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Two problems on doubling measures

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Doubling measures appear in relation to quasiconformal mappings of the unit disk of the complex plane onto itself. Each such map determines a homeomorphism of the unit circle on itself, and the problem arises, which mappings f can occur as boundary mappings? A famous theorem of Beurling and Ahlfors [2] states a necessary and sufficient condition: the Lebesgue measures $|f(I)|$ and $|f(J)|$ are comparable, $|f(I)| \simeq |f(J)|$, whenever I and J are adjacent arcs of equal length. Denoting by μ the measure on the unit circle such that $\mu(I) = |f(I)|$, this can be expressed by the inequality $\mu(2I) \leq c\,\mu(I)$, where $2I$ denotes an arc on the circle, concentric with I, of twice the length. The measure μ in the Beurling-Ahlfors theorem is the harmonic measure for a certain elliptic operator in divergence form, whence the problem of null-sets for doubling measures is closely related to that of null-sets for harmonic measure [3].

Certain estimations, such as those for singular integrals and maximal functions, which are classical in the case of Lebesgue measure, can be obtained for doubling measure in Euclidean space ([4], [5] and [6]). Doubling measures also appear in relation to inner functions in several complex variables [1].

The definition of doubling measure has meaning for any metric space (X, ρ), *i.e.* $\mu(B(x, 2r)) \leq c\,\mu(B(x, r))$, and it is natural to ask which compact metric spaces (X, ρ) carry non-trivial doubling measures. A necessary and sufficient condition was found by Vol'berg and Konyagin [8], and called finite uniform metric dimension: in each ball

$B(x, 2r)$ at most N points can be found with mutual distances at least r.

In view of the original interest in singular mappings and singular measures, mutually singular doubling measures on the same metric space are of interest. We prove that such measures exist provided (X, ρ) carries a doubling measure and is perfect. This answers a question stated in [8, p. 637].

A measure μ on $\mathbb{R}^1$, is called *dyadic-doubling* if $\mu(I) \leq c\,\mu(J)$ whenever I and J are adjacent dyadic intervals of the same length, whose union is also dyadic. These measures occur in the theory of weights and are completely characterized [7]. It is hardly surprising that the class of doubling measures and the class of dyadic doubling measures are different, but less trivial that the corresponding classes of null-sets (which we abbreviate as $\mathcal{N}$ and $\mathcal{N}_d$) are different. The class $\mathcal{N}$ is bilipschitz invariant. The class $\mathcal{N}_d$ lacks an invariance property of $\mathcal{N}$: we find a closed set E, not in $\mathcal{N}_d$, and a set T of full measure in $\mathbb{R}^1$, $|\mathbb{R}^1 \backslash T| = 0$, such that $t + E$ is in $\mathcal{N}_d$ for each t in T. A previous example [9] accomplished this with a set T of dimension 1. The class $\mathcal{N}_d$ is not invariant under multiplication by positive numbers t, but our example is not as strong as the one for addition.

1. Singular doubling measures on compact metric space.

Vol'berg and Konyagin proved [8] that *a compact metric space (X, ρ) carries a nontrivial doubling measure μ on X:*

$$(1.1) \qquad \mu(B(x, 2R)) \leq \Lambda\, \mu(B(x, R)), \qquad \textit{for all } x \in X,\ R > 0,$$

where $\Lambda \geq 1$ and $B(x, R) = \{y \in X : \rho(x, y) < R\}$, if and only if it has finite uniform metric dimension. In particular *any compact set X in $\mathbb{R}^n$ carries a nontrivial doubling measure.*

They also raised the question: *on which compact metric spaces (X, ρ) are all doubling measures mutually absolutely continuous?* It follows from a well-known example of Beurling and Ahlfors [2] that this is not the case even for the unit circle. We prove the following.

Theorem 1. *Let (X, ρ) be a compact metric space and μ be a doubling measure on X having no atoms. Then there exists a doubling measure on X singular with respect to μ.*

We emphasize that a doubling measure on X satisfies the doubling condition on X only; that is, only balls with centers in X figure in the definition.

We say that (X,ρ) has *finite uniform metric dimension* if there exists a finite $N = N(X,\rho)$ such that for any $x \in X$ and $R > 0$, there are at most N points in $B(x, 2R)$ separated from one another by a distance at least R.

PROOF OF THEOREM 1. Let (X,ρ) satisfy all conditions in Theorem 1, and μ be a doubling measure on X with $\mu(X) = 1$. To construct a doubling measure on X singular with respect to μ, we invoke the idea of Riesz product on the measure space (X,μ). The functions w_k in the next lemma play the role of $1 + a_k \cos kx$ in the usual Riesz product.

Lemma 1. *There exist measurable functions w_k on X taking values $1/2$ and $3/2$ only, so that*

$$\mu(w_k = 1/2) = \mu(w_k = 3/2) = 1/2\,,$$

$$w_k \longrightarrow 1 \qquad \textit{weakly in } L^2(d\mu)\,, \tag{1.2}$$

and

$$w_k^{1/2} \longrightarrow \frac{1}{2}(\sqrt{1/2} + \sqrt{3/2}) \qquad \textit{weakly in } L^2(d\mu)\,. \tag{1.3}$$

PROOF. We observe that every measurable set E of measure $\mu(E) > 0$, can be divided into two subsets, each of measure $\mu(E)/2$. This is a general property of measure spaces with no atoms. Hence there is a measurable function w such that $\mu(w < t) = t$, $0 \le t \le 1$. Let $g(t)$ have period 1 on $[0, +\infty)$ with $g = 1/2$ on $[0, 1/2)$, $g = 3/2$ on $[1/2, 1]$. We set $w_k = g(2^k w)$.

