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Convex domains and unique continuation at the boundary

Vilhelm Adolfsson, Luis Escauriaza and Carlos Kenig

Abstract. We show that a harmonic function which vanishes continuously on an open set of the boundary of a convex domain cannot have a normal derivative which vanishes on a subset of positive surface measure. We also prove a similar result for caloric functions vanishing on the lateral boundary of a convex cylinder.

1. Introduction.

Bourgain and Wolff [BW] have constructed a counterexample of a C^1-harmonic function u in $\mathbb{R}^d_+ = \{X \in \mathbb{R}^d : X_d \geq 0\}$, $d \geq 3$, for which u and its gradient vanish on a set of positive measure on $\partial\mathbb{R}^d_+$. On the other hand, it has been shown (see [F2]) that when D is a $C^{1,1}$ domain in $\mathbb{R}^d$, $d \geq 2$, and u is a non-constant harmonic function in D with $u = 0$ on an open set V contained in the boundary ∂D of D, then the Hausdorff measure of the set $\{Q \in V : \nabla u(Q) = 0\}$ is less or equal than $d - 2$.

In general, the following conjecture still remains an open question: if u is a harmonic function on a connected Lipschitz domain D in $\mathbb{R}^d$ vanishing continuously on an open subset V of ∂D and whose normal derivative vanishes on a subset of V of positive measure, then u is identically zero on D.

When u is non-negative, we have from the comparison principle for harmonic funtions vanishing continuously on an open subset of ∂D, [D], that the normal derivative of u is pointwise comparable to the density of the harmonic measure with respect to surface measure $d\sigma$ on any compact subset K contained in V, and it is well known that the harmonic measure is mutually absolutely continuous with respect to $d\sigma$, [D]. Therefore, in this case the answer to the conjecture is positive.

Let D denote a Lipschitz domain in $\mathbb{R}^d$ and w be a non-negative function defined on ∂D. We recall that a nonnegative function w is a $B_2(d\sigma)$-weight provided that there is a constant C such that for all $Q \in \partial D$ and $r > 0$ the following holds

$$\Big(\frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} w^2 \, d\sigma\Big)^{1/2} \le C \, \frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} w \, d\sigma \,,$$

where $\Delta_r(Q) = \partial D \cap B_r(Q)$.

In this note, we will prove the following regularity theorem.

Theorem 1. *Let D be a Lipschitz domain in $\mathbb{R}^d$, $d \ge 2$, $Q_0 \in \partial D$, and u be harmonic in D vanishing continuously on $\Delta_6(Q_0)$. Assume that there exists a constant M, possibly depending on u, such that for all $Q \in \Delta_3(Q_0)$ and $0 < r < 2$ the following doubling property holds,*

$$\int_{\Gamma_{2r}(Q)} u^2 \, dX \le M \int_{\Gamma_r(Q)} u^2 \, dX \,, \tag{1.1}$$

where $\Gamma_r(Q) = B_r(Q) \cap D$. Then, there exists a constant C depending on M, the Lipschitz character D and d, such that for all $Q \in \Delta_2(Q_0)$ and $0 < r < 1$

$$\Big(\frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} \Big|\frac{\partial u}{\partial N}\Big|^2 d\sigma\Big)^{1/2} \le C \, \frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} \Big|\frac{\partial u}{\partial N}\Big| \, d\sigma \,.$$

In particular, the absolute value of the normal derivative of u is a B_2-weight when restricted to $\Delta_2(Q_0)$.

In [F2] it is shown that the doubling property (1.1) holds for such a harmonic function when D is a $C^{1,1}$ domain, with M depending on the $C^{1,1}$ character and u. In this note we will show that the doubling property (1.1) also holds when the domain D is convex, obtaining the following theorem.

Theorem 2. *Let D, u and Q_0 be as in the previous theorem, and assume that either D is a $C^{1,1}$ or a convex domain. Then, the absolute value of the normal derivative of u on $\Delta_1(Q_0)$ is a $B_2(d\sigma)$-weight.*

It is well known that the above $B_2(d\sigma)$ condition implies that the set $\{Q \in \Delta_2(Q_0) : \partial u(Q)/\partial N = 0\}$ has zero surface measure unless $\partial u/\partial N = 0$ almost everywhere on $\Delta_2(Q_0)$. Therefore, if u is harmonic in D, vanishing continuously on an open subset V of ∂D, and $\{Q \in V : \partial u(Q)/\partial N = 0\}$ has positive surface measure, both u and $\partial u/\partial N$ must vanish identicaly on V. Extending u as zero outside of D, we obtain a new function which is harmonic in an open set Ω of $\mathbb{R}^d$ containing V and identically zero on $\Omega \setminus D$. It is well known that this implies that u must be identically zero in the connected component of its domain of definition containing V. Hence we obtain the following theorem.

