# An improved bound for Kakeya type maximal functions

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The purpose of this paper is to improve the known results (specifically [1]) concerning $L^p$ boundedness of maximal functions formed using $1 \times \delta \times \cdots \times \delta$ tubes. We briefly recall the problem: let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then for $0 < \delta < 1$ one defines (we follow [1] in notation and in the definition of $f_\delta^*$)

$$f_\delta^* : \mathbb{P}^{d-1} \to \mathbb{R}, \qquad f_\delta^*(e) = \sup_T \frac{1}{|T|} \int_T |f|,$$

where $\mathbb{P}^{d-1}$ is projective space and $T$ runs through all cylinders with length 1, cross section radius $\delta$ and axis in the $e$ direction. Also

$$f_\delta^{**} : \mathbb{R}^d \to \mathbb{R}, \qquad f_\delta^{**}(x) = \sup_T \frac{1}{|T|} \int_T |f|,$$

where $T$ runs through cylinders containing $x$ with length 1 and cross section radius $\delta$. A conjecture (some years old for $f_\delta^{**}$, formulated in [1] for $f_\delta^*$) is that for any $\varepsilon > 0$

$$\|M_\delta f\|_d \leq C_\varepsilon \, \delta^{-\varepsilon} \|f\|_d \,,$$

where $M_\delta$ denotes either $f_\delta^*$ or $f_\delta^{**}$. Interpolating this conjectured bound with the trivial $L^1 \to L^\infty$ bound, one obtains the equivalent conjecture

$$\|M_\delta f\|_q \leq C_\varepsilon \, \delta^{-(d/p-1+\varepsilon)} \|f\|_p \,, \qquad 1 \leq p \leq d, \ q = (d-1)p' \,.$$

In either case $M_\delta$ takes functions supported in a disc of radius 1 to functions whose support has measure bounded by a fixed constant , so for fixed $p$ the range of $q$ must be an interval $[1, q_0(p)]$ and the conjecture

$$(1) \quad \|M_\delta f\|_q \leq C_\varepsilon \, \delta^{-(d/p-1+\varepsilon)} \, \|f\|_p \,, \qquad 1 \leq p \leq d, \ q \leq (d-1)p'$$

is again equivalent. Our purpose is to extend the range of $p$ and $q$ for which (1) is known to hold. We assume throughout the paper that $d \geq 3$. When $d = 2$, Theorem 1 below is well-known; see [6], and [1] for the case of $f_\delta^*$.

Results like (1) with $p = q = 2$ go back to A. Cordoba's work in the mid-1970's, e.g. [6]. When $p = (d+1)/2$, $q = (d-1)p' = d+1$, (1) follows from S. Drury's result [7] and a somewhat stronger result is proved for $f_\delta^{**}$ in Christ-Duoandikoetxea-Rubio de Francia [8]. The exponent $p = (d+1)/2$ plays a natural role, and getting beyond it was accomplished only recently by Bourgain [1] who proved (1) with $p = p(d) \in ((d+1)/2, (d+2)/2)$ given by the recursion

$$p(2) = 2, \qquad p(d) = \frac{(d+2)p(d-1) - d}{2p(d-1) - 1}$$

for $d \geq 3$, and $q = p$. Our result is the following further improvement.

**Theorem 1.** (1) *holds for* $M_\delta f = f_\delta^*$ *or* $f_\delta^{**}$, $p = (d+2)/2$, $q = (d-1)p'$.

Thus we improve the $L^p$ exponent from e.g. $7/3$ to $5/2$ in three dimensions; our argument also gives the correct value of $q$. An immediate corollary (cf. [1]) is that any Besicovitch or Nikodym set in $\mathbb{R}^d$ has Hausdorff dimension at least $(d+2)/2$.

We wanted to avoid giving separate arguments for $f_\delta^*$ and $f_\delta^{**}$ and will therefore base the proof of Theorem 1 on certain axioms which are satisfied by both of them. In order to do this we first have to make a couple of (well-known) reductions in the problem. To begin with, it clearly suffices to prove Theorem 1 for functions $f$ which are supported in a fixed compact set. Next, instead of $f_\delta^{**}$ it is more convenient to work with a certain variant. If $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ then define

$$f_\delta^{***} : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$$

via

$$f_\delta^{***}(x) = \sup_T \frac{1}{|T|} \int_T |f|,$$

where $T$ runs through all cylinders with length 1 and cross section radius $\delta$ whose axis makes an angle less than $\pi/100$ with the $d$-th coordinate direction. We then have:

*Any estimate of the form $\|f_\delta^{***}\|_q \leq A(\delta)\|f\|_p$ with $q \geq p$, valid for all functions $f$ with fixed compact support, implies a corresponding estimate $\|f_\delta^{**}\|_q \leq C\,A(\delta)\,\|f\|_p$ for functions $f$ with fixed compact support.*

This is proved as follows: the assumption means that if

$$M_\delta f(x) = \sup_T \frac{1}{|T|} \int_T |f|$$

with $T$ running through $1 \times \delta$ cylinders containing $x$ whose axis makes an angle less than $\pi/100$ with $e_d$, then

$$\int_{\mathbb{R}^{d-1}} (M_d f(\alpha, 0))^q \, d\alpha \lesssim \|f\|_p^q.$$

Clearly 0 here could be replaced by $t$ for any $t$. Integrating $dt$ over a suitable compact set we obtain $\|M_\delta f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_p$. It remains only to remove the restriction that the axis of $T$ make an angle less or equal than $\pi/100$ with $e_d$. However this is easily done by using finitely many different choices of coordinates, so the proof is complete. It follows that in order to prove Theorem 1 it suffices to prove

**Theorem 1b.** (1) *holds for $M_\delta f = f_\delta^*$ or $f_\delta^{***}$, $p = (d+2)/2$, $q = (d-1)p'$, assuming $f$ is supported in the unit disc.*

Next, let $M(d, 1)$ be all lines in $\mathbb{R}^d$. We mean all lines here and not all lines through the origin, *i.e.* $M(d, 1)$ is a $(2d - 2)$-dimensional manifold. We can map $M(d, 1)$ onto $\mathbb{P}^{d-1}$ via $\ell \to e_\ell$ where $e_\ell$ is the line through the origin parallel to $\ell$. It is convenient to fix a Riemannian metric on $M(d, 1)$ and let $\text{dist}(\ell_1, \ell_2)$, $\ell_1, \ell_2 \in M(d, 1)$ be the associated distance function. Since we will work locally we do not care what the metric is and just note the following. Let $D$ be a disc in $\mathbb{R}^d$, $\tilde{D}$ the concentric disc with radius$(\tilde{D}) = 100\,\text{radius}\,(D)$. If $\ell_1$ and $\ell_2$ are

lines which intersect $D$ then dist $(\ell_1, \ell_2)$ is comparable to the Hausdorff distance between $\ell_1 \cap \tilde{D}$ and $\ell_2 \cap \tilde{D}$ and therefore also satisfies

$$(2) \qquad \operatorname{dist}(\ell_1, \ell_2) \approx \theta(\ell_1, \ell_2) + d_{\min}(\ell_1, \ell_2),$$

where $\theta(\ell_1, \ell_2) \in [0, \pi/2]$ is the angle between $e_{\ell_1}$ and $e_{\ell_2}$ and

$$d_{\min}(\ell_1, \ell_2) = \inf\{|x - y| : \; x \in \ell_1 \cap \tilde{D}, \; y \in \ell_2 \cap \tilde{D}\}$$

(constants depend on $D$).

Now let $(\mathcal{A}, d)$ be a metric space (necessarily bounded, by (4) below) with a measure $\mu$ satisfying

$$(3) \qquad \mu(D(\alpha, \delta)) \approx \delta^m, \qquad \alpha \in \mathcal{A}, \quad \delta \leq \operatorname{diam} \mathcal{A},$$

for a certain $m \in \mathbb{R}^+$. Here $D(\alpha, \delta)$ is the $\delta$-disc centered at $\alpha$, and we will also use the notation $|E|$ for $\mu(E)$. Suppose that for each $\alpha \in \mathcal{A}$ a subset $F_\alpha \subset M(d, 1)$ is given, with $\overline{\cup_\alpha F_\alpha}$ compact and with

$$(4) \qquad d(\alpha, \beta) \lesssim \inf_{\substack{\ell \in F_\alpha \\ m \in F_\beta}} \operatorname{dist}(\ell, m), \qquad \text{for all } \alpha, \beta \in \mathcal{A}.$$

If $f : \mathbb{R}^d \to \mathbb{R}$ then we define $M_\delta f : \mathcal{A} \to \mathbb{R}$ by

$$(5) \qquad M_\delta f(\alpha) = \sup_{\ell \in F_\alpha} \sup_{a \in \ell} \frac{1}{|T_\ell^\delta(a)|} \int_{T_\ell^\delta(a)} |f|,$$

where $T_\ell^\delta(a)$ is the cylinder with length 1, radius $\delta$, axis $\ell$ and center $a$.

