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Følner Sequences in Polycyclic Groups

Christophe Pittet

Abstract. The isoperimetric inequality

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{\text{constant}}{\log|\Omega|}$$

for finite subsets Ω in a finitely generated group Γ with exponential growth is optimal if Γ is polycyclic.

1. Introduction and statements.

Let Γ be an infinite group generated by a finite set $S = S^{-1}$. If $\gamma \in \Gamma$ we denote by $\|\gamma\|_S$ the smallest number $k \in \mathbb{N}$ such that there exist $s_1, \dots, s_k \in S$ with $\gamma = s_1 \cdots s_k$. The distance between $\gamma, \gamma' \in \Gamma$ is defined as

$$d_S(\gamma, \gamma') = \|\gamma^{-1}\gamma'\|_S \,.$$

This distance on Γ, called the *word metric associated to* S, is left-invariant. We denote by $B(n) = \{\gamma \in \Gamma : \ \|\gamma\|_S \leq n\}$ the ball of radius n in Γ with center the identity. If $\Omega \subset \Gamma$ is a finite subset we denote by $|\Omega|$ its cardinal. Its boundary (relative to S) is defined by

$$\partial\Omega = \{\gamma \in \Gamma : \ \text{there exists } s \in S \text{ such that } \gamma s \notin \Omega\} \,.$$

Let $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{N}$ be the "inverse growth function of Γ"

$$\Phi(\lambda) = \min\{n \in \mathbb{N} : \ |B(n)| > \lambda\} \,.$$

A fundamental relation between the isoperimetric properties of the group and its growth is expressed in the following result (see [CSC93] and [Var91]).

Theorem 1.1 (Coulhon, Saloff-Coste). *Any finite non-empty subset* $\Omega \subset \Gamma$ *satisfies*

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{1}{4\,|S|\,\Phi(|\Omega|)}\,.$$

If Γ has polynomial growth of degree d this implies the existence of a constant $c > 0$, such that

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{|\Omega|^{1/d}}\,, \qquad \text{for all } \Omega \subset \Gamma\,. \tag{1}$$

This result is due to Varopoulos [Var86]. Up to the changing of the value of c (which depends anyway on the choice of a generating set for Γ) this inequality is optimal (see [Gro93, 5. Eb]).

If Γ has exponential growth, Theorem 1.1 implies the existence of a constant $c > 0$, such that

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{\log|\Omega|}\,, \qquad \text{for all } \Omega \subset \Gamma\,. \tag{2}$$

A finitely generated group Γ is non-amenable if and only if there is a constant $c > 0$ such that

$$\frac{|\partial\Omega|}{|\Omega|} \geq c\,, \qquad \text{for all } \Omega \subset \Gamma\,.$$

Hence the inequality of Theorem 1.1 is not optimal in this case.

The aim of this paper is to show that the inequality of Theorem 1.1 is optimal for polycyclic groups. A polycyclic group is solvable hence according to Milnor and Wolf (see [Mil68] and [Wol68]) its growth is either polynomial or exponential. Therefore, in view of (1) and (2) it is sufficient to prove the following statement.

Theorem 1.2. *Let* Γ *be an infinite polycyclic group. Let* $S = S^{-1}$ *be a finite generating set for* Γ*. There is a constant* $C > 1$ *and a family* Ω_n, $n \in \mathbb{N}$, *of finite subsets of* Γ *such that* $|\Omega_n| < |\Omega_{n+1}|$ *and*

$$\frac{C}{\log|\Omega_n|} \geq \frac{|\partial\Omega_n|}{|\Omega_n|}\,.$$

An inequality of the type

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{\log|\Omega|}(\log|\Omega|)^{\epsilon}, \qquad \text{for all } \Omega \subset \Gamma,$$

for some $\epsilon > 0$ was known to be impossible (see [CSC93, 2.2]). This is an immediate consequence of Theorem 1.2.

