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Bilipschitz extensions from smooth manifolds

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Abstract. We prove that every compact C^1-submanifold of $\mathbb{R}^n$, with or without boundary, has the bilipschitz extension property in $\mathbb{R}^n$.

1. Introduction.

Let $X \subset \mathbb{R}^n$ and let $f : X \to \mathbb{R}^n$ be a map. We say that f is *L-bilipschitz* (abbreviated L-BL) if $L \geq 1$ and if

$$\frac{|x-y|}{L} \leq |f(x) - f(y)| \leq L\,|x-y|\,,$$

for all $x, y \in X$. Thus 1-BL maps preserve distances, and we call them *isometries*. Every isometry $f : X \to \mathbb{R}^n$ is the restriction of a unique affine isometry $g : \mathrm{aff}(X) \to \mathbb{R}^n$; we let $\mathrm{aff}(X)$ denote the affine subspace of $\mathbb{R}^n$ generated by a nonempty subset X of $\mathbb{R}^n$. Hence every isometry f has an extension to an isometry of $\mathbb{R}^n$.

In general, an L-BL map $f : X \to \mathbb{R}^n$ need not have an extension to a bilipschitz map of $\mathbb{R}^n$, even if X is a very simple set. For example, X may be the unit circle and $f : X \to \mathbb{R}^3$ a homeomorphism onto a knotted curve. The situation changes, however, if L is required to be close to 1. The following concept was introduced in [V, p. 239]:

A set $X \subset \mathbb{R}^n$ is said to have the *bilipschitz extension property* (abbreviated BLEP) in $\mathbb{R}^n$ if there is $L_0 > 1$ such that if $1 \leq L \leq L_0$, then every L-BL map $f: X \to \mathbb{R}^n$ has an L_1-BL extension $g: \mathbb{R}^n \to \mathbb{R}^n$, where $L_1 = L_1(L, X, n) \to 1$ as $L \to 1$.

In [V, 5.17] it was proved that every compact $(n-1)$-dimensional C^1-submanifold X of $\mathbb{R}^n$ has the BLEP in $\mathbb{R}^n$. In the present paper we are going to prove the same result for all compact p-dimensional C^1-submanifolds X of $\mathbb{R}^n$, $1 \le p \le n$, with or without boundary. The result is given as Theorem 3.14. For $p = 1$, a proof was recently given in [HP]. We shall modify the method of [HP] to cover the technically more challenging remaining dimensions as well. Our proof is based on the BLEP of compact polyhedra, which was established in [PV].

1.1. NOTATIONS. We let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of real numbers, integers, and positive integers, respectively. If $1 \le p \le n-1$, we identify $\mathbb{R}^p$ with the subset $\{x : x_{p+1} = \cdots = x_n = 0\}$ of $\mathbb{R}^n$. The distance between two sets $A, B \subset \mathbb{R}^n$ is written as $\operatorname{dist}(A, B)$ with the convention that $\operatorname{dist}(A, B) = \infty$ if A or B is empty. The diameter of A is $\operatorname{diam}(A)$ with $\operatorname{diam}(\varnothing) = 0$. For $r > 0$ we set

$$B^n(A, r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < r\},$$
$$\bar{B}^n(A, r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \le r\},$$

where we have simplified the notation by writing x for $\{x\}$.

If f and g are two functions defined in a set A and with values in $\mathbb{R}^n$, we set

$$|f - g|_A = \sup_{x \in A} |fx - gx|.$$

We often omit parentheses writing fx instead of $f(x)$. A map $f : A \to \mathbb{R}^n$ ($A \subset \mathbb{R}^n$) is called a *similarity* if there is $\lambda > 0$ such that

$$|fx - fy| = \lambda |x - y|$$

for all $x, y \in A$. The *similarity class* of a set $A \subset \mathbb{R}^n$ consists of all the images fA of A under similarities $f : A \to \mathbb{R}^n$.

We use the notation $\mathcal{P}(X)$ for the set of subsets of a set X. The cardinality of X will be denoted by $\# X$. The symbol id is used to denote various inclusion maps.

Let $Q \subset \mathbb{R}^n$ be a closed (or open) p-cube, $1 \le p \le n$, and let $t > 0$. We use the notation $Q(t)$ to denote the closed (or open) p-cube of $\mathbb{R}^n$ with the same center as Q, with side length t times that of Q, and with edges parallel to those of Q. The interior of Q is written as $\mathring{Q}$; it is the topological interior of Q in $\operatorname{aff}(Q)$.

2. Preparations.

We begin with a purely set-theoretic lemma. Perhaps surprisingly, it has an important role in the sequel.

2.1. Lemma. *Let $\varphi_j :]1, a_j] \to]1, \infty[$ $(j \in \mathbb{N}, a_j > 1)$ be a sequence of functions satisfying $\lim_{t \to 1} \varphi_j(t) = 1$ for all $j \in \mathbb{N}$. Then there exists a function $m :]1, a_1] \to \mathbb{N}$ with the following properties*

1) $a_{m(t)} \geq t\,,$ *for all* $t \in]1, a_1]\,,$
2) $m(t) \to \infty$ *as* $t \to 1\,,$
3) $\varphi_{m(t)}(t) \to 1$ *as* $t \to 1\,.$

PROOF. Set $b_1 = a_1$ and construct inductively a sequence $b_1 > b_2 > b_3 > \cdots$ of numbers $b_j > 1$ such that for $j \geq 2$ we have $b_j \leq a_j$, $b_j \leq 1 + 1/j$, and $\varphi_j]1, b_j] \subset]1, 1 + 1/j]$. Define $m :]1, a_1] \to \mathbb{N}$ by setting

$$m(t) = \max\{j \in \mathbb{N} : \ b_j \geq t\}$$

for $t \in]1, a_1]$. Since $a_{m(t)} \geq b_{m(t)} \geq t$ for all $t \in]1, a_1]$, (1) holds. Since $m(t) = j$ for $t \in]b_{j+1}, b_j]$, (2) holds. Since $\varphi_{m(t)}(t) \leq 1 + 1/m(t)$ for $t \in]1, b_2]$, (3) follows now from (2).

We now introduce the relative BLEP, which can be considered a generalization of the ordinary BLEP. In Theorem 2.4 we derive a useful property of the relative BLEP. Our notation in Definition 2.2 and remarks 2.3 is chosen to suit the application in Theorem 2.4.

2.2. Definition. *Let $X \subset \mathbb{R}^n$, let $K_0 > 1$, and let $A :]1, K_0] \to \mathcal{P}(X)$ be a function. We say that X has the bilipschitz extension property relative to A in $\mathbb{R}^n$ (abbreviated* BLEP *rel A) if there is $K' \in]1, K_0]$ such that if $1 < L \leq K'$, then every L-BL map $f : X \to \mathbb{R}^n$ with $f|_{A(L)} = \mathrm{id}$ has a K_1-BL extension $g : \mathbb{R}^n \to \mathbb{R}^n$, where $K_1 = K_1(L, X, A, n) \to 1$ as $L \to 1$.*

2.3. REMARKS 1). Let $A, B :]1, K_0] \to \mathcal{P}(X)$ be two functions as in Definition 2.2. If there is $L_0 \in]1, K_0]$ such that $A(L) \subset B(L)$ for

$L \in \,]1, L_0]$, we write $A \subset B$. If $A \subset B$ and if X has the BLEP rel A, then X has the BLEP rel B.

2). If $C \subset X$, we write BLEP rel C for BLEP rel A, where $A : \,]1, K_0] \to \mathcal{P}(X)$ is the constant function with value $A(L) = C$. The ordinary BLEP defined in the introduction is then the same concept as BLEP rel $\varnothing$.

3). Trivially, X always has the BLEP rel X.

4). To simplify the notation, we usually write $K_1 = K_1(L)$ without explicitly mentioning the data X, A, n, on which K_1 also depends.

