Revista Matemática Iberoamericana
Vol. 11, N.° 3, 1995

A criterion of Petrowsky's kind for a degenerate quasilinear parabolic equation

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Abstract. The celebrated criterion of Petrowsky for the regularity of the latest boundary point, originally formulated for the heat equation, is extended to the so-called p-parabolic equation. A barrier is constructed by the aid of the Barenblatt solution.

Little is known about the "Dirichlet boundary value problem" of genuinely nonlinear parabolic partial differential equations in arbitrary domains in space-time. Equations akin to the p-parabolic equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

are notoriously difficult to study in domains that are not space-time cylinders, that is, not of the form $G \times (0,T)$, $G \subset \mathbb{R}^n$. The aim of this note is to exhibit some interesting domains in $\mathbb{R}^n \times (-\infty, 0)$ for which the origin $(0,0)$ is a regular boundary point with the normal parallel to the time axis. This is the result of my efforts to extend the celebrated criterion of Petrowsky to a nonlinear situation.

In 1933 Petrowsky obtained a sharp criterion for the regularity of "the latest moment" in connexion with the heat equation, *cf.* [P]. For example, for the "one-dimensional" equation $u_t = u_{xx}$ the origin is a regular boundary point of the domain defined by

$$(1) \qquad -\frac{x^2}{4t} < \log|\log(-t)|, \qquad -T < t < 0,$$

while the origin is not a regular boundary point of any domain defined by

$$(2) \qquad -\frac{x^2}{4t} < (1+\varepsilon)\log|\log(-t)|, \qquad -T < t < 0,$$

if $\varepsilon > 0$. If continuous boundary values are prescribed on the Euclidean boundary of the domain (1) in the (x,t)-plane, then there is a solution to the heat equation taking these boundary values, in particular, at the origin. Notice that the boundary values are prescribed, as it were, for an elliptic problem, no special attention being paid to the parabolic boundary.

The boundary behaviour is a delicate question, indeed. A boundary point can be regular for the equation $u_t = \Delta u$ and, at the same time, irregular for the equation $2u_t = \Delta u$. Such a domain can be constructed with Petrowsky's criterion. A necessary and sufficient geometric condition for the regularity of an *arbitrary* boundary point, the so-called parabolic Wiener criterion, was proved in 1980 by Evans and Gariepy, *cf.* [EG]. The generalizations of the Wiener criterion to nonlinear parabolic equations have not been completely successful, *cf.* [G] and [Z]. They do not include equations like the p-parabolic and the porous medium equation.

The objective of our study is the p-parabolic equation

$$(3) \qquad \frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad 2 \le p < \infty.$$

The singular case $1 < p < 2$ would require modifications in the calculations to come and, for simplicity, we take $p > 2$. This is a prototype for a vast class of equations of the type $u_t = \operatorname{div} \mathbb{A}(x,t,\nabla u)$. The p-parabolic equation is also of interest for non-Newtonian fluids, *cf.* [B][1].

[1] NOTE ADDED IN NOVEMBER 1994. It has come to my attention that the p-parabolic equation has a strong application. Its solution represents the temperature in the atmosphere after the explosion of a hydrogen bomb, and the finite speed of propagation is essential.

In general, the equation ought to be interpreted in the weak sense. We refer the reader to the book [D]. The gradient ∇u of a solution is known to be Hölder continuous, but, in general, the time derivative u_t is merely a distribution. See [C], [Y], and [KV], for example.

Our result is the following.

Theorem. *Let $p > 2$. For the p-parabolic equation the origin is a regular boundary point of the domain*

$$(4) \qquad \frac{|x|^{p/(p-1)}}{(-t)^{p/\lambda(p-1)}} < K(-t)^{n(p-2)/\lambda}\,|\ln(-t)|^{\alpha(p-2)}, \quad -T < t < 0,$$

K and α denoting arbitrarily large constants, $\lambda = n(p-2)+p$, and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

REMARKS: 1°) The origin is *a fortiori* a regular boundary point of any subdomain of (4), if it is a boundary point at all.

2°) By *the exterior sphere condition* all the other boundary points of (4) are regular. It is the origin that is crucial.

3°) The geometric situation is interesting, because the tangent plane at $(0,0)$ is perpendicular to the time axis.

