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Pluriharmonic interpolation and hulls of C^1 curves in the unit sphere

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1. Introduction.

Let Γ be a simple closed C^1 curve lying in the unit sphere $b\mathbb{B}^n$, $\mathbb{B}^n$ being the unit ball in $\mathbb{C}^n$. By Stolzenberg's theorem [12], either Γ is polynomially convex or $\widehat{\Gamma} \setminus \Gamma$ is a 1-dimensional analytic subvariety of $\mathbb{B}^n$. Čirka [6] and Forstneric [7] showed that if Γ is C^2 and not polynomially convex, then $\widehat{\Gamma} \setminus \Gamma$ is smooth near Γ (*i.e.*, singularities do not accumulate at Γ) and Γ is transverse at each of its points-in the sense that the tangents to Γ never lie in the complex tangent space to $b\mathbb{B}^n$. In particular, if Γ is C^2 and has at least one complex tangent, then Γ is polynomially convex. This is no longer true when Γ is only C^1: Rosay [9] constructed a C^1 Jordan curve in $b\mathbb{B}^2$, bounding an analytic disk in $\mathbb{B}^2$, and having a complex tangent at a single point. Motivated by this, the first author proved in [1] that a rectifiable curve Γ is polynomially convex if the set of points of Γ where its tangent (these exist almost everywhere) is complex-tangential has positive linear measure.

On the other hand, Berndtsson and Bruna [4] showed that, when Γ is of class C^3 (in fact, $C^{2+\epsilon}$ was enough), the functions in $C(\Gamma)$ which can be interpolated by functions pluriharmonic on $\mathbb{B}^n$ and continuous on $\overline{\mathbb{B}^n}$ form a closed subspace of $C(\Gamma)$ of finite codimension. When Γ is polynomially convex, this codimension is zero. When Γ is not polynomially convex, then by Forstneric's result, $\widehat{\Gamma} \setminus \Gamma$ is smooth near

Γ and the Berndtsson-Bruna theorem is related with the solvability of the Dirichlet problem in $\widehat{\Gamma} \setminus \Gamma$. In fact, Shcherbina [11] later used this approach to characterize the codimension, for $n = 2$ and for C^∞ curves, in terms of the topology of Γ.

Here we shall study the C^1 case of both types of problems-hulls and pluriharmonic interpolation. Our first result (Theorem 2.1) states that if Γ is not polynomially convex, then $\widehat{\Gamma} \setminus \Gamma$ is still nice near Γ. In fact, close to $Q \in \Gamma$, $\widehat{\Gamma} \setminus \Gamma$ is the graph, over its projection on a suitable complex line, of a holomorphic function -the complex line being the normal to the sphere in the case when the tangent to Γ is transverse and being a complex tangent line to the sphere at Q otherwise. From this local parametrization we deduce in Section 3 our second result: If Γ is not polynomially convex and $T(s)$ is the so-called index of transversality of Γ (*i.e.*, $i\,T(s)$ is the complex normal component of the unit tangent to Γ) then $T(s)$ is greater or equal than 0 (after a possible change of orientation of Γ) and

$$\int \frac{ds}{T(s)^p} < \infty ,$$

for all $p > 0$. This captures both Forstneric's result (because, in the C^2 case, $T(s) = O(|s - s_0|)$ close to a complex-tangential point $\gamma(s_0)$) and the C^1 version of the Theorem in [1] for rectifiable curves.

Finally, in Section 4, we prove, under the hypotheses that T has constant sign and that $\int T^{-1} ds$ converges, that the Berndtsson-Bruna result on pluriharmonic interpolation carries over to C^1 curves; in particular, this holds for all non-polynomially convex C^1 curves.

To simplify the exposition, we assume in the rest of the paper that $n = 2$. It is routinely checked that all proofs generalize to $n > 2$ with straightforward modifications.

2. The local structure of the hull.

Let Γ be a simple closed curve of class C^1 lying on the unit sphere $S = b\,\mathbb{B}^2$, with arc-length parametrization $\gamma(s)$. We assume that Γ is not polynomially convex. By Stolzenberg's theorem (see [12] or [13, Theorem 30.1], [15, Chapter 13]), $V = \widehat{\Gamma} \setminus \Gamma$ is a one-dimensional analytic variety. We will prove here:

Theorem 2.1. *For each point $Q \in \Gamma$ there is a neighbourhood N and a complex line L through Q such that if π is the projection on L one* *has:*

a) π *is one-to-one from* $\overline{V} \cap N$ *onto a domain* $\overline{U} \subset L$ *of class* C^1, *and* π *maps* $\Gamma \cap N$ *onto an arc* $\tau \subset bU$.

b) There is an holomorphic function f *in* U *of class* $C^1(\overline{U})$ *such that* $\overline{V} \cap N$ *is the graph of* f *over* $\overline{U}$.

Thus V is locally a graph in the neighbourhood of Γ. In the proof of Theorem 2.1 we need the following "general principle" (see [12], [2], [3], [5] for the original argument; see also [15, Theorem 10.7], and [13, Lemmas 30.7 and 30.9]):

Lemma 2.2. *Let* $X \subset \mathbb{C}^2$ *be compact and* p *a polynomial. Let* Ω_∞ *be the unbounded component of* $\mathbb{C} \setminus p(X)$. *Suppose that there is an open Jordan arc* σ, *open in* $p(X)$, *such that*

a) $\sigma \subset b\,\Omega_\infty \cap b\,\Omega$, *where* Ω *is a bounded component of* $\mathbb{C} \setminus p(X)$.

b) $p^{-1}(\lambda) \cap X$ *contains exactly one point for all* $\lambda \in \sigma$.

Then, either $p^{-1}(\Omega) \cap \widehat{X}$ *is empty or* $p^{-1}(\Omega) \cap \widehat{X}$ *is single sheeted. In the later case, there exists* $\phi \in H^\infty(\Omega, \mathbb{C}^2)$ *such that*

$$p^{-1}(\Omega) \cap \widehat{X} = \{\phi(\lambda) : \ \lambda \in \Omega\}.$$

Moreover, there are no points of $\widehat{X} \setminus X$ *over* σ, *and* ϕ *has a continuous extension to* σ.

In case p is a coordinate function, say $p(z) = z_1$, then $\phi(\lambda) = (\lambda, f(\lambda))$, so over Ω, $\widehat{X} \setminus X$ is the graph of $f \in H^\infty(\Omega)$.

PROOF OF THEOREM 2.1. We shall distinguish two cases:

Case A: Γ *is transverse at* Q, *i.e. the tangent to* Γ *at* Q *has a non-zero complex normal component.* We can assume without loss of generality that $Q = (1, 0) = \gamma(0)$ and transversality means that $\gamma_1'(0)$ is a non-zero (pure imaginary) number.

