$. If this happens, Theorem 8.2 implies that $(1 - \overline{z}_0 b)\, \mathcal{H}(\overline{b}) = \mathcal{H}(b)$. On the other hand, if this equality holds, then by Theorem 8.2,

$$(1 - \overline{z}_0\, b)\, \mathcal{H}(\overline{b}) = \mathcal{H}(b) = (1 - \overline{z}_0\, b)\, \mathcal{H}(b_0)\,.$$

Thus $\mathcal{H}(\overline{b}) = \mathcal{H}(b_0)$, and Theorem 8.2 again, shows that $\mathcal{H}(\overline{b}_0) = \mathcal{H}(b_0)$.

2) implies 3). $\mathcal{H}(b) \supset \mathcal{H}(\overline{b}) = (1 - \overline{z}_0\, b)^{-1} \mathcal{H}(b)$.

3) implies 2). Let $f \in \mathcal{H}(b)$; then $(1 - \overline{z}_0\, b)^{-1} f = g \in \mathcal{H}(b)$. Therefore

$$g - \overline{z}_0\, b\, g = (1 - \overline{z}_0\, b)\, g = f\,.$$

Since $g \in \mathcal{H}(b)$, we have that $b\,g$ must be in $\mathcal{H}(b)$; but for a function $g \in H^2$ it is well known that $b\,g \in \mathcal{H}(b)$ if and only if $g \in \mathcal{H}(\bar{b})$ (see Section 1). Hence,

$$f = (1 - \overline{z}_0\, b)\, g \in (1 - \overline{z}_0\, b)\, \mathcal{H}(\bar{b})\,.$$

For $z_0 = 0$ condition 3) is trivial. The equivalence of 1) and 2) for this case is proved in Theorem 7.1 of [13]. More can be said now. Suppose that $z_0 \in \mathbb{D} \setminus \mathrm{sp}\,(b)$, then b_0 and b_0^{-1} are multipliers of $\mathcal{H}(b_0)$. Since by Theorem 8.2 $\mathcal{M}(b_0) = \mathcal{M}(b)$, we also have $b_0^{-1} \in \mathcal{M}(b)$. Besides, by Corollary 8.3 $(1 - \overline{z}_0\, b)^{-1} \in \mathcal{M}(b)$, then $b_0^{-1}(1 - \overline{z}_0\, b)^{-1} = (b - z_0)^{-1} \in \mathcal{M}(b)$.

Corollary 8.4. *Let b be extreme. If u is an inner function such that* $\mathrm{sp}\,(ub)$ *is not the whole closed disc, then $\mathcal{M}(ub) = \mathcal{M}(b) = \mathcal{M}(\bar{b})$.*

Proof. Since $\mathrm{sp}\,(ub)$ is compact, there must be some point $z_0 \neq 0$ such that $z_0 \in \mathbb{D} \setminus \mathrm{sp}\,(ub)$. By Corollary 8.3 $(1 - \overline{z}_0 u\, b)\mathcal{H}(\overline{ub}) = \mathcal{H}(ub)$; then clearly $\mathcal{M}(\overline{ub}) = \mathcal{M}(ub)$. The assertion now follows from Section 1, taking into account that $\mathcal{H}(\overline{ub}) = \mathcal{H}(\bar{b})$.

9. Continuity conditions.

Theorem 9.1. *Let $b \in B(H^\infty)$ with $d\mu_b = \sigma\, d\theta/2\pi + d\mu_S$. If $\varepsilon > 0$ and $0 < r < 1$, then*

$$\|\mu_S\| = \lim_{\varepsilon \to 0} \lim_{r \to 1} \int_{|1 - b(e^{i\theta})| < \varepsilon} \frac{1 - r^2\, |b(e^{i\theta})|^2}{|1 - r\, b(e^{i\theta})|^2} \, \frac{d\theta}{2\pi}\,.$$