To see that $w_k \to 1$ weakly in $L^2(d\mu)$, we observe that the functions w_k are independent. Therefore w_k tends weakly to its mean, as does $w_k^{1/2}$.

For $x \in X$ and $r > 0$, define

$$h_{x,r}(y) = \begin{cases} 1\,, & \text{if } \rho(x,y) \le r\,, \\ 0\,, & \text{if } \rho(x,y) \ge 3r/2\,, \\ 3 - 2\,\rho(x,y)\,r^{-1}\,, & \text{if } r < \rho(x,y) < 3r/2\,. \end{cases}$$

By the doubling property of μ, there exists $A > 1$ independent of x and r, so that

$$\int_X h_{x,2r}(y)\,d\mu(y) \le A \int_X h_{x,r}(y)\,d\mu(y)\,, \tag{1.4}$$

for all $x \in X$ and $r > 0$.

Let $\alpha = (\sqrt{1/2}+\sqrt{3/2})/2$, $\beta = 21\,\alpha/20$ and $\{w_k\}$ be the functions in Lemma 1. Note that $\beta < 1$.

We shall construct continuous functions $\{u_n\}_1^\infty$ and $\{v_n\}_1^\infty$ on X, so that the following inequalities are true:

$$\frac{1}{2} - \frac{1}{100\,(n+1)} \le u_n \le \frac{3}{2} + \frac{1}{100\,(n+1)}\,, \tag{1.5}$$

$$\int_X \prod_0^n u_i\,d\mu = 1\,, \tag{1.6}$$

$$\int_X \Big(\prod_0^n u_i\Big)^{1/2} d\mu \le \beta^n\,, \tag{1.7}$$

$$\int_X h_{x,2r} \prod_0^n u_i\,d\mu \le \Big(7 - \frac{1}{n+1}\Big) A \int_X h_{x,r} \prod_0^n u_i\,d\mu\,, \tag{1.8}$$

for all $x \in X$ and $r > 0$;

$$0 \le v_n \le 1\,, \tag{1.9}$$

$$\int_X v_n\,d\mu \le \beta^{n/2}\,, \tag{1.10}$$

and for all $0 \le j \le n$,

$$\int_X (1 - v_j) \prod_0^n u_i\,d\mu \le \Big(3 - \frac{1}{n+1}\Big)\beta^j\,. \tag{1.11}$$

Let $u_0 \equiv 1$ and $v_0 \equiv 0$ on X.

Assume that $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ have been chosen so that (1.5) to (1.11) are satisfied; we shall construct u_{n+1} and v_{n+1}.

Because of (1.2), (1.3), (1.7) and (1.11), for sufficiently large $k > k(u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_n)$,

$$\int_X \Big(\prod_0^n u_i\Big)^{1/2} w_k^{1/2}\, d\mu \le \Big(1 + \frac{1}{100\,(n+1)}\Big)\, \alpha\, \beta^n\,, \tag{1.12}$$

and

$$\int_X (1 - v_j)\Big(\prod_0^n u_i\Big) w_k\, d\mu \le \Big(3 - \frac{1}{\sqrt{(n+1)(n+2)}}\Big)\, \beta^j\,, \tag{1.13}$$

for all $0 \le j \le n$.

Because $u_i, 0 \le i \le n$, are uniformly continuous on X with values in $[1/4, 2]$, it follows from (1.4) that for all $x \in X$,

$$\int_X h_{x,2r} \prod_0^n u_i\, d\mu \le A\Big(1 + \frac{1}{n+1}\Big) \int_X h_{x,r} \prod_0^n u_i\, d\mu\,,$$

provided that $0 < r < r(u_0, u_1, \ldots, u_n)$. Since $1/2 \le w_k \le 3/2$,

$$\int_X h_{x,2r} \prod_0^n u_i\, w_k\, d\mu \le 3A\Big(1 + \frac{1}{n+1}\Big) \int_X h_{x,r} \prod_0^n u_i\, w_k\, d\mu \tag{1.14}$$

for all $x \in X$ and $0 < r < r(u_0, u_1, \ldots, u_n)$.

Now we can see by (1.2), the compactness of X and the continuity of $h_{x,r}(y)$ with respect to the variables x, y and r that

$$\lim_{k\to\infty} \int_X h_{x,r}(y) \prod_0^n u_i(y)\, w_k(y)\, d\mu(y) = \int_X h_{x,r}(y) \prod_0^n u_i(y)\, d\mu(y)$$

uniformly for $x \in X$, $r \ge r(u_0, \ldots, u_n)$. Moreover the integrals on the right have a positive lower bound for all x and $r \ge r(u_0, \ldots, u_n)$. We deduce from (1.8) and (1.14) that for sufficiently large k,

$$\begin{aligned} &\int_X h_{x,2r} \prod_0^n u_i\, w_k\, d\mu \\ &\qquad \le \Big(7 - \frac{1}{\sqrt{(n+1)(n+2)}}\Big) A \int_X h_{x,r} \prod_0^n u_i\, w_k\, d\mu\,, \end{aligned} \tag{1.15}$$

for all $x \in X$ and $r > 0$.

Now choose and fix one $w_{k(n)}$, so that (1.12), (1.13) and (1.15) are satisfied.