REMARK. This setup arises as follows. Suppose $m$ is an integer, $U$ is an open subset of $M(d, 1)$ with compact closure, $M$ is an $m$-dimensional manifold (with a Riemannian metric) and $F$ a smooth map from $U$ into $M$ which can be extended smoothly to a neighborhood of $\overline{U}$. Assume that the range $F(U)$ contains a certain open subset $\mathcal{A}$ of $M$ with smooth boundary. For $\alpha \in \mathcal{A}$ define $F_\alpha = F^{-1}(\alpha)$. It is easy to see that (3) and (4) will then hold.

In practice, $m$ is always less or equal than $d - 1$. Examples with $m = d - 1$ are

I) $U =$ all lines intersecting $D(0, 1)$, $M = \mathcal{A} = \mathbb{P}^{d-1}$, $F(\ell) = e_\ell$. Then $M_\delta f = f_\delta^*$.

II) $U$ = all lines intersecting $D(0,1)$ and making an angle less than $\pi/100$ with the $x_d$-axis, $M = \mathbb{R}^{d-1}$, $F(\ell) = \ell \cap \mathbb{R}^{d-1}$, $\mathcal{A} = F(U)$, where of course we are identifying $\mathbb{R}^{d-1}$ with the points in $\mathbb{R}^d$ with last coordinate zero. Then $M_\delta f = f_\delta^{***}$.

The natural analogue of the conjecture (1) in the general case where $m$ is not necessarily $d-1$ is that

(6)    $\|M_\delta f\|_q \leq C_\varepsilon \, \delta^{-(d/p-1+\varepsilon)} \, \|f\|_p \, , \quad 1 \leq p \leq m+1 \, , \, 1 \leq q \leq mp' \, .$

If $p \leq (m+2)/2$ then (6) is true in the general context (3), (4), (5). This is implicit both in [1] and in [7], [8]; we will give an argument based on [1] in Section 2 below. In many cases no improvement is possible as we will explain in Section 5. However, in examples I), II) the following additional property (∗) is satisfied. Here if $\Pi$ is a 2-plane in $\mathbb{R}^d$ we let $M(\Pi,1)$ be the lines contained in $\Pi$, and dist $(\ell, M(\Pi,1))$ = $\inf_{m \in M(\Pi,1)}$ dist $(\ell, m)$. A $\delta$-separated subset of $\mathcal{A}$ means of course a subset $\{\alpha_j\}$ such that $j \neq k$ implies $d(\alpha_j, \alpha_k) \geq \delta$.

**Property** (∗). *If $\ell_0 \in \cup_\alpha F_\alpha$ and $\Pi$ is a 2-plane containing $\ell_0$, and if $\sigma \geq \delta$, and if $\{\alpha_j\}_{j=1}^N$ is a $\delta$-separated subset of $\mathcal{A}$ and for each $j$ there is $\ell_j \in F_{\alpha_j}$ with dist $(\ell_j, M(\Pi,1)) < \delta$ and dist $(\ell, \ell_0) < \sigma$, then $N \leq C\sigma/\delta$.*

REMARK (intended as motivation): When $M_\delta$ arises from a foliation as discussed in the preceding Remark, property (∗) is roughly the statement that there is no 2-plane $\Pi$ such that each line contained in $\Pi$ belongs to a different fiber $F_\alpha$. More precisely, one should require the infinitesimal version of this condition -see Section 5.

To verify property (∗) in examples I), II) it suffices to show that the set $\{\alpha_j\}$ in question is contained in the intersection of a $C\sigma$-disc with a $C\delta$-neighborhood of a curve. It is clearly contained in a $C\sigma$-disc centered at $F(\ell_0)$, so it suffices to show that it is contained in a $C\delta$-neighborhood of a curve. This in turn will follow if the set

$$\gamma \stackrel{\text{def}}{=} \{\alpha : \ F^{-1}\alpha \cap M(\Pi,1) \neq \varnothing\}$$

is a curve. However, in cases I) and II) $\gamma$ is respectively the great circle on the sphere (mod $\pm 1$) obtained by intersecting the sphere with the translate of $\Pi$ to the origin, and the intersection of $\Pi$ with $\mathbb{R}^{d-1}$; note

that in case II) $\Pi$ and $\mathbb{R}^{d-1}$ intersect transversally since $\Pi$ contains $\ell_0$ which makes an angle less or equal than $\pi/100$ with the $x_d$ axis.

Our proof of Theorem 1 works in the abstract context (3), (4), (5) provided property (∗) is satisfied. Namely, we will prove the following result.

**Theorem 1c.** *Assume* $2 \leq m \leq d-1$, (3), (4) *and property* (∗). *Then the estimate* (6) *is valid for the maximal function defined by* (5) *provided* $p \leq (m+3)/2$.

As indicated above, this result includes Theorems 1 and 1b. We had trouble deciding whether to use the axiomatic setup but eventually decided to do so since it gives a simultaneous proof for $f_\delta^*$ and $f_\delta^{**}$ and complicates the arguments only in technical respects. We also hope it may be of some interest. However, when $m = d-1$ (which would seem to be the main case) property (∗) and also the conclusion of Theorem 1c hold essentially only in examples I) and II); see Proposition 5.1.[1]

We finish this introduction by making some standard remarks about the definition (3), (4), (5). If $E \subset \mathcal{A}$ then a maximal $\delta$-separated subset of $E$ is of course a subset $\{\alpha_k\} \subset E$ which is $\delta$-separated and is maximal with respect to this property. If $\{\alpha_k\}$ is a maximal $\delta$-separated subset of $E$, then the discs $D(\alpha_k, \delta)$ cover $E$ by the maximality property, and furthermore the concentric $\delta/2$-discs are disjoint. It follows that the cardinality of a maximal $\delta$-separated subset of $E$ is greater or equal than $C^{-1} |E|/\delta^m$ for a certain constant $C$, and also that if $\delta < \sigma$ then any $\delta$-separated set $\{\alpha_k\}_{k=1}^{M}$ has a $\sigma$-separated subset $\{\alpha_{k_j}\}_{j=1}^{\overline{M}}$ with $\overline{M} \geq C^{-1}(\delta/\sigma)^m M$. Finally, if $\{\alpha_k\}$ is any $\delta$-separated subset (maximal or not) and if $A$ is any constant then there is a constant $C_A$ such that no point $\alpha \in A$ belongs to more than $C_A$ discs of the form $D(\alpha_k, A\delta)$. This "bounded overlap" follows from the doubling property of $\mu$ (a consequence of (3)) in a standard way using disjointness of the discs $D(\alpha_k, \delta/2)$.

## 2. Preliminaries.

This section is expository. Its purpose is to give convenient forms of some known results. We first introduce some more notation. If $\rho \geq$

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$\delta > 0$ and $\ell \in M(d,1)$, $a \in \ell$, then $T_\ell^{\rho\delta}(a)$ will mean the cylinder with length $\rho$, cross section radius $\delta$, axis contained in $\ell$, and center $a$. $\tilde{T}_\ell^{\rho\delta}(a)$ will mean the same as $T_\ell^{(100\rho)(100\delta)}(a)$. We will usually take $\rho = 1$, and we abbreviate $T_\ell^{1\delta}(a)$ and $\tilde{T}_\ell^{1\delta}(a)$ by $T_\ell^\delta(a)$ and $\tilde{T}_\ell^\delta(a)$ respectively. We will also drop the $a$ argument when no confusion will result. Thus $T_\ell^\delta$ means "$T_\ell^\delta(a)$ for some $a \in \ell$", etc. Finally if $\Pi$ is a 2-plane and $\delta > 0$ then

$$\Pi^\delta \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : |x - y| < \delta\}.$$

The first result we want to recall is a simple geometrical fact and was already used e.g. in [6].

(7)    For any $T_{\ell_1}^\delta$ and $T_{\ell_2}^\delta$, $T_{\ell_1}^\delta \cap T_{\ell_2}^\delta$ is contained in a $T_{\ell_1}^{(C\delta/\theta(\ell_1,\ell_2))\delta}$

In particular,

(8)        For any $T_{\ell_1}^\delta$ and $T_{\ell_2}^\delta$, $|T_{\ell_1}^\delta \cap T_{\ell_2}^\delta| \lesssim \dfrac{\delta^d}{\theta(\ell_1,\ell_2) + \delta}$.