2. Preliminaries to the proof.

Definition 2.1. *Two metric spaces X, Y are quasi-isometric if there exist a constant $\lambda > 1$ and applications $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that*

a) $d(f(x), f(x')) \leq \lambda\, d(x, x') + \lambda$, *for all* $x, x' \in X$.

b) $d(g(y), g(y')) \leq \lambda\, d(y, y') + \lambda$, *for all* $y, y' \in Y$.

c) $d(g \circ f(x), x) \leq \lambda$, *for all* $x \in X$.

d) $d(f \circ g(y), y) \leq \lambda$, *for all* $y \in Y$.

f is a quasi-isometry, g (which is also a quasi-isometry) is a quasi-inverse of f and λ is a quasi-isometry constant for f.

EXAMPLE 2.1. Let Γ be a group generated by a finite set $S = S^{-1}$. Let $H \subset \Gamma$ be a finite index subgroup. Let $T = T^{-1}$ be a finite generating set for H. Then H and Γ with the word metrics associated to T and S are quasi-isometric.

Let X be a metric space and let $R > 0$. Let $\Omega \subset X$. Let

$$V_R(\Omega) = \{x \in X : \text{ there exists } x' \in \Omega \text{ such that } d(x, x') \leq R\}$$

be the R-neighborhood of Ω. If $x \in X$ we denote by

$$d(x, \Omega) = \inf_{x' \in \Omega} d(x, x')$$

the distance between x and Ω.

Proposition 2.1. *Let X and Y be two finitely generated groups with word metrics. Let $f : X \longrightarrow Y$ be a quasi-isometry. Then they are constants $C > 1$ and $R > 0$ such that, for all finite subsets $\Omega \subset X$,*

$$|\partial V_R(f(\Omega))| \leq C\,|\partial\Omega|$$

and

$$|f(\Omega)| \leq |\Omega| \leq C\,|f(\Omega)|.$$

PROOF. We prove the first inequality. Let g be a quasi-inverse of f and let λ be a constant of quasi-isometry. We can assume $\lambda \in \mathbb{N}$. We choose $R = \lambda + 1$. We want to define an application

$$h : \partial V_R(f(\Omega)) \longrightarrow \partial\Omega$$

which is "almost injective". First, we notice that if $y \in \partial V_R(f(\Omega))$ then $g(y) \notin \Omega$. This is because if $g(y) \in \Omega$ then

$$d(y, f(\Omega)) \leq d(y, f \circ g(y)) \leq \lambda < R$$

and this contradicts $y \in \partial V_R(f(\Omega))$. We choose $x \in \Omega$ such that

$$d(g(y), x) = d(g(y), \Omega).$$

As $g(y) \notin \Omega$ it follows that $x \in \partial\Omega$. We put $h(y) = x$. Now we check that there is a constant $C > 1$ such that, if $x \in \partial\Omega$ then $|h^{-1}(x)| \leq C$. Let $y \in h^{-1}(x)$. Then

$$\begin{aligned} d(g(y), x) = d(g(y), \Omega) &\leq \lambda\, d(f \circ g(y), f(\Omega)) + \lambda \\ &\leq \lambda\,(\lambda + d(y, f(\Omega))) + \lambda \leq \lambda^2 + \lambda\, R + \lambda = M. \end{aligned}$$

Hence

$$d(y, f(x)) \leq d(f \circ g(y), f(x)) + \lambda \leq \lambda\, M + 2\,\lambda.$$

We choose

$$C = |B(\lambda\, M + 2\,\lambda)|.$$

This proves the first inequality of the proposition. The others are obvious.

Lemma 2.1. *Let N be a group generated by a finite set $B = B^{-1}$. Assume that a group G with finite generating set $A = A^{-1}$ acts on N by automorphisms. Then there is an integer $q > 1$, such that, for all $w \in G$,*

$$\|w(x)\|_B \leq q^{\|w\|_A}\, \|x\|_B\,, \qquad \textit{for all } x \in N.$$

ROOF. Let $q = \sup_{a \in A,\, b \in B} \|a(b)\|_B$. If $x \in N$ and $a \in A$, then

$$\|a(x)\|_B \leq q\, \|x\|_B\,.$$

·onclude by induction on $\|w\|_A$.