Then $\gamma_1(s)$ determines s for $|s|$ small enough, say $|s| < \varepsilon$. Since Γ is simple, $\gamma_1(s) \neq 1$ for $|s| \geq \varepsilon$. Hence, shrinking ε if needed we see that the points of

$$\sigma \overset{\text{def}}{=} \{\gamma_1(s) : \ |s| < \varepsilon\}$$

are covered only once by z_1 on Γ. We also assume that ε is small enough so that σ is a C^1-curve (because $\gamma_1'(0) \neq 0$). We apply Lemma 2.2 with $X = \Gamma$ (note that $\sigma \subset b\,\Omega_\infty$ because $1 \in b\,\Omega_\infty$), $p(z) = z_1$ and therefore over Ω, the bounded component of $z_1(\Gamma)$ having σ in the boundary, V is the graph of some holomorphic function f. On σ

$$f(\gamma_1(s)) = \gamma_2(s)\,, \qquad |s| < \varepsilon\,.$$

Thus f is of class C^1 on σ. Now we take as U a C^1 domain in the z_1-plane contained in Ω and such that $b\,U \cap b\,\Omega \overset{\text{def}}{=} \tau \subset \sigma$.

At this point we need:

Lemma 2.3. *Let U be a C^1-domain in the complex plane, let f be holomorphic in U, continuous on $\overline{U}$. Let $\tau \subset bU$ be an arc on which $f|_{bU}$ is of class C^1. Then f' extends continuously to the points of τ.*

PROOF. Let $g : \overline{\Delta} \to \overline{U}$ be the Riemann mapping function from the unit disk $\overline{\Delta}$ to $\overline{U}$. Let $I \subset \mathbb{T} = b\,\Delta$ the arc mapped onto τ. Let $\tau' \subset \tau$ be a closed subarc of τ and $I' \subset I$ its corresponding arc in $\mathbb{T}$. It is well-known ([8, Theorem 10.1]) that $\arg g'$ has a continuous extension to $\overline{\Delta}$, hence $\log g \in$ VMOA, and so g' and $1/g'$ are in $L^p(\mathbb{T})$ for all $p > 0$. Let $h = f \circ g$, which is in the disc algebra. The hypothesis implies that h is absolutely continuous in I with derivative

$$h' = (f' \circ g)\, g'$$

at almost all points of I, and h' is in $L^p_{\text{loc}}(I)$ because f' is continuous on τ. Assume without loss of generality that $I = \mathbb{T} \cap D(1, r)$, $I' = \mathbb{T} \cap \overline{D}(1, r')$. Let χ be a C^∞ function supported in $D(1, r)$ equal to 1 on $D(1, r')$. We consider

$$H(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\chi(\zeta)\, h(\zeta)}{\zeta - z}\, d\zeta\,, \qquad z \in \Delta\,.$$

Note that

$$H(z) = \chi(z)\, h(z) - \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\chi(z) - \chi(\zeta)}{\zeta - z}\, h(\zeta)\, d\zeta$$

and that this last integral defines a smooth function on $\overline{\Delta}$. Therefore H is in the disk algebra and on $\mathbb{T}$ it is an absolutely continuous function with derivative in $L^p(\mathbb{T})$. This implies that H' is in the Hardy class H^p, $p > 0$, or which is the same, the non-tangential maximal function over the Stolz angle $S(\theta)$

$$(H')^*(\theta) = \sup\{|H'(z)| : z \in S(\theta)\}$$

belongs to $L^p(\mathbb{T})$, $p > 0$. From this it follows that

$$(h')^*(\theta) = \sup\{|h'(z)| : z \in S(\theta)\}$$

is in $L^p(I')$ for all $p > 0$ and hence is in $L^p_{\mathrm{loc}}(I)$ for all $p > 0$.

Since also $(g')^{-1}$ has non-tangential maximal function in $L^p(\mathbb{T})$ for all $p > 0$, we conclude that $(f' \circ g)^* \in L^1_{\mathrm{loc}}(I)$. We will show now that $f' \circ g$ extends continuously to all points of I, which obviously implies the lemma. Fix a closed subarc $J \subset I$ and let D be a C^∞-domain in Δ such that $J \subset bD \cap \mathbb{T} \subset I$. Then $f' \circ g$ has non-tangential maximal function (with respect to D) in $L^1(bD)$ and therefore belongs to $H^1(D)$. Now, $f' \circ g$ is continuous in $bD \cap \mathbb{T}$ and so $f' \circ g|_D$ extends continuously to the closure of D (here we use the fact that D being a C^∞ domain the holomorphic function theory of D is analogous to the one of Δ). By the choice of D, it then follows that $f' \circ g$ extends continuously to all points of J.

Shrinking the domain U a bit we conclude the proof of Theorem 2.1 in the case A.

Case B. *Γ is complex-tangential at Q, i.e. the tangent to Γ at Q points in the complex-tangential direction.* We can assume, without loss of generality, that $Q = (1,0) = \gamma(0)$ and that $\gamma_2'(0) = 1$, $\gamma_1'(0) = 0$. It follows immediately that there is $\varepsilon > 0$ such that

$$s \mapsto |\gamma_2(s)|$$

is strictly increasing in $(0,\varepsilon)$ and strictly decreasing in $(-\varepsilon, 0)$.

Since $|\gamma_1(s)|^2 + |\gamma_2(s)|^2 = 1$,

$$s \mapsto |\gamma_1(s)|$$

is strictly decreasing in $(0,\varepsilon)$ and strictly increasing in $(-\varepsilon, 0)$. Let us define

$$\sigma_+ = \{\gamma_1(s) : 0 < s < \varepsilon\},$$

$$\sigma_- = \{\gamma_1(s) : \ -\varepsilon < s < 0\} ,$$
$$\sigma = \{\gamma_1(s) : \ -\varepsilon < s < \varepsilon\} .$$

As before, since Γ is simple, and shrinking ε if needed we may assume that $\gamma_1(s) \notin \sigma$ for $|s| \geq \varepsilon$.