The second is a version of an argument of Bourgain [1, p. 153-154]. We formulate it as a lemma:

**Lemma 2.1.** *Suppose* $\{T_{\ell_j}^\beta\}_{j=1}^M$ *are tubes and $E$ a set. Assume that $\varepsilon > \beta$, that*

(9)        $T_{\ell_j}^\beta \cap T_{\ell_k}^\beta \neq \varnothing$    *implies*    $\theta(\ell_j, \ell_k) \geq C^{-1}\varepsilon$

*and that for $a \in \mathbb{R}^d$, $1 \leq j \leq M$,*

(10)        $\left|T_{\ell_j}^\beta \cap E \cap (\mathbb{R}^d \backslash D(a, \dfrac{\beta}{\varepsilon}))\right| \geq \dfrac{\lambda}{2} |T_{\ell_j}^\beta|.$

*Then* $|E| \gtrsim \lambda \beta^{d-1} \sqrt{M}$, *where the constant depends on $C$.*

PROOF. ([1]) We have $\sum_j |T_{\ell_j}^\beta \cap E| \gtrsim \lambda\beta^{d-1}M$, so there must be a point $a \in E$ which belongs to $\gtrsim (\lambda\beta^{d-1}M/|E|) T_{\ell_j}^\beta$'s. The sets $\tau_j \stackrel{\text{def}}{=} T_{\ell_j}^\beta \cap (\mathbb{R}^d \backslash D(a, \beta/\varepsilon))$, where $T_{\ell_j}^\beta$ runs over this set of tubes, have bounded overlap (i.e. no point belongs to more than a fixed finite number of them) for the following reason. First, it is easy to see using the hypothesis (9) that for any constant $A$ and any given $j$ there are at

least $C_A$ tubes $T_{\ell_k}^\beta$ containing $a$ and with $\theta(\ell_j, \ell_k) \leq A\varepsilon$. Thus it suffices to show that if $\theta(\ell_j, \ell_k) \geq A\varepsilon$ for a large fixed constant $A$, then $\tau_j \cap \tau_k = \varnothing$, i.e. $T_{\ell_j}^\beta \cap T_{\ell_k}^\beta \subset D(a, \beta/\varepsilon)$. On the other hand by (7), $\theta(\ell_j, \ell_k) \geq A\varepsilon$ implies $\text{diam}(T_{\ell_j}^\beta \cap T_{\ell_k}^\beta) \leq \beta/\varepsilon$ for large $A$. Since the point $a$ belongs to each $T_{\ell_j}^\beta$ the bounded overlap follows. By assumption

$$|\tau_j \cap E| \geq \frac{\lambda}{2} |T_{\ell_j}^\beta| \approx \lambda \beta^{d-1}, \qquad \text{for each } j.$$

Hence

$$\frac{\lambda \beta^{d-1} M}{|E|} \lambda \beta^{d-1} \lesssim |E|$$

and the lemma follows.

REMARK. In [1], this lemma is used together with additional combinatorial arguments to obtain the result we mentioned in the introduction, that the conjecture (1) is true for certain $p > (d+1)/2$ (and $q = p$). If one reads [1] carefully then one sees that the $p = (d+1)/2$ case follows directly from Lemma 2.1. The argument proves the following[2]:

**Proposition 2.1.** *Assume* (3), (4). *Then the estimate* (6) *is valid for the maximal function defined by* (5) *provided* $p \leq (m+2)/2$.

PROOF. ([1]) It suffices to prove the following restricted weak type estimate at the endpoint $p = (m+2)/2$, $q = mp' = m+2$,

$$(11) \qquad |\{\alpha \in \mathcal{A} : M_\delta f(\alpha) \geq \lambda\}| \lesssim \left( \frac{|E|}{\delta^{d-(m+2)/2} \lambda^{(m+2)/2}} \right)^2,$$

where $E$ denotes a set and $f$ its characteristic function. To prove (11), let $\varepsilon = A\delta/\lambda$ for a large fixed constant $A$, and choose a maximal $\varepsilon$-separated subset $\{\alpha_j\}_{j=1}^M$ of the set $\{\alpha \in \mathcal{A} : M_\delta f(\alpha) \geq \lambda\}$ and, for each $j$, a tube $T_{\ell_j}^\delta$ with $\ell_j \in F_{\alpha_j}$ and $|T_{\ell_j}^\delta \cap E| \geq \lambda |T_{\ell_j}^\delta|$. Then $M \gtrsim \varepsilon^{-m} |\{\alpha : M_\delta f(\alpha) \geq \lambda\}|$. The choice of $\varepsilon$ implies that $|T_{\ell_j}^\delta \cap D(a, \delta/\varepsilon)| \leq \lambda |T_{\ell_j}^\delta|/2$, in particular (10) holds (with $\beta$ replaced by $\delta$).

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[2] Actually the argument proves a slightly sharper result: the $\delta^{-\varepsilon}$ factor is only needed at the endpoint $p=(m+2)/2$. Proposition 2.1 also follows from the results in [7] or [4] (at least if $m=d-1$), in fact the argument based on [4] gives a still sharper result where the $\delta^{-\varepsilon}$ factor is not needed at all.

Also (9) holds for a certain constant $C$ by (4) and (2). We therefore obtain $M \lesssim (|E|/\lambda\delta^{d-1})^2$, *i.e.*

$$|\{\alpha : M_\delta f(\alpha) \geq \lambda\}| \lesssim \left( \frac{|E|}{\delta^{d-(m+2)/2} \lambda^{(m+2)/2}} \right)^2 ,$$

as claimed.

In order to prove Theorem 1c we will combine this type of argument with the $L^2$ argument of Cordoba and we now state a convenient form of the latter.

**Lemma 2.2.** *Assume $E \subset \mathbb{R}^d$ is a set, $\Pi$ is a 2-plane in $\mathbb{R}^d$, and $\{T^\delta_{\ell_j}\}_{j=1}^M$ are tubes which are contained in $\Pi^{C_0\delta}$. Assume the following conditions:*

i) $|E \cap T^\delta_{\ell_j}| \geq \lambda |T^\delta_{\ell_j}|$, *for all $j$.*

ii) $\mathrm{card}(\{j : T^\delta_{\ell_j} \subset \tilde{T}^\sigma_{\ell_k}\}) \leq C_0 \, \sigma/\delta$, *for all $\sigma \in (\delta, 1)$ and all $k$.*

*Then*

$$(12) \qquad M \, \delta^{d-1} \leq C \, \frac{|E \cap \Pi^{C_0\delta}|}{\lambda^2} \log\frac{1}{\delta} ,$$

*where $C$ depends on $C_0$.*

PROOF. ([6]) We may assume that $E \subset \Pi^{C_0\delta}$, and then we have

$$M \lambda \delta^{d-1} \leq \sum_j |E \cap T^\delta_{\ell_j}|$$

$$= \int_E \sum \chi_{T^\delta_{\ell_j}}$$

$$\leq |E|^{1/2} \left\| \sum \chi_{T^\delta_{\ell_j}} \right\|_2$$

$$= |E|^{1/2} \left( \sum_i \sum_j |T^\delta_{\ell_i} \cap T^\delta_{\ell_j}| \right)^{1/2}$$

$$\leq |E|^{1/2} \left( C M \delta^{d-1} + \sum_i \sum_{j \neq i} |T_{\ell_i} \cap T_{\ell_j}| \right)^{1/2}$$

$$\leq |E|^{1/2} \left( C M \delta^{d-1} \right.$$

$$+ \sum_i \sum_{k=0}^{\log_2 \pi/\delta} \frac{\delta^{d-1}}{2^k} \operatorname{card}(\{j \,:\, T^\delta_{\ell_i} \cap T_{\ell_j} \neq \varnothing ,$$

$$2^k \delta \leq \theta(\ell_i, \ell_j) \leq 2^{k+1} \delta\}))^{1/2}$$

$$\leq C |E|^{1/2} \left( M \delta^{d-1} + \sum_i \sum_{k=0}^{\log_2 \pi/\delta} \delta^{d-1} \right)^{1/2},$$

where the last line follows from the assumption ii). So

$$M \lambda \delta^{d-1} \leq C |E|^{1/2} \left( M \log \frac{1}{\delta} \delta^{d-1} \right)^{1/2}$$

which is equivalent to (12).

## 3. Main argument.

This section will be the proof of the following Lemma 3.1.

**Lemma 3.1.** *Assume* $d \geq 3$, $2 \leq m \leq d-1$. *Suppose that a set* $E \subset \mathbb{R}^d$ *and tubes* $\{T^{\rho\delta}_{\ell_j}\}_{j=1}^M$, $\delta \leq \rho/100$ *are given and that with a sufficiently large constant* $C_0$ *and some constant* $C$,

i) $|E \cap T^{\rho\delta}_{\ell_j}| \geq \lambda |T^{\rho\delta}_{\ell_j}|$, *for all* $j$.

ii) *If* $x \in \mathbb{R}^d$ *then, for all* $j$,

$$\left| E \cap T^{\rho\delta}_{\ell_j} \cap D\left(x, \left(\log \frac{\rho}{\delta}\right)^{-\nu} \rho\right) \right| \leq C_0^{-1} \frac{\lambda}{\log \frac{\rho}{\delta}} | T^{\rho\delta}_{\ell_j} |,$$

iii) *For any* $k$ *and any* $\sigma \in (\delta, \rho)$ *the set*

$$\{j \,:\, \tilde{T}^{\rho\sigma}_{\ell_j} \cap \tilde{T}^{\rho\sigma}_{\ell_k} \neq \varnothing , \; \theta(\ell_j, \ell_k) \leq \sigma/\rho\}$$

*has cardinality less or equal than* $C (\sigma/\delta)^m$.

iv) *For any* $k$, *any 2-plane* $\Pi$ *with* $\ell_k \subset \Pi$ *and any* $\sigma \in (\delta, \rho)$, *the set*

$$\{j \,:\, T^{\rho\delta}_{\ell_j} \subset \tilde{T}^{\rho\sigma}_{\ell_k} \cap \Pi^{C_0 \delta}\}$$

*has cardinality less or equal than* $C \sigma/\delta$.