Lemma 2.2. *Let $F(a,b)$ be the free group on two letters. Let $k \in \mathbb{N}$. Using the notation $[b,a] = b\,a\,b^{-1}\,a^{-1}$ we have in $F(a,b)$ that*

$$b^k\,a = \Big(\prod_{j=1}^{k} b^{k-j}\,[b,a]\,b^{j-k}\Big)\,a\,b^k\,. \tag{3}$$

PROOF. Let $x_j = b^{k-j}\,[b,a]\,b^{j-k}$. We obtain the equality

$$b^{k+1-j}\,a = x_j\,b^{k-j}\,a\,b \tag{4}$$

by induction on j (where $1 \le j \le k$). We deduce the equality of the lemma by successively applying (4).

REMARK 2.1. If $k \in \mathbb{Z}^*$ the equality (3) generalizes to

$$b^k\,a = \Big(\prod_{j=1}^{|k|} b^{\epsilon(k)(|k|-j)}\,[b^{\epsilon(k)},a]\,b^{\epsilon(k)(j-|k|)}\Big)\,a\,b^k \tag{5}$$

where $\epsilon(k) = \pm 1$ is the sign of k.

Lemma 2.3. *Let $0 \to N \to \Gamma \to \mathbb{Z}^r \to 0$ be an exact sequence of groups where N is finitely generated. Let $B = B^{-1}$ be a finite generating set for N. Let $a_1, \dots, a_r$ be elements of Γ which project respectively on the canonical basis vectors $e_1, \dots, e_r$ of $\mathbb{Z}^r$. Then there is an integer $q > 1$, such that for each r-uple $K = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and for each integer ν where $1 \le \nu \le r$, there exists a corresponding $x \in N$ with the following properties:*

a) $a_1^{k_1} \cdots a_\nu^{k_\nu} \cdots a_r^{k_r}\,a_\nu^{\pm 1} = x\,a_1^{k_1} \cdots a_\nu^{k_\nu \pm 1} \cdots a_r^{k_r}\,.$

b) $\|x\|_B \le q^{|K|}\,.$

(*Where* $|K| = \sum_{i=1}^{r} |k_i|$.)

PROOF. We assume the exponent of a_ν is positive (the other case is analogous). We assume that $K \ne 0$ (if $K = 0$ we choose $x = e$ and there is nothing to show). If $\nu = r$ then $x = e$. If $\nu < r$ we define for each $1 \le i \le r$

$$A_i = a_1^{k_1} \cdots a_i^{k_i}$$

and, if $k_i \neq 0$,

$$X_i = \prod_{j=1}^{|k_i|} x_{i,j}$$

where

$$x_{i,j} = a_i^{\epsilon(k_i)(|k_i|-j)}\,[a_i^{\epsilon(k_i)}, a_\nu]\,a_i^{\epsilon(k_i)(j-|k_i|)}\,.$$

If $k_i = 0$ we put $X_i = e$. For $2 \leq i \leq r$ the equality

$$A_i\,a_\nu = A_{i-1}\,X_i\,a_\nu\,a_i^{k_i} \tag{6}$$

follows from (5). We obtain the equality

$$a_1^{k_1}\cdots a_r^{k_r}\,a_\nu = \Big(\prod_{i=1}^{r-\nu} A_{r-i}\,X_i\,A_{r-i}^{-1}\Big)\,a_1^{k_1}\cdots\,a_\nu^{k_\nu+1}\cdots\,a_r^{k_r}$$

by successively applying (6) and by putting in terms of the form $A_{r-i}^{-1}A_{r-i}$;

$$x = \prod_{i=1}^{r-\nu} A_{r-i}\,X_i\,A_{r-i}^{-1}$$

belongs to N because it belongs to the derived group of Γ. Now we want an upper bound for $\|x\|_B$. According to Lemma 2.1 there is a constant $q > 1$ such that

$$\|x_{i,j}\| \leq q^{|K|} \sup_{1\leq i\leq r} \|[a_i^{\epsilon(k_i)}, a_\nu]\|_B\,.$$

Hence, up to increasing q we obtain

$$\|x_{i,j}\| \leq q^{|K|}\,.$$

Using this last inequality and Lemma 2.1 again we have

$$\|x\|_B \leq r \sup_{1\leq i\leq r-1} \|A_{r-i}\,X_i\,A_{r-i}^{-1}\|_B \leq r\,q^{|K|}\,\|X_i\|_B \leq r\,q^{|K|}\,|K|\,q^{|K|}\,.$$

By increasing q again we obtain the wanted inequality.