Denote by $d\nu_n = \prod_0^n u_i \, d\mu$. It follows from Lusin's theorem that there exists a continuous $\tilde{w}_{k(n)}$ on X taking values in $[1/2, 3/2]$ that agrees with $w_{k(n)}$ on X outside a set E_n of small ν_n measure. And let

$$u_{n+1} = \Big(\int_X \tilde{w}_{k(n)} \, d\nu_n \Big)^{-1} \tilde{w}_{k(n)} \, .$$

Clearly

$$\int \prod_0^{n+1} u_i \, d\mu = \int u_{n+1} \, d\nu_n = 1 \, ,$$

and

$$\frac{1}{2} - \frac{1}{100\,(n+2)} \le u_{n+1} \le \frac{3}{2} + \frac{1}{100\,(n+2)}$$

if $\nu_n(E_n)$ is sufficiently small. Moreover, $\nu_n(E_n)$ can be chosen small enough, so that (1.12), (1.13) and (1.15) remain true for slightly bigger constants when w_k is replaced by u_{n+1} :

$$\int_X \Big(\prod_0^{n+1} u_i \Big)^{1/2} d\mu \le \beta^{n+1} \, , \tag{1.16}$$

$$\int_X (1 - v_j) \prod_0^{n+1} u_i \, d\mu \le \Big(3 - \frac{1}{n+2} \Big) \beta^j \, , \qquad 0 \le j \le n \, ,$$

(the case $j = n + 1$ shall be provided later) and

$$\int_X h_{x,2r} \prod_0^{n+1} u_i \, d\mu \le \Big(7 - \frac{1}{n+2} \Big) A \int_X h_{x,r} \prod_0^{n+1} u_i \, d\mu \, .$$

Finally, choose v_{n+1} continuous on $X, 0 \le v_{n+1} \le 1$, and

$$v_{n+1} = \begin{cases} 0 \, , & \text{on the set where } \prod_0^{n+1} u_i \le \beta^{n+1} \, , \\ 1 \, , & \text{on the set where } \prod_0^{n+1} u_i \ge 2\,\beta^{n+1} \, . \end{cases}$$

It follows from (1.16) that

$$\int_X v_{n+1}\, d\mu \le \mu\Big(\prod_0^{n+1} u_i \ge \beta^{n+1}\Big) \le \beta^{(n+1)/2},$$

and

$$\int_X (1 - v_{n+1}) \prod_0^{n+1} u_i\, d\mu \le \int_{\prod_0^{n+1} u_i \le 2\beta^{n+1}} \prod_0^{n+1} u_i\, d\mu \le 2\,\beta^{n+1}.$$

Hence u_{n+1} and v_{n+1} satisfy all properties (1.5) to (1.11).

Finally let ν be a w^* limit of $\prod_0^n u_i\, d\mu$, or of some subsequence.

In view of (1.10) and (1.11), $\int v_j\, d\mu \le \beta^{j/2}$ and $\int (1-v_j)\, d\nu \le 3\beta^j$ for all j. Thus $v_j \to 0$ almost everywhere with respect to $d\mu$ and $v_j \to 1$ almost everywhere with respect to $d\nu$. Therefore μ and ν are mutually singular.

From (1.8), it follows that for all $x \in X$ and $r > 0$,

$$\nu(B(x, 2r)) \le \int h_{x,2r}\, d\nu \le 7A \int h_{x,r}\, d\nu \le 7A\, \nu\Big(B\Big(x, \frac{3}{2} r\Big)\Big).$$

Therefore ν is a doubling measure on X. This completes the proof of Theorem 1.

Let (X, ρ) be a compact metric space of finite uniform metric dimension. Let E_X be the set of accumulation points in X and F_X be the set of isolated points in X. Then $X = E_X \cup F_X$.

Lemma 2. *Let μ be any doubling measure on X. Then every point in F_X has positive μ-measure, and every point in E_X has zero μ-measure.*

PROOF. It is clear that every isolated point has positive μ-measure by the doubling condition. Let $x \in E_X$, and pick $\{x_n\} \subseteq X$ so that $0 < \rho(x, x_n) < \rho(x, x_{n-1})/10$. Then $\{B(x_n, 2\,\rho(x, x_n)/3)\}$ are mutually disjoint, and $x \in B(x_n, 4\,\rho(x, x_n)/3)$. Therefore

$$\begin{aligned}\mu(X) &\ge \sum_n \mu\Big(B\big(x_n, \frac{2}{3}\,\rho(x, x_n)\big)\Big) \\ &\ge c \sum_n \mu\Big(B\big(x_n, \frac{4}{3}\,\rho(x, x_n)\big)\Big) \ge c \sum_n \mu(\{x\}).\end{aligned}$$

Since $\mu(X) < \infty$, $\mu(\{x\})$ must be zero.

Therefore, for a doubling measure on (X, ρ) we may call $\mu|_{E_X}$ the *continuous part of* μ *and* $\mu|_{F_X}$ *the atomic part of* μ.

Corollary 1. *If X is a perfect set, then with respect to each doubling measure on X there exists a singular one.*

The following statement, which follows easily from the proof of Theorem 1, answers the question of Vol'berg and Konyagin.

Corollary 2. *All doubling measures on (X, ρ) are mutually absolutely continuous if and only if every doubling measure on X is purely atomic. A necessary topological condition is that $\overline{F}_X = X$.*

Although $\overline{F}_X = X$ is a necessary condition, it is far from being a sufficient condition for the mutual absolute continuity of all doubling measures on X; see the example below.

EXAMPLE. There are compact subsets X, Y and Z of $\mathbb{R}^1$, so that the sets of accumulation points E_X, E_Y and E_Z are all perfect sets, and the closures of isolated points $\overline{F}_X$, $\overline{F}_Y$ and $\overline{F}_Z$ equal to X, Y and Z respectively. However, all doubling measures on X are purely atomic; every doubling measure on Y contains a nontrivial continuous part; some doubling measures on Z are purely atomic and others have a nontrivial continuous part.