Theorem 3. *Let D be a convex connected domain in $\mathbb{R}^d$, $d \geq 2$, and u be harmonic in D. Then, if u vanishes on an open subset V contained in ∂D and the set $\{Q \in V : \partial u(Q)/\partial N = 0\}$ has a positive surface measure, u must be identically zero on D.*

We want to remark that in [F2], the author claimed that his methods also applied to prove the above doubling property in the case of $C^{1,\alpha}$ domains, $0 < \alpha < 1$. But in a personal communication we learnt that his claim was incorrect.

This article is divided in two sections. In Section 2 we prove the theorems 1, 2 and 3, and in Section 3 we show that a similar result holds for solutions to the heat operator in convex cylinders.

2. Proofs of the main results.

To prove Theorem 1 we will need the following inequality.

Lemma 1. *There exists a constant C depending only on d, such that if $r > 0$, n is a positive integer, $0 < \beta < 1$, and $f \in C_0^\infty(B_{4r} \setminus B_{\beta r}(0))$, the following holds*

$$\int_{B_r} |f(X)|\, dX \leq C\, n\, \beta^{3-d-n}\, r^2 \int_{B_{2r}} |\Delta f(X)|\, dX$$

$$+ C\, 2^{-n} r^2 \int\limits_{B_{4r}\setminus B_{2r}} |\Delta f(X)|\, dX\,.$$

PROOF. By rescaling we may assume that $r = 1$. Let $f \in C_0^\infty(B_r \setminus B_\beta)$ and n be an integer greater than 1. If $\Gamma(X,Y)$ denotes the fundamental solution for the Laplace operator on $\mathbb{R}^d$, we have

$$f(X) = \int \Gamma_n(X,Y)\, \Delta f(Y)\, dY\,, \qquad \text{for all } \; X \in \mathbb{R}^d\,,$$

where

$$\Gamma_n(X,Y) = \Gamma(X,Y) - \sum_{k=0}^{n-1} \sum_{|\alpha|=k} \frac{1}{\alpha!}\, D_X^\alpha \Gamma(0,Y)\, X^\alpha\,.$$

Since $|D_X^\alpha \Gamma(0,Y)| \le C(d)^{|\alpha|}\, |Y|^{-(d-2+|\alpha|)}$, we have for $|X| \le 1$ and $|Y| \ge 2$

$$|\Gamma_n(X,Y)| \le C(d) \sum_{k\ge n} 2^{-k} \le C(d)\, 2^{-n}\,,$$

and, for $|X| \le 1$ and $\beta \le |Y| \le 2$,

$$|\Gamma_n(X,Y)| \le |\Gamma(X,Y)| + C(d)\, n\, \beta^{3-d-n}\,.$$

From these estimates and the support properties of f we obtain

$$\int_{B_1} |f(X)|\, dX \le C\, n\, \beta^{3-d-n} \int_{B_2} |\Delta f(X)|\, dX \\ + C\, 2^{-n} \int\limits_{B_4\setminus B_2} |\Delta f(X)|\, dX\,.$$

PROOF OF THEOREM 1. Let u and Q_0 be as in Theorem 1, $Q \in \Delta_3(Q_0)$ and $0 < r < 1$. Let β denote a vector field supported in $\Gamma_{2r}(Q)$ with $|\nabla\beta| \le C\, r^{-1}$, $\beta \cdot N \ge C^{-1}$ on $\Delta_r(Q)$ for some constant C depending on the Lipschitz character of D and $\beta \cdot N \ge 0$ on $\Delta_{2r}(Q)$, where N denotes the exterior unit normal to D at points of ∂D.