2.4. Theorem. *Let $X \subset \mathbb{R}^n$, let $K_0 > 1$, and let $A, B : \,]1, K_0] \to \mathcal{P}(X)$ be two functions. Suppose also that X has the* BLEP rel A. *Then X has the* BLEP rel B *if and only if the following condition holds*

(*) *There exists $L_0 \in \,]1, K_0]$ such that if $1 < L \leq L_0$ and if $f : X \to \mathbb{R}^n$ is an L*-BL *mapping satisfying $f|_{B(L)} = \mathrm{id}$, then the map $f|_{A(L)} : A(L) \to \mathbb{R}^n$ has an L_1*-BL *extension $g_A : \mathbb{R}^n \to \mathbb{R}^n$, where $L_1 = L_1(L) = L_1(L, X, A, B, n) \to 1$ as $L \to 1$.*

PROOF. It is obvious that if X has the BLEP rel B, then (*) is true. To prove the converse, assume (*) and let $K' \in \,]1, K_0]$ be the number and $K_1 : \,]1, K'] \to \,]1, \infty[$ the function given by Definition 2.2 for the BLEP of X rel A. We must find the corresponding objects $K'' = K'_B$ and $K_2 = K_1^B$ for the BLEP of X rel B. By choosing K' small enough, we may assume that $K' \leq L_0$.

Choose $K'' > 1$ such that $L\,L_1(L) \leq K'$ for all $L \in \,]1, K'']$. Let $1 < L \leq K''$ and let $f : X \to \mathbb{R}^n$ be an L-BL mapping satisfying $f|_{B(L)} = \mathrm{id}$. Then (*) implies that the map $f|_{A(L)} : A(L) \to \mathbb{R}^n$ has an $L_1(L)$-BL extension $g_A : \mathbb{R}^n \to \mathbb{R}^n$. The map $f_A = g_A^{-1} \circ f : X \to \mathbb{R}^n$ is $L\,L_1(L)$-BL and satisfies $f_A|_{A(L)} = \mathrm{id}$. Since $L\,L_1(L) \leq K'$, we may apply the BLEP of X rel A to find an L_2-BL extension $g_1 : \mathbb{R}^n \to \mathbb{R}^n$ of f_A with $L_2 = K_1(L\,L_1(L))$. Now, the map $g = g_A \circ g_1 : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $g|_X = f$ and is K_2-BL with $K_2 = L_1 L_2$ satisfying $K_2 \to 1$ as $L \to 1$. It follows that X has the BLEP rel B.

2.5. Flatness.

Let $\Delta = v_0 \cdots v_p \subset \mathbb{R}^n$ be a p-simplex with vertices $v_0, \ldots, v_p$,

$p \geq 1$. As in [V, 2.6], we define the *flatness* $\rho(\Delta)$ of Δ by

$$\rho(\Delta) = \frac{\operatorname{diam}(\Delta)}{b(\Delta)},$$

where $b(\Delta)$ is the smallest height of Δ. Explicitly, we have

$$b(\Delta) = \min_{0 \leq i \leq p} \operatorname{dist}(v_i, \operatorname{aff}(\Delta_i)),$$

where Δ_i is the $(p-1)$-face of Δ opposite to v_i. The simplex $\Delta = v_0 \cdots v_p$ is called a *corner* if there is $i \in \{0, \ldots, p\}$ such that the vectors $v_j - v_i$ $(0 \leq j \leq p,\ j \neq i)$ are mutually orthogonal and of equal length $|v_j - v_i| = t$. The number $t > 0$ is called the *size* of the corner Δ. The flatness of a corner Δ is

$$\rho(\Delta) = \begin{cases} 1, & p = 1, \\ \sqrt{2p}, & p \geq 2. \end{cases}$$

Since $\rho(\Delta)$ is a continuous function of $(v_0, \ldots, v_p)$, we can choose an integer $m_n \geq 2$ with the following property: If $1 \leq p \leq n-1$, if $\Delta_0 = u_0 \cdots u_p \subset \mathbb{R}^n$ is a p-corner with size t, and if $v_0, \ldots, v_p$ are points in $\mathbb{R}^n$ with $|u_j - v_j| \leq t/m_n$ for all $j \in \{0, \ldots, p\}$, then $\Delta = v_0 \cdots v_p$ is a p-simplex with $\rho(\Delta) \leq 2p$.

3. The main result.

3.1. Basic assumptions.

Let $1 \leq p \leq n$ and let X be a compact p-dimensional C^1-submanifold of $\mathbb{R}^n$. The purpose of this paper is to prove that X has the BLEP in $\mathbb{R}^n$. In Section 3 we give a detailed exposition of the case where X has no boundary. The modifications needed to cover the case of manifolds with boundary will be briefly discussed in Section 4.

We begin by giving our assumptions on X more explicitly. Thus, in Section 3, $1 \leq p \leq n-1$ and X is a compact subset of $\mathbb{R}^n$ such that for every point y of X there is an open set U of $\mathbb{R}^p$ and an embedding $f : U \to \mathbb{R}^n$ satisfying the following conditions 1)-3):

1) $y \in fU \subset X$ and fU is open in X,

2) f is continuously differentiable in U,

3) for $x \in U$, the linear map $f'(x) : \mathbb{R}^p \to \mathbb{R}^n$ is injective.

As usual, such a map $f : U \to \mathbb{R}^n$ is called a *chart* of X at y.

3.2. The cube family $\mathcal{K}$.

Let $\delta > 0$ and let $\mathcal{J} = \mathcal{J}(\delta)$ be the family of all closed n-cubes $Q \subset \mathbb{R}^n$ with side length δ and with vertices in $\delta\mathbb{Z}^n$. Let $m_n \in \mathbb{N}$ be as in 2.5, and define

$$N_0 = N_0(n) = 2\,n\,(m_n + 1)\,, \qquad N = N(n) = (N_0 + 1)^n\,,$$
$$W = \{0, 1, \ldots, N_0\}^n\,.$$

For $w \in W$ we set

$$\mathcal{J}_w = [0, \delta]^n + \delta w + (N_0 + 1)\,\delta\mathbb{Z}^n\,.$$

Then the N subfamilies $\mathcal{J}_w$, $w \in W$, of $\mathcal{J}$ are disjoint, and

$$\mathcal{J} = \bigcup_{w \in W} \mathcal{J}_w\,.$$

Moreover, we have

$$\operatorname{dist}(Q, R) \geq N_0\,\delta$$

whenever $Q, R \in \mathcal{J}_w$, $Q \neq R$, $w \in W$. Choose an arbitrary enumeration

$$W = \{w(1), \ldots, w(N)\}$$

of W, set $\mathcal{J}_i = \mathcal{J}_{w(i)}$ for $i \in \{1, \ldots, N\}$, and note that

$$\mathcal{J} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_N\,.$$

Set

$$\mathcal{K} = \mathcal{K}(\delta) = \{Q \in \mathcal{J} :\ Q \cap X \neq \varnothing\}\,, \qquad \mathcal{K}_i = \mathcal{K}_i(\delta) = \mathcal{K}(\delta) \cap \mathcal{J}_i\,,$$

and observe that $\mathcal{K}$ is the disjoint union

$$\mathcal{K} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N\,.$$

3.3. The numbers $\delta(L)$ and the sets $D(L)$.

Let $j \in \mathbb{N}$ and choose $\delta = 1/j$ in Paragraph 3.2. For every $Q \in \mathcal{K}$, choose a point $x_Q \in Q \cap X$. Since X is compact, the set

$$D_j = \{x_Q : \ Q \in \mathcal{K}\}$$

is finite. By *e.g.* [P, 2.4], D_j has the BLEP in $\mathbb{R}^n$. Hence there are $L_0^j > 1$ and a function $L_1^j :]1, L_0^j] \to]1, \infty[$ such that $L_1^j(L) \to 1$ as $L \to 1$ and such that if $1 < L \leq L_0^j$, then every L-BL map $f : D_j \to \mathbb{R}^n$ has an $L_1^j(L)$-BL extension $g : \mathbb{R}^n \to \mathbb{R}^n$.