CONSTRUCTION OF X. Let E be the Cantor ternary set on $[0,1]$, F be the centers of all maximal intervals in $[0,1]\backslash E$, and $X = E \cup F$. Let $\{a_{n,j}\}_{j=1}^{2^{n-1}}$ be all points in F of distance $3^{-n}/2$ to E and $I_{n,j} = [a_{n,j} - 3^{-n+1}/2, a_{n,j} + 3^{-n+1}/2]$, thus $\{I_{n,j}\}_{j=1}^{2^{n-1}}$ forms a covering of E. Let μ be a doubling measure on X. Then there exists $c > 0$, so that

$$\sum_j \mu(\{a_{n,j}\}) \geq c \sum_j \mu(I_{n,j} \cap X) \geq c\,\mu(E),$$

for each $n \geq 1$. Since $\mu(X) < \infty$, $\mu(E)$ must be zero.

CONSTRUCTION OF Y. Let E be the Cantor ternary set on $[0,1]$, and $[0,1]\backslash E = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} I_{n,j}$, where $\{I_{n,j}\}_{j=1}^{2^{n-1}}$ are the maximal intervals in $[0,1]\backslash E$ of length exactly 3^{-n}, arranged in ascending order with respect to j. Given $0 < \beta_n < 1/4$, let $a_{n,j}$ and $b_{n,j}$ $(a_{n,j} > b_{n,j})$ be the two points in $I_{n,j}$ of distance $\beta_n 3^{-n}$ to E,

$$F = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} \{a_{n,j}, b_{n,j}\}$$

and $Y = E \cup F$.

Suppose that

$$\sum \left(log \frac{1}{\beta_n}\right)^{-1} < \infty . \tag{1.17}$$

Then every doubling measure μ on Y has a nontrivial continuous part.

Denote by $\{E_{n,j}\}_{j=1}^{2^n}$ the intervals in $[0,1]\backslash \bigcup_{k=1}^{n} \bigcup_{j=1}^{2^{k-1}} I_{k,j}$ in ascending order with respect to j. Note that $|E_{n,j}| = 3^{-n}$, and that

$$\bigcup_{j=1}^{2^n} E_{n,j} \cap F = \bigcup_{k=n+1}^{\infty} \bigcup_{j=1}^{2^{k-1}} \{a_{k,j}, b_{k,j}\} . \tag{1.18}$$

Moreover for each (n,j),

$$\begin{aligned} \operatorname{dist}(a_{n,j}, E) &= \operatorname{dist}(a_{n,j}, E_{n,2j}) \\ &= \operatorname{dist}(b_{n,j}, E) \\ &= \operatorname{dist}(b_{n,j}, E_{n,2j-1}) = \beta_n 3^{-n} . \end{aligned}$$

Suppose that μ is a doubling measure on Y which is purely atomic, *i.e.* $\mu(E) = 0$. Then there exists $c > 0$ depending only on μ, so that

$$\mu(E_{n,2j} \cap F) \geq c\left(\log \frac{1}{\beta_n}\right) \mu(\{a_{n,j}\}) , \tag{1.19}$$

and

$$\mu(E_{n,2j-1} \cap F) \geq c\left(\log \frac{1}{\beta_n}\right) \mu(\{b_{n,j}\}) . \tag{1.20}$$

In fact, fix (n,j) and write $E_{n,2j}$ as $[p/3^n, (p+1)/3^n]$ for some integer p and $a_{n,j}$ as $p/3^n - \beta_n/3^n$. Let $x_q = p/3^n + 1/3^{n+2q} \in E$, and $B_q = B(x_q, 2 \cdot 3^{-n-2q-1})$. Note that $\{B_q\}_1^\infty$ are mutually disjoint and

$B_q \subseteq E_{n,2j}$; moreover if $\beta_n 3^{-n} < 3^{-n-2q-1}$, then $a_{n,j} \in 2B_q$ and $2B_q \cap F = \{a_{n,j}\} \cup (2B_q \cap E_{n,2j})$, where $2B_q$ is the interval $B(x_q, 4\cdot 3^{-n-2q-1})$. Therefore there exist $c' > 1$, so that for $1 \le q \le (1/3) \log 1/\beta_n$,

$$\mu(\{a_{n,j}\}) \le \mu(2\,B_q) \le c'\,\mu(B_q) = c'\,\mu(B_q \cap F)\,.$$

Summing over q, $1 \le q \le (1/3) \log 1/\beta_n$, we obtain

$$\mu(\{a_{n,j}\}) \log \frac{1}{\beta_n} \le 3\,c'\,\mu(E_{n,2j} \cap F)\,.$$

The proof of (1.20) is similar.

Denote by $m_n = \mu(\bigcup_{j=1}^{2^{n-1}} \{a_{n,j}, b_{n,j}\})$ and recall that $\sum_1^\infty m_n = \mu(Y) < \infty$. Summing over j's in (1.19) and (1.20), we deduce from (1.18) that $\sum_{n+1}^\infty m_k \ge c\,(\log 1/\beta_n)\, m_n$, for each $n \ge 1$. Denote $\sum_n^\infty m_k$ by r_n and $\log 1/\beta_n$ by N_n, we have $r_{n+1} \ge r_n\,(c\,N_n)/(1 + c\,N_n)$ for $n \ge 1$. Thus

$$r_{n+1} \ge \prod_{k=1}^{n} \frac{cN_k}{1 + cN_k}\, \mu(Y)\,.$$

As $n \to \infty$, the left hand side approaches 0, and the right hand side has a positive limit under the hypothesis (1.17), which is impossible.

Therefore every doubling measure on Y must have a nontrivial continuous part.

The construction of Z uses Whitney modification of measures. Let E be a closed set on $\mathbb{R}^1$ and μ be a measure on $\mathbb{R}^1$. We call μ^E, a measure on $\mathbb{R}^1$, a *Whitney modification of* μ if $\mu^E \equiv \mu$ on E, and for some Whitney decomposition $\mathcal{W} = \{I\}$ of $\mathbb{R}^1 \backslash E$, $\mu^E(\{x_I\}) = \mu^E(I) = \mu(I)$ for every $I \in \mathcal{W}$ and x_I the center of I.