We know that both σ_+, σ_- meet each circle $|\lambda| = \rho$ at most once. We claim that σ_+, σ_- do not intersect. This is seen as in [9]: suppose that σ_+ and σ_- meet at $\lambda_0 = \gamma_1(a) = \gamma_1(b)$, with $0 < a < \varepsilon$, $-\varepsilon < b < 0$. Let $\nu = \gamma_1(b, a)$. Let R be a smooth simply connected domain in the z_1 plane separating ν from $z_1(\Gamma) \setminus \nu$ and containing $z_1(\Gamma) \setminus \nu$. The domain R admits a peaking function $H(\lambda)$ for the point λ_0. The function equal to H on $\overline{R}$ and to 1 in the domain bounded by ν can be uniformly approximated, by Mergelyan's theorem, by a sequence of polynomials $p_n(\lambda)$. This shows that the arc $\Gamma_1 = \gamma([b, a]) \subset \Gamma$ is a peak set for the algebra $P(\Gamma)$. Analogously, $\Gamma_2 = \Gamma \setminus \Gamma_1$ is also a peak set for $P(\Gamma)$. But Γ_1, Γ_2 are smooth arcs and so $P(\Gamma_1) = C(\Gamma_1)$, $P(\Gamma_2) = C(\Gamma_2)$. By general theory of uniform algebras (in fact an easy duality argument works), it follows that $P(\Gamma) = C(\Gamma)$ and Γ would be polynomially convex.

Therefore, σ_+ and σ_- do not meet, which means that γ_1 is one to one in $(-\varepsilon, \varepsilon)$ and Lemma 2.2 applies again as before. The main difference here with respect the situation in case A is that here the curve σ is in general not smooth at 1. This is why the z_1 projection does not work in this case and we shall look now to the z_2 projection.

Let g be the holomorphic function on Ω given by Lemma 2.2 so that $\lambda \mapsto (\lambda, g(\lambda))$ parametrizes V over Ω. Note that on σ

$$g(\gamma_1(s)) = \gamma_2(s) , \qquad |s| < \varepsilon ,$$

defines a curve τ in the z_2-plane which we can assume smooth because $\gamma_2'(0) = 1$.

We denote, for small δ,

$$\Omega_\delta = \{\lambda \in \Omega : \ |\lambda| > 1 - \delta\} .$$

The function g extends continuously to $\overline{\Omega}_\delta$, and

$$V_\delta = \{(\lambda, g(\lambda)) : \ \lambda \in \overline{\Omega}_\delta\}$$

is a neighbourhood of $Q = (1, 0)$ in $\widehat{\Gamma}$. The boundary $b\Omega_\delta$ consists of

$$C_\delta = \{\lambda \in \Omega : \ |\lambda| = 1 - \delta\}$$

and two arcs σ_+^δ, σ_-^δ included respectively in σ_+, σ_-. We denote $\sigma^\delta = \sigma_+^\delta \cup \sigma_-^\delta \cup \{1\} = \gamma_1(I_\delta)$ and $\tau^\delta = \gamma_2(I_\delta) = g(\sigma^\delta)$, a smooth subarc of τ.

We claim that for small enough δ, z_2 does not vanish at V_δ except at Q, *i.e.* g does not have zeros in Ω_δ. To see this, choose first δ such that g does not vanish on $\overline{C_\delta}$. Ω_δ is a simply-connected rectifiable domain and $g(b\,\Omega_\delta)$ is a closed piecewise smooth curve containing the arc $\tau^\delta = g(\sigma^\delta)$. Since $g(C_\delta)$ does not pass through 0 in the neighbourhood of 0 there are two components of $\mathbb{C} \setminus g(b\Omega_\delta)$, which we call R_+^δ and R_-^δ. Let m_+, m_- be the number of preimages (counting multiplicities) in Ω_δ of points of R_+^δ, R_-^δ, respectively. Let $N = \max\{m_+, m_-\}$.

If λ is a zero of g in Ω_δ, as g is an open mapping, the image of a neighbourhood of λ is a neighbourhood of 0 and hence meets both R_+^δ and R_-^δ. Therefore there are at most N zeros of g in Ω_δ, and g has no zeros in Ω_δ for small enough δ.

We will show next that $m_+ - m_-$ is either $+1$ or -1. We have

$$m_+ = \frac{1}{2\pi}\,\Delta_{b\Omega_\delta} \arg(g(\lambda) - a)\,, \qquad a \in R_+^\delta\,,$$

$$m_- = \frac{1}{2\pi}\,\Delta_{b\Omega_\delta} \arg(g(\lambda) - b)\,, \qquad b \in R_-^\delta\,,$$

or

$$\begin{aligned} 2\pi m_+ &= \Delta_{g(b\Omega_\delta)} \arg(\zeta - a) \\ &= \Delta_{g(C_\delta)} \arg(\zeta - a) + \Delta_{\tau^\delta} \arg(\zeta - a)\,, \end{aligned}$$

$$\begin{aligned} 2\pi m_- &= \Delta_{g(b\Omega_\delta)} \arg(\zeta - b) \\ &= \Delta_{g(C_\delta)} \arg(\zeta - b) + \Delta_{\tau^\delta} \arg(\zeta - b)\,. \end{aligned}$$

Recall that τ^δ is a smooth curve. Now substract both equations and make $a, b \to 0$ to get

$$2\pi(m_+ - m_-) = \lim_{a\to 0} \Delta_{\tau^\delta} \arg(\zeta - a) - \lim_{b\to 0} \Delta_{\tau^\delta} \arg(\zeta - b) = \pm 2\pi\,.$$

If instead we add the equations we get

$$2\pi(m_+ + m_-) = 2\,\Delta_{g(C_\delta)} \arg\zeta + \lim_{a\to 0} \Delta_{\tau^\delta} \arg(\zeta - a) + \lim_{b\to 0} \Delta_{\tau^\delta} \arg(\zeta - b)\,.$$

Since τ^δ is smooth the limits

$$\lim_{\delta\to 0}\lim_{a\to 0} \Delta_{\tau^\delta} \arg(\zeta - a)\,, \qquad \lim_{\delta\to 0}\lim_{b\to 0} \Delta_{\tau^\delta} \arg(\zeta - b)\,,$$

are π, $-\pi$ or $-\pi, \pi$ respectively. Hence we obtain

$$\lim_{\delta \to 0} \Delta_{C_\delta} \arg g(\lambda) = \lim_{\delta \to 0} \Delta_{g(C_\delta)} \arg \zeta = \pi (m_+ + m_-) .$$

Let $m = m_+ + m_-$, an odd integer; m is positive, because g is not constant. Since g does not vanish in Ω_δ we can consider $h = g^{1/m}$. Then

$$\lim_{\delta \to 0} \Delta_{C_\delta} \arg h(\lambda) = \pi .$$

As the argument shows, this holds for all arcs in Ω_δ joining σ_+^δ and σ_-^δ.

Next we will see that for small enough δ, h is a one-to-one map from Ω_δ to a domain R_δ, which is smooth in the neighbourhood of $0 \in b R_\delta$. Recall that $g(\gamma_1(s)) = \gamma_2(s)$, $\sigma^\delta = \gamma_1(I_\delta)$ and $\gamma_2(0) = 1$. Without loss of generality we can assume that $h(\gamma_1(s))$, $s > 0$, is the principal determination of $\gamma_2(s)^{1/m}$, so that $h(\sigma_+^\delta)$ is a C^1 arc having limiting tangent $(1,0)$ at 0, as it easily seen using polar coordinates. In the same way, $h(\sigma_-^\delta)$ is a C^1 arc having as tangent at 0 the opposite of some m-root of (-1). Since π is the variation of the argument, and m is odd, this root must be of course -1 and hence $h(\sigma^\delta)$ is a smooth arc. The fact that h is one-to-one follows then from the argument principle.