We set $K_0 = L_0^1$. Applying Lemma 2.1 with $a_j = L_0^j$, $\varphi_j = L_1^j$, we get a function $m :]1, K_0] \to \mathbb{N}$ with the following properties:

1) $L_0^{m(L)} \geq L$, for all $L \in]1, K_0]$,

2) $m(L) \to \infty$ as $L \to 1$,

3) $L_1^{m(L)}(L) \to 1$ as $L \to 1$.

If $1 < L \leq K_0$, we define

$$\delta(L) = \frac{1}{m(L)}, \qquad L_2(L) = L_1^{m(L)}(L), \qquad D(L) = D_{m(L)}.$$

Then $\delta(L) \to 0$ and $L_2(L) \to 1$ as $L \to 1$, and $D :]1, K_0] \to \mathcal{P}(X)$ is a function. Moreover, if $1 < L \leq K_0$, then every L-BL map $f : D(L) \to \mathbb{R}^n$ has an $L_2(L)$-BL extension $g : \mathbb{R}^n \to \mathbb{R}^n$. This fact, together with Theorem 2.4 (with $A = D$, $B = \varnothing$) and Remark 2.3.2, immediately implies the next lemma, which reduces our task to that of proving the BLEP of X rel D.

3.4. Lemma. *If X has the* BLEP rel D *in $\mathbb{R}^n$, then X has the* BLEP *in $\mathbb{R}^n$.*

Since X and n are fixed, we mostly do not indicate the dependence of various quantities on them in our notation. In many considerations we may also think of $L \in]1, K_0]$ as being fixed, at least temporarily. Then we simplify the notation by dropping the parameter L out of it. For example, from now on we write

$$(3.5) \quad \delta = \delta(L), \qquad D = D(L), \qquad \mathcal{K} = \mathcal{K}(\delta(L)) = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N,$$

whenever $1 < L \leq K_0$.

3.6. Constructions.

Let $L \in \,]1, K_0]$, and let δ, D, and $\mathcal{K}$ be as in (3.5). Then $D = \{x_Q : Q \in \mathcal{K}\}$, where $x_Q \in Q \cap X$ is the point chosen in Paragraph 3.3 with $\delta = 1/m(L)$.

If $Q \in \mathcal{K}$, we let T_Q be the tangent plane of X at x_Q. Explicitly, if $f : U \to \mathbb{R}^n$ is a chart of X at x_Q as in Paragraph 3.1 and if $x_Q = f(x)$, then

$$T_Q = x_Q + \operatorname{im} f'(x) = x_Q + T_Q^0 ,$$

where $T_Q^0 = \operatorname{im} f'(x)$ is a p-dimensional linear subspace of $\mathbb{R}^n$, which can be shown to be independent of the chart f.

For each $Q \in \mathcal{K}$ we choose a closed n-cube Q^* of $\mathbb{R}^n$ with center x_Q, with side length

$$\lambda = \lambda(L) = 2\,(m_n + 1)\sqrt{n}\,\delta(L),$$

and such that Q^* has a p-face parallel to T_Q. Figure 1 illustrates the situation with $n = 2$, $p = 1$. The maps π_Q and ψ_Q will be defined below.

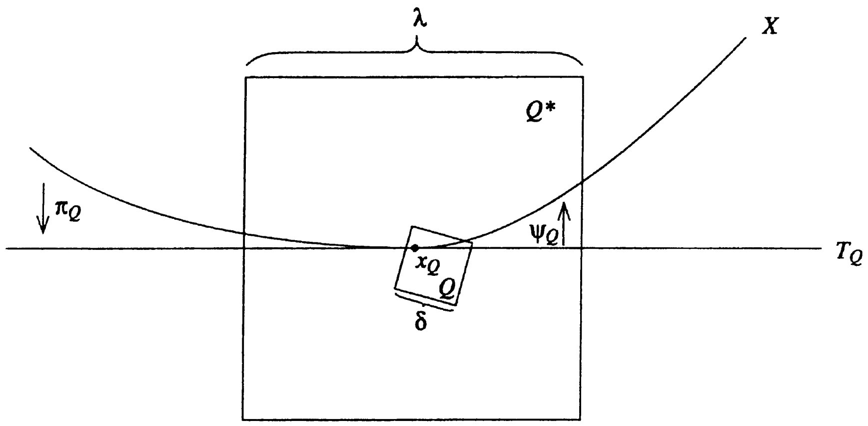

Figure 1

For $Q \in \mathcal{K}$ we let $\pi_Q : \mathbb{R}^n \to T_Q$ be the orthogonal projection. We omit the elementary but long proof of the following lemma. The result is geometrically obvious, because X is a compact C^1-manifold without boundary and because $\delta \to 0$ as $L \to 1$.

3.7. Lemma. *There exists a number $K_1 \in]1, K_0]$ such that if $1 < L \le K_1$, then for every $Q \in \mathcal{K}$ the map*

$$\pi_Q^* : X \cap Q^* \longrightarrow T_Q \cap Q^*$$

defined by π_Q is a homeomorphism with inverse

$$\psi_Q = (\pi_Q^*)^{-1} : T_Q \cap Q^* \longrightarrow X \cap Q^*$$

satisfying the following conditions

1) *If $x, y \in T_Q \cap Q^*$, then we have*

$$|x - y| \le |\psi_Q x - \psi_Q y| \le M\,|x - y|,$$

where $M = M(L) \in [1, 2]$ and $M(L) \to 1$ as $L \to 1$. In particular, ψ_Q is M-BL.

2) *$|\psi_Q - \mathrm{id}\,|_{T_Q \cap Q^*} \le \varepsilon_1\,\delta$, where $\varepsilon_1 = \varepsilon_1(L) \in [0, 1/2]$ and $\varepsilon_1(L) \to 0$ as $L \to 1$.*

3.8. The maps φ_Q.

Let $1 < L \le K_1$. Next we extend the maps $\psi_Q : T_Q \cap Q^* \to X \cap Q^*$ obtained from Lemma 3.7 to homeomorphisms $\varphi_Q : Q^* \to Q^*$ as follows:

Let $Q \in \mathcal{K}$, let $y \in T_Q \cap Q^*$, and set

$$R_y = \pi_Q^{-1}(y) \cap Q^*, \qquad B_y = \bar{B}^n(y, \delta) \cap R_y\,.$$

Then R_y is an $(n-p)$-cube, and B_y is an $(n-p)$-ball with center y. Let S_y be the boundary $(n-p-1)$-sphere of B_y. Since $|y - \psi_Q y| \le \varepsilon_1\,\delta \le \delta/2$ by Lemma 3.7.2), we can represent B_y as a cone in two ways: $B_y = yS_y = \psi_Q(y)S_y$. Let $\varphi_Q^y : B_y \to B_y$ be the $\psi_Q(y)$-cone of the identity map of S_y with vertex y, *i.e.*, φ_Q^y maps each segment $[y, z]$, $z \in S_y$, affinely onto the segment $[\psi_Q y, z]$. By the proof of [P, 2.3] we deduce that φ_Q^y is M_1-BL with

$$M_1 = M_1(L) = \frac{1}{1 - \varepsilon_1(L)}$$

satisfying $M_1 \to 1$ as $L \to 1$. We extend φ_Q^y to a map $\varphi_Q^y : R_y \to R_y$ by letting $\varphi_Q^y = \mathrm{id}$ in $R_y \setminus B_y$. The desired map $\varphi_Q : Q^* \to Q^*$ can now be defined by letting φ_Q agree with φ_Q^y in R_y for all $y \in T_Q \cap Q^*$. Then φ_Q is a homeomorphism and $\varphi_Q|_{T_Q \cap Q^*} = \psi_Q$. Moreover, by the construction of φ_Q the following assertions are clearly true with M_1 and ε_1 as above:

1) φ_Q is M_1-BL,

2) $|\varphi_Q - \mathrm{id}\,|_{Q^*} \le |\psi_Q - \mathrm{id}\,|_{T_Q \cap Q^*} \le \varepsilon_1 \, \delta$.