*Then for large $\nu$*

$$
(13) \qquad \begin{aligned} & \rho^{-d}\,|E| \\ & \geq C_\nu^{-1}\,\lambda^2\,(\rho^{-1}\delta)^{d-(m+3)/2}\,\Big(M\big(\tfrac{\delta}{\rho}\big)^m\Big)^{(1+1/m)/2}\,\Big(\log\tfrac{\rho}{\delta}\Big)^{-d\nu}. \end{aligned}
$$

REMARKS. 1) Here $\tilde{T}_{\ell_k}^{\rho\sigma}$ is taken concentric with $T_{\ell_k}^{\rho\delta}$, *i.e.* if $T_{\ell_k}^{\rho\delta} = T_{\ell_k}^{\rho\delta}(a)$ then $\tilde{T}_{\ell_k}^{\rho\sigma} = T_{\ell_k}^{(100\rho)(100\sigma)}(a)$.

2) In the proof we may assume $\rho = 1$ since we can reduce to this case by scaling. We will in fact assume $\rho = 1$ throughout this section and will denote $T_{\ell_j}^{\rho\delta}$ by $T_{\ell_j}^\delta$, etc, as mentioned at the beginning of Section 2. We may then also assume that $E$ is contained in a fixed compact set, say, the unit disc. It then follows by iii) with $\sigma \approx 1$ that $M \lesssim \delta^{-m}$.

3) This remark is intended as motivation. An immediate corollary of Lemma 3.1 is that any Besicovitch set in $\mathbb{R}^d$ has Minkowski dimension greater or equal than $(d+2)/2$, *i.e.* the following statement: *suppose $E$ is a compact subset of $\mathbb{R}^d$ which contains a unit line segment in every direction. Let $E_\delta = \{x : \text{dist}(x, E) < \delta\}$. Then*

$$
(14) \qquad |E_\delta| \geq C_\nu^{-1}\,\delta^{(d-2)/2}\,\Big(\log\frac{1}{\delta}\Big)^{-d\nu}
$$

*for large $\nu$.* To prove this statement set $m = d - 1$, $\rho = 1$. Let $\{e_j\}_{j=1}^M$ be a maximal $\delta$-separated subset of $\mathbb{P}^{d-1}$. For each $j$ there is a line in the $e_j$ direction with a unit segment on it belonging to $E$, and therefore a tube $T_{\ell_j}^\delta$ with $e_{\ell_j} = e_j$ which is contained in $E_\delta$. Thus i) of Lemma 3.1 holds with $E$ replaced by $E_\delta$ and $\lambda = 1$, and then ii) holds tautologically if $\nu > 1$ and $\delta$ is small. Also iii) holds since the directions of the $\ell_j$ must belong to a $C\sigma$-disc in $\mathbb{P}^{d-1}$, and iv) holds since the directions of the $\ell_j$ must also belong to a $C\delta$-neighborhood of the great circle determined by $\Pi$. We conclude that (13) holds with $E$ replaced by $E_\delta$, and $\rho = 1$, $\lambda = 1$, $m = d - 1$. Since $M \approx \delta^{-(d-1)}$ we obtain (14). Theorem 1 is a more refined result and requires an additional argument which we will give in Section 4.[3]

We start the proof by observing that we can make the following additional assumption:

$$
(15) \qquad \text{If } T_{\ell_j}^\delta \cap T_{\ell_k}^\delta \neq \varnothing \text{ then } \theta(\ell_j, \ell_k) \geq \delta.
$$

---

[3] Actually, for the statement about the dimension of Besicovitch sets we only needed Lemma 3.1 in the case $\lambda = 1$. In this case the proof can be simplified.

Namely, if we let $\{T^\delta_{\ell_{j_i}}\}^{\overline{M}}_{i=1}$ be a subset of $\{T^\delta_{\ell_j}\}$ which satisfies (15) and is maximal with respect to this property, then every tube $T^\delta_{\ell_j}$ must satisfy $T^\delta_{\ell_j} \cap T^\delta_{\ell_{j_i}} \neq \varnothing$ and $\theta(\ell_j, \ell_{j_i}) \leq \delta$ for some $i$. It follows by iii) with $\sigma = \delta$ that $\overline{M} \geq A^{-1}M$. We could replace $\{T^\delta_{\ell_j}\}$ by $\{T^\delta_{\ell_{j_i}}\}$, so we may assume that $\{T^\delta_{\ell_j}\}$ satisfies (15).

We now fix a number $N$ and consider the following possibilities.

I. (low multiplicity) There are at least $M/2$ values of $j$ such that

$$|\{x \in T^\delta_{\ell_j} \cap E : \text{card}(\{i : x \in T^\delta_{\ell_i}\}) \leq N\}| \geq \frac{\lambda}{2}\, |T^\delta_{\ell_j}|.$$

$II_\sigma$. (high multiplicity at angle $\sigma$) There are at least $C_1^{-1} M$ $(\log 1/\delta)^{-1}$ values of $j$ such that

$$
\begin{aligned}
\left|\left\{x \in T^\delta_{\ell_j} \cap E : \right.\right. & \text{card}(\{i : x \in T^\delta_{\ell_i} \text{ and } \sigma \leq \theta(\ell_i, \ell_j) \leq 2\sigma\}) \\
& \left.\left. \geq \left(C_1 \log \frac{1}{\delta}\right)^{-1} N\right\}\right| \\
& \geq \left(C_1 \log \frac{1}{\delta}\right)^{-1} \lambda\, |T^\delta_{\ell_j}|.
\end{aligned}
$$

(16)

**Lemma 3.2.** *There is a number $N$ for which we have both I, and also $II_\sigma$ for some $\sigma \in [\delta, \pi]$.*

PROOF. Take the smallest $N \in \mathbb{Z}^+$ for which I holds. Then there are $M/2$ values of $j$ for which

$$(17) \qquad |\{x \in T^\delta_{\ell_j} \cap E : \text{card}(\{i : x \in T^\delta_{\ell_i}\}) \geq N\}| \geq \frac{\lambda}{2}\, |T^\delta_{\ell_j}|.$$

For any $j$ as in (17) and any $x \in T^\delta_{\ell_j}$ with $\text{card}(\{i : x \in T^\delta_{\ell_i}\}) \geq N$, (15) implies there is some $k \in \{1, \ldots, \log_2 \pi/\delta\}$ such that $\text{card}(\{i : x \in T^\delta_{\ell_i}, \theta(\ell_i, \ell_j) \in [2^{k-1}\delta, 2^k\delta]\}) \geq (\log_2 \pi/\delta)^{-1}N$. Thus for any $j$ as in (17) there is some $k \in \{1, \ldots, \log_2 \pi/\delta\}$ such that

$$
\begin{aligned}
\left|\left\{x \in T^\delta_{\ell_j} \cap E : \right.\right. & \text{card}(\{i : x \in T^\delta_{\ell_i}, \theta(\ell_i, \ell_j) \in [2^{k-1}\delta, 2^k\delta]\}) \\
& \left.\left. \geq \left(\log_2 \frac{\pi}{\delta}\right)^{-1} N\right\}\right| \\
& \geq \left(\log_2 \frac{\pi}{\delta}\right)^{-1} \frac{\lambda}{2}\, |T^\delta_{\ell_j}|.
\end{aligned}
$$

It follows that $\mathrm{II}_\sigma$ holds for some $\sigma = 2^k\delta$.

The logic will now be as follows: we make separate estimates in the cases I and $\mathrm{II}_\sigma$ and then apply them both with $N$ given by Lemma 3.2 to obtain (13). The estimate in case I is very simple:

**Lemma 3.3.** *If* i) *of Lemma 2.1 and* I *hold then* $|E| \gtrsim \lambda M \delta^{d-1}/N$ .

PROOF. Let $\tilde{E} = \{x \in E : \; \mathrm{card}(\{i : \; x \in T^\delta_{\ell_i}\}) \leq N\}$. Then $|T^\delta_{\ell_j} \cap \tilde{E}| \geq \lambda |T^\delta_{\ell_j}|/2$ for $M/2$ values of $j$, and

$$|E| \geq |\tilde{E}| \geq |\tilde{E} \cap (\cup_j T^\delta_{\ell_j})|$$

$$\geq N^{-1} \sum_j |\tilde{E} \cap T^\delta_{\ell_j}| \geq \frac{\lambda}{2N} \sum_j |T^\delta_{\ell_j}| \gtrsim \frac{\lambda M \delta^{d-1}}{N}$$

and the lemma is proved.