To understand the strange quantity in (4) we mention the *Barenblatt solution*

$$\mathcal{B}_p(x,t) = t^{-n/\lambda}\left(C - \frac{p-2}{p}\,\lambda^{-1/(p-1)}\left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}$$

defined when $t > 0$ and $x \in \mathbb{R}^n$. Here $p > 2$ and $\lambda = n(p-2)+p$. The positive constant C is usually determined so that $\int \mathcal{B}_p(x,t)\,dx = 1$, when $t > 0$, *i.e.*, $\mathcal{B}_p(x,0+) = \delta(x)$, the Dirac measure. See [B], [D, Section V.4, Equation (4.7), p. 125], or [KV]. When $p \to 2+$, the normalized Barenblatt solution approaches the ordinary heat kernel

$$(4\pi t)^{-n/2}\, e^{-|x|^2/4t},$$

obtained by Weierstrass. The key point in deriving our theorem is to construct a barrier (a supersolution of a specific kind) by the aid of the Barenblatt solution. This approach counts for our difficulties in obtaining the asymptotically right formulas, as $p \to 2+$. There are too many quantities in the calculations blowing up, as $p \to 2+$, to lead to Petrowsky's inequality (1).

Condition (4) is rather good, when $p > 2$. *We conjecture that the origin is an irregular boundary point of the domain*

$$\frac{|x|^{p/(p-1)}}{(-t)^{p/\lambda(p-1)}} < K(-t)^{n(p-2)/\lambda-\varepsilon}, \qquad -T < t < 0,$$

if $\varepsilon > 0$. It would be interesting to know the truth in this matter.[2]

It seems to be well-known that a boundary point is regular if and only if there exists a barrier at this point. Especially, a boundary point satisfying the exterior sphere condition (the earliest moment of the sphere being excluded as a tangent point) has a barrier and hence it is regular. Thus our theorem means that, given continuous boundary values on the boundary of the domain defined by (4), there exists a unique p-parabolic function taking the prescribed values in the classical sense. For all this we refer the reader to [KL].

Let Ω be a domain in $\mathbb{R}^n \times \mathbb{R}$ having the Euclidean boundary $\partial\Omega$ and $(0,0) \in \partial\Omega$. To be on the safe side, we remind the reader that a function $u : \Omega \longrightarrow \mathbb{R}$ satisfying the conditions

i) $u_t \geq \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ in Ω,

ii) $u > 0$ in Ω and $\liminf_{\zeta\to\xi} u(\zeta) > 0$ for all $\xi \in \partial\Omega$, $\xi \neq (0,0)$,

and

iii) $\lim_{\zeta\to(0,0)} u(\zeta) = 0$,

will do as a barrier at the origin (with respect to the domain Ω), see [KL]. Our barrier will be so smooth that (i) is satisfied in the classical sense. It will be constructed as a function of the form

$$(5)\quad \begin{aligned} u(x,t) = f(t)\left(C + \frac{p-2}{p}\,\lambda^{-1/(p-1)}\left(\frac{|x|}{(-t)^{1/\lambda}}\right)^{p/(p-1)}\right)^{(p-1)/(p-2)} \\ + \phi(t) \end{aligned}$$

when $x \in \mathbb{R}^n$ and $t < 0$. As in the Barenblatt solution $\lambda = n(p-2)+p$ and C is any positive constant. We will later choose

$$(6)\qquad f(t) = -\varepsilon\,|\ln(-t)|^{\alpha}, \qquad \phi(t) = -C^{(p-1)/(p-2)} f(t) + \rho(t),$$

[2] It is not too difficult to show that, if the right-hand member of the inequality is replaced by $K(-t)^{\beta}$ where $\beta = n(p-2)/\lambda(p-1)$, then the origin is irregular, indeed. Here $p > 2$.

where $\varepsilon > 0$ and $\rho(t) > 0$.

We shall select $\rho(t)$ so that u is a supersolution in the domain where $u > 0$ and this domain is to contain the domain (4). We do not care about what happens when $u \leq 0$. Notice that u is positive precisely when

$$\Big(C + \frac{p-2}{p}\,\lambda^{-1/(p-1)}\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\Big)^{(p-1)/(p-2)} < C^{(p-1)/(p-2)} + \frac{\rho(t)}{-f(t)}\,. \tag{7}$$

This inequality is at our disposal in the proof of i).