Let $f : R_\delta \to \Omega_\delta$ be the inverse map of h, $h(\lambda) = \zeta$, $g(\lambda) = \zeta^m$. We thus get the parametrization

$$\begin{aligned} R_\delta &\longrightarrow V_\delta \\ \zeta &\mapsto (f(\zeta), \zeta^m) , \end{aligned}$$

$f(0) = 1$, and R_δ is smooth near 0. Also, $f \in C(\overline{R}_\delta)$.

The final step is to show that m must, in fact, be 1. Suppose that $m \geq 3$. Let $F : \Delta \to R_\delta$ be the Riemann mapping function, $F(1) = 0$ from the unit disk to R_δ. Then we have a parametrization

$$\begin{aligned} \overline{\Delta} &\longrightarrow V_\delta \\ z &\mapsto (f(F(z)), F(z)^m) = (G(z), F(z)^m) . \end{aligned}$$

Shrinking R_δ we may suppose that R_δ is a C^1-domain so the mapping F satisfies, as said in the proof of Lemma 2.3, that $F' \in H^p$ for all p. In particular, F satisfies a Lipschitz condition of order β for all $\beta < 1$ and G is then in the disk algebra. Let $\alpha \subset b\,\Delta$ be the arc parametrizing $V_\delta \cap \Gamma$, $1 \in \alpha$.

We will show that $G(e^{it})$ has a non-zero derivative μ at $t = 0$. Once this is seen, since

$$|F(e^{it})| = O(|t|^{\beta})$$

for all $\beta < 1$, taking $\beta > 1/m$ we see that $F^m(e^{it})$ has zero-derivative and then $(\mu, 0)$ is tangent to Γ at $Q = (1,0)$, in contradiction with the assumption that Γ is complex-tangential at Q.

Let $G = IK$ be the inner-outer factorization of G,

$$K(z) = \exp\Big(\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{e^{it}+z}{e^{it}-z}\log|G(e^{it})|\,dt\Big).$$

Since $G(1) = 1$, the inner part I has an analytic continuation near 1; using that $|I(re^{it})|^2 \le 1$ and $|I(e^{it})|^2 = 1$ it is immediate to obtain that $I'(1)\,\overline{I(1)} \ge 0$. We want to show now that the formal rule

$$K'(1) = -K(1)\frac{1}{4\pi}\int_{-\pi}^{+\pi} \frac{1}{\sin^2\dfrac{t}{2}}\log|G(e^{it})|\,dt$$

(which makes sense because $\log|G(e^{it})| \simeq 1 - |G(e^{it})| \simeq |F(e^{it})|^{2m} = O(t^2)$ near $t = 0$) obtained by differentiating K under the integral sign is fully justified. This will give

$$K'(1) = k\,K(1)$$

with $k > 0$, because $\log|G| \le 0$ and then

$$\begin{aligned}\mu = G'(1) &= I'(1)\,K(1) + I(1)\,K'(1)\\ &= I'(1)\,\overline{I(1)} + K'(1)\,\overline{(K(1))} = k + I'(1)\,\overline{I(1)} > 0\,.\end{aligned}$$

It remains thus to show that the formal rule above holds true. For this it is enough to show that the part of the integral over α satisfies the rule. Now on α, $|G(e^{it})|^2 = 1 - |F(e^{it})|^{2m}$. Writing

$$u(t) = \frac{1}{2}\log(1 - |F(e^{it})|^{2m})$$

and

$$Hu(z) = \frac{1}{2\pi}\int_{\alpha} \frac{e^{it}+z}{e^{it}-z}u(t)\,dt$$

we shall show that $Hu(z)$ has an unrestricted derivative at 1, *i.e.*

$$\frac{Hu(z)-Hu(1)}{z-1}-\frac{1}{2\pi}\int_\alpha \frac{2e^{it}}{(e^{it}-1)^2}\,u(t)\,dt \underset{z\to 1}{\longrightarrow} 0 .$$

The last expression can be written

$$\frac{1}{\pi}\int_\alpha \frac{u(t)}{e^{it}-1}\Big(\frac{e^{it}}{e^{it}-z}-\frac{e^{it}}{e^{it}-1}\Big)\,dt = C(v)(z)-C(v)(1),$$

where $v(t)=u(t)/(e^{it}-1)$ and $C(v)$ denotes the Cauchy integral of v over the arc α. Now

$$v'(t)=\frac{u'(t)}{e^{it}-1}-\frac{i\,e^{it}\,u(t)}{(e^{it}-1)^2}.$$

But $u' = m\,|F^{2(m-1)}|(\mathrm{Re}\,\overline{F}F')(1-|F|^{2m})^{-1}$, so that $v'\in L^p$ for all p. Thus v satisfies a Lipschitz condition of order β for all β, hence so does $C(v)$ and we are done.

In conclusion we have proved that $m=1$. This means, with the notations used before, that for small δ, g is a one-to-one map from Ω_δ to $R_\delta = g(\Omega_\delta)$ with inverse f. We now take as U a C^1 domain included in R_δ so that $b\,U\cap b\,R_\delta=\tau^\delta$. On τ^δ

$$f(\gamma_2(s))=\gamma_1(s)$$

and hence f is C^1 on τ^δ. With Lemma 2.3 we conclude as before.

This completes the proof of Theorem 2.1.

3. Analytical properties of the curve Γ.

Theorem 2.1 has several consequences regarding the curve Γ itself. Here we will draw one of them, to be used in the next section. If $\gamma(s)$ is the arc-length parametrization of Γ, as mentioned in the introduction

$$\gamma(s)\,\overline{\gamma'(s)}=i\,T(s)$$

with T a real-valued continuous function (we use the notation $a\,\overline{b}$ for $\sum_j a_j\,\overline{b_j}$ if $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$). T can be said to measure the transversality of Γ. We prove:

Theorem 3.1. *If Γ is a C^1 simple closed curve which is not polynomially convex, then T has constant sign and*

$$\int \frac{ds}{|T(s)|^p} < +\infty\,, \qquad \textit{for all } p\,.$$

Corollary 3.2. *If* $\displaystyle\int \frac{ds}{T(s)} = +\infty$, *$\Gamma$ is polynomially convex.*

The Corollary implies Forstneric' result [7] according to which a C^2 curve which is complex-tangential at one point is polynomially convex and should be compared as well with Alexander's in [1], stating that a rectifiable curve whose set of complex tangencies has positive length is also polynomially convex.