The idea behind the estimate in case $\mathrm{II}_\sigma$ is as follows. We are evidently in a situation where many tubes intersect some given tube $T$. Each one of these tubes is then contained in a $C_0\delta$-neighborhood of some 2-plane containing the axis of $T$. The latter sets have nice intersection properties *i.e.* the same intersection properties as tubes through the origin in $\mathbb{R}^{d-1}$. This allows us to use Lemma 2.2 separately for each 2-plane and then sum the resulting estimates.

We carry this out as follows:

**Lemma 3.4.** *Assume* i), ii) *and* iv) *of Lemma 3.1, and suppose that* $T^\delta_{\ell_j}$ *is a tube for which* (16) *holds. Then*

$$(18) \qquad |E \cap \tilde{T}^\sigma_{\ell_j}| \gtrsim \lambda^3 \sigma \delta^{d-2} N \left(\log \frac{1}{\delta}\right)^{-(d-2)\nu-3}$$

PROOF. Let $\mathcal{F}$ be the set of all tubes $T^\delta_{\ell_i}$ such that $T^\delta_{\ell_i} \cap T^\delta_{\ell_j} \cap E \neq \varnothing$ and $\sigma \leq \theta(\ell_i, \ell_j) \leq 2\sigma$ and let $|\mathcal{F}|$ be the cardinality of $\mathcal{F}$. If $T^\delta_{\ell_i} \in \mathcal{F}$ and $x \in \mathbb{R}^d$ then for a suitable constant $C_2$ the set

$$\left\{x \in T^\delta_{\ell_i} : \; \mathrm{dist}\,(x, \ell_j) \leq C_2^{-1}\,\sigma \left(\log \frac{1}{\delta}\right)^{-\nu}\right\}$$

is contained in $T_{\ell_i}^{\delta} \cap \tilde{T}_{\ell_j}^{\sigma(\log 1/\delta)^{-\nu}}$ and therefore by (7) is contained in a disc of radius $C(\log 1/\delta)^{-\nu}$. Consequently by ii)

$$(19) \qquad \left| \left\{ x \in T_{\ell_i}^{\delta} \cap E : \ \mathrm{dist}\,(x,\ell_j) \geq C_2^{-1}\,\sigma\Big( \log\frac{1}{\delta} \Big)^{-\nu} \right\} \right| \geq \frac{\lambda}{2}\,|T_{\ell_i}^{\delta}|,$$

provided $C_0$ has been chosen large enough. We choose 2-planes $\Pi_k$ containing $\ell_j$ so that

A) any tube $T_{\ell_i}^{\delta} \in \mathcal{F}$ is contained in $\Pi_k^{C_0\delta}$ for some $k$,

B) any point $x$ with $\mathrm{dist}\,(\ell_j,x) \geq C_2^{-1}\sigma(\log 1/\delta)^{-\nu}$ belongs to at most $C\,(\log 1/\delta)^{(d-2)\nu}\Pi_k^{C_0\delta}$'s.

For this, it suffices to choose a maximal $\delta/\sigma$-separated set $\{\alpha_k\}$ in the unit sphere (modulo $\pm 1$) in the $d-1$-dimensional space $e_j^{\perp} \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R}^d : \ \langle x,e_j \rangle = 0\}$, where $e_j = e_{\ell_j}$ is the direction of $e_j$ and consider the planes through $\ell_j$ spanned by the $\alpha_k$ directions. It is easy to see that A) and B) will then hold; we will now prove this. For A), we let $e_i = e_{\ell_i} \in \mathbb{P}^{d-1}$ be the direction of $\ell_i$ and write $e_i = ae_j + be$ with $e \perp e_j$. Then $|b|$ is clearly the sine of the angle between $e_i$ and $e_j$ and is therefore $\lesssim \sigma$. Now choose $k$ with $|e \pm \alpha_k| \leq \delta/\sigma$ (this is possible by maximality of $\{\alpha_k\}$). We assume for notational purposes that $|e - \alpha_k| \leq \delta/\sigma$. Since $T_{\ell_i}^{\delta} \cap T_{\ell_j}^{\delta} \neq \varnothing$ there are $x_1 \in \ell_j$ and $x_2 \in \ell_i$ with $|x_1 - x_2| \lesssim \delta$. Given $x \in T_{\ell_i}^{\delta}$, choose $y \in \ell_i$ with $|x - y| \lesssim \delta$. Then we have

$$x = x_1 + (y - x_2) + (x - y) + (x_2 - x_1).$$

Here $y - x_2 = te_i$ with $t \lesssim 1$. So we may further write

$$x = (x_1 + t\,a\,e_j + t\,b\,\alpha_k) + t\,b\,(e - \alpha_k) + (x - y) + (x_2 - x_1).$$

The term $x_1 + tae_j + tb\alpha_k$ belongs to $\Pi_k$, and the remaining terms are all $\lesssim \delta$ in absolute value, so we have proved A).

For B), we may clearly assume $\ell_j$ passes through the origin. We fix $x$ as in B) and let $x_{\perp}$ be the orthogonal projection of $x$ on $e_j^{\perp}$. Thus $x_{\perp} = te$ with $e \in e_j^{\perp}$, $|e| = 1$, $t \geq C_2^{-1}\sigma(\log 1/\delta)^{-\nu}$. If $x \in \Pi_k^{C_0\delta}$ then it follows that $\mathrm{dist}\,(x_{\perp},\mathrm{span}\,\{\alpha_k\}) \leq C_0\delta$, so $|e \pm \alpha_k| \lesssim t^{-1}\delta \lesssim \delta(\log 1/\delta)^{\nu}/\sigma$. Since the $\alpha_k$ are $\delta/\sigma$- separated there are $\lesssim (\log 1/\delta)^{(d-2)\nu}$ such values of $k$, so B) is proved.

For each $k$, let $\mathcal{F}_k$ be those tubes $T_{\ell_i}^{\delta} \in \mathcal{F}$ which are contained in $\Pi_k^{C_0 \delta}$, and let $|\mathcal{F}_k|$ be the cardinality of $\mathcal{F}_k$. Note that by A) above, $\mathcal{F} = \cup_k \mathcal{F}_k$. Also, since $\theta(\ell_i, \ell_j) \leq 2\sigma$, $T_{\ell_i}^{\delta}$ is contained in $\tilde{T}_{\ell_j}^{\sigma}$. We will apply Lemma 2.2 with $E$ replaced by $E \cap \tilde{T}_{\ell_j}^{\sigma} \cap \{x : \text{dist}\,(x, \ell_j) \geq C_2^{-1}\sigma(\log 1/\delta)^{-\nu}\}$, and with the 2-plane $\Pi_k$. Assumption i) of Lemma 2.2 holds (with $\lambda/2$ instead of $\lambda$) because of (19), and assumption ii) of Lemma 2.2 holds because of our current assumption iv). We conclude that

$$
|\mathcal{F}_k|\,\delta^{d-1}
$$
$$
\leq C\,\frac{|E \cap \tilde{T}_{\ell_j}^{\sigma} \cap \Pi_k^{C_0 \delta} \cap \{x : \text{dist}\,(x, \ell_j) \geq C_2^{-1}\sigma(\log 1/\delta)^{-\nu}\}|}{\lambda^2}\,\log\frac{1}{\delta}\,,
$$

for suitable $C$. It follows on summing over $k$ that

$$
|\mathcal{F}|\,\delta^{d-1} \leq \frac{C}{\lambda^2}\,\log\frac{1}{\delta}
$$
$$
(20) \qquad \cdot \sum_k \left| E \cap \tilde{T}_{\ell_j}^{\sigma} \cap \Pi_k^{C_0 \delta} \cap \left\{ x : \text{dist}\,(x, \ell_j) \geq C_2^{-1}\,\sigma\left(\log\frac{1}{\delta}\right)^{-\nu} \right\} \right|
$$
$$
\lesssim \frac{C}{\lambda^2}\,|E \cap \tilde{T}_{\ell_j}^{\sigma}|\left(\log\frac{1}{\delta}\right)^{(d-2)\nu}\log\frac{1}{\delta}
$$

by B). On the other hand, since (16) holds we know that

$$
\sum_{T_{\ell_i}^{\delta} \in \mathcal{F}} \chi_{T_{\ell_i}^{\delta}}(x) \geq \left(C_1\,\log\frac{1}{\delta}\right)^{-1} N\,,
$$

for all $x$ in a subset of $T_{\ell_j}^{\delta}$ with measure at least $\lambda(C_1 \log 1/\delta)^{-1}|T_{\ell_j}^{\delta}|$. Accordingly

$$
\left(C_1\,\log\frac{1}{\delta}\right)^{-1}\lambda\,|T_{\ell_j}^{\delta}| \leq N^{-1}C_1\,\log\frac{1}{\delta}\int_{T_{\ell_j}^{\delta}}\sum_{T_{\ell_i}^{\delta} \in \mathcal{F}} \chi_{T_{\ell_i}^{\delta}}(x)\,dx
$$
$$
= N^{-1}C_1\,\log\frac{1}{\delta}\sum_{T_{\ell_i}^{\delta} \in \mathcal{F}} |T_{\ell_i}^{\delta} \cap T_{\ell_j}^{\delta}|
$$
$$
\lesssim N^{-1}\,\frac{\delta^d}{\sigma}\,|\mathcal{F}|\,\log\frac{1}{\delta}\,,
$$

where the last line follows from (8). Thus $|\mathcal{F}| \gtrsim \sigma(\log 1/\delta)^{-2}N/\delta$. This inequality and (20) imply (18).