Observe that

$$u(x,t) \leq f(t)\,C^{(p-1)/(p-2)} - C^{(p-1)/(p-2)}\,f(t) + \rho(t) = \rho(t)\,.$$

Thus iii) is valid, if $\rho(t) \to 0$ as $t \to 0-$. This requirement restricts the choice of $\rho(t)$ in a decisive way.

Our aim is to show that u *is a supersolution in the domain defined by* (7), as required in i). Using the abbreviation

$$Q = C + \frac{p-1}{p}\,\lambda^{-1/(p-1)}\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)} \tag{8}$$

we have

$$\nabla Q = \frac{p-2}{p-1}\,\lambda^{-1/(p-1)}\,\frac{|x|^{(2-p)/(p-1)}x}{(-t)^{p/\lambda(p-1)}},$$

$$\frac{\partial Q}{\partial t} = \frac{p-2}{p-1}\,\lambda^{-p/(p-1)}\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\,\frac{1}{-t}\,,$$

since $\nabla|x|^q = q\,|x|^{q-2}\,x$. Recall that

$$u(x,t) = f(t)\,Q^{(p-1)/(p-2)} + \phi(t)\,. \tag{9}$$

Thus

$$\nabla u = f(t)\,Q^{1/(p-2)}\,\lambda^{-1/(p-1)}\,\frac{|x|^{(2-p)/(p-1)}x}{(-t)^{p/\lambda(p-1)}}\,,$$

$$|\nabla u|^{p-2}\,\nabla u = |f(t)|^{p-2}\,f(t)\,\lambda^{-1}Q^{(p-1)/(p-2)}\,\frac{x}{(-t)^{p/\lambda}}\,.$$

We obtain

$$
\begin{aligned}
&\operatorname{div}(|\nabla u|^{p-2}\,\nabla u)\\
(10)\qquad &= |f(t)|^{p-2}\,f(t)\,\lambda^{-1}\,Q^{(p-1)/(p-2)}\,\frac{n}{(-t)^{p/\lambda}}\\
&\quad + |f(t)|^{p-2}\,f(t)\,\lambda^{-p/(p-1)}\,Q^{1/(p-2)}\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\frac{1}{(-t)^{p/\lambda}}
\end{aligned}
$$

after some arithmetic. The last term in (10) can be written as

$$
\frac{p\,|f(t)|^{p-2}\,f(t)}{\lambda\,(p-2)\,(-t)^{p/\lambda}}\,Q^{1/(p-2)}(Q-C)\,.
$$

Further we have

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = f'(t)\,Q^{(p-1)/(p-2)}\\
(11)\qquad &\quad + \phi'(t) + f(t)\,\lambda^{-p/(p-1)}\,Q^{1/(p-2)}\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\frac{1}{-t}\,,
\end{aligned}
$$

where the last term can be written as

$$
\frac{p\,f(t)}{\lambda\,(p-2)\,(-t)}\,Q^{1/(p-2)}\,(Q-C)\,.
$$

Combining equations (10) and (11) we finally arrive at the expression

$$
\begin{aligned}
&u_t - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)\\
(12)\qquad &= \phi'(t) - \frac{pC}{\lambda\,(p-2)}\Big(\frac{1}{-t} - \frac{|f(t)|^{p-2}}{(-t)^{p/\lambda}}\Big)\,f(t)\,Q^{1/(p-2)}\\
&\quad + \Big(f'(t) + \frac{p\,f(t)}{\lambda\,(p-2)\,(-t)} - \frac{|f(t)|^{p-2}f(t)}{(p-2)\,(-t)^{p/\lambda}}\Big)\,Q^{(p-1)/(p-2)}\,,
\end{aligned}
$$

where we have used the identity

$$
\frac{1}{p-2} = \frac{n}{\lambda} + \frac{p}{\lambda\,(p-2)}\,.
$$

For $f(t) = -\varepsilon\,|\ln(-t)|^{\alpha}$ we have

$$
\begin{aligned}
f'(t) &+ \frac{p\,f(t)}{\lambda\,(p-2)\,(-t)} - \frac{|f(t)|^{p-2}\,f(t)}{(p-2)\,(-t)^{p/\lambda}}\\
&= \frac{-\varepsilon\,\alpha\,|\ln(-t)|^{\alpha-1}}{-t} - \frac{\varepsilon\,p\,|\ln(-t)|^{\alpha}}{\lambda\,(p-2)\,(-t)} + \frac{\varepsilon^{p-1}\,|\ln(-t)|^{\alpha(p-1)}}{(p-2)\,(-t)^{p/\lambda}}
\end{aligned}
$$