PROOF OF THEOREM 3.1. Of course it is enough to prove the result locally around a complex-tangent point $Q \in \Gamma$. Let's consider the parametrization described in Theorem 2.1. Without loss of generality we assume $= (1,0)$, that U is a C^1-domain in the z_2-plane, f is holomorphic in U, of class C^1 up to $\overline{U}$ and

$$\begin{aligned} \overline{U} &\longrightarrow \overline{V} \cap N \\ \lambda &\mapsto (f(\lambda), \lambda) \end{aligned}$$

is the parametrization. Let $\tau \subset bU$ be the arc parametrizing $\Gamma \cap N$.

As before, let $F : \overline{\Delta} \to \overline{U}$ be the Riemann mapping function from the unit disk to U, $F(1) = 0$, and the parametrization

$$\begin{aligned} \overline{\Delta} &\overset{\phi}{\longrightarrow} \overline{V} \cap N \\ z &\mapsto (G(z), F(z))\,. \end{aligned}$$

Let $\alpha \subset \mathbb{T}$ be the arc mapped onto τ by F. We know that $F' \in H^p$ for all p. In particular $F(e^{it})$ is absolutely continuous and G is in the disk algebra. If $e^{it_0} \in \alpha$ is a point where $F(e^{it})$ is differentiable, since f is C^1 on τ, $G(e^{it})$ is also differentiable at t_0. Hence $G(e^{it})$ is differentiable almost everywhere on α and moreover $d(G(e^{it}))/dt$ is in $L^p(\alpha)$. By [14, Theorem IV.5], the non-tangential limit at such point

$$\lim_{\substack{z \to e^{it_0} \\ \text{n.t.}}} G'(z) \overset{\text{def}}{=} G'(e^{it_0})$$

exists and equals $-i\,e^{-it_0}\,d(G(e^{it}))/dt\,|_{t=t_0}$. It obviously follows that this limit also equals the non-tangential limit

$$\lim_{\substack{z\to e^{it_0}\\ \text{n.t.}}} \frac{G(z)-G(e^{it_0})}{z-e^{it_0}}\,.$$

Therefore at almost all points of α the tangential and radial derivatives of F and G exist and belong to $L^p(\alpha)$ for all p. In particular,

$$ds = (|G'|^2+|F'|^2)^{1/2}\,dt$$

and T is given almost everywhere on $\phi(\alpha)$ by

$$\begin{aligned} i\,T &= \frac{1}{(|G'|^2+|F'|^2)^{1/2}}\,\overline{\langle i\,e^{it}\,G', i\,e^{it}\,F'\rangle}\,\langle G,F\rangle \\ &= -\frac{i\,e^{-it}}{(|G'|^2+|F'|^2)^{1/2}}\,(\overline{G'}\,G+\overline{F'}\,F)\,. \end{aligned}$$

We claim that there is a constant $c>0$ such that

$$e^{it}(G'\,\overline{G}+F'\,\overline{F})\geq c\,,\qquad \text{almost everywhere on } \alpha\,.$$

Lemma 3.3. *Let $\phi:\overline{\Delta}\to\overline{\mathbb{B}^2}$, $\phi=\langle G,F\rangle$ be an analytic disk such that the tangential and radial derivatives of F, G exist almost everywhere on an arc $\alpha\subset\mathbb{T}$, with $\phi(\alpha)\subset S$. Let $a=\phi(0)\in\mathbb{B}^2$. Then*

$$e^{it}\,(G'\,\overline{G}+F'\,\overline{F})\geq\frac{1-|a|}{1+|a|}\,,\qquad \textit{almost everywhere on } \alpha\,.$$

PROOF. Since $|F|^2+|G|^2=1$ on α and $|F(re^{it})|^2+|G(re^{it})|^2\leq 1$, at one point where everything makes sense, one has

$$\begin{aligned} 0 &= \frac{d}{dt}(|F(e^{it})|^2+|G(e^{it})|^2) \\ &= 2\,\mathrm{Re}\;i\,e^{it}\left(F'(e^{it})\,\overline{F(e^{it})}+G'(e^{it})\,\overline{G(e^{it})}\right)\,, \end{aligned}$$

$$0\leq \left.\frac{d}{dr}\right|_{r=1}\left(|F(re^{it})|^2+|G(re^{it})|^2\right)=2\,\mathrm{Re}\;e^{it}\,(F'\,\overline{F}+G'\,\overline{G})\,.$$

Therefore $e^{it}(F'\overline{F}+G'\overline{G})$ is real and non-negative.

Assume now that $a=0$. Then we can write $F(z)=z\,F_0(z)$, $G(z)=z\,G_0(z)$ and apply the above to $\langle G_0,F_0\rangle$ because $|G_0|^2+|F_0|^2=1$ on α. Then

$$e^{it}(G'\overline{G}+F'\overline{F})=1+e^{it}(G_0'\overline{G_0}+F_0'\overline{F}_0)\geq 1$$

and the result is proved when $a=0$.

Assume now $a\neq 0$. We can choose complex orthonormal coordinates such that $a=(\lambda,0)$. Let $\varphi_a=\langle\varphi_1,\varphi_2\rangle$ be the automorphism of B with

$$\varphi_1=\frac{(s-1)\,z_1+\lambda-s\,z_1}{1-\overline{\lambda}\,z_1}\,,\qquad \varphi_2=-\frac{s\,z_2}{1-\overline{\lambda}\,z_1}$$

where $s^2=1-|\lambda|^2$, so that $\varphi_a(a)=0$, $\varphi_a^{-1}=\varphi_a$ (see [10, Chapter 2]).

If $\varphi_a\circ\phi=\phi_0=\langle G_0,F_0\rangle$, then $G=\varphi_1\langle G_0,F_0\rangle$, $F=\varphi_2\langle G_0,F_0\rangle$. Therefore, with $D_i=\partial/\partial z_i$

$$\begin{aligned} G' &= (D_1\varphi_1)\,G_0'+(D_2\varphi_1)\,F_0'\,,\\ F' &= (D_1\varphi_2)\,G_0'+(D_2\varphi_2)\,F_0'\,, \end{aligned}$$

and

$$\begin{aligned} G'\,\overline{G}+F'\,\overline{F} = {} & \big((D_1\varphi_1)\overline{\varphi_1}+(D_1\varphi_2)\overline{\varphi_2}\big)\,G_0'\\ & +\big((D_2\varphi_1)\overline{\varphi_1}+(D_2\varphi_2)\overline{\varphi}_2\big)\,F_0'\,. \end{aligned}$$

A computation shows that the brackets at (z_1,z_2) equal

$$\frac{s^2\,\overline{z_1}}{|1-\overline{\lambda}\,z_1|^2}\,,\qquad \frac{s^2\,\overline{z_2}}{|1-\overline{\lambda}\,z_1|^2}\,,$$

respectively. Hence,

$$e^{it}(G'\,\overline{F}+F'\,\overline{F})=\frac{s^2}{|1-\overline{a}\,\phi_0|^2}\,e^{it}(G_0'\,\overline{G}_0+F_0'\,\overline{F_0})\,.$$

Since $\phi_0(0)=0$, $e^{it}(G_0'\overline{G_0}+F_0'\overline{F_0})\geq 1$ almost everywhere on α and the lemma is proved.