Integrating the Rellich-Necas identity

$$\operatorname{div}(\beta \cdot |\nabla u|^2) = 2 \operatorname{div}((\beta \cdot \nabla u)\, \nabla u) + O(|\nabla\beta|\, |\nabla u|^2)$$

over $\Gamma_{2r}(Q)$, and since $\nabla u = \partial u/\partial N$ almost everywehre on $\Delta_5(Q_0)$, we obtain

$$\int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma \le C\, r^{-1} \int_{\Gamma_{2r}(Q)} |\nabla u|^2 \, dX\,,$$

and from Cacciopoli's inequality and the doubling property of u

$$\int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma \le C\, r^{-3} \int_{\Gamma_{r/40}(Q)} u^2 \, dX\,,$$

where C depends on the Lipschitz character of D, and M.

From standard estimates for subharmonic functions vanishing at the boundary [GT, Theorem 8.25] we can bound the L^2 averages of u by L^1 averages, obtaining

$$\left(\int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma\right)^{1/2} \le C\, r^{-(3+d)/2} \int_{\Gamma_{r/30}(Q)} |u| \, dX\,.$$

We claim that for some constant C as above

$$\int_{\Gamma_{r/30}(Q)} |u| \, dX \le C\, r^2 \int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right| d\sigma\,,$$

and assuming the claim, the theorem follows from the last two inequalities.

To prove the last claim, we may assume without loss of generality that $Q = 0$ and that near 0, ∂D coincides with $\{(x, y) :\ x \in \mathbb{R}^{d-1},\ y = \varphi(x)\}$ for some Lipschitz function φ with $\varphi(0) = 0$.

Let Z denote the point whose coordinates with respect to this coordinate system are $x = 0$ and $y = -r/2$. From the Lipschitz character of D we can find $0 < \beta < 1/8$ such that $B_{2\beta r}(Z)$ is contained in the complement of D. We extend u to be zero outside D and define $u_\varepsilon = u * \theta_\varepsilon$, where θ_ε is a regularization of the identity. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on $B_{2r}(Z)$ and whose support is contained in $B_{4r}(Z)$.

Setting $f_\varepsilon = \varphi u_\varepsilon$, we have that $f_\varepsilon \in C_0^\infty(B_{4r} \setminus B_{\beta r}(Z))$ for $\varepsilon > 0$ sufficiently small, and for X in $B_{4r}(Z)$

$$\begin{aligned} |\Delta f_\varepsilon(X)| \le 2\, |\Delta\varphi|\, |(\nabla u) * \theta_\varepsilon| + |u_\varepsilon|\, |\Delta\varphi| \\ + \int_{\partial D} \left|\frac{\partial u}{\partial N}(Q)\right| \varphi(X)\, \theta_\varepsilon(X - Q)\, d\sigma(Q)\,. \end{aligned}$$

Applying to f_ε the translation to Z of the inequality in Lemma 1, and letting ε tend to zero, we obtain from the support properties of φ and standard estimates for harmonic functions the following estimate

$$\int_{\Gamma_{r/2}(Q)} |u|\, dX \le C(d,\beta,n)\, r^2 \int_{\Delta_{5r}(Q)} \left|\frac{\partial u}{\partial N}\right| d\sigma \\ + C(d)\, 2^{-n} \int_{\Gamma_{5r}(Q)} |u|\, dX \,.$$

Using the doubling property of u, the second term above can be hidden on the left hand side of the inequality after choosing n large enough, getting

$$\int_{\Gamma_{r/2}(Q)} |u|\, dX \le C\, r^2 \int_{\Delta_{5r}(Q)} \left|\frac{\partial u}{\partial N}\right| d\sigma \,,$$

where C depends on d, the Lipschitz character of D and M; and this proves the claim.

PROOF OF THE DOUBLING PROPERTY. Assume now that D is convex and let u be as in the statement of Theorem 2. For $Q \in \Delta_3(Q_0)$ we define

$$H(r,Q) = \int_{\partial B_r(Q)\cap D} u^2\, d\sigma \qquad \text{and} \qquad D(r,Q) = \int_{\Gamma_r(Q)} |\nabla u|^2\, dX \,.$$

As Almgren, [A], we consider the frequency function

$$N(r,Q) = \frac{2\,r\,D(r,Q)}{H(r,Q)} \,.$$

We will prove that

$$r\,\frac{d}{dr}\,(\log(H(r,Q)\,r^{1-d})) = \frac{2\,r\,D(r,Q)}{H(r,Q)} = N(r,Q) \,,$$

and that the frequency function $N(\,\cdot\,,Q)$ is non-decreasing for $Q \in \Delta_3(Q_0)$. Therefore, if $0 < r < 2$, we have

$$r\,\frac{d}{dr}\,(\log(H(r,Q)\,r^{1-d})) \le N(2,Q) \,.$$

Standard arguments imply that the doubling constant M is bounded by $2^{d+\beta}$, where β is an upper bound of $N(2,Q)$ on $\Delta_3(Q_0)$.