The choice of $N_0 = 2n(m_n + 1)$ in Paragraph 3.2 was made to guarantee that the interiors $\mathring{Q}^*$ of the cubes Q^*, $Q \in \mathcal{K}_i$, are disjoint. In the next lemma we verify this. We also derive an estimate for the cardinality of the set $\mathring{Q}^* \cap D$, $Q \in \mathcal{K}$.

3.9. Lemma. *Let $i \in \{1, \dots, N\}$, let $1 < L \le K_0$, and let $Q, R \in \mathcal{K}_i$, $Q \ne R$. Then we have*

1) $\mathring{Q}^* \cap \mathring{R}^* = \varnothing$,

2) $\#(\mathring{Q}^* \cap D) \le N$.

PROOF. To prove 1), observe that $\mathring{Q}^* \subset B^n(x_Q, N_0\delta/2)$ and $\mathring{R}^* \subset B^n(x_R, N_0\delta/2)$, because $\mathrm{diam}\,(Q^*) = \lambda\sqrt{n} = N_0\,\delta = \mathrm{diam}\,(R^*)$. Since $|x_Q - x_R| \ge \mathrm{dist}\,(Q, R) \ge N_0\,\delta$ by Paragraph 3.2, we get $\mathring{Q}^* \cap \mathring{R}^* = \varnothing$.

For 2), note that $\mathring{Q}^* \subset B^n(x_Q, N_0\delta/2) \subset \mathring{Q}(N_0+1)$, where $\mathring{Q}(N_0+1)$ is the interior of $Q(N_0+1)$ (*cf.* 1.1). If $x = x_S \in \mathring{Q}^* \cap D$, $S \in \mathcal{K}$, then obviously $S \subset Q(N_0+1)$. Since

$$\#\{S \in \mathcal{J} : \; S \subset Q(N_0+1)\} = (N_0+1)^n = N$$

and since $\mathcal{K} \subset \mathcal{J}$, we get 2).

3.10. The basic polyhedra Z.

In [PV] it was proved that every compact polyhedron $Z \subset \mathbb{R}^n$ has the BLEP in $\mathbb{R}^n$. We are going to apply this result to some basic polyhedra, which belong to a finite number of similarity classes. Here we choose a set of representatives Z for these classes.

If $k \in \mathbb{N}$ and $t > 0$, we set $I^k(t) = [-t,t]^k$. We let $N_1 = N_1(n)$ be the unique integer satisfying

$$\frac{2}{3}(m_n+1)N\sqrt{n} \le N_1 < \frac{2}{3}(m_n+1)N\sqrt{n}+1\,,$$

where $N = (N_0+1)^n$ is as above. We divide $I^p(3)$ into $(6N_1)^p$ closed p-cubes R with side length $1/N_1$. Let $\mathcal{R}$ be the family of these p-cubes R, and let $\mathcal{Y}$ be the family of all polyhedra Y satisfying the conditions

1) $Y = \bigcup \mathcal{R}_1$ for some $\mathcal{R}_1 \subset \mathcal{R}$,
2) $I^p(2) \subset Y$.

Since $I^p(3) \subset I^n(3)$ by the identification of 1.1, we can now define a finite family $\mathcal{F}$ of compact polyhedra $Z \subset I^n(3)$ by setting

$$\mathcal{F} = \{Z:\ Z = Y \cup \partial I^n(3)\,,\ Y \in \mathcal{Y}\}\,.$$

3.11. The sets E_i.

Let $1 < L \le K_1$. For $Q \in \mathcal{K}$ we set

$$P_Q = (Q^* \cap T_Q)(2/3)\,,$$

i.e., P_Q is the closed p-cube with center x_Q, side length $\frac{2}{3}\lambda$, and edges parallel to those of $Q^* \cap T_Q$. If $1 \le i \le N$, we define a subset $E_i = E_i(L)$ of X by setting

$$E_i = \bigcup\{\psi_Q P_Q:\ Q \in \mathcal{K}_i\}\,,$$

where $\psi_Q : T_Q \cap Q^* \to X \cap Q^*$ is the homeomorphism of Lemma 3.7. We also set

$$q = q(L) = \frac{\delta(L)\sqrt{n}}{N}\,.$$

The next lemma is the decisive tool in our proof of the BLEP of X.

3.12. Lemma. *Let $A, B :]1,K_1] \to \mathcal{P}(X)$ be two functions satisfying the following conditions:*

1) $A(L) \subset B(L)$, *for all* $L \in]1,K_1]$,

2) $\operatorname{dist}(A(L), X \setminus B(L)) \geq q(L)$, *for all* $L \in]1, K_1]$.

Let $1 \leq i \leq N$, *and suppose that* X *has the* BLEP rel $A \cup D \cup E_i$ *in* $\mathbb{R}^n$. *Then* X *has the* BLEP rel $B \cup D$ *in* $\mathbb{R}^n$.

PROOF. To begin with, the reader should be informed that we allow the case where $A(L) = \varnothing$ or $B(L) = \varnothing$. In fact, we shall apply Lemma 3.12 with $A = B = \varnothing$ in the proof of Theorem 3.14.

Applying Theorem 2.4 with the substitution $K_0 \mapsto K_1$, $A(L) \mapsto A(L) \cup D(L) \cup E_i(L)$, $B(L) \mapsto B(L) \cup D(L)$, we first observe that it suffices to prove the following statement:

(3.13) *There exists* $L_0 \in]1, K_1]$ *such that if* $1 < L \leq L_0$ *and if* $f : X \to \mathbb{R}^n$ *is an* L-BL *mapping satisfying* $f|_{B(L) \cup D(L)} = \mathrm{id}$, *then* $f|_{A(L) \cup D(L) \cup E_i(L)}$ *has an* L_1-BL *extension* $F : \mathbb{R}^n \to \mathbb{R}^n$, *where* $L_1 = L_1(L) \to 1$ *as* $L \to 1$.

Let $1 < L \leq K_1$, and let $f : X \to \mathbb{R}^n$ be an L-BL map satisfying $f|_{B(L) \cup D(L)} = \mathrm{id}$. To be able to construct the desired L_1-BL extension F of $f|_{A(L) \cup D(L) \cup E_i(L)}$ we shall introduce new restrictions on L of the type $L \leq K_j = K_j(X, n) > 1$ ($j \geq 2$) whenever need arises. The proof below will imply (3.13) with $L_0 = \min_j K_j$. In it we use the notation $\varepsilon_j = \varepsilon_j(L)$, $j \geq 2$, for positive functions depending only on (X, n) and satisfying $\varepsilon_j(L) \to 0$ as $L \to 1$.

Fix $Q \in \mathcal{K}_i$, and let $\psi_Q = (\pi_Q^*)^{-1} : T_Q \cap Q^* \to X \cap Q^*$ be as in Lemma 3.7. Define a map $f_Q : T_Q \cap Q^* \to \mathbb{R}^n$ by setting

$$f_Q x = f \psi_Q x = f \varphi_Q x$$

for all $x \in T_Q \cap Q^*$. We prove that f_Q has the following properties:

a) f_Q is LM-BL with $M = M(L)$ as in Lemma 3.7.1),

b) $f_Q(x) = \psi_Q(x)$ for all $x \in \pi_Q(D \cap Q^*)$,

c) $|f_Q - \mathrm{id}|_{T_Q \cap Q^*} \leq \varepsilon_2(L)\,\delta$.

Since a) and b) are obvious, we only need to verify c). We first construct a not too flat p-simplex Δ such that the set Δ^0 of vertices of Δ is contained in $\pi_Q(D \cap Q^*)$.