Recall that intervals in $\mathcal{W}$ have mutually disjoint interiors, $\cup_{\mathcal{W}} I = \mathbb{R}^1 \backslash E$ and $\operatorname{dist}(I, E)/4 \le |I| \le 4 \operatorname{dist}(I, E)$ for each $I \in \mathcal{W}$. A measure μ is said to have the doubling property on a closed set S, if (1.1) is satisfied for all $x \in S$ and $R > 0$.

Lemma 3. *If μ is a doubling measure on $\mathbb{R}^1$, then any Whitney modification μ^E of μ has the doubling property on $E \cup F$, where F consists of the centers of intervals in $\mathcal{W}$, and $\mathcal{W}$ is the Whitney decomposition associated with μ^E.*

PROOF. For $x \in E$, let $I_x = \{x\}$, and for $x \in F$, let I_x be the interval in $\mathcal{W}$ centered at x. For any $x \in E \cup F$ and $R > 0$, we claim that

$$\mu^E(B(x,R)) \cong \mu(B(x, \operatorname{dist}(x,E))), \tag{1.21}$$

if $B(x,R) \cap (E \cup F) = \{x\}$, and

$$\mu^E(B(x,R)) \cong \mu(B(x,R)), \tag{1.22}$$

if $B(x,R) \cap (E \cup F)$ has at least two points.

By $c \cong d$ we mean c/d is bounded above and below by positive numbers depending only on the constant Λ in the doubling property of μ.

Let $a = \inf\{y \in E \cup F : \ y > x - R\}$ and $b = \sup\{y \in E \cup F : \ y < x + R\}$. Note that $a = x - R$ if $x - R \in E$, and $a \in F$ otherwise; and that $b = x + R$ if $x + R \in E$, and $b \in F$ otherwise.

If $a = b$, then $a = b = x$. In this case,

$$\mu^E(B(x,R)) = \mu^E(\{x\}) = \mu(I_x) \cong \mu(B(x, \operatorname{dist}(x,E))).$$

If $a \neq b$, then $b - a \geq \max\{\operatorname{dist}(a,E), \operatorname{dist}(b,E)\}/64$. Note that $x + R - b \leq 64 \operatorname{dist}(b,E)$ and $a - (x - R) \leq 64 \operatorname{dist}(a,E)$. Therefore $2R \geq b - a \geq 2^{-12}R$. In this case

$$\mu^E(B(x,R)) = \mu^E([a,b]) = \mu([a - |I_a|/2, b + |I_b|/2]) \cong \mu(B(x,R)).$$

Doubling property of μ^E on $E \cup F$ follows immediately from (1.21) and (1.22).

This property of the Whitney modification has a natural generalization to $\mathbb{R}^n$. For the converse, we raise the following question.

QUESTION. For which $(X, \mu), X$ perfect in $\mathbb{R}^n$ and μ doubling on X, is μ the restriction of a doubling measure in $\mathbb{R}^n$?

CONSTRUCTION OF Z. It follows from Lusin's theorem and an example of Beurling and Ahlfors [1] that there exist a nontrivial doubling measure μ on $\mathbb{R}^1$ and a perfect set $E \subseteq [0,1]$ of positive length so that $\mu(E) = 0$. Let $\mathcal{W}$ be any Whitney decomposition of $\mathbb{R}^1 \backslash E, F$ be the centers of intervals in $\mathcal{W}$ and $Z = E \cup (F \cap [-100, 100])$. Then the Whitney modification μ^E has the doubling property on Z (a modification of Lemma 3) and is purely atomic.

Let σ be the Lebesgue measure on $\mathbb{R}^1$ and σ^E be a Whitney modification. Then $\sigma^E(E) = \sigma(E) > 0$, and σ^E has the doubling property on Z.

QUESTION. Do there exist X compact in $\mathbb{R}^1$, a doubling measure μ on X, such that $\overline{F}_X = X$ and that $\mu|_{E_X}$ is also a nontrivial doubling measure on E_X? (Recall that E_X is the set of accumulation points and F_X is the set of isolated points.)

QUESTION. Given a compact set X on $\mathbb{R}^n$, and $\alpha > 0$, does there exist a measure μ doubling on X such that μ has full measure on a Borel set of Hausdorff dimensions less or equal than α?

We believe that the answer is positive when $n = 1$.

2. Null sets for dyadic doubling measures.

A measure μ on $\mathbb{R}^1$ is called a *doubling measure* if (1.1) holds for all $x \in \mathbb{R}^1$ and $R > 0$, equivalently, there exists $\lambda \geq 1$ so that $\mu(I) \leq \lambda\,\mu(J)$ for all neighboring intervals I and J of the same length. A measure μ on $\mathbb{R}^1$ is called a *dyadic doubling measure* if there exists $\lambda \geq 1$ so that $\mu(I) \leq \lambda\,\mu(J)$ whenever I and J are two dyadic neighboring intervals of same length and $I \cup J$ is also a dyadic interval. We shall refer to the constant λ above as $\lambda(\mu)$.

Denote by $\mathcal{D}$ the collection of all doubling measures on $\mathbb{R}^1$ and by $\mathcal{D}_d$ the collection of all dyadic doubling measures on $\mathbb{R}^1$. Denote by $\mathcal{N}$ the collection of null sets for $\mathcal{D}$, *i.e.*, $\mathcal{N} = \{E \subseteq \mathbb{R}^1 : \mu(E) = 0$ for all $\mu \in \mathcal{D}\}$, and $\mathcal{N}_d$ its dyadic counterpart $\{E \subseteq \mathbb{R}^1 : \mu(E) = 0$ for all $\mu \in \mathcal{D}_d\}$. Clearly $\mathcal{N}_d \subseteq \mathcal{N}$, and $\mathcal{N}$ is invariant under any bilipschitz mapping on $\mathbb{R}^1$. However $\mathcal{N}_d$ is not invariant under translation, or under multiplication.