To prove our claim, we may assume that $Q = 0$, $H(r,0) = H(r)$, $B_r = B_r(0)$, and $D(r,0) = D(r)$. Then,

$$\frac{d}{dr} H(r) = \frac{d-1}{r} H(r) + 2\, D(r)\,. \tag{2.1}$$

From the Rellich-Necas identity with vector field X, *i.e.*,

$$\operatorname{div}(X\,|\nabla u|^2) = 2\operatorname{div}((X\cdot\nabla u)\,\nabla u) + (d-2)\,|\nabla u|^2\,,$$

and the fact that the tangential derivative of u is zero on $\Delta_3(Q_0)$ we get

$$\frac{d}{dr} D(r) = 2\int_{\partial B_r\cap D}\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma + \frac{d-2}{r} D(r) + \frac{1}{r}\int_{\Delta_r}(Q\cdot N)\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma.$$

But in a convex domain with $0 \in \partial D$, $Q\cdot N$ is non-negative on ∂D. Hence,

$$\frac{d}{dr} D(r) \ge 2\int_{\partial B_r\cap D}\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma + \frac{d-2}{r} D(r)\,.$$

From the above inequality (2.1), and the quotient rule we obtain

$$\frac{d}{dr}\Big(\frac{r\,D(r)}{H(r)}\Big) \ge 2\,r\,H(r)^{-2}\bigg(\int_{\partial B_r\cap D}\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma \int_{\partial B_r\cap D} u^2\, d\sigma - \Big(\int_{\partial B_r\cap D} u\,\frac{\partial u}{\partial N}\, d\sigma\Big)^2\bigg),$$

and from Hölder's inequality we get

$$\frac{d}{dr}\Big(\frac{r\,D(r)}{H(r)}\Big) \ge 0\,,$$

as we wanted.

3. The parabolic case.

Here we show that a similar result holds for caloric functions vanishing continuously on the lateral boundary of a convex cylinder $D\times(0,\infty)$. The reader will observe in the next proof, that in general, the same result can be obtained when D is just a Lipschitz domain and the corresponding "unique continuation property" holds at points $(Q,0)$ with $Q \in \partial D$, for harmonic functions defined on $W \cap D \times (0,\infty)$ and vanishing continuously on $W \cap \partial(D \times (0,\infty))$, where W is an open set in R^{d+1} containing the boundary point $(Q,0)$.

Theorem 4. *Let D be a convex connected domain in $\mathbb{R}^d$, $d \geq 2$ and $u(X,t)$ satisfy*

$$\begin{cases} \Delta u - \partial_t u = 0\,, & \text{on } D \times (0,\infty)\,, \\ u(X,0) = f(X)\,, & \\ u(Q,t) = 0\,, & \text{for } Q \in \partial D \text{ and } t > 0\,, \end{cases}$$

for some f in a suitable class. Assume that the set

$$E = \Big\{(Q,t) \in \partial D \times (0,\infty) : \frac{\partial u}{\partial N}(Q,t) = 0\Big\}$$

has positive surface measure on $\partial D \times (0,\infty)$. Then, u must be identically zero.

PROOF. Without loss of generality we may assume that (Q_0,τ), where $Q_0 \in \partial D$ and $\tau > 0$, is a density point of E, *i.e.*,

$$\lim_{r\to 0} \frac{m(E \cap (\Delta_r(Q_0) \times [\tau - r^2, \tau]))}{\sigma(\Delta_r(Q_0))r^2} = 1\,, \tag{3.1}$$

where $dm = d\sigma\, dt$ on $\partial D \times (0,\infty)$.

We claim that this implies that $u(\,\cdot\,,\tau)$ vanishes to infinity order at Q_0:

$$\int_{\Gamma_r(Q_0)} |u(X,\tau)|\, dX = O(r^k)\,, \qquad \text{for all } k \geq 1 \text{ as } r \to 0\,.$$

Assuming this claim, we have

$$u(X,t) = \sum_{k\geq 1} a_k\, \varphi_k(X)\, e^{-\lambda_k t}\,,$$

where

$$f(X) = \sum_{k\geq 1} a_k\, \varphi_k(X)$$

and $\{\varphi_1, \varphi_2, \dots, \varphi_k, \dots\}$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \cdots$ denote respectively the eigenfunctions and eigenvalues for the Laplace operator on D.