Choose an orthonormal basis $(v_1, \ldots, v_p)$ of T_Q^0 (see Paragraph 3.6), and set

$$t = m_n \sqrt{n}\,\delta, \qquad z_j = x_Q + t\,v_j$$

for $1 \le j \le p$. Since $|z_j - x_Q| = m_n\sqrt{n}\,\delta < \lambda/2$, we have $z_j \in T_Q \cap Q^*$, $\psi_Q z_j \in X \cap Q^*$. Hence we can choose cubes $R_j \in \mathcal{K}$ so that $\psi_Q z_j \in R_j$, $1 \le j \le p$. We set $y_j = \pi_Q x_{R_j}$, $\Delta_0 = x_Q z_1 \cdots z_p$. Then Δ_0 is a p-corner with size t, see Paragraph 2.5. Since π_Q decreases distances, we have

$$|y_j - z_j| \le |x_{R_j} - \psi_Q z_j| \le \operatorname{diam}(R_j) = \sqrt{n}\,\delta = t/m_n \ ,$$

for all $j \in \{1, \ldots, p\}$. By the definition of m_n in Paragraph 2.5, this implies that $\Delta = x_Q y_1 \cdots y_p$ is a p-simplex with $\rho(\Delta) \le 2p$. Moreover, we have $\Delta^0 \subset Q^*$, because

$$|y_j - x_Q| \le |y_j - z_j| + |z_j - x_Q| \le \sqrt{n}\,\delta + m_n\sqrt{n}\,\delta = \lambda/2$$

for all j. Since $|x_{R_j} - \psi_Q z_j| \le \sqrt{n}\,\delta$ and since $\varepsilon_1(L) \le 1/2$ in Lemma 3.7.2), it easily follows that $x_{R_j} \in Q^*$ for all j. Hence $\Delta^0 \subset \pi_Q(D \cap Q^*)$, and Δ is the desired p-simplex.

Next we observe that $f_Q|_{\Delta^0} = \psi_Q|_{\Delta^0}$ by (b). Applying this and Lemma 3.7.2) we get

$$|f_Q - \mathrm{id}\,|_{\Delta^0} \le |\psi_Q - \mathrm{id}\,|_{T_Q \cap Q^*} \le \varepsilon_1(L)\,\delta\,.$$

By the approximation theorem [V, 3.1] there exists an isometry $h : T_Q \to \mathbb{R}^n$ such that we have

$$|f_Q - h|_{T_Q \cap Q^*} \le \varepsilon_3(L)\operatorname{diam}(T_Q \cap Q^*) < \varepsilon_4(L)\,\delta\,,$$

where $\varepsilon_4(L) = 2n\,(1 + m_n)\,\varepsilon_3(L)$. Then h satisfies

$$|h - \mathrm{id}\,|_{\Delta^0} \le |h - f_Q|_{\Delta^0} + |f_Q - \mathrm{id}\,|_{\Delta^0} \le \varepsilon_5(L)\,\delta$$

with $\varepsilon_5(L) = \varepsilon_4(L) + \varepsilon_1(L)$. From [V, 2.11] it follows that for all $x \in T_Q$ we have

$$|hx - x| \le \varepsilon_5(L)\,\delta\,\left(1 + \operatorname{diam}(\Delta)^{-1} M_0\,|x - x_Q|\right)\,,$$

where

$$M_0 = 4 + 6\,\rho(\Delta)\,p\,(1 + \rho(\Delta))^{p-1} \le M'$$

with $M' = 4 + 12\,p^2\,(1 + 2p)^{p-1}$. Since $m_n \ge 2$, we have

$$\begin{aligned}\operatorname{diam}(\Delta) &\ge |y_1 - x_Q| \ge |z_1 - x_Q| - |z_1 - y_1| \\ &\ge t - \frac{t}{m_n} \ge \frac{t}{2} = \frac{m_n\sqrt{n}\,\delta}{2}\,.\end{aligned}$$

Applying this estimate and the fact that $Q^* \subset \bar{B}^n(x_Q, \lambda\sqrt{n}/2)$ we get

$$|h - \mathrm{id}\,|_{T_Q \cap Q^*} \le \varepsilon_5(L)\,\delta\,\left(1 + 2\,M'\,\sqrt{n}\,\frac{m_n+1}{m_n}\right) \le \varepsilon_6(L)\,\delta\,,$$

where $\varepsilon_6(L) = (1 + 3\,M'\,\sqrt{n})\,\varepsilon_5(L)$. We now get the desired estimate

$$|f_Q - \mathrm{id}\,|_{T_Q \cap Q^*} \le |f_Q - h|_{T_Q \cap Q^*} + |h - \mathrm{id}\,|_{T_Q \cap Q^*} \le \varepsilon_2(L)\,\delta$$

with $\varepsilon_2(L) = \varepsilon_4(L) + \varepsilon_6(L)$. Hence c) is true.

Writing $A = A(L)$, $B = B(L)$ we set

$$A_Q = \pi_Q(Q^* \cap A)\,, \qquad B_Q = \pi_Q(Q^* \cap B)\,, \qquad D_Q = \pi_Q(Q^* \cap D)\,.$$

Since π_Q^* is M-BL with $M \le 2$ as in Lemma 3.7, the assumption 2) in Lemma 3.12 implies that we have

d) $$\mathrm{dist}\,(A_Q, T_Q \cap Q^* \setminus B_Q) \ge q/2\,.$$

Let N_1 be the integer defined in Paragraph 3.10. We divide the p-cube $T_Q \cap Q^*$ into $(6N_1)^p$ closed p-cubes R with side length $\lambda/6N_1$. Let $\mathcal{L}$ be the family of all these p-cubes R. If $R \in \mathcal{L}$, then by paragraphs 3.6, 3.10 and the definition of q before Lemma 3.12 we get the estimate

$$\mathrm{diam}\,(R) = \frac{\lambda\,\sqrt{p}}{6\,N_1} < \frac{(m_n+1)\,n\,\delta}{3\,N_1} \le \frac{q}{2}\,.$$

Applying this together with d) we see that the implication

e) $$R \cap A_Q \ne \varnothing \qquad \text{implies} \qquad R \subset B_Q$$

is true for all $R \in \mathcal{L}$. We set

$$\mathcal{L}_A = \{R \in \mathcal{L} :\ R \cap A_Q \ne \varnothing\}\,, \qquad \mathcal{L}_D = \{R \in \mathcal{L} :\ R \cap D_Q \ne \varnothing\}\,.$$

We divide the set $\mathring{Q}^* \cap T_Q \setminus \mathring{P}_Q$ into N_1 disjoint sets

$$H_j = \mathring{P}_Q\left(1 + \frac{j}{2\,N_1}\right) \setminus \mathring{P}_Q\left(1 + \frac{j-1}{2\,N_1}\right)\,, \qquad 1 \le j \le N_1\,,$$

as illustrated in Figure 2. For notation, see 1.1.

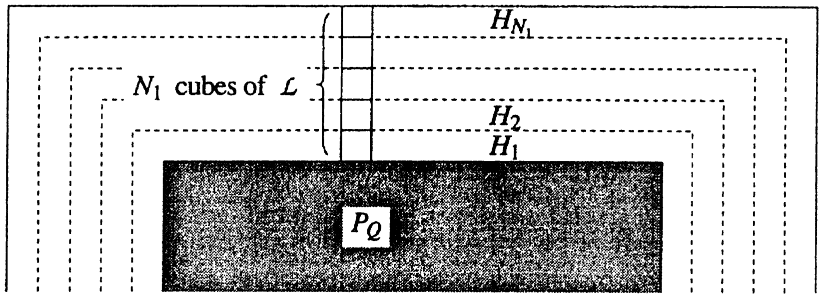

Figure 2

By Lemma 3.9.2) and Paragraph 3.10 we have

$$\#(\mathring{Q}^* \cap D) \leq N < N_1 \ .$$

Since the sets $H_j \subset \mathring{Q}^* \cap T_Q = \pi_Q(\mathring{Q}^* \cap X)$, $1 \leq j \leq N_1$, are disjoint, we can choose $j_0 \in \{1, \dots, N_1\}$ such that

$$H_{j_0} \cap D_Q = \varnothing \, .$$

We define

$$\mathcal{L}_1 = \{R \in \mathcal{L}_D : \ \mathring{R} \subset P_Q \cup H_1 \cup \cdots \cup H_{j_0-1}\} \, ,$$
$$\mathcal{L}_2 = \{R \in \mathcal{L}_D : \ \mathring{R} \subset H_{j_0+1} \cup \cdots \cup H_{N_1}\} \, .$$