**Lemma 3.5.** *With the same assumptions as in Lemma* 3.4 *we also have*

$$|E \cap (\mathbb{R}^d \backslash D(a, (\log 1/\delta)^{-\nu})) \cap \tilde{T}_{\ell_j}^\sigma| \gtrsim \lambda^3 \sigma \delta^{d-2} N \left(\log \frac{1}{\delta}\right)^{-(d-2)\nu-3},$$

*for any* $a \in \mathbb{R}^d$.

PROOF. We need only to observe that the hypotheses of Lemma 3.4 still hold (with $\lambda$ replaced by $\lambda/2$) if $E$ is replaced by

$$E \cap (\mathbb{R}^d \backslash D(a, (\log 1/\delta)^{-\nu})),$$

provided $C_0$ has been chosen large enough. The main point is that (16) holds; this follows from the corresponding statement for $E$ since $|T_{\ell_j}^\delta \cap E \cap D(a, (\log 1/\delta)^{-\nu})|$ is small by assumption ii).

**Lemma 3.6.** *Assume the hypotheses of Lemma* 3.1, *and also* $II_\sigma$ *for some* $\sigma$. *Then*

$$(21) \qquad |E| \gtrsim \lambda^3 N \delta^{d-2} (M \delta^m)^{1/m} \left(\log \frac{1}{\delta}\right)^{-2d\nu}.$$

PROOF. If $\sigma \geq (M\delta^m)^{1/m}(\log 1/\delta)^{-\nu}$ this follows directly from Lemma 3.4, so we assume $\sigma \leq (M\delta^m)^{1/m}(\log 1/\delta)^{-\nu} \lesssim (\log 1/\delta)^{-\nu}$. By hypothesis there are $(C_1 \log 1/\delta)^{-1} M$ $\ell_j$'s for which (16) holds. Choose a subset $\{\ell_{j_k}\}_{k=1}^{\overline{M}}$ which is maximal with respect to the following property:

$$\tilde{T}_{\ell_{j_k}}^\sigma \cap \tilde{T}_{\ell_{j_i}}^\sigma \neq \varnothing \quad \text{implies} \quad \theta(\ell_{j_k}, \ell_{j_i}) \geq \sigma \left(\log \frac{1}{\delta}\right)^\nu.$$

Then

$$\overline{M} \gtrsim M \left(\log \frac{1}{\delta}\right)^{-1} \left(\frac{\delta}{\sigma \left(\log \frac{1}{\delta}\right)^\nu}\right)^m;$$

this follows from the maximality using assumption iii) with $\sigma$ replaced by $\sigma(\log 1/\delta)^\nu$, as in the argument after formula (15). We claim that Lemma 2.1 is applicable to the tubes $\tilde{T}_{\ell_{j_k}}^\sigma$ with $\beta$, $\lambda$, and $\varepsilon$ there equal to $\sigma$,

$$C^{-1} \lambda^3 \frac{\delta^{d-2}}{\sigma^{d-2}} N \left(\log \frac{1}{\delta}\right)^{-(d-2)\nu-3} \qquad \text{and} \qquad \sigma \left(\log \frac{1}{\delta}\right)^\nu,$$

respectively. Namely, hypothesis (9) follows by construction, and (10) follows from Lemma 3.5. We conclude that

$$|E| \gtrsim \lambda^3 \sigma \, \delta^{d-2} N \left( \log \frac{1}{\delta} \right)^{-(d-2)\nu-3} \sqrt{M \left( \log 1/\delta \right)^{-1} \left( \frac{\delta}{\sigma \, (\log 1/\delta)^\nu} \right)^m}$$

$$\gtrsim \lambda^3 \sigma \, N \, \delta^{d-2} \left( \log \frac{1}{\delta} \right)^{-2d\nu} \sqrt{M \frac{\delta^m}{\sigma^m}} \, ,$$

for large $\nu$. The lemma now follows since $\sigma \, (M \, \delta^m / \sigma^m)^{1/2}$ is a decreasing function of $\sigma$ when $m \geq 2$, hence minorized by its value at $(M \, \delta^m)^{1/m}$.

COMPLETION OF PROOF OF LEMMA 3.1. We need only choose $N$ by Lemma 3.2 and then take the geometric mean of (21) and the estimate in Lemma 3.3.

## 4. Completion of the proof.

In order to prove Theorem 1c it suffices of course to prove the corresponding restricted weak type estimate at the endpoint, $i.e.$ the estimate

$$|\{\alpha : M_\delta \chi_E(\alpha) \geq \lambda\}| \leq C_\varepsilon \left( \delta^{-\varepsilon} \frac{|E|}{\delta^{d-p} \lambda^p} \right)^{q/p} ,$$

where $p = (m+3)/2$ and $q = m \, p'$. We may also assume $E$ is contained in the unit disc $D$. Furthermore (consider a maximal $\delta$-separated subset) it suffices to prove the discrete analogue, $i.e.$ that if $\{\alpha_j\}_{j=1}^M$ are $\delta$-separated and if $\ell_j \in F_{\alpha_j}$ and $|E \cap T_{\ell_j}^\delta| \geq \lambda |T_{\ell_j}^\delta|$ then

$$(22) \qquad M \, \delta^m \leq C_\varepsilon \left( \delta^{-\varepsilon} \frac{|E|}{\delta^{d-p} \lambda^p} \right)^{q/p} .$$

We will actually prove that if $\delta \leq \rho \leq 1$ and $\{\alpha_j\}_{j=1}^M$ are $\delta/\rho$-separated and $\ell_j \in F_{\alpha_j}$ and if $|E \cap T_{\ell_j}^{\rho\delta}| \geq \lambda \, |T_{\ell_j}^{\rho\delta}|$, then

$$(23) \qquad \rho^{-d} |E| \geq C_\varepsilon^{-1} \left( M \left( \frac{\delta}{\rho} \right)^m \right)^{p/q} \left( \frac{\delta}{\rho} \right)^{d-p} \lambda^p \left( \frac{\delta}{\rho} \right)^\varepsilon .$$

The case $\rho = 1$ gives (22).

In order to prove (23) we note first that any $T_{\ell_j}^{\rho\delta}$'s as there will automatically satisfy iii) and iv) of Lemma 3.1. Namely, to prove iii) fix $k$ and $\sigma$. Let

$$J = \{j : \tilde{T}_{\ell_j}^{\rho\sigma} \cap \tilde{T}_{\ell_k}^{\rho\sigma} \neq \varnothing, \; \theta(\ell_j, \ell_k) \leq \sigma/\rho\}.$$

If $j \in J$ then $\operatorname{dist}(\ell_j, \ell_k) \lesssim \sigma/\rho$ by (2), so $d(\alpha_j, \alpha_k) \lesssim \sigma/\rho$ by (4). Thus the $\{\alpha_j\}_{j \in J}$ form a $\delta/\rho$-separated subset of a $C\sigma/\rho$-disc, so card $J \lesssim (\delta/\sigma)^m$, i.e. iii) holds.

The argument for iv) is similar. Fix $k$, a 2-plane $\Pi$ and $\sigma$ and let

$$J = \{j : T_{\ell_j}^{\rho\delta} \subset \tilde{T}_{\ell_k}^{\rho\sigma} \cap \Pi^{C_0\delta}\}.$$

If $j \in J$ then by assumption there is a segment of length $\rho$ on $\ell_j$ which is contained in $\tilde{D}$ and is at distance less or equal than $C_0\delta$ from $\Pi$. Consequently by similar triangles, $\ell_j \cap \tilde{D}$ is at distance $\lesssim \delta/\rho$ from $\Pi$. We conclude that if $j \in J$ then $\operatorname{dist}(\ell_j, M(\Pi, 1)) \lesssim \delta/\rho$. We also know that $\operatorname{dist}(\ell_j, \ell_k) \lesssim \sigma/\rho$. Since the $\{\alpha_j\}_{j \in J}$ are $\delta/\rho$-separated, property ($*$) implies that card $J \lesssim \sigma/\delta$, i.e. iv) holds.

We now fix $\varepsilon$, and will prove (23) for an appropriate constant $C_\varepsilon$ by induction on $\rho$. The choice of $C_\varepsilon$ requires some care. We first let $B$ be a constant with the following property: *if $\delta < \varepsilon < 1$, then any $\delta$-separated subset $Y$ of $A$ has an $\varepsilon$-separated subset $Z$ with* card $(Z) \geq B^{-1}(\delta/\varepsilon)^m$ card $(Y)$ (see the remarks at the end of the introduction). Next we fix $\nu$ large enough that $\nu\varepsilon > p$. (23) is trivial when $\rho \leq A\delta$ for any fixed constant $A$, provided $C_\varepsilon$ is large enough.[4] We take $A$ to satisfy $A \geq 100$, $3(\log A)^{-\nu} < 1$ and $(\log A)^{\nu\varepsilon - p} \geq (2B)^{p/q} C_0^p 3^\varepsilon$ ($C_0$ is the constant in Lemma 3.1), and determine $C_\varepsilon$ by the following requirements: (23) should hold when $\rho \leq 3A\delta$, and $C_\varepsilon \geq 2^{(1+1/m)/2} C_\nu \sup_{t > A} t^{-\varepsilon}(\log t)^\nu$ where $C_\nu$ is the constant in (13).