Proof. Since the Poisson kernel

$$P_r(z) = \frac{1}{2\pi}\, \frac{1 - r^2\, |z|^2}{|1 - r\, z|^2}$$

is harmonic (for $z \in \mathbb{D}$ and $0 \le r \le 1$), then

$$P_r(b(z)) = \frac{1}{2\pi}\, \frac{1 - r^2\, |b(z)|^2}{|1 - r\, b(z)|^2}$$

is harmonic. Thus

$$\int_0^{2\pi} \frac{1 - r^2\,|b(e^{i\theta})|^2}{|1 - r\,b(e^{i\theta})|^2}\,\frac{d\theta}{2\pi} = \frac{1 - r^2\,|b(0)|^2}{|1 - r\,b(0)|^2}\,,$$

which tends to $(1 - |b(0)|^2)/|1 - b(0)|^2$ when $r \to 1$. By formula (1) of Section 1, this is the norm of μ_b. On the other hand, for $\varepsilon > 0$,

$$\lim_{r \to 1} \int_{|1-b(e^{i\theta})| \geq \varepsilon} P_r(b(e^{i\theta}))\,d\theta = \int_{|1-b(e^{i\theta})| \geq \varepsilon} P_1(b(e^{i\theta}))\,d\theta\,,$$

because the integrand converges uniformly in $|1 - b(e^{i\theta})| \geq \varepsilon$. Since $P_1 \circ b = \sigma/2\pi \in L^1$, the last integral tends to $\int_0^{2\pi} \sigma(e^{i\theta})\,d\theta/2\pi = \|\sigma\,d\theta/2\pi\|$ when ε tends to 0. Substracting, we obtain

$$\begin{aligned} \|\mu_S\| &= \|\mu_b\| - \|\sigma\,d\theta/2\pi\| \\ &= \lim_{r \to 1} \int_0^{2\pi} P_r(b(e^{i\theta}))\,d\theta - \lim_{\varepsilon \to 0} \lim_{r \to 1} \int_{|1-b(e^{i\theta})| \geq \varepsilon} P_r(b(e^{i\theta}))\,d\theta \\ &= \lim_{\varepsilon \to 0} \lim_{r \to 1} \int_{|1-b(e^{i\theta})| < \varepsilon} P_r(b(e^{i\theta}))\,d\theta\,. \end{aligned}$$

Corollary 9.2. *If $(1 - b)^{-1} \in L^2$, then μ_b is absolutely continuous.*

Proof. Since $|1 - r\,b(e^{i\theta})| \geq |1 - b(e^{i\theta})|/2$ almost everywhere with respect to $d\theta$, then

$$\frac{1 - r^2\,|b(e^{i\theta})|^2}{|1 - r\,b(e^{i\theta})|^2} \leq \frac{4}{|1 - b(e^{i\theta})|^2} \in L^1\,.$$

Hence, by the dominated convergence theorem,

$$\lim_{r \to 1} \int_{|1-b(e^{i\theta})| < \varepsilon} P_r(b(e^{i\theta}))\,d\theta = \int_{|1-b(e^{i\theta})| < \varepsilon} \sigma(e^{i\theta})\,\frac{d\theta}{2\pi}\,,$$

and since $\sigma \in L^1$, the last integral tends to 0 when $\varepsilon \to 0$.

Notice that the above result also holds for b nonextreme. We keep assuming that b is not an inner function.

Theorem 9.3. *Let b be an extreme point of $B(H^\infty)$, continuous on $\partial\mathbb{D}$. Then $\mathcal{M}(b) = \mathcal{M}(\overline{b})$.*

Proof. Factorize $b = ub_0$, where u is the inner factor of b and b_0 is its outer factor. Since b is continuous, b_0 is continuous (see [11 , p. 69]); and $\overline{u}\, b_0$ is also continuous. It is well known ([9, IV]) that for a function f continuous on $\partial\mathbb{D}$ there is a unique best approximation $g \in H^\infty$, and that $|f(e^{i\theta}) - g(e^{i\theta})| = \operatorname{dist}\{f, H^\infty\}$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. Therefore, $\operatorname{dist}\{\overline{u}\, b_0, H^\infty\} < 1$, because otherwise since $\|\overline{u}\, b_0\| = 1$, the best approximation for $\overline{u} b_0$ in H^∞ must be the trivial function. So $|\overline{u}\, b_0| = 1$ almost everywhere, which is not the case. Thus, $\operatorname{dist}\{b_0, uH^\infty\} < 1$ and then Theorem 13.5 of [13] implies $\mathcal{M}(ub_0) = \mathcal{M}(b_0)$. Now it is clear from the equality $\mathcal{H}(\overline{ub_0}) = \mathcal{H}(\overline{b}_0)$ that we can assume $b = b_0$ outer.