Theorem 2. *There exist a perfect set $S \subseteq [0,1]$ which is in $\mathcal{N}\backslash\mathcal{N}_d$, and a set $T \subseteq \mathbb{R}^1$ of full measure (i.e., $\mathbb{R}^1\backslash T$ has zero length) such that $t + S \in \mathcal{N}_d$ for each $t \in T$.*

A weaker version of Theorem 2 was proved in [9] with $\dim T = 1$. The present proof has the same structure, but uses more refined estimations.

The analogue of Theorem 2 under multiplication is more difficult. We are only able to find perfect sets S and T with $\dim T = 0$, so that $S \in \mathcal{N}\backslash\mathcal{N}_d$ but $tS \in \mathcal{N}_d$ for each $t \in T$. We shall report this elsewhere.

The following lemmas from [9] are needed in our proof.

Lemma 4. *Let μ be a dyadic doubling measure on $\mathbb{R}^1$. Then there exists $c > 1$ depending on $\lambda(\mu)$ only, so that for any dyadic interval S and any subinterval T of S,*

$$\frac{1}{4}\left(\frac{|T|}{|S|}\right)^c \mu(S) \le \mu(T) \le 4\left(\frac{|T|}{|S|}\right)^{1/c} \mu(S).$$

Lemma 5. *Let S be any dyadic interval and μ and ν be two dyadic doubling measures on $\mathbb{R}^1$ satisfying $\mu(S) = \nu(S)$. Then the new measure $\omega \equiv \nu$ on S, $\omega \equiv \mu$ on $\mathbb{R}^1\backslash S$ is a dyadic doubling measure on $\mathbb{R}^1$ with $\lambda(\omega) \le \max\{\lambda(\mu), \lambda(\nu)\}$.*

Lemma 6. *Given a, ε, $\delta \in (0,1)$ with $\varepsilon + \delta^a < 1/16$, then there exists a measure $\tau \in \mathcal{D}_d$, with $\lambda(\tau) \le 10^{1/a}$, which satisfies $\tau([0,1]) = 1$, $\tau([0,\varepsilon]) = \varepsilon$ and $\tau([1-\delta, 1]) = \delta^a$.*

We shall use $\tau_{a,\varepsilon,\delta}$ to denote the restriction of this τ measure to the interval $[0,1]$.

PROOF OF THEOREM 2. Let $\alpha > 1$ and choose $\beta > \alpha$, $0 < a < \min\{1/5, \alpha\beta^{-1}\}$, $0 < c_m < 1/4$ and positive integers L_m ($m \ge 1$) so that the following are true:

$$c_m^{-1} m^{\alpha-\beta} = o(1), \qquad \text{as } m \to \infty, \tag{2.1}$$

$$\sum m^{\beta a - \alpha} < \infty, \tag{2.2}$$

$$\sum (1 - m^{-\beta a})^{L_m} < \infty, \tag{2.3}$$

and

$$\sum (1 - 4c_m)^{L_m} = \infty. \tag{2.4}$$

For example, choose $\beta = 4\alpha$, $a = (\alpha-1)/5\alpha$, $c_m = (4m^{2\alpha})^{-1}$ and $L_m = [m^{2\alpha}]$, with $[\cdot]$ the greatest integer function.

Let $K_1 = 0$, and $K_{m+1} = K_m + L_m$ for $m \geq 1$. Define n_k inductively by letting $n_0 = 10$ and

$$n_{k+1} = n_k + 10 + [\beta \log_2 m - \log_2 c_m] \tag{2.5}$$

when $K_m \leq k < K_{m+1}$.

Given $m \geq 1$, $1+K_m \leq k \leq K_{m+1}$ and integer j, denote by $L's, I's$ and $J's$ the dyadic intervals:

$$L_{k,j} = \left[\frac{j}{2^{n_k}}, \frac{j+1}{2^{n_k}}\right],$$

$$I_{k,j} = \left[\frac{j}{2^{n_k}}, \frac{j}{2^{n_k}} + \frac{1}{2^{n_k+5}\dot{m}^\alpha}\right]$$

and

$$J_{k,j} = \left[\frac{j+1}{2^{n_k}} - \frac{1}{2^{n_k+5}\dot{m}^\beta}, \frac{j+1}{2^{n_k}}\right],$$

where $\dot{m}^\alpha = 2^{[\alpha \log_2 m]}$, $\dot{m}^\beta = 2^{[\beta \log_2 m]}$ and $[\cdot]$ is the greatest integer function. Note that for $1 + K_m \leq k \leq K_{m+1}$,

$$|J_{k,j}|/|I_{k,j'}| = O(m^{\alpha-\beta}) \to 0, \qquad \text{as } m \to \infty \tag{2.6}$$

and

$$|J_{k,j}|/|L_{k+1,j'}| = 2^{n_{k+1}-n_k-[\beta \log_2 m]-5} \geq m^{2\alpha}.$$

To *construct* S, first we permanently remove from $S_1 \equiv [0,1]$ a group of mutually disjoint intervals $I_{k,j}$ of different sizes, with k ranging from $1+K_1$ to K_2, and collect some of the J intervals from the remaining part of S_1; call the union S_2. Next we permanently remove from S_2 a group of intervals $I_{k,j}$ with k ranging from $1 + K_2$ to K_3, and collect some of the J intervals from the remaining part of S_2; call the union S_3, etc. Finally let $S = \cap S_m$.