3. Proof of Theorem 1.2.

Let Γ be an infinite polycyclic group. According to Example 2.1 and Proposition 2.1 it is sufficient to prove the theorem for a finite index subgroup of Γ. According to a theorem of Mal'cev (see for example [Rob82, 15.1.6]) a polycyclic group has a finite index subgroup with nilpotent derived group. In order to avoid torsion elements in the abelianisation we again consider a finite index subgroup. Therefore it is sufficient to prove the theorem for polycyclic groups Γ of the form

$$0 \longrightarrow N \longrightarrow \Gamma \longrightarrow \mathbb{Z}^r \longrightarrow 0$$

where the sequence is exact and where N is nilpotent. As N is a subgroup in a polycyclic group it is polycyclic and hence finitely generated. Let $B = B^{-1}$ be a finite generating set for N. Choose elements $a_1, \ldots, a_r \in \Gamma$ which project respectively on the canonical basis vectors $e_1, \ldots, e_r$ of $\mathbb{Z}^r$. The set

$$S = B \cup \{a_1^{\pm 1}, \ldots, a_r^{\pm 1}\}$$

generates Γ. Any $\gamma \in \Gamma$ can be written in a unique way

$$\gamma = x\, a_1^{k_1} \cdots a_r^{k_r} ,$$

where $x \in N$ and $(k_1, \ldots, k_r) \in \mathbb{Z}^r$. Let q_1 be the constant of Lemma 2.1 (with $G = \mathbb{Z}^r$) and let q_2 be the constant of Lemma 2.3. Let $q = \max\{q_1, q_2\}$. For each $n \in \mathbb{N}$ we define

$$\Omega_n = \{x\, a_1^{k_1} \cdots a_r^{k_r} : \|x\|_B \leq q^{2n} , \ |K| \leq n\} ,$$

$$\omega_n = \{x\, a_1^{k_1} \cdots a_r^{k_r} : \|x\|_B \leq q^{2n} - q^n , \ |K| \leq n-1\} .$$

We want to show that

$$\partial \Omega_n \cap \omega_n = \varnothing . \tag{7}$$

That is, if $\gamma \in \omega_n$ and $s \in S$ then $\gamma s \in \Omega_n$.

a) Assume $s \in B$. If

$$x\, a_1^{k_1} \cdots a_r^{k_r} \in \omega_n$$

then

$$x\, a_1^{k_1} \cdots a_r^{k_r}\, s = x\, a_1^{k_1} \cdots a_r^{k_r}\, s\, a_r^{-k_r} \cdots a_1^{-k_1}\, a_1^{k_1} \cdots a_r^{k_r}$$

and according to Lemma 2.1

$$\begin{aligned} \|x\, a_1^{k_1} \cdots a_r^{k_r}\, s\, a_r^{-k_r} \cdots a_1^{-k_1}\|_B &\le \|x\|_B + q^{|K|} \\ &\le q^{2n} - q^n + q^{n-1} \le q^{2n} . \end{aligned}$$

b) If $s \in \{a_1^{\pm 1}, \ldots, a_r^{\pm 1}\}$, we can assume $s = a_\nu$ where $1 \le \nu \le r$. Let

$$x\, a_1^{k_1} \ldots a_r^{k_r} \in \omega_n .$$

According to Lemma 2.3 there exists $x' \in N$, such that

$$x\, a_1^{k_1} \cdots a_\nu^{k_\nu} \cdots a_r^{k_r}\, a_\nu = x\, x'\, a_1^{k_1} \cdots a_\nu^{k_\nu + 1} \cdots a_r^{k_r}$$

and such that

$$\|x'\|_B \le q^{|K|} .$$

We have

$$\|x\, x'\|_B \le \|x\|_B + \|x'\|_B \le q^{2n} - q^n + q^{|K|} \le q^{2n} - q^n + q^{n-1} \le q^{2n} .$$