Then $\mathcal{L}_1$ consists of the cubes $R \in \mathcal{L}_D$ inside H_{j_0} and $\mathcal{L}_2$ of those outside H_{j_0} in Figure 2. We set

$$Y_A = \bigcup \mathcal{L}_A \, , \qquad Y_1 = \bigcup \mathcal{L}_1 \, , \qquad Y_2 = \bigcup \mathcal{L}_2 \, .$$

Then Y_1 and Y_2 satisfy the conditions

$$D_Q \subset Y_1 \cup Y_2 \cup (T_Q \cap \partial Q^*) \, , \qquad \operatorname{dist}(Y_1, Y_2) \geq \frac{\lambda}{6 N_1} \, ,$$
$$\operatorname{dist}(Y_1, \partial Q^*) \geq \frac{\lambda}{6 N_1} \, , \qquad \operatorname{dist}(Y_2, P_Q) \geq \frac{\lambda}{6 N_1} \, .$$

We define two polyhedra Y and Z_Q by setting

$$Y = Y_A \cup Y_1 \cup Y_2 \cup P_Q \ , \qquad Z_Q = Y \cup \partial Q^* \, .$$

Obviously Z_Q is similar to some member Z_0 of $\mathcal{F}$, *cf.* Paragraph 3.10. Since $\mathcal{F}$ is finite and since every $Z \in \mathcal{F}$ has the BLEP in $\mathbb{R}^n$ by [PV, 1.1], there exists $L_0^* > 1$ and a function $L_1^* :]1, L_0^*] \to]1, \infty[$ satisfying $L_1^*(K) \to 1$ as $K \to 1$ and such that if $Z \in \mathcal{F}$ and $1 < K \leq L_0^*$, then every K-BL map $g : Z \to \mathbb{R}^n$ has an $L_1^*(K)$-BL extension $G : \mathbb{R}^n \to \mathbb{R}^n$. Since Z_Q and Z_0 are similar, the same is true for all K-BL maps $g : Z_Q \to \mathbb{R}^n$, $1 < K \leq L_0^*$.

We next show that after restricting L we can define a function $g_Q : Z_Q \to \mathbb{R}^n$ by setting

$$\text{f)} \qquad g_Q x = \begin{cases} \varphi_Q^{-1} f \psi_Q x = \varphi_Q^{-1} f_Q x\,, & \text{if } x \in P_Q \cup Y_A \cup Y_1\,, \\ x\,, & \text{if } x \in Y_2 \cup \partial Q^*\,. \end{cases}$$

If $R \in \mathcal{L}_A$, then $R \subset B_Q$ by e). Hence $\psi_Q R \subset Q^* \cap B$, and we get $(\varphi_Q^{-1} \circ f_Q)|_R = \mathrm{id}$, because $f|_B = \mathrm{id}$ by the assumption of (3.13). It follows that $\varphi_Q^{-1} \circ f_Q = \mathrm{id}$ in the set Y_A, which contains the intersection of $P_Q \cup Y_A \cup Y_1$ and $Y_2 \cup \partial Q^*$. Since φ_Q^{-1} is defined in Q^* only, we must yet verify that if L is chosen small enough, then $f_Q(P_Q \cup Y_1) \subset Q^*$.

Let $x \in P_Q \cup Y_1$. Since $\mathrm{dist}\,(x, \partial Q^*) \geq \lambda/6\,N_1$, the desired condition $f_Q x \in Q^*$ would follow if we had $|f_Q x - x| < \lambda/6N_1$. To arrange this we let $L \leq K_2$, where $K_2 \in]1, K_1]$ is such that the function ε_2 of c) satisfies for all $K \in]1, K_2]$ the estimate

$$\text{g)} \qquad \varepsilon_2(K) < \mu\,, \qquad \mu = \mu(n) = \frac{(m_n + 1)\sqrt{n}}{3\,N_1}\,.$$

By c) and Paragraph 3.6 we then indeed have

$$|f_Q x - x| \leq \varepsilon_2(L)\,\delta < \mu\,\delta = \frac{\lambda}{6\,N_1}\,.$$

Hence we get $f_Q(P_Q \cup Y_1) \subset Q^*$, if $L \leq K_2$.

From now on we always assume $L \leq K_2$. Then g_Q is well defined by f). Next we show that after restricting L once more, g_Q is M_2-BL, where $M_2 = M_2(L) \to 1$ as $L \to 1$. For this, let $x, y \in Z_Q$. We must derive suitable estimates for the number

$$\alpha = \frac{|g_Q x - g_Q y|}{|x - y|}\,.$$

If $\{x, y\} \subset P_Q \cup Y_A \cup Y_1$, then $1/M_1\,L\,M \leq \alpha \leq M_1\,L\,M$ by f), 3.8.1) and a). If $\{x, y\} \subset Y_A \cup Y_2 \cup \partial Q^*$, then $\alpha = 1$ by f) and the above observation that $g_Q|_{Y_A} = \mathrm{id}$.

It remains to consider the case $x \in P_Q \cup Y_1$, $y \in Y_2 \cup \partial Q^*$. Then we have

$$|x-y| \geq \frac{\lambda}{6\,N_1} = \mu\,\delta$$

with μ as in g). Hence by 3.8.2) and c) we get

$$\begin{aligned}|g_Q x - g_Q y| &= |\varphi_Q^{-1} f_Q x - y| \\ &\leq |\varphi_Q^{-1} f_Q x - f_Q x| + |f_Q x - x| + |x-y| \\ &\leq \varepsilon_1(L)\,\delta + \varepsilon_2(L)\,\delta + |x-y| \\ &\leq (1+\varepsilon_7(L))\,|x-y|\,,\end{aligned}$$

where $\varepsilon_7(L) = (\varepsilon_1(L) + \varepsilon_2(L))/\mu$. Similarly we get

$$|g_Q x - g_Q y| \geq (1-\varepsilon_7(L))\,|x-y|\,.$$

Hence $1-\varepsilon_7(L) \leq \alpha \leq 1+\varepsilon_7(L)$.

Let $K_3 \in\,]1, K_2]$ be such that $\varepsilon_7(K) < 1$ for all $K \in\,]1, K_3]$. From now on we assume that $L \leq K_3$. By the above estimates, g_Q is then M_2-BL, where

$$M_2 = M_2(L) = \max\left\{L\,M_1(L)\,M(L),\ \frac{1}{1-\varepsilon_7(L)}\right\}$$

satisfies $M_2 \to 1$ as $L \to 1$.

Choose $K_4 \in\,]1, K_3]$ so that $M_2(K) \leq L_0^*$ for all $K \in\,]1, K_4]$. From now on we assume that $L \leq K_4$. Then g_Q has an L_3-BL extension

$$G_Q : \mathbb{R}^n \to \mathbb{R}^n\,,$$

where $L_3 = L_3(L) = L_1^*(M_2(L)) \to 1$ as $L \to 1$. Obviously, we have $G_Q|_{\partial Q^*} = \mathrm{id}$ and $G_Q Q^* = Q^*$.

We define a homeomorphism $F_Q : Q^* \to Q^*$ by letting $F_Q = \varphi_Q \circ G_Q \circ \varphi_Q^{-1}$. We prove that F_Q has the following properties:

h) F_Q is L_4-BL with $L_4 = L_4(L) = L_3(L)\,M_1(L)^2$,

i) $F_Q|_{\partial Q^*} = \mathrm{id}$,

j) $F_Q x = fx$, for all $x \in Q^* \cap (A \cup D \cup E_i)$.