Note that the lemma also gives a proof of the fact that an analytic disk $\phi : \overline{\Delta} \to \overline{\mathbb{B}^2}$ with $\phi(\mathbb{T}) \subset S$ passing through $a \in \mathbb{B}^2$ must have a boundary of length $\geq 2\pi(1-|a|)/(1+|a|)$.

This already shows that T has constant sign. Finally

$$\int_{\phi(\alpha)} \frac{ds}{|T(s)|^p} = \int_\alpha \frac{(|G'|^2+|F'|^2)^{(p+1)/2}}{|\overline{G'}\,G+\overline{F'}\,F|^p}\,dt \leq C\int_\alpha (|G'|^2+|F'|^2)^{(p+1)/2}\,dt < +\infty\,,$$

which ends the proof of Theorem 3.1.

4. Pluriharmonic interpolation from Γ.

In this section we assume that Γ is a simple closed C^1-curve on $S = b\,\mathbb{B}^2$, with arc-length parametrization $\gamma(s)$, such that its index of transversality defined by

$$i\,T(s) = \overline{\gamma'(s)}\,\gamma(s)$$

satisfies

$$T(s) \geq 0 \qquad \text{and} \qquad \int \frac{ds}{T(s)} < +\infty\,.$$

We may say that Γ is *close to transverse*. As seen in the previous section, this is the case if Γ is not polynomially convex, but we don't assume this here. Our purpose is to prove

Theorem 4.1. *With the assumptions above, the space* PHC *of pluriharmonic functions in* $\mathbb{B}^2$, *continuous up to* $b\,\mathbb{B}^2$ *has a closed trace of finite codimension in* $C(\Gamma)$. *In particular, if* Γ *is polynomially convex, any continuous function on* Γ *can be interpolated by a pluriharmonic function in* PHC.

This was proved in [4] for C^3 curves without any other assumption.

PROOF. The scheme of the proof is the same as that in [4], but each of the steps needs substantial modifications due to the lack of extra smoothness. Let $E \subset \Gamma$ be the set of complex-tangential points of Γ

and let $C_0(\Gamma)$ be the space of continuous functions in Γ vanishing on E. The first step is to construct an operator

$$K : C_0(\Gamma) \longrightarrow \mathrm{PHC}$$

such that

$$(K\varphi)(\gamma(s)) = \int L(t,s)\,\varphi(t)\,dt + \varphi(s)\,,$$

where the integral operator of the right-hand side is compact. The second step consists in showing that E is an interpolation set for the ball algebra, so that by a general result in [13, Theorem. 22.2], there is a linear continuous operator

$$I : C(E) \longrightarrow A(B)$$

such that $I\psi|_E = \psi$. Then the operator

$$P : C(\Gamma) \longrightarrow C(\Gamma)$$

defined by

$$P\varphi = K(\varphi - I\varphi) + I\varphi$$

satisfies

$$(P\varphi)(\gamma(s)) = \varphi(s) + \int L(t,s)\,(\varphi(t) - I\varphi(t))\,dt\,.$$

Now, Range P consists of boundary values of pluriharmonic functions in PHC. Moreover, by Fredholm theory, Range P is closed and of finite codimension. Then, a functional analysis argument ends the proof of the theorem (see [4, Section 6]).

To start with, let

$$K(t,z) = \frac{1}{\pi}\,\mathrm{Im}\,\frac{\gamma(t)\overline{\gamma'(t)}}{1 - z\overline{\gamma(t)}}\,, \qquad K\varphi(z) = \int K(t,z)\,\varphi(t)\,dt\,,$$

$$L(t,x) = K(t,\gamma(x)) = \frac{1}{\pi}\,T(t)\,\mathrm{Re}\,\frac{1}{1 - \gamma(x)\,\overline{\gamma(t)}}\,.$$

Note that $K(t,z)$ is positive and that

$$\mathrm{Re}\,(1 - \gamma(x)\,\overline{\gamma(t)}) = \frac{1}{2}\,|\gamma(x) - \gamma(t)|^2 \simeq |t - x|^2\,,$$

$$\mathrm{Im}\,(1 - \gamma(x)\,\overline{\gamma(t)}) = \mathrm{Im}\int_x^t \gamma'(s)\,\overline{\gamma(t)}\,ds = \int_x^t T(s)\,ds + O(|x - t|^2)\,.$$

Hence

$$\left|1-\gamma(x)\overline{\gamma(t)}\right| \simeq (t-x)^2 + \left|\int_t^x T(s)ds\right| .$$

Lemma 4.2. *With the assumption of Theorem 4.1,*

a) $L(t,x)$ *satisfies*

$$\sup_x \int_{|x-t|\le\delta} L(t,x)\,dt \to 0\,, \qquad \text{as } \delta \to 0\,.$$

b) $\int K(t,z)\,dt \le C\,,$ *for all* $z \in \mathbb{B}^2$.

PROOF.

$$|L(t,x)| \simeq T(t)\frac{|t-x|^2}{\left(|t-x|^2+\left|\int_t^x T(\xi)\,d\xi\right|\right)^2} \le T(t)\frac{|t-x|^2}{\left|\int_t^x T(\xi)\,d\xi\right|^2} .$$

Let $\phi(t)=\int_0^t T(\xi)\,d\xi$, ϕ is strictly increasing; let $\psi=\phi^{-1}$, we make the change of variables $u=\phi(t)$. If $v=\phi(x)$, since $T(t)\,dt=du$.