when $-1 < t < 0$. This expression is certainly negative, if

$$\frac{\varepsilon p\,|\ln(-t)|^{\alpha}}{\lambda\,(p-2)\,(-t)} \geq \frac{\varepsilon^{p-1}\,|\ln(-t)|^{\alpha(p-1)}}{(p-2)\,(-t)^{p/\lambda}}$$

when $-1 < t < 0$. This yields the condition

$$\frac{p}{\lambda} = \left(\frac{\alpha\,\lambda}{n\,e}\right)^{\alpha(p-2)} \varepsilon^{p-2} \tag{13}$$

for the largest possible ε. Let us fix ε this way. Then

$$-\frac{p\,C}{\lambda\,(p-2)}\left(\frac{1}{-t} - \frac{|f(t)|^{p-2}}{(-t)^{p/\lambda}}\right) f(t) > 0$$

in (12). Thus we have, using (7),

$$\begin{aligned}
&u_t - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)\\
&\quad\geq \phi'(t) - \frac{p\,C}{\lambda\,(p-2)}\left(\frac{1}{-t} - \frac{|f(t)|^{p-2}}{(-t)^{p/\lambda}}\right) f(t)\,C^{1/(p-2)}\\
&\qquad+ \left(f'(t) + \frac{p\,f(t)}{\lambda\,(p-2)\,(-t)} - \frac{|f(t)|^{p-2}\,f(t)}{(p-2)\,(-t)^{p/\lambda}}\right)\\
&\qquad\quad\cdot \left(C^{(p-1)/(p-2)} + \frac{\rho(t)}{-f(t)}\right)\\
&\quad= \rho'(t) - \frac{n\,C^{(p-1)/(p-2)}\,|f(t)|^{p-2}\,f(t)}{\lambda\,(-t)^{p/\lambda}}\\
&\qquad+ \rho(t)\left(\frac{f'(t)}{-f(t)} - \frac{p}{\lambda\,(p-2)\,(-t)} + \frac{|f(t)|^{p-2}}{(p-2)\,(-t)^{p/\lambda}}\right),
\end{aligned}$$

where we have used that

$$\phi'(t) = -C^{(p-1)/(p-2)}\,f'(t) + \rho'(t)\,.$$

Substituting the expression for $f(t)$ we obtain

$$\begin{aligned}
&u_t - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)\\
&\quad\geq \rho'(t) + \frac{n\,C^{(p-1)/(p-2)}\,\varepsilon^{p-1}\,|\ln(-t)|^{\alpha(p-1)}}{\lambda(-t)^{p/\lambda}}\\
&\qquad+ \rho(t)\left(\frac{-\alpha}{(-t)\,|\ln(-t)|} - \frac{p}{\lambda\,(p-2)\,(-t)}\right.\\
&\qquad\qquad\left.+ \frac{\varepsilon^{p-2}\,|\ln(-t)|^{\alpha(p-2)}}{(p-2)\,(-t)^{p/\lambda}}\right),
\end{aligned} \tag{14}$$

when $-1 < t < 0$ and $u > 0$.

Let us choose

$$\rho(t) = A(-t)^{1-p/\lambda}\,|\ln(-t)|^{\alpha(p-1)}.$$

Notice that

$$1 - \frac{p}{\lambda} = \frac{n\,(p-2)}{\lambda} > 0$$

so that $\rho(t) \to 0$, as $t \to 0-$. Inserting $\rho(t)$ into (14) we obtain

$$\begin{aligned} &u_t - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) \\ &\geq \Big(n\,C^{(p-1)/(p-2)}\,\varepsilon^{p-1} - A\,n\,(p-2) - \frac{A\,p}{p-2}\Big)\,\frac{|\ln(-t)|^{\alpha(p-1)}}{\lambda(-t)^{p/\lambda}} \\ &\quad + A\,\alpha\,(p-2)\,|\ln(-t)|^{\alpha(p-1)-1}\,(-t)^{-p/\lambda} \\ &\quad + \frac{A\,\varepsilon^{p-2}\,|\ln(-t)|^{\alpha(2p-3)}\,(-t)^{1-2p/\lambda}}{p-2}\,. \end{aligned}$$