By 3.8.1), h) is obvious. Since $\varphi_Q \partial Q^* = \partial Q^*$ and since $G_Q|_{\partial Q^*} = \mathrm{id}$, we get i). To prove j), let $x \in Q^* \cap (A \cup D \cup E_i)$ and set $y =$

$\varphi_Q^{-1}(x) = \pi_Q x$. Then we have $G_Q y = g_Q y$. If $x \in A$, then $g_Q y = y$, because $A_Q \subset Y_A$ and because $g_Q|_{Y_A} = \mathrm{id}$. Since $f|_A = \mathrm{id}$ by (3.13) and Lemma 3.12.1), we get

$$F_Q x = \varphi_Q y = x = fx$$

as desired. If $x \in D$, then $fx = x$, and by f) we have $g_Q y = y$. Hence we get $F_Q x = x = fx$. If $x \in E_i$, then $y \in P_Q$ by Paragraph 3.11 and Lemma 3.9.1). Hence f) implies that

$$g_Q y = \varphi_Q^{-1} f \psi_Q y = \varphi_Q^{-1} fx\,,$$

and we obtain j) in this last case as well:

$$F_Q x = \varphi_Q g_Q y = fx\,.$$

Letting $Q \in \mathcal{K}_i$ vary, we get a family of maps $F_Q : Q^* \to Q^*$, $Q \in \mathcal{K}_i$, as above. By Lemma 3.9.1) and by i) these maps can be glued together into a homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$Fx = \begin{cases} F_Q x\,, & \text{if } x \in Q^* \text{ with } Q \in \mathcal{K}_i \\ x\,, & \text{if } x \in \mathbb{R}^n \setminus \bigcup\{Q^* : Q \in \mathcal{K}_i\}\,. \end{cases}$$

By h) and j) it is easy to see that F is L_4-BL and satisfies $Fx = fx$ for all $x \in A \cup D \cup E_i$. Hence (3.13) is true with $L_1 = L_4$, and Lemma 3.12 is proved.

We are now ready to prove our main theorem.

3.14. Theorem. *Let $1 \le p \le n$ and let X be a compact p-dimensional C^1-submanifold of $\mathbb{R}^n$ with or without boundary. Then X has the* BLEP *in $\mathbb{R}^n$.*

PROOF. As before in Section 3, we assume that $1 \le p \le n-1$ and that X has no boundary. Manifolds with boundary will be considered in Section 4.

Let $1 < L \le K_1$, where $K_1 \in \,]1, K_0]$ is as in Lemma 3.7. We shall freely use the definitions and results of 3.3-3.11. The parameter L will often be dropped out of the notation as *e.g.* in (3.5).

Letting N be as in Paragraph 3.2, we set

$$r_j = \sqrt{n}\,\delta + j\,q = (1 + j/N)\sqrt{n}\,\delta\,, \qquad 1 \le j \le N\,,$$
$$B_{ij} = \bigcup\{\psi_Q(\bar{B}^n(x_Q, r_j) \cap T_Q) : \ Q \in \mathcal{K}_i\}\,, \qquad 1 \le i, j \le N\,.$$

Note that we have $B_{ij} \subset E_i$, because $\bar{B}^n(x_Q, r_j) \cap T_Q \subset P_Q$ by the definitions of P_Q and Q^* in paragraphs 3.11 and 3.6 and by the fact that

$$\frac{1}{3}\,\lambda = \frac{2}{3}\,(m_n + 1)\sqrt{n}\,\delta \geq 2\sqrt{n}\,\delta \geq r_j\,.$$

We define N sets $B_i = B_i(L) \subset X$, $1 \leq i \leq N$, by setting

$$\begin{aligned} B_1 &= B_{1N}\,, \\ B_2 &= B_{1,N-1} \cup B_{2N}\,, \\ &\vdots \\ B_i &= B_{1,N-i+1} \cup B_{2,N-i+2} \cup \cdots \cup B_{iN} \\ &\vdots \\ B_N &= B_{11} \cup B_{22} \cup \cdots \cup B_{NN}\,. \end{aligned}$$

Let $Q \in \mathcal{K}$. Since $Q \subset \bar{B}^n(x_Q, \sqrt{n}\,\delta)$, since π_Q decreases distances, and since $\pi_Q x_Q = x_Q$, we have

$$\pi_Q(X \cap Q) \subset T_Q \cap \bar{B}^n(x_Q, \sqrt{n}\,\delta) \subset T_Q \cap Q^*\,.$$

Applying the map ψ_Q we get the inclusion

$$X \cap Q \subset \psi_Q(\bar{B}^n(x_Q, \sqrt{n}\,\delta) \cap T_Q)\,.$$

Since this holds for all $Q \in \mathcal{K} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N$ and since $r_j \geq \sqrt{n}\,\delta$ for all $j \in \{1, \ldots, N\}$, it follows that $X \subset B_N$. Hence X trivially has the BLEP rel $B_N \cup D$, see Remark 2.3.3.

We continue by induction. Suppose that $1 \leq i \leq N-1$ and that X has the BLEP rel $(B_{i+1} \cup D)$. With the aid of Lemma 3.12 we want to prove that X has the BLEP rel $(B_i \cup D)$.

We define a set $A_i = A_i(L) \subset X$ by

$$A_i = B_{1,N-i} \cup B_{2,N-i+1} \cup \cdots \cup B_{i,N-1}\,.$$

Obviously, we have $A_i \subset B_i$. Moreover, we get

$$\operatorname{dist}(A_i, X \setminus B_i) \geq q\,,$$

because ψ_Q increases distances by Lemma 3.7.1). Observe that the set

$$B_{i+1} = B_{1,N-i} \cup B_{2,N-i+1} \cup \cdots \cup B_{i,N-1} \cup B_{i+1,N}$$

satisfies $B_{i+1} \subset A_i \cup E_{i+1}$, because $B_{i+1,N} \subset E_{i+1}$, as noted above. Hence $B_{i+1} \cup D \subset A_i \cup D \cup E_{i+1}$, and X has the BLEP rel $(A_i \cup D \cup E_{i+1})$ by the inductive hypothesis and Remark 2.3.1). Applying Lemma 3.12 with the substitution $A \mapsto A_i$, $B \mapsto B_i$, $i \mapsto i+1$, we deduce that X indeed has the BLEP rel $(B_i \cup D)$.

By induction, we see that X has the BLEP rel $(B_1 \cup D)$. Since $B_1 = B_{1N} \subset E_1$, X also has the BLEP rel $(E_1 \cup D)$. Applying Lemma 3.12 again, now with the substitution $A \mapsto \varnothing$, $B \mapsto \varnothing$, $i \mapsto 1$, we see that X has the BLEP rel D in $\mathbb{R}^n$. By Lemma 3.4, this implies that X has the BLEP in $\mathbb{R}^n$.

3.15. REMARK. An analysis of our method in the proof of Theorem 3.14 reveals that the actual extension of an L-BL map $f : X \to \mathbb{R}^n$ with $L-1$ small enough can be done in $N+1$ steps. The first step begins by extending $f|_D$ to an $L_2(L)$-BL map $g : \mathbb{R}^n \to \mathbb{R}^n$ as in Paragraph 3.3. Replacing f by $g^{-1} \circ f$ a normalization $f|_D = \mathrm{id}$ is then obtained. In the second step, the restriction $f|_{B_1 \cup D}$ (or even $f|_{E_1 \cup D}$) of this normalized map f is then extended and the stronger normalization $f|_{B_1 \cup D} = \mathrm{id}$ is seen to be possible. The remaining $N-1$ steps correspond to the inductive steps above in reverse order. We proved that the BLEP of X rel $B_{i+1} \cup D$ implies the BLEP of X rel $B_i \cup D$, $1 \leq i \leq N-1$. This corresponds to extending $f|_{B_{i+1} \cup D}$ (or even $f|_{A_i \cup D \cup E_{i+1}}$), where $f : X \to \mathbb{R}^n$ is normalized by $f|_{B_i \cup D} = \mathrm{id}$, and using the extended map for a new normalization $f|_{B_{i+1} \cup D} = \mathrm{id}$. Since $B_N \cup D = X$, we finally see that f can be normalized by $f|_X = \mathrm{id}$. This implies that f indeed has an extension.

3.16. The case $p = 1$.

If X is one-dimensional, the proof of Theorem 3.14 can be essentially simplified. By [P, 2.5] X can be assumed to be connected. Then X is a C^1 arc or a C^1 Jordan curve. Assuming that it is a Jordan curve, we present an outline of the method used in [HP] for the extension of an L-BL map $f : X \to \mathbb{R}^n$ with $L-1$ small enough.