$$\int_{|t-x|\le\delta} T(t)\frac{|t-x|^2}{\left|\int_t^x T(\xi)\,d\xi\right|^2}\,dt \le \int_{|u-v|\le\varepsilon(\delta)} \frac{|\psi(u)-\psi(v)|^2}{|u-v|^2}\,du$$

with $\varepsilon(\delta)\to 0$ as $\delta\to 0$. Now we apply Hardy's inequality, or rather its proof:

$$\begin{aligned}
|\psi(u)-\psi(u)|^2 &= \left|\int_v^u \psi'(\xi)\,d\xi\right|^2\\
&\le \left(\int_v^u |\psi'(\xi)|\,|\xi-v|^{1/2}\,|\xi-v|^{-1/2}\,d\xi\right)^2\\
&\le 2\,|u-v|^{1/2}\int_v^u |\psi'(\xi)|^2\,|\xi-v|^{1/2}\,d\xi
\end{aligned}$$

by Holder's inequality with the measure $|\xi - v|^{-1/2}\, d\xi$. Then,

$$\begin{aligned}
&\int\limits_{|u-v|\le\varepsilon(\delta)} \frac{|\psi(u)-\psi(v)|^2}{|u-v|^2}\, du \\
&\quad\le 2 \int\limits_{|u-v|\le\varepsilon(\delta)} |u-v|^{-3/2} \left| \int_v^u |\psi'(\xi)|^2 \, |\xi - v|^{1/2}\, d\xi \right| du \\
&\quad\le 2 \int\limits_{|\xi-v|\le\varepsilon(\delta)} |\psi'(\xi)|^2 \, |\xi - v|^{1/2} \Bigg(\int\limits_{|u-v|\ge|\xi-v|} |u-v|^{-3/2}\, du \Bigg)\, d\xi \\
&\quad\le K \int\limits_{|\xi-v|\le\varepsilon(\delta)} |\psi'(\xi)|^2\, d\xi\,.
\end{aligned}$$

Hence for $a)$ it is enough to have $\int |\psi'(\xi)|^2\, d\xi < +\infty$. But changing variables again, $u = \psi(\xi)$, $du = \psi'(\xi)\, d\xi$, $\psi'(\xi) = 1/T(u)$ this is

$$\int \frac{du}{T(u)} < +\infty\,.$$

This proves part $a)$ of Lemma 4.2. For $b)$, let for fixed z, $s = s(z)$ be such that

$$|1 - \overline{\gamma(s)}\, z| = \min\{|1 - \overline{\gamma(t)}\, z| : \text{ all } t\}\,.$$

Then

$$|1 - \overline{\gamma(t)}\, z| \simeq |1 - \overline{\gamma(s)}\, z| + |1 - \overline{\gamma(t)}\, \gamma(s)|\,.$$

The inequality $\lesssim$ is immediate because $|1 - \overline{a} b|^{1/2}$ satisfies a triangle inequality. On the other hand,

$$\begin{aligned}
|1 - \overline{\gamma(t)}\, \gamma(s)| + |1 - \overline{\gamma(s)}\, z| &\lesssim |1 - \overline{\gamma(t)}\, z| + 2|1 - \overline{\gamma(s)}\, z| \\
&\lesssim 3|1 - \overline{\gamma(t)}\, z|
\end{aligned}$$

by the choice of s. Hence

$$|1 - \overline{\gamma(t)}\, z| \simeq |1 - \overline{\gamma(s)}\, z| + |t - s|^2 + \left| \int_s^t T(\xi)\, d\xi \right|,$$

$$K(t, z) = \frac{1}{\pi}\, T(t)\, \mathrm{Re}\, \frac{1}{1 - \overline{\gamma(t)}\, z}$$

$$
\begin{aligned}
&= \frac{1}{\pi} T(t) \frac{\operatorname{Re}(1-\overline{\gamma(t)}\,z)}{|1-\overline{\gamma(t)}\,z|^2} \\
&\simeq T(t) \frac{\operatorname{Re}(1-\overline{\gamma(t)}\,z)}{\left(|1-\overline{\gamma(s)}\,z| + |t-s|^2 + \left|\int_s^t T(\xi)\,d\xi\right|\right)^2}\,.
\end{aligned}
$$

Next,

$$
\begin{aligned}
2\operatorname{Re}(1-\overline{\gamma(t)}\,z) &= 1-\overline{\gamma(t)}\,z + 1 - \gamma(t)\,\overline{z} \\
&= |\gamma(t)-z|^2 + 1 - |z|^2 \\
&\lesssim |\gamma(t)-\gamma(s)|^2 + |z-\gamma(s)|^2 + 1 - |z|^2 \\
&\lesssim |t-s|^2 + |z-\gamma(s)|^2 + 1 - |z|^2\,.
\end{aligned}
$$

Write $r = r(z) = |z-\gamma(s)|^2 + 1 - |z|^2$, $R = |1-\overline{\gamma(s)}\,z|$. Then

$$
\begin{aligned}
K(t,z) &\lesssim T(t) \frac{|t-s|^2}{\left(|t-s|^2 + \int_0^t T(\xi)\,d\xi\right)^2} + T(t) \frac{r(z)}{\left(R + \int_s^t T(\xi)\,d\xi\right)^2} \\
&= K_1(t,s) + K_2(t,z)\,.
\end{aligned}
$$

In proving *a*) we have already seen that $\int K_1(t,s)\,dt = O(1)$.

Next, with l equal to the length of γ, assuming $s=0$, and with the change of variables $u = \int_0^t T(\xi)\,d\xi$,

$$
\int_0^l \frac{T(t)\,dt}{\left(R + \int_0^t T(\xi)\,d\xi\right)^2} = \int_0^M \frac{du}{(R+u)^2} \le \int_0^{+\infty} \frac{du}{(R+u)^2} = \frac{1}{R}\,.
$$

But $r \lesssim R$. Hence $\int K_2(t,z)\,dt \le C$, for all $z \in \mathbb{B}^2$, and part *b*) is also proved.

Lemma 4.3. *If φ vanishes whenever T vanishes, then*

$$
\lim_{z\to\gamma(x)} K\varphi(z) = \varphi(x) + \int \varphi(t)\,L(t,x)\,dt\,.
$$

PROOF.

$$\left|\int K(t,z)\,\varphi(t)\,dt - \varphi(x) - \int \varphi(t)\,L(t,x)\,dt\right|$$
$$\leq \int_{|x-t|\leq\delta} K(t,z)\,|\varphi(t)-\varphi(x)|\,dt + |\varphi(x)|\left|\int_{|x-t|\leq\delta} K(t,z)\,dt - 1\right|$$
$$+ \int_{|x-t|\geq\delta} |\varphi(t)|\,|K(t,z)-L(t,x)|\,dt + \int_{|x-t|\leq\delta} |\varphi(t)|\,|L(t,x)|\,dt$$
$$\stackrel{\text{def}}{=} T_1 + T_2 + T_3 + T_4\,.$$