Defining $v(X, y)$ for $X \in D$ and $y > 0$ as

$$v(X,y) = \sum_{k\geq 1} a_k\, \varphi_k(X)\, e^{-\lambda_k \tau}\, \frac{\sinh(\lambda_k^{1/2} y)}{\lambda_k^{1/2}}$$

we have that v is a harmonic function on $\Omega = D \times (0, \infty)$ with $v = 0$ on the bottom and sides of Ω. From the convexity of D, Ω is a convex domain in $\mathbb{R}^{d+1}$, and from our previous estimates for harmonic functions in convex domains vanishing on a boundary portion we have for $0 < r < 1$

$$\int_{\Gamma_r(Q_0,0)} |v(X,y)|\, dX\, dy \leq C\, r^2 \int_{\Delta_r(Q_0,0)} \left|\frac{\partial v}{\partial N}\right|\, d\gamma\,,$$

where $\Gamma_r(Q_0, 0) = B_r(Q_0, 0) \cap \Omega$, $\Delta_r(Q_0, 0) = B_r(Q_0, 0) \cap \partial\Omega$, $d\gamma$ denotes surface measure on $\partial\Omega$ and $C > 0$ is a constant depending on d, the Lipschitz character of D and v. From the doubling property of the absolute value of the normal derivative of v on $\partial\Omega$ and the fact that $\partial v(X, 0)/\partial N = -\partial_y v(X, 0) = -u(X, \tau)$ on the bottom of Ω, we obtain

$$\int_{\Delta_r(Q_0,0)} \left|\frac{\partial v}{\partial N}\right|\, d\gamma \leq C \int_{\Gamma_r(Q_0)} |u(X,\tau)|\, dX = O(r^k)\,, \quad \text{for all } \ k \geq 1\,.$$

From the above inequalities we conclude that v vanishes to infinity order at $(Q_0, 0)$ and since v is doubling with respect to balls centered at $(Q_0, 0)$, it can only happen when v is identically zero, which implies that u must be equal to zero.

Therefore, to finish our proof we must prove the above claim. It will follow from the analogue of the Lemma 1 in the parabolic case.

Lemma 2. *There exists a constant C depending on $d \geq 2$, such that if $r > 0$, $0 < \beta < 1$, $f \in C_0^\infty((B_{4r} \setminus B_{\beta r}) \times (-(4r)^2, 0])$, and n is an integer greater than* 1, *the following holds*

$$\int_{-r^2}^0 \int_{B_r} |f|\, dX\, dt \leq C\, n\, \beta^{-(d+2(n-1))}\, r^2 \int_{-(2r)^2}^0 \int_{B_{2r}} |(\Delta - \partial_t) f|\, dX\, dt$$
$$+ C\, 2^{-n}\, r^2 \int\!\!\int_{H_r} |(\Delta - \partial_t) f|\, dX\, dt\,,$$

where $H_r = B_{4r} \setminus B_{2r} \times (-(4r)^2, 0] \cup B_{2r} \times (-(4r)^2, -(2r)^2]$.

PROOF. As usual we may assume that $r = 1$. Setting $\Gamma(X,t,Y,s) = (4\pi(t-s))^{-d/2}\exp(-|X-Y|^2/4(t-s))$ for $t > s$ and $\Gamma(X,t,Y,s) = 0$ for $t \leq s$ we obtain using the fact that $D^\alpha \partial_t^j f(0,0) = 0$ for all d-tuples and $j \geq 0$, and a simple argument of integration by parts, that for $t \leq 0$ and X in $\mathbb{R}^d$

$$f(X,t) = \int_{-\infty}^{0} \int \Gamma_n(X,t,Y,s)\,(\Delta - \partial_s)\, f(Y,s)\, dY\, ds\,,$$

where

$$\Gamma_n(X,t,Y,s) = \Gamma(X,t,Y,s) - \sum_{k=0}^{n-1} \sum_{|\alpha|+j=k} \frac{1}{\alpha! j!} D_X^\alpha \partial_t^j \Gamma(0,0,Y,s)\, X^\alpha t^j\,.$$