Then b has square roots, and we will show that $\mathcal{M}(b) = \mathcal{M}(b^{2^n})$ for every integer n. We only have to prove that $\mathcal{M}(b) = \mathcal{M}(b^2)$. By Section 1, $\mathcal{H}(b^2) = \mathcal{H}(b) + b\,\mathcal{H}(b)$, thus $\mathcal{M}(b) \subset \mathcal{M}(b^2)$. Let $m \in \mathcal{M}(b^2)$ and $f \in \mathcal{H}(b)$. Then $bf \in \mathcal{H}(b^2)$ and therefore $m\, bf = g_1 + bg_2$ with $g_1,\ g_2 \in \mathcal{H}(b)$. Hence,

$$g_1 = b\,(mf - g_2) \in bH^2 \cap \mathcal{H}(b) = b\,\mathcal{H}(\overline{b}) \subset b\,\mathcal{H}(b)\,.$$

Thus $b\,mf = g_1 + b\,g_2 \in b\,\mathcal{H}(b)$, that is, $mf \in \mathcal{H}(b)$. Also, $\mathcal{H}(\overline{b}) = \mathcal{H}(\overline{b}^{2^n})$ for every integer n. As before, it is enough to take $n = 1$. This immediately follows from the inequalities

$$1 - |b|^2 \le 1 - |b^2|^2 \le 2\,(1 - |b^2|)$$

and the Cauchy transform representations of $\mathcal{H}(\overline{b})$ and $\mathcal{H}(\overline{b}^2)$.

It will therefore be enough to prove that there is an integer n such that $\mathcal{M}(b^{2^n}) = \mathcal{M}(\overline{b}^{2^n})$. Since the argument of b is continuous on the compact set $F = \{z :\ |z| \le 1,\ |b(z)| = 1\}$, there is some negative integer n such that the argument of $b^{2^n}(z)$ lives in $(-\pi/4, \pi/4)$ for $z \in F$. The continuity of b^{2^n} implies that for $\lambda \in \partial\mathbb{D}$ with $\operatorname{Re}\lambda < 0$,

$$|1 - \overline{\lambda}\, b^{2^n}(z)| \ge \delta > 0\,, \qquad \text{for all } z,\ |z| \le 1\,. \tag{17}$$

Therefore $|1 - \overline{\lambda}\, b^{2^n}(e^{i\theta})|^{-1} \in L^2$, and Corollary 9.2 implies that $\mu_{\overline{\lambda} b}$ is absolutely continuous, say $d\mu_{\overline{\lambda} b} = \sigma\, d\theta/2\pi$. Also, if $\rho = 1 - |b^{2^n}|^2$, condition (17) implies that the spaces $K_{\rho^{1/2}}(L^2(\chi_E))$ and $K_{\sigma^{1/2}}(L^2(\chi_E))$

coincide. Then,

$$\begin{aligned}(1-\overline{\lambda}\, b^{2^n})\, \mathcal{H}(\overline{b}^{2^n}) &= (1-\overline{\lambda}\, b^{2^n})\, K_{\rho^{1/2}}(L^2(\chi_E)) \\ &= (1-\overline{\lambda}\, b^{2^n})\, K_{\sigma^{1/2}}(L^2(\chi_E)) = \mathcal{H}(\overline{\lambda}\, b^{2^n}) = \mathcal{H}(b^{2^n})\,.\end{aligned}$$

Hence, $\mathcal{M}(b^{2^n}) = \mathcal{M}(\overline{b}^{2^n})$ and the theorem follows.

The argument to reduce the preceding theorem to the case in which b is an outer function is by D. Sarason (personal communication). My original proof of this fact was slightly more complicated.

The equality $\mathcal{M}(b^{2^n}) = \mathcal{M}(\overline{b}^{2^n})$ for n a suitable negative integer can be also proved using Corollary 8.4. Of course, Theorem 9.3 implies that the preceding algebras coincide for all integers n.