Let $w(\delta)$ be the modulus of continuity of φ. Then,

$$T_1 \leq w(\delta)\int K(t,z)\,dt \leq C\,w(\delta)\,,$$

$$T_4 \leq \|\varphi\|_\infty \int_{|x-t|\leq\delta} L(t,x)\,dt\,.$$

We break the integral in T_2, accordingly to

$$\begin{aligned} K(t,z) &= \frac{1}{\pi}\,\mathrm{Im}\,\frac{z\,\overline{\gamma'(t)}}{1-z\,\overline{\gamma(t)}} + \frac{O(|z-\gamma(t)|)}{|1-z\,\overline{\gamma(t)}|} \\ &= -\frac{1}{\pi}\frac{d}{dt}\,\mathrm{Im}\,\log(1-z\,\overline{\gamma(t)}) + O(|1-z\,\overline{\gamma(t)}|^{-1/2})\,. \end{aligned}$$

Note that

$$|1-z\,\overline{\gamma(t)}| \gtrsim \int_s^t T(\xi)\,d\xi\,.$$

If z is close to $\gamma(x)$, with $T(x)\neq 0$, and δ is small, so that T is bounded below by some constant $C(x)$ between s and t, one has $|1-z\,\overline{\gamma(t)}| \geq C(x)\,|s-t|$. Hence

$$\int_{|x-t|\leq\delta} |1-z\,\overline{\gamma(t)}|^{-1/2} \leq C(x)\,\delta^{1/2}\,,$$

for δ small and z close to $\gamma(x)$. All these gives, taking into account that $\lim_{z\to\gamma(x)} T_3 = 0$ by dominated convergence ($K(t,z)$ is singular at

$z = \gamma(t)$ only), and Lemma 4.2,

$$\lim_{z\to\gamma(x)} \sup \left| K\varphi(z) - (\varphi(x) + \int \varphi(t)\, L(t,x)\, dt) \right|$$
$$\le C\, w(\delta) + C\, \delta^{1/2} + |\varphi(x)| \Big(\frac{1}{\pi} \Big(\arg\,(1 - \gamma(x)\, \overline{\gamma(x-\delta)}) - \arg\,(1 - \gamma(x)\, \overline{\gamma(x+\delta)}) \Big) - 1 \Big)$$
$$+ \|\varphi\|_\infty \int_{|x-t|\le\delta} L(t,x)\, dt\,.$$

Since

$$1 - \gamma(x)\, \overline{\gamma}(x+\delta) = -\, i\, T(x)\, \delta + o(\delta)\,,$$
$$1 - \gamma(x)\, \overline{\gamma}(x-\delta) = i\, T(x)\, \delta + o(\delta)\,,$$

we obtain Lemma 4.3 by making $\delta \to 0$. When $T(x) = 0$ this argument does not control the term T_2, but it vanishes because $\varphi(x) = 0$.

Lemmas 4.2 and 4.3 complete the first step of the proof. Indeed, obviously $K\varphi$ is pluriharmonic and $K\varphi$ has a continuous extension to $\overline{\mathbb{B}^2} \setminus \Gamma$. Lemma 4.3 shows that $K\varphi \in$ PHC if $\varphi \in C_0(\Gamma)$. Finally, the operator $\varphi \mapsto \psi$ where

$$\psi(x) = \int \varphi(t)\, L(t,x)\, dt$$

is compact in $C(\Gamma)$. For this, we must show, to prove equicontinuity, that

$$\int |L(t,x) - L(t,y)|\, dt$$

is small for $|x - y|$ small, and this follows from part *a*) of Lemma 4.2 and the continuity of L off the diagonal.

It only remains to prove that E is an interpolation set for the ball algebra. This is a well-known result that can be proved for instance applying the Davie-Øksendal theorem ([10, Theorem. 10.4.3]): it is enough to see that for each ε there are Koranyi balls $V(\xi_1, \delta_1), \dots, V(\xi_m, \delta_m)$ where

$$V(\xi, \delta) = \{z \in S :\ |1 - \xi\, \overline{z}| < \delta\}$$

such that $\sum_{i=1}^{m} \delta_i < \varepsilon$ and $E \subset V(\xi_1, \delta_1) \cup \cdots \cup V(\xi_m, \delta_m)$.

Given ε, let $\gamma(s_1), \ldots, \gamma(s_m) \in E$ be such that

$$E \subset \gamma\Big(\bigcup_{i=1}^{m}(s_i - \varepsilon, s_i + \varepsilon)\Big)$$

with $m = O(1/\varepsilon)$. By the mean value theorem

$$\gamma(s_i - \varepsilon, s_i + \varepsilon) \subset V(\gamma(s_i), \delta_i)$$

with

$$\delta_i \simeq \varepsilon\, w(\varepsilon)\,,$$

where w is the modulus of continuity of γ'. Hence

$$\sum_{i=1}^{m} \delta_i \leq m\, \varepsilon\, w(\varepsilon) = O(w(\varepsilon))$$

can be made arbitrarily small.

This ends the proof of Theorem 4.1.

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References.

[1] Alexander, H., Ends of varieties. *Bull. Soc. Math. France* **120** (1992), 523-547.

[2] Alexander, H., Polynomial approximation and hulls in sets of finite linear measure in $\mathbb{C}^n$. *Amer. J. Math.* **93** (1971), 65-74.

[3] Alexander, H., The polynomial hull of a rectifiable curve in $\mathbb{C}^n$. *Amer. J. Math.* **110** (1988), 629-640.

[4] Berndtsson, B., Bruna, J., Traces of pluriharmonic functions on curves. *Arkiv Math.* **28** (1990), 221-230.

[5] Bishop, E., Holomorphic completions, analytic continuation, and the interpolation of semi-norms. *Ann. of Math.* **78** (1963), 468-500.

[6] Čirka, E. M., Regularity of boundaries of analytic sets. *Math. Sb.* **117** (1982), 291-336. *Math. USSR Sbor.* **45** (1983), 291-335.

[7] Forstneric, F., Regularity of varieties in strictly pseudoconvex domains. *Publ. Mat.* **32** (1988), 145-150.

[8] Pommerenke, Ch., *Univalent Functions.* Vandenhoeck and Ruprecht, 1975.

[9] Rosay, J. P., A remark on a theorem by F. Forstneric. *Arkiv. Math.* **28** (1990), 311-314.

[10] Rudin, W., *Function Theory in the unit ball of* $\mathbb{C}^n$. Springer-Verlag, 1980.

[11] Shcherbina, N., Traces of pluriharmonic functions on the boundaries of analytic varieties. *Math. Z.* **213** (1993), 171-177.

[12] Stolzenberg, G., Uniform approximation on smooth curves. *Acta Math.* **115** (1966), 185-198.

[13] Stout, E. L., *The theory of uniform algebras.* Bogden and Quigleg, 1971.

[14] Tsuji, M., *Potential Theory in modern Function Theory.* Chelsea Publ., 1959.

[15] Wermer, J., *Banach algebras and several complex variables.* Graduate Texts in Mathematics, Springer-Verlag, 1976.

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