In proving (23) we may suppose that $\rho \geq A\delta$ and (by the second requirement on A) that (23) has already been proved for parameters $\bar{\rho} \leq 3\rho(\log \delta/\rho)^{-\nu}$. We consider two cases: 1) There are at least $M/2$ values of $j$ for which ii) of Lemma 3.1 holds; 2) There are at least $M/2$ values of $j$ for which ii) fails.

In case 1) we simply apply (13) after deleting those $\ell_j$ for which

---

[4]   In fact, if $\rho \leq A\delta$ then the $\delta/\rho$-separation property implies an upper bound on $M$ so that (23) follows from the obvious inequality $|E| \gtrsim \delta^d \lambda$.

ii) fails. Thus

$$\rho^{-d}|E| \geq C_\nu^{-1}\lambda^2(\rho^{-1}\delta)^{d-(m+3)/2}\left(\frac{M}{2}\left(\frac{\delta}{\rho}\right)^m\right)^{(1+1/m)/2}\left(\log\frac{\delta}{\rho}\right)^{-d\nu}$$

$$\geq C_\varepsilon^{-1}\lambda^2(\rho^{-1}\delta)^{d-(m+3)/2}\left(M\left(\frac{\delta}{\rho}\right)^m\right)^{(1+1/m)/2}\left(\frac{\delta}{\rho}\right)^\varepsilon,$$

where the last inequality holds by the second requirement on $C_\varepsilon$. Note that $(1+1/m)/2$ is identical with $p/q$. Also $p \geq 2$ and we may of course assume that $\lambda \leq 1$, so we may replace $\lambda^2$ by $\lambda^p$. Thus we obtain (23).

In case 2) we define $\overline{\rho} = 3\rho\,(\log\delta/\rho)^{-\nu}$, $\overline{\lambda} = \lambda\,(\log\delta/\rho)^{\nu-1}/3C_0$. We also drop the values of $j$ for which ii) holds and choose a maximal $\delta/\overline{\rho}$-separated subset of the remaining $\{\alpha_j\}$; this sequence will still be denoted $\{\alpha_j\}$, and has cardinality greater or equal than

$$\overline{M} \stackrel{\text{def}}{=} (2B)^{-1}(\overline{\rho}/\rho)^m M\,.$$

We claim that for each $j$ there is a tube $T_{\ell_j}^{\overline{\rho}\delta}$ such that $|E \cap T_{\ell_j}^{\overline{\rho}\delta}| \geq \overline{\lambda}\,|T_{\ell_j}^{\overline{\rho}\delta}|$. Namely, by the assumption that ii) fails there is a disc $D(x,\overline{\rho}/3)$ with $|E \cap T_{\ell_j}^{\rho\delta} \cap D(x,\overline{\rho}/3)| \geq C_0^{-1}\lambda\,|T_{\ell_j}^{\rho\delta}|/\log(\rho/\delta)$. It is easy to see that $T_{\ell_j}^{\rho\delta} \cap D(x,\overline{\rho}/3)$ is contained in a tube of the form $T_{\ell_j}^{\overline{\rho}\delta}$, and with this $T_{\ell_j}^{\overline{\rho}\delta}$ we have

$$|T_{\ell_j}^{\overline{\rho}\delta} \cap E| \geq C_0^{-1}\,\frac{\lambda}{\log\dfrac{\delta}{\rho}}\,|T_{\ell_j}^{\rho\delta}| = \overline{\lambda}\,|T_{\ell_j}^{\overline{\rho}\delta}|,$$

proving the claim. By the inductive assumption

$$\overline{\rho}^{-d}|E| \geq C_\varepsilon^{-1}\left(\overline{M}\left(\frac{\delta}{\overline{\rho}}\right)^m\right)^{p/q}\left(\frac{\delta}{\overline{\rho}}\right)^{d-p}\overline{\lambda}^p\left(\frac{\delta}{\overline{\rho}}\right)^\varepsilon.$$

A calculation shows this is equivalent with

$$\rho^{-d}|E| \geq (2B)^{-p/q}C_0^{-p}\,3^{-\varepsilon}\left(\log\frac{\delta}{\rho}\right)^{\nu\varepsilon-p}C_\varepsilon^{-1}$$

$$\cdot\left(M\left(\frac{\delta}{\rho}\right)^m\right)^{p/q}\left(\frac{\delta}{\rho}\right)^{d-p}\lambda^p\left(\frac{\delta}{\rho}\right)^\varepsilon.$$

Since $\rho/\delta \geq A$, the choice of $A$ implies that the factor

$$(2B)^{-p/q}C_0^{-p}3^{-\varepsilon}(\log\delta/\rho)^{\nu\varepsilon-p}$$

appearing here is greater or equal than 1 and can be dropped. So we obtain (23).

## 5. Further remarks.

This section will consist of negative results, primarily the following.

**Proposition 5.1.** *Assume $d \geq 3$, let $U$ be an open subset of $M(d, 1)$, $\mathcal{A}$ a $d - 1$-manifold, $F : U \to \mathcal{A}$ a submersion. Assume that there is $\alpha \in \mathcal{A}$ such that the following hold: $F^{-1}\alpha$ is connected, there is no vector $e \in \mathbb{P}^{d-1}$ such that $e_\ell = e$ for all $\ell \in F^{-1}\alpha$, and there is no point $p \in \mathbb{R}^d$ such that $p \in \ell$ for all $\ell \in F^{-1}\alpha$. Then property $(*)$ fails, and furthermore estimate $(1)$ fails for the maximal function defined by $(5)$ and the subsequent remark if $(d - 3)/q < (p - 2)/p$.*

REMARKS. 1) The relationship $(d - 3)/q < (p - 2)/p$ is satisfied if $p > (d + 1)/2$ and $q = (d - 1)p'$, so this shows in particular that $(1)$ cannot hold for the full range of $q$ for any $p > (d + 1)/2$. If $d = 3$, then $(1)$ cannot hold for any $q$ if $p > 2$.

2) Proposition 5.1 and its proof are related to the counterexamples in Bourgain [2].

3) If $\mathcal{A}$ is an $m$-manifold with $m > d - 1$ then property $(*)$ always fails. This follows from an abbreviated version of the proof below. When $m < d - 1$ there are various examples where $(*)$ holds although it still fails generically.

The proof is easy, but it is convenient to split it into several lemmas. If $(s, t) \in \mathbb{P}^1$ then we define $Y_{st} \subset \mathbb{R}^n \times \mathbb{R}^n$ by $Y_{st} = \{(sx, tx) : x \in \mathbb{R}^n\}$, and if $x \in \mathbb{R}^n$ then we define $E_x \subset \mathbb{R}^n \times \mathbb{R}^n$ by $E_x = \text{span}\{(x, 0), (0, x)\}$.

**Lemma 5.1.** *Suppose $Y$ is an $n$-dimensional subspace of $\mathbb{R}^n \times \mathbb{R}^n$, $n \geq 2$. Then either*

  i) *$Y = Y_{st}$ for some $(s, t) \in \mathbb{P}^1$, or else*

  ii) *There is $x \in \mathbb{R}^n \backslash \{0\}$ such that $E_x \cap Y = \{0\}$.*

PROOF. Let $P_1$ and $P_2$ be the projections of $Y \subset \mathbb{R}^n \times \mathbb{R}^n$ on $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ respectively. If ii) fails then it is clear that every $x \in \mathbb{R}^n$ is in the range of either $P_1$ or $P_2$. By linearity it follows that either $P_1$ or $P_2$ is onto, and therefore also 1-1. We may assume then that $P_1$ is 1-1 and onto. Since ii) fails and $P_1$ is 1-1 it follows that for every $x \in \mathbb{R}^n$ there is $\tau(x) \in \mathbb{R}$ such that $(x, \tau(x) x) \in Y$. The map $x \to (x, \tau(x) x)$ is a right inverse for $P_1$, hence linear, which implies $\tau$ is constant, *i.e.* $Y = Y_{1t}$ for some $t$.

From Lemma 5.1 we obtain a similar statement for submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$ which we now formulate. If $M$ is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ and $y \in M$ then $T_y M$ is the tangent space to $M$ at $y$ which we may of course identify with a subspace of $\mathbb{R}^n \times \mathbb{R}^n$ .