10. Inner factors in $\mathcal{H}(\overline{b}) + \mathbb{C}$.

Denote by $\mathcal{H}(\overline{b})_+$ the linear space $\mathcal{H}(\overline{b}) + \mathbb{C}$. The map $a \mapsto a_*$ defines a conjugation on $\mathcal{H}(\overline{b})_+$, where, for $a = K_\rho(q) + c \in \mathcal{H}(\overline{b})_+$, the function a_* is defined by $a_*(z) = -K_\rho(\overline{q})(z) + K_\rho(\overline{q})(0) + \overline{c} = \overline{a(1/\overline{z})}$ (see Section 1).

Theorem 10.1. *Let $a \in \mathcal{H}(\overline{b})_+$ and let u be an inner function. Then $ua \in \mathcal{H}(\overline{b})_+$ if and only if a_* is in uH^2. In this case, $(ua)_* = a_*/u$.*

PROOF. We can assume that u is not a constant function. If $a \in \mathcal{H}(\overline{b})_+$, then $a = K_\rho(q) + c$, with $q \in L^2(\rho)$ and $c \in \mathbb{C}$.

Sufficiency. The inner boundary function of $a - \overline{a}_*$ is $q\,\rho$, so the boundary function of $ua - u\overline{a}_*$ is $u\,q\,\rho$. By hypothesis a_*/u is in H^2, so $u(z)\,a(z) - (\overline{a_*(z)/u(z)})$ is harmonic, and since $\overline{u(z)}^{-1}$ and $u(z)$ have the same nontangential limit almost everywhere in $\partial\mathbb{D}$, the boundary function of $ua - (\overline{a_*/u})$ is also $u\,q\,\rho$. Hence, Lemma 2.1 gives

$$u(z)\,a(z) = K_\rho(uq)(z) + \overline{(a_*/u)(0)} \in \mathcal{H}(\overline{b})_+$$

and

$$a_*(z)/u(z) = \overline{K_\rho(uq)(1/\overline{z})} + (a_*/u)(0)\,.$$

Thus, $a_*(z)/u(z) = (ua)_*$.

Necessary condition. If $ua \in \mathcal{H}(\overline{b})_+$, then also $d = (ua)_* \in \mathcal{H}(\overline{b})_+$. Further, $d_* = ua \in uH^2$; so by the other implication of the theorem, $ud \in \mathcal{H}(\overline{b})_+$ and

$$(ud)_* = d_*/u = ua/u = a\,.$$

Hence, $a_* = ud \in u\mathcal{H}(\overline{b})_+ \subset uH^2$.

Corollary 10.2. *If m belongs to any of the algebras $\mathcal{M}(b)$, $\mathcal{M}(\overline{b})$ or $K^\infty(\rho)$, and u is an inner function, then $u\,m$ belongs to the same algebra as m if and only if $m_* \in uH^2$.*

Proof. The necessary condition is immediate from the above theorem, since all the algebras are contained in $\mathcal{H}(\overline{b})_+$. For the other implication, the argument for $\mathcal{M}(b)$ and $\mathcal{M}(\overline{b})$ is the same. So, suppose that $m \in \mathcal{M}(b)$, $m_* \in uH^2$, and take $a \in \mathcal{H}(b)$. Since $m_* \in \mathcal{M}(b)$, then $m_* a \in \mathcal{H}(b)$. Thus, $(m_*/u)a = T_{\overline{u}}(m_* a) \in \mathcal{H}(b)$. That is, $m_*/u \in \mathcal{M}(b)$ and then $(m_*/u)_*$ also belongs to $\mathcal{M}(b)$. Besides, $(m_*/u)_* = um$ by Theorem 10.1.

If $m \in K^\infty(\rho) \subset \mathcal{H}(\overline{b})_+$ and $m_* \in uH^2$, then $m_* \in uH^2 \cap H^\infty = uH^\infty$. By Theorem 10.1, $u\,m \in \mathcal{H}(\overline{b})_+ \cap H^\infty$ and $(u\,m)_* = m_*/u \in \mathcal{H}(\overline{b})_+ \cap H^\infty$. Thus, $u\,m$ and $(u\,m)_*$ belong to H^∞, which means that $(u\,m)(z)$ is bounded for all $z \in \mathbb{C} \setminus \partial\mathbb{D}$. Consequently $u\,m \in K^\infty(\rho)$.

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