It is plain that $u_t \geq \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$, if A does not exceed the value

$$A = \frac{n\,(p-2)\,C^{(p-1)/(p-2)}\,\varepsilon^{p-1}}{n\,(p-2)^2 + p} \tag{15}$$

and (7) holds. Using (13) and (15), we can now write (7) in the form

$$\begin{aligned} &\Big(1 + \frac{p-1}{p\,C}\,\lambda^{-1/(p-1)}\,\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\Big)^{(p-1)/(p-2)} \\ &\qquad < 1 + \frac{n\,p\,(p-2)}{\lambda\,(n\,(p-2)^2 + p)}\,\Big(\frac{n\,e}{\alpha\,\lambda}\Big)^{\alpha(p-2)} \\ &\qquad\quad \cdot (-t)^{n(p-2)/\lambda}\,|\ln(-t)|^{\alpha(p-2)}\,. \end{aligned} \tag{16}$$

Here the constant C is at our disposal.

In order to conclude the proof we have only to observe that (4) implies (16). Indeed, suppose that (4) holds for $-T < t < 0$, where $T < 1$. The right-hand member of (4) is less than $K(\alpha\lambda/ne)^{\alpha(p-2)}$. Hence

$$\begin{aligned} &\Big(1 + \frac{p-2}{p\,C}\,\lambda^{-1/(p-1)}\,\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\Big)^{(p-1)/(p-2)} \\ &\qquad < 1 + \frac{p-1}{p\,C}\,\lambda^{-1/(p-1)}\,\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{(p-1)/(p-2)} \\ &\qquad\quad \cdot \Big(1 + \frac{p-2}{p\,C}\,\lambda^{-1/(p-1)}\,K\Big(\frac{\alpha\,\lambda}{n\,e}\Big)^{\alpha(p-2)}\Big)^{1/(p-2)} \end{aligned}$$

$$= 1 + \frac{B}{C}\left(\frac{|x|}{(-t)^{1/\lambda}}\right)^{p/(p-1)}$$
$$\leq 1 + \frac{B}{C}K(-t)^{n(p-2)/\lambda}\,|\ln(-t)|^{\alpha(p-2)}$$

where B denotes a constant. It is plain that this estimate implies (16), if C is large enough.

Enlarging C further, if necessary, we have ii) valid. This concludes our proof.

REMARK. Forgetting the logarithm we can write (4) as

$$|x| = O((-t)^{\kappa}),$$

where $\kappa = \kappa(n,p)$. Now $\kappa \to 1/2$ as $p \to 2+$ and $\kappa \to n/(n+1)$ as $p \to \infty$. Moreover $\kappa < 1/2$ in the range $2 < p < 2n/(n-1)$. If $n = 1$, then $\kappa < 1/2$ for all $p > 2$. The smaller κ is, the better the condition (4). The smallest value of κ occurs, when

$$p - 2 = \frac{2n(\sqrt{2}-1)+2}{n^2+2n-1}\,.$$

For this rather strange value of p the exponent κ is slightly less than $1/2$ ($\kappa = \sqrt{2}-1$, when $n = 1$ and $\kappa = 1/2 - (3-2\sqrt{2})/4$, when $n = 2$).

The result has a natural extension to domains in the (x,t)-plane bounded by two Hölder continuous curves and two characteristic lines:

$$s_1(t) < x < s_2(t)\,, \qquad t_1 < t < t_2\,.$$

Suppose that the curves $x = s_1(t)$ and $x = s_2(t)$ are Hölder continuous with the aforementioned exponent κ, when $t_1 \leq t \leq t_2$. That is

$$|s_j(t+\tau) - s_j(t)| \leq K\,|\tau|^{\kappa}\,, \qquad j = 1,2\,.$$

Then *the boundary points lying on the curves are regular.* Some auxiliary constructions are needed to deduce this from the Theorem. We will not pursue the matter any further.

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Recibido: 25 de abril de 1.994
Revisado: 24 de noviembre de 1.994

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