**Lemma 5.2.** *Suppose $\Omega$ is an open set in $\mathbb{R}^n \times \mathbb{R}^n$, $n \geq 2$, and $M$ is a (connected) $n$-dimensional submanifold of $\Omega$. Then either*

i) *There are $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(s, t) \in \mathbb{R} \times \mathbb{R}$, $(s, t) \neq (0, 0)$ such that $M$ is the affine space $\{(a + sx, b + tx) : x \in \mathbb{R}^n\}$, or else*

ii) *There are $y \in M$ and $v \in \mathbb{R}^n$ such that $T_y M \cap E_v = \{0\}$ .*

PROOF. We assume that ii) does not hold, and will show that then i) holds. Since ii) fails we know by Lemma 5.1 that for each $y \in M$ there is $(s(y), t(y)) \in \mathbb{P}^1$ such that $T_y M = Y_{s(y)t(y)}$. The map $y \to (s(y), t(y))$ is continuous from $M$ into $\mathbb{P}^1$ so since the question is local and is symmetric with respect to $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ we may assume it does not take the value $(0, 1)$. Then locally $M$ is a graph over $\mathbb{R}^n \times \{0\}$, $M = \{(x, F(x)) : x \in U\}$, where $U \subset \mathbb{R}^n$, and furthermore $DF(x) = \tau(x) \cdot$ identity, where $\tau(x) = t(y)/s(y)$, $y = (x, F(x))$. It is well known (and easy to prove using equality of mixed second partials) that this property of $F$ implies $\tau$ is constant, *i.e.* $F$ has the form $F(x) = b + \tau x$ with $\tau \in \mathbb{R}$ and $b \in \mathbb{R}^n$. Thus i) holds, locally in a neighborhood of every point of $M$. The set where i) holds for a given $a, b, s, t$ is then open-closed so a connectedness argument completes the proof.

**Lemma 5.3.** *Assume $\Omega \subset M(d, 1)$ is open and $M$ is a connected $(d - 1)$-dimensional submanifold of $\Omega$. Then either*

i) *there is $e \in \mathbb{P}^{d-1}$ such that $e_\ell = e$ for all $\ell \in M$, or*

ii) *there is $p \in \mathbb{R}^d$ such that $p \in \ell$ for all $\ell \in M$, or*

iii) *there are $\ell \in M$ and a 2-plane $\Pi$ with $\ell \subset \Pi$ such that $T_\ell M(\Pi, 1) \cap T_\ell M = \{0\}$ .*

PROOF. We let $\ell_0$ be the $x_d$ axis, and may assume that $\ell_0 \in M$. It suffices to prove that either i), ii) or iii) holds in some neighborhood of $\ell_0$. Introduce local coordinates near $\ell_0$ on $M(d, 1)$ as follows: any line close to $\ell_0$ is uniquely $\{(x, 0) + t(y, 1) : t \in \mathbb{R}\}$, with $x \in \mathbb{R}^{d-1}$, $y \in$

$\mathbb{R}^{d-1}$, and our coordinate system is defined by

$$\ell \to (x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}.$$

We will apply Lemma 5.2 in these coordinates with $n = d-1$. First we note the following: if $\ell \in M(d,1)$ is a line close to $\ell_0$ with coordinates $(x,y)$ and if $v \in \mathbb{R}^{d-1}$, then $v$ determines a 2-plane $\Pi_v$ containing $\ell$, namely the 2-plane $\Pi_v = \{(x + tv + sy, s) : s,t \in \mathbb{R} \subset \mathbb{R}^{d-1} \times \mathbb{R} \approx \mathbb{R}^d$. It is clear that $M(\Pi_v, 1)$ consists of all lines with coordinates $(x+tv, y+sv)$, where $s,t$ are arbitrary real numbers, and therefore the tangent space to $M(\Pi_v, 1)$ at $\ell$ coincides (in our local coordinates) with the space $E_v$. We therefore conclude by Lemma 5.2 that either iii) of Lemma 5.3 holds, or else there are $s,t \in \mathbb{R}$, $(s,t) \neq (0,0)$, and $a \in \mathbb{R}^{d-1}$, $b \in \mathbb{R}^{d-1}$ such that $M$ is all lines with coordinates $(a + sx, b + tx)$, $x \in \mathbb{R}^{d-1}$. If the latter possibility and if $t = 0$, then i) of Lemma 5.3 holds with $e = (b,1) \in \mathbb{P}^{d-1}$. On the other hand if $t \neq 0$ then ii) of Lemma 5.3 holds with $p = (a - sb/t, -s/t) \in \mathbb{R}^{d-1} \times \mathbb{R} \approx \mathbb{R}^d$.

PROOF OF PROPOSITION 5.1. We know by Lemma 5.3 that there are $\ell \in F^{-1}\alpha \cap U$ and a 2-plane $\Pi \supset \ell$ with $T_\ell(F^{-1}\alpha) \cap T_\ell M(\Pi, 1) = \{0\}$. It follows that the restriction of $F$ to $M(\Pi, 1)$ is an immersion in a neighborhood of $\ell$, and therefore $F$ maps a $\delta$-neighborhood of $M(\Pi, 1)$ onto a set $S_\delta$ containing a $C^{-1}\delta$-neighborhood of a surface $S$. This shows in the first place that property (*) fails, let $\{\alpha_j\}$ be a maximal $\delta$-separated subset of $S$. Next let $K \subset \mathbb{R}^d$ be a sufficiently large fixed compact set and let $\chi_\delta$ be the characteristic function of the intersection of $K$ with a $\delta$-neighborhood of $\Pi$. Then $\|\chi_\delta\|_p \approx \delta^{(d-2)/p}$. On the other hand, $M_\delta\chi_\delta(\alpha) \geq C^{-1}\alpha$ if $\alpha \in S_\delta$. So $\|M_\delta\chi_\delta\|_q \gtrsim \delta^{(d-3)/q}$, whence (1) fails if $(d-3)/q < (d-2)/p - (d/p-1)$, i.e. if $(d-3)/q < (p-2)/p$.

FINAL REMARKS. 1) The results in this paper can of course be applied to oscillatory integrals. We have nothing serious to say in this connection and will just record what follows by plugging Theorem 1 into the numerology in sections 4 and 5 of [3]. Namely, the Bochner-Riesz means are bounded on $L^p$ for the optimal range of parameters provided that

$$p \in \left( \frac{2\,(d^2 + 3d + 3)}{d^2 + 5d + 7}, \frac{2\,(d^2 + 3d + 3)}{d^2 + d - 1} \right),$$

and furthermore the adjoint of the restriction operator maps $L^p$ to $L^p$ if $p > 2(d^2 + 3d + 3)/(d^2 + d - 1)$.

2) Proposition 5.1 suggests (as does [2]) that it should not be possible to prove Theorem 1 via space-time estimates for the x-ray transform, *i.e.* estimates for the x-ray transform from $L^p(\mathbb{R}^d)$ functions with fixed compact support to $W^{q\alpha}_{\text{loc}}(M(d,1))$ where $W^{q\alpha}$ is the Sobolev space with $\alpha$ derivatives in $L^q$, since such estimates do not distinguish situations where property (∗) is satisfied. Compare [8], where such estimates are used in the context of the circular maximal function, as well as [5]. We now sketch a concrete proof of this fact when $p = q$.

**Proposition.** *If the x-ray transform R maps*

$$L^q_{\text{comp}}(\mathbb{R}^d) \quad to \quad W^{q\alpha}_{\text{loc}}(M(d,1))$$

*then* $\alpha \leq 1/q$.

We are concerned with the case $q \geq 2$ here. Conversely, if $q \geq 2$ and $\alpha \leq 1/q$ then it is known that $R$ maps $L^q_{\text{comp}}$ to $W^{q\alpha}_{\text{loc}}$. This follows *e.g.* from formula (4.36) of [1] (if $q = 2$, which suffices since the $q = \infty$ case is trivial), or alternatively as pointed out in [10] from corresponding results [9] for general Fourier integral operators. As indicated by the proposition no improvement is possible.

We now sketch the proof of the proposition. Let $E = \{(\overline{x}, \overline{\overline{x}}) \in \mathbb{R}^{d-1} \times \mathbb{R} : |\overline{x}| \leq 1, |\overline{\overline{x}}| \leq \delta\}$, where $\delta$ is small. Let $X$ be the set of lines in $M(d,1)$ which make an angle less or equal than $\delta$ with $\mathbb{R}^{d-1} \times \{0\}$ and intersect $D(0, 1/2) \times \{0\} \subset \mathbb{R}^{d-1} \times \mathbb{R}$, and let $Y$ be the set of lines obtained by translating the lines in $X$ by $100\,\delta$ in the direction of the positive $x_d$ axis. Then it is clear that $R\chi_E \geq$ constant $> 0$ on $X$, whereas $R\chi_E = 0$ on $Y$. In suitable local coordinates $Y$ is a translate of $X$ by an amount $\approx \delta$. This implies a lower bound: const $\delta^{-\alpha} |X|^{1/q}$ for the $W^{q,\alpha}$ Sobolev norm of $R\chi_E$. Since $|X| \approx \delta^2$ and $\|\chi_E\|_q = |E|^{1/q} \approx \delta^{1/q}$ we obtain the proposition.

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