Let $S_1 = [0,1]$,

$$\mathcal{C}_1^I = \{I_{1,j} : \ J_{1,j-1} \cup I_{1,j} \subseteq S_1\},$$

$$\mathcal{C}_1^J = \{J_{1,j} : \ J_{1,j} \cup I_{1,j+1} \subseteq S_1\}.$$

For $1 = 1 + K_1 \leq k < K_2$, let

$$\begin{aligned}\mathcal{C}_{k+1}^I = \mathcal{C}_k^I \cup \{I_{k+1,j} : \ & J_{k+1,j-1} \cup I_{k+1,j} \text{ is contained in } S_1 , \\ & \text{but neither } J_{k+1,j-1} \text{ nor } I_{k+1,j} \\ & \text{is contained in any interval in } \mathcal{C}_k^I \cup \mathcal{C}_k^J\} ,\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{k+1}^J = \mathcal{C}_k^J \cup \{J_{k+1,j} : \ & J_{k+1,j} \cup I_{k+1,j+1} \text{ is contained in } S_1 , \\ & \text{but neither } J_{k+1,j} \text{ nor } I_{k+1,j+1} \\ & \text{is contained in any interval in } \mathcal{C}_k^I \cup \mathcal{C}_k^J\} .\end{aligned}$$

Note that all intervals in $\mathcal{C}_{K_2}^I \cup \mathcal{C}_{K_2}^J$ have mutually disjoint interiors, and that intervals in $\mathcal{C}_{K_2}^I$ and those in $\mathcal{C}_{K_2}^J$ appear in pairs sharing common end points.

Let

$$S_2^I = \text{union of all intervals in } \mathcal{C}_{K_2}^I ,$$

and

$$S_2 = \text{union of all intervals in } \mathcal{C}_{K_2}^J .$$

We keep the interior of S_2^I in the complement of S permanently, and construct S_3^I and S_3 as subsets of S_2. Let

$$\mathcal{C}_{1+K_2}^I = \{I_{1+K_2,j} : \ J_{1+K_2,j-1} \cup I_{1+K_2,j} \subseteq S_2\} ,$$

and

$$\mathcal{C}_{1+K_2}^J = \{J_{1+K_2,j} : \ J_{1+K_2,j} \cup I_{1+K_2,j+1} \subseteq S_2\} .$$

And define for $1 + K_2 \leq k < K_3$,

$$\begin{aligned}\mathcal{C}_{k+1}^I = \mathcal{C}_k^I \cup \{I_{k+1,j} : \ & J_{k+1,j-1} \cup I_{k+1,j} \text{ is contained in } S_2 , \\ & \text{but neither } J_{k+1,j-1} \text{ nor } I_{k+1,j} \\ & \text{is contained in any interval in } \mathcal{C}_k^I \cup \mathcal{C}_k^J\} ,\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{k+1}^J = \mathcal{C}_k^J \cup \{J_{k+1,j} : \ & J_{k+1,j} \cup I_{k+1,j+1} \text{ is contained in } S_2 , \\ & \text{but neither } J_{k+1,j} \text{ nor } I_{k+1,j+1} \\ & \text{is contained in any interval in } \mathcal{C}_k^I \cup \mathcal{C}_k^J\} ;\end{aligned}$$

and let

$$S_3^I = \text{union of all intervals in } \mathcal{C}_{K_3}^I ,$$

and

$$S_3 = \text{union of all intervals in } \mathcal{C}_{K_3}^J .$$

We keep the interior of S_3^I in the complement of S permanently, and construct S_4^I and S_4 from S_3 as above. Continue this process to obtain $\mathcal{C}_{K_m}^I$, $\mathcal{C}_{K_m}^J$, S_m^I and S_m for all $m \geq 5$. Let

$$S = \bigcap_1^\infty S_m \, .$$

The rest of the proof is based on the simple fact that $J_{k,j}$ and $I_{k,j+1}$ are two adjacent intervals of very uneven sizes (2.6), whose common boundary point $(j+1)/2^{n_k}$ is a dyadic number.

To prove $S \in \mathcal{N}$, we note from (2.6) that for any $\nu \in \mathcal{D}$, and $1 + K_m \leq k \leq K_{m+1}$,

$$\nu(J_{k,j}) \leq m^{c(\alpha-\beta)} \, \nu(J_{k,j} \cup I_{k,j+1}) \, ,$$

for some $c > 0$ depending only on the doubling constant of ν. Summing over all $J_{k,j}$ in $\mathcal{C}_{K_m+1}^J$ we have $\nu(S) \leq \nu(S_{m+1}) \leq m^{c(\alpha-\beta)}\nu([0,1])$. Thus $\nu(S) = 0$. Alternatively, S is a porous set with large holes, therefore it is in $\mathcal{N}$, see [9].

To show $S \notin \mathcal{N}_d$, we apply scaled versions of Lemmas 5 and 6 repeatedly, to obtain a measure $\mu \in \mathcal{D}_d$ on $\mathbb{R}^1$, periodic with period 1, such that for $1 + K_m < k \leq K_{m+1}$ and all integers j

$$\mu(I_{k,j}) = (32\,\dot{m}^\alpha)^{-1} \, \mu(L_{k,j}) \tag{2.7}$$

and

$$\mu(J_{k,j}) = (32\,\dot{m}^\beta)^{-a} \, \mu(L_{k,j}) \, . \tag{2.8}$$

More precisely, μ is the weak limit of a subsequence of measures $\{\mu_{k_m}\}$ to be constructed as follows. Let μ_0 be the Lebesgue measure on $\mathbb{R}^1$. Assume that $\mu_{k_m} \in \mathcal{D}_d$, has been constructed with period 1. Then inductively for $1 + K_m \leq k \leq K_{m+1}$, let $f_{k,j}$ be the linear map that maps $L_{k,j}$ onto $[0,1]$, and define for $E \subseteq L_{k,j}$,

$$\mu_{k_{m+1}}(E) = \mu_{k_m}(L_{k,j}) \, \tau_{a,(32\dot{m}^\alpha)^{-1},(32\dot{m}^\beta)^{-1}}(f_{k,j}(E)) \, ,$$

where τ is the measure in Lemma 6. In view of Lemma 5, the measure $\mu_{k_{m+1}}$ is in $\mathcal{D}_d$ and satisfies (2.7) and (2.8) with μ replaced by $\mu_{k_{m+1}}$.