Let $B(n) \subset N$ be the ball of radius n with respect to B. Let d be the degree of the growth of N. According to Grunewald (see [Gri90, 7.2]) we have

$$|B(n)| = \alpha\, n^d + O(n^{d-1/2}) ,$$

where $\alpha > 0$ is a constant. We define

$$f(n) = \frac{|B(q^{2n} - q^n)|}{|B(q^{2n})|} .$$

Hence

$$f(n) = \frac{\alpha + \dfrac{O(1)}{q^n}}{\alpha + \dfrac{O(1)}{q^n}} .$$

Let $P(n)$ be the number of elements in $\mathbb{Z}^r$ of word norm less or equal than n with respect to the canonical generating set. The function $P(n)$ is polynomial of degree r. According to (7) we have

$$\frac{|\partial \Omega_n|}{|\Omega_n|} \le \frac{|\Omega_n| - |\omega_n|}{|\Omega_n|} .$$

This last term is equal to

$$1 - f(n)\,\frac{P(n-1)}{P(n)} = (1 - f(n)) + f(n)\,\frac{P(n) - P(n-1)}{P(n)}$$
$$= \frac{O(1)}{q^n} + O(\frac{1}{n}) = O(\frac{1}{n}).$$

Hence there is a constant $C_1 > 0$ such that

$$\frac{|\partial\Omega_n|}{|\Omega_n|} \leq \frac{C_1}{n}\,, \qquad \text{for all } n \in \mathbb{N}.$$

On the other hand, there exists a constant $C_2 > 0$ such that

$$\log|\Omega_n| \leq C_2\, n\,, \qquad \text{for all } n \in \mathbb{N}.$$

Eventually, $C = C_1 C_2$ is the constant we were looking for.

4. Remarks and questions.

a) Theorem 1.1 generalizes a result of Varopoulos (see [VCSC82, VI.3.1]) which shows that a group with superpolynomial growth has infinite isoperimetric dimension. As a solvable group is amenable, a solvable Lie group (with any left-invariant metric) containing a lattice with exponential growth (the group *Sol* for example [Thu82]) has infinite isoperimetric dimension but is not open at infinity (see [GLP81, Chapter 6]).

b) Theorem 1.1 combined with the Milnor-Wolf theorem on the growth of solvable groups [Mil68], [Wol68], shows that a finitely generated solvable group with finite isoperimetric dimension contains a finite index nilpotent subgroup (see [GLP81, 6.29]).

c) The isoperimetric profile of a finitely generated group (with a given generating set) is defined as (the asymptotic behaviour of) the function

$$I_\Gamma(n) = \inf_{|\Omega|=n} |\partial\Omega|$$

(see [Gro93, 5.E]). If Γ has exponential growth, Theorem 1.1 implies the existence of a constant $c > 0$ such that

$$I_\Gamma(n) \geq c\,\frac{n}{\log n}\,, \qquad \text{for all } n \in \mathbb{N}.$$

If moreover, the group Γ is polycyclic, it follows from the proof of Theorem 1.2 that there exist constants $p, q > 1$ and $C > 1$ such that for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$\frac{p^n}{C} \leq m \leq C\, q^n\,, \qquad I_\Gamma(m) \leq C\, \frac{m}{\log m}\,.$$

Can we replace p^n and q^n by n?

c) Theorem 1.2 is true for the solvable non-polycyclic group

$$\langle a, b \mid aba^{-1} = b^2 \rangle\,.$$

Is Theorem 1.2 true for finitely generated solvable groups?

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Christophe Pittet
Laboratoire de Topologie et Géométrie
Université Paul Sabatier
118 rte. de Narbonne
31062 Toulouse Cedex, FRANCE