Interior estimates for caloric functions imply that for $s < 0$ we have

$$|D_X^\alpha \partial_t^j \Gamma(0,0,Y,s)| \leq C(d)^{|\alpha|+2j}\, |s|^{-(|\alpha|+2j+d)/2}\,, \qquad \text{for } \ |Y|^2 \leq |s|\,,$$

$$|D_X^\alpha \partial_t^j \Gamma(0,0,Y,s)| \leq C(d)^{|\alpha|+2j}\, |Y|^{-(|\alpha|+2j+d)}\,, \qquad \text{for } \ |Y|^2 \geq |s|\,.$$

These estimates imply that for $|X| < 1$, $-1 < t < 0$ we have

$$|\Gamma_n(X,t,Y,s)| \leq C(d)\,(\Gamma(X,t,Y,s) + n\,\beta^{-(d+2(n-1))})\,,$$

for $\beta < |Y| < 2$ and $-4 < s < 0$, and for $(Y,s) \in H_1$ we can estimate Γ_n using the generalized mean value theorem, obtaining

$$|\Gamma_n(X,t,Y,s)| \leq C(d)\,2^{-n}\,, \qquad \text{for } \ (Y,s) \in H_1\,.$$

The inequality follows from these estimates and the support properties of f.

Let now u be as in Theorem 4. Without loss of generality we may assume that $(Q_0, \tau) = (Q_0, 0)$, and that u is caloric for $X \in D$, $-2 < t < 0$. As before, for $0 < r < 1$ we let Z denote a point outside of D such that $|Z - Q_0| = r/2$, and β be a number with $0 < \beta < 1/8$, and such that $B_{2\beta r}(Z)$ does not intersect D.

We extend u to be zero outside the cylinder with base D and define $f_\varepsilon(x,t) = u_\varepsilon(x,t)\,\varphi(x,t)$, where $u_\varepsilon(X,t) = u(\,\cdot\,,t) * \theta_\varepsilon(X)$, and where $\varphi \in C_0^\infty(B_{4r}(Z) \times (-(4r)^2, 0])$ is such that $\varphi = 1$ on $B_{2r}(Z) \times (-(2r)^2, 0]$.

Applying to f_ε the translation of the inequality in Lemma 2 to $(Z, 0)$, observing that

$$|(\Delta - \partial_t) f_\varepsilon(X,t)| \le 2\,|\nabla\varphi|\,|(\nabla u) * \theta_\varepsilon| + |u_\varepsilon|\,|(\Delta - \partial_t)\varphi| + \int_{\partial D} \left|\frac{\partial u}{\partial N}(Q,t)\right| \varphi(X,t)\,\theta_\varepsilon(X - Q)\,d\sigma(Q)\,,$$

for X in $\mathbb{R}^d$ and $-1 < t < 0$, letting ε tend to zero, and using standard estimates for caloric functions we obtain

$$\int_{-r^2}^0 \int_{\Gamma_{r/2}(Q_0)} |u|\,dX\,dt \le C(d,n,\beta)\,r^2 \int_{-(5r)^2}^0 \int_{\Delta_{5r}(Q_0)} \left|\frac{\partial u}{\partial N}\right| d\sigma\,dt + C(d)\,2^{-n} \int_{-(5r)^2}^0 \int_{\Gamma_{5r}(Q_0)} |u|\,dX\,dt\,.$$

On the other hand, the first term of the right hand side of the last inequality can be bounded by

$$r^2\, m((\Delta_{5r}(Q_0) \times [-(5r)^2, 0]) \setminus E)^{1/2} \Big(\int_{-(5r)^2}^0 \int_{\Delta_{5r}(Q_0)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma\,dt \Big)^{1/2}$$

If v denotes the solution to $\Delta v - \partial_t v = 0$ on $\Gamma_{6r}(Q_0) \times (-(6r)^2, 0]$ satisfying $v(Q,t) = 0$ for $Q \in B_{6r}(Q_0) \cap \partial D$ and $t \in (-(6r)^2, 0]$, and $v(Q,t) = 1$ on the remaining part of the parabolic boundary of $\Gamma_{6r}(Q_0) \times (-(6r)^2, 0]$, we have from [FS] that for some constant C depending on d, and the Lipschitz character of D,

$$\Big(\int_{-(5r)^2}^0 \int_{\Delta_{5r}(Q_0)} \left|\frac{\partial v}{\partial N}\right|^2 d\sigma\,dt \Big)^{1/2} \le C\, r^{(d+1)/2}\,.$$

On the other hand, from the parabolic maximum principle and standard estimates for caloric functions vanishing on the lateral boundary of a Lipschitz cylinder we have

$$|u(X,t)| \le C\,v(X,t)\,r^{-(d+2)} \int_{-(8r)^2}^0 \int_{\Gamma_{8r}(Q_0)} |u|\,dX\,dt\,,$$

for all (X,t) in $\Gamma_{6r}(Q_0) \times (-(6r)^2, 0]$.