We note from the construction that

$$\mu_{k_2}([0,1] \backslash (S_2^I \cup S_2)) \leq \Big(1 - \frac{1}{2}\,(32^{-1} + 32^{-a})\Big)^{K_2 - K_1} \, .$$

The occurrence of 1/2 above is due to the fact that each $J \in \mathcal{C}_2^J$ and its companion I interval are not contained in the same L interval, but rather in two adjacent L intervals. Therefore

$$\begin{aligned}\mu_{k_2}(S_2) &\geq \mu_{k_2}\left(S_2^I \cup S_2\right)\left(1-32^{a-1}\right) \\ &\geq \left(1-\left(1-\frac{1}{2}\left(32^{-1}+32^{-a}\right)\right)^{K_2-K_1}\right)\left(1-32^{a-1}\right) \\ &\geq \left(1-\left(1-\frac{1}{2}32^{-a}\right)^{K_2-K_1}\right)\left(1-32^{a-1}\right).\end{aligned}$$

From the construction of S,

$$\mu(S) \geq \prod_{m=1}^{\infty}\left(1-\left(1-\frac{1}{2}32^{-a}m^{-\beta a}\right)^{K_{m+1}-K_m}\right)\left(1-32^{a-1}m^{\beta a-\alpha}\right),$$

which is positive in view of (2.2) and (2.3). Therefore $S \notin \mathcal{N}_d$.

For $x \in \mathbb{R}^1$, denote by $\|x\|$ the distance from x to the nearest integer. Let T be the set of t's such that there are infinitely many m's so that

$$\text{(2.9)} \quad \| t\,2^{5+n_k}\dot{m}^{\alpha} \| > c_m\,, \qquad \text{for every } k,\ 1+K_m \leq k \leq K_{m+1}\,.$$

Denote points t in $[0,1]$ by their binary expansion $\sum_{n=1}^{\infty} t_n 2^{-n}$ with $t_n = 1$ or 0. Then $\| t\,2^{5+n_k}\dot{m}^{\alpha} \| > c_m$ provided that not all t_n equal 0 for those n in the interval $(5+n_k+[\alpha\log_2 m], 7+n_k+[\alpha\log_2 m]-\log_2 c_m)$, and not all t_n equal 1 for the same range of $n's$. In view of (2.5), $n_{k+1} > n_k + [\alpha\log_2 m] - \log_2 c_m + 7$; thus for $m \geq 1$,

$$|\{t \in [0,1] :\ (2.9)\text{ holds}\}| \geq (1-4c_m)^{K_{m+1}-K_m}\,.$$

Since $[0,1]\backslash T = \bigcup_{M\geq 10}^{\infty}$ (2.9) fails for every $m \geq M$,

$$|[0,1]\backslash T| \leq \sum_{M=10}^{\infty}\prod_{m\geq M}\left(1-(1-4c_m)^{K_{m+1}-K_m}\right) = 0\,,$$

because of (2.4). Similar argument show that $|\mathbb{R}^1\backslash T| = 0$.

Given $t \in T$, assume that for a certain m,

$$\| t\,2^{n_k+5}\dot{m}^{\alpha} \| > c_m\,, \qquad \text{for every } k,\ 1+K_m \leq k \leq K_{m+1}\,;$$

then for each integer j,

$$\text{(2.10)} \qquad \frac{p}{2^{n_k+5}\dot{m}^\alpha} + \frac{c_m}{2^{n_k+5}\dot{m}^\alpha} \le t + \frac{j+1}{2^{n_k}} \le \frac{p+1}{2^{n_k+5}\dot{m}^\alpha} - \frac{c_m}{2^{n_k+5}\dot{m}^\alpha}$$

for some integer p. Note that $t + (j+1)/2^{n_k}$ is the common boundary point for intervals $t + J_{k,j}$ and $t + I_{k,j+1}$, and that in view of (2.10),

$$t + J_{k,j} \subseteq \left[\frac{p}{2^{n_k+5}\dot{m}^\alpha}, t + \frac{j+1}{2^{n_k}}\right],$$

$$t + I_{k,j+1} \supseteq \left[t + \frac{j+1}{2^{n_k}}, \frac{p+1}{2^{n_k+5}\dot{m}^\alpha}\right] \equiv I'_{k,j+1}\,.$$

Suppose ν is in $\mathcal{D}_d$. Because

$$\left[\frac{p}{2^{n_k+5}\dot{m}^\alpha}, \frac{p+1}{2^{n_k+5}\dot{m}^\alpha}\right]$$

is a dyadic interval, it follows from Lemma 4 that

$$\frac{\nu(t + J_{k,j})}{\nu(t + I_{k,j+1})} \le \left(\frac{|J_{k,j}|}{|I'_{k,j+1}|}\right)^c \le \left(\frac{\dot{m}^{\alpha-\beta}}{c_m}\right)^c$$

for some $c > 0$ depending on ν only. Summing over all $J_{k,j}$ in $\mathcal{C}^J_{K_m}$, we have

$$\nu(t + S) \le (m^{\alpha-\beta}\, c_m^{-1})^c\, \nu([0,1])\,.$$

Because t is in T, m can be made arbitrarily large. Therefore $\nu(t+S) = 0$ by (2.1). This proves that $t + S \in \mathcal{N}_d$ for every $t \in T$.

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