The sets D_j of Paragraph 3.3 with j odd are not needed. For j even we simply let D_j consist of j points, which divide X into j subarcs of equal length. Then the number $m = m(L) \in 2\mathbb{N}$ and the set $D = D(L) = D_{m(L)}$ are obtained as in Paragraph 3.3; for this we have to replace $\mathbb{N}$ by $2\mathbb{N}$ in Lemma 2.1. The set D so defined consists of m points $a_1, \dots, a_m, a_{m+1} = a_1$, which divide X into m equally long subarcs C_j joining a_j to a_{j+1}, $1 \le j \le m$. As in Remark 3.15 we obtain the normalization $f|_D = \text{id}$ by making use of an extension of $f|_D$. However, in the rest of the proof only the two steps described below are necessary. Moreover, only two rather simple similarity types of compact polyhedra are needed. The following constructions are possible if $L-1$ is small enough.

Let J_j denote the line segment joining a_j to a_{j+1}, $1 \le j \le m$. Choose closed n-cubes Q_j, $1 \le j \le m$, of $\mathbb{R}^n$ in such a way that a_j and a_{j+1} are the centers of two opposite $(n-1)$-faces of Q_j. Define the polyhedra

$$X_j = \partial Q_j\,(5/4) \bigcup J_j\,, \qquad Y_j = \partial Q_j \cup J_j\,,$$

and let $\psi_j : J_j \to C_j$ be the inverse of the orthogonal projection $C_j \to J_j$. Consider the maps $F_j : X_j \to \mathbb{R}^n$ and $\Psi_j : X_j \to \mathbb{R}^n$ ($1 \le j \le m$, j odd) defined by letting F_j coincide with $(f|_{C_j}) \circ \psi_j$ and Ψ_j with ψ_j in J_j and letting $F_j = \text{id} = \Psi_j$ in $\partial Q_j\,(5/4)$. Extending these maps with the aid of the BLEP of X_j and using the extensions glued together we are able to complete the first step by obtaining the normalization

$$(*) \qquad f|_{\cup\{C_j:\ j \text{ odd}\}} = \text{id}\,.$$

In the second step we then apply the same argument for the subarcs C_j of X with $j \in \{1, \dots, m\}$ even. From the normalization $(*)$ it here follows that we can use the polyhedra Y_j (j even) in the same role as the polyhedra X_j had above for j odd. Hence this step actually leads to the normalization $f|_X = \text{id}$, showing that f can be extended.

4. Manifolds with boundary.

In this section we give an outline of the proof of Theorem 3.14 in the case where the compact p-dimensional C^1-manifold $X \subset \mathbb{R}^n$ has boundary. The case $p = n$ follows easily from the BLEP of ∂X, which was proved in Section 3 and in [V, 5.17]. Suppose that $p \le n-1$.

As in Section 3 we consider a number $\delta > 0$ and the cube families $\mathcal{J}$ and $\mathcal{K}$. However, the numbers N_0 and N are larger. We now let N_0 and N be the integers satisfying the conditions

$$6\,(m_n + 3)\,n^{3/2} < N_0 \le 6\,(m_n + 3)\,n^{3/2} + 1\,, \qquad N = (N_0 + 1)^n\,.$$

In Paragraph 3.3 the points $x_Q \in Q \cap X$, $Q \in \mathcal{K}$, are chosen so that $x_Q \in Q \cap \partial X$ whenever Q meets the boundary ∂X of X. As in Paragraph

3.3, we apply Lemma 2.1 to find the number $K_0 > 1$ and for $L \in \,]1, K_0]$ the numbers $\delta(L) > 0$, $L_2(L) > 1$, and the finite set $D(L) \subset X$. Then Lemma 3.4 holds verbatim.

Let Q^* be as in Paragraph 3.6. We define the new cube families

$$\mathcal{K}^1 = \{Q \in \mathcal{K} : \; Q^* \cap \partial X = \varnothing\},$$
$$\mathcal{K}^2 = \{Q \in \mathcal{K} : \; Q \cap \partial X \neq \varnothing\},$$
$$\mathcal{K}^0 = \mathcal{K}^1 \cup \mathcal{K}^2 .$$

In the sequel, only the cubes of $\mathcal{K}^0$ will be used.

If $Q \in \mathcal{K}^2$, we let T_Q and T'_Q denote the tangent planes of X and ∂X at x_Q, respectively. Let H_Q be the closed half plane of T_Q with $\partial H_Q = T'_Q$ such that H_Q and $\pi_Q X$ are in a natural sense on the same side of T'_Q near x_Q.

We set

$$\alpha = (m_n + 3)\, n\, \delta .$$

If $Q \in \mathcal{K}^2$, we replace Q^* by the larger cube $\tilde{Q}$ with center x_Q, side 6α, and having p-dimensional and $(p-1)$-dimensional faces parallel to T_Q and T'_Q, respectively.

For $Q \in \mathcal{K}^1$, the homeomorphism $\varphi_Q : Q^* \to Q^*$ and the p-cube P_Q are defined as in paragraphs 3.8 and 3.11. For $Q \in \mathcal{K}^2$, the corresponding homeomorphism $\varphi_Q : \tilde{Q} \to \tilde{Q}$ is defined in two steps. First, we define a homeomorphism $\varphi'_Q : \tilde{Q} \to \tilde{Q}$ such that $\varphi'_Q(\tilde{Q} \cap H_Q) = \pi_Q(\tilde{Q} \cap X)$. Next, we extend the map $\psi_Q : \pi_Q(\tilde{Q} \cap X) \to \tilde{Q} \cap X$ to a homeomorphism $\varphi''_Q : \tilde{Q} \to \tilde{Q}$. These maps are chosen in such a way that the homeomorphism $\varphi_Q = \varphi''_Q \circ \varphi'_Q : \tilde{Q} \to \tilde{Q}$ has the properties 1) and 2) of Paragraph 3.8. The sets P_Q, $Q \in \mathcal{K}^2$, are defined by setting

$$P_Q = \frac{2}{3}\,(H_Q \cap \tilde{Q} - x_Q) + x_Q .$$

The number N_1 of 3.10 is replaced by the larger number $N_1 = 2N\,(m_n + 3)\,n$. We divide $\mathcal{K}$ into disjoint subfamilies $\mathcal{K}_1, \dots, \mathcal{K}_N$ as in (3.5), and we set

$$\mathcal{K}_i^0 = \mathcal{K}^0 \cap \mathcal{K}_i , \qquad E_i = \bigcup\{\varphi_Q P_Q : \; Q \in \mathcal{K}_i^0\} .$$

Then Lemma 3.12 holds verbatim with $q = \delta\sqrt{n}/N$. Its proof requires some modifications when considering the cubes Q of $\mathcal{K}^2$. For example, we define the maps

$$f_Q : \tilde{Q} \cap H_Q \to \mathbb{R}^n , \qquad Q \in \mathcal{K}^2 ,$$

by $f_Q x = f\varphi_Q x$. To prove that f_Q is close to the identity mapping in $\tilde{Q} \cap H_Q$ we again need a basis $(v_1, \ldots, v_p)$ of $T_Q - x_Q$. This basis is now chosen so that $v_1 + \cdots + v_p$ is a normal vector of T'_Q in T_Q, pointing to H_Q.

In the final proof of the BLEP of X, we define the sets B_{ij}, $1 \leq i, j \leq N$, as follows: If $Q \in \mathcal{K}^1$, we set

$$r_j = \sqrt{n}\,\delta + j\,q\,, \qquad U(Q,j) = \bar{B}^n(x_Q, r_j) \cap T_Q\,.$$

If $Q \in \mathcal{K}^2$, we write

$$s_j = \alpha + 2j\,q\,, \qquad U(Q,j) = \bar{B}^n(x_Q, s_j) \cap H_Q\,.$$

Then we define

$$B_{ij} = \bigcup \{\varphi_Q U(Q,j) : \ Q \in \mathcal{K}_i^0\}\,.$$

The rest of the proof remains essentially unchanged.

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