Thus,

$$\Big(\int_{-(5r)^2}^0 \int_{\Delta_{5r}(Q_0)} \Big|\frac{\partial u}{\partial N}\Big|^2 \, d\sigma \, dt\Big)^{1/2} \le C\, r^{-(d+5)/2} \int_{-(8r)^2}^0 \int_{\Gamma_{8r}(Q_0)} |u| \, dX \, dt\,,$$

where C depends only on the Lipschitz character of D, and d.

From the above chain of inequalities we have that for $0 < r < 1$

$$\int_{-r^2}^0 \int_{\Gamma_{r/2}(Q_0)} |u| \, dX \, dt \le \Big(C(d,n,\beta)\Big(\frac{m((\Delta_{5r}(Q_0) \times [-(5r)^2, 0]) \setminus E)}{\sigma(\Delta_{5r}(Q_0))(5r)^2}\Big)^{1/2} + C(d)\, 2^{-n}\Big) \int_{-(8r)^2}^0 \int_{\Gamma_{8r}(Q_0)} |u| \, dX \, dt\,.$$

Therefore, from (3.1) and choosing n large enough, we find that for all $\varepsilon > 0$ there exists $r(\varepsilon) > 0$, such that for $0 < r < r(\varepsilon)$

$$\int_{-r^2}^0 \int_{\Gamma_r(Q_0)} |u| \, dX \, dt \le \varepsilon \int_{-(12r)^2}^0 \int_{\Gamma_{12r}(Q_0)} |u| \, dX \, dt\,.$$

This is well known to imply that

$$\int_{-r^2}^0 \int_{\Gamma_r(Q_0)} |u| \, dX \, dt = O(r^k)\,, \qquad \text{for all } \; k \ge 1\,.$$

On the other hand, estimates for caloric functions vanishing on the lateral boundary give

$$\int_{\Gamma_r(Q_0)} |u(X,0)| \, dX \le C\, r^{-2} \int_{-(2r)^2}^0 \int_{\Gamma_{2r}(Q_0)} |u| \, dX \, dt\,,$$

where C depends on d and the Lipschitz character of D, and this implies our claim.

References.

[A] Almgren F. J., Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. *Minimal submanifolds and geodesics* (ed. by M. Obata), North Holland (1979), 1-6.

[BW] Bourgain J., Wolff T., A remark on gradients of harmonic functions in dimension ≥ 3. *Colloquium Mathematicum* **60/61** (1990), 253-260.

[D] Dahlberg, B. E. J., Estimates fo harmonic measure. *Arch. Rational Mech. Anal.* **65** (1977), 275-285.

[F1] Fang Hua Lin, A uniqueness theorem for parabolic equations. *Comm. Pure Appl. Math.* **43** (1990), 127-136.

[F2] Fang Hua Lin, Nodal sets of solutions of elliptic and parabolic equations. *Comm. Pure Appl. Math.* **45** (1991), 287-308.

[FS] Fabes E. B., Salsa, S., Estimates of caloric measure and the initial Dirichlet problem for the heat equation in Lipschitz cylinders. *Trans. Amer. Math. Soc.* **279** (1983), 635-650.

[GT] Gilbarg, D., Trudinger N. S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

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Vilhelm Adolfsson
Matematiska Institutionen
CTH
S-412 96 Göteborg
SWEDEN
vilhelm@math.chalmers.se

Luis Escauriaza
Universidad del Pais Vasco
Departamento de Matematicas
Apartado 644, 48080 Bilbao, Leioa
SPAIN
mtpeszul@lg.ehu.es

Carlos Kenig
University of Chicago
Department of Mathematics
University Avenue 54037
Chicago, IL, 60637, U.S.A.
cek@math.uchicago.edu