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Wavelets obtained by continuous deformations of the Haar wavelet

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Abstract. One might obtain the impression, from the wavelet literature, that the class of orthogonal wavelets is divided into subclasses, like compactly supported ones on one side, band-limited ones on the other side. The main purpose of this work is to show that, in fact, the class of low-pass filters associated with "reasonable" (in the localization sense, not necessarily in the smooth sense) wavelets can be considered to be an infinite dimensional manifold that is arcwise connected. In particular, we show that any such wavelet can be connected in this way to the Haar wavelet.

0. Introduction.

The aim of this paper is to show that, in some sense, any "localized", or of "polynomial decrease" (see below) wavelet may be obtained by a continuous deformation from the Haar function. The case of compactly supported wavelets is due to P. G. Lemarié-Rieusset and G. Malgouyres [6]. More precisely, we shall consider those wavelets which are obtained from a multiresolution analysis (MRA). Let us recall that an MRA is given by an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$, whose union is dense in $L^2(\mathbb{R})$. The space V_{j+1} is obtained from V_j by a dilation by 2; that is, $f \in V_{j+1}$ if and only if $f(2^{-1}x) \in V_j$. One also assumes that there exists a function φ such that $\{\varphi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_0. This function φ is generaly called a *scaling function* or a *father wavelet.*

From $V_0 \subset V_1$, we have

$$\varphi(x) = 2\sum_{k\in\mathbb{Z}} c_k\, \varphi(2x-k)\,. \tag{1}$$

That is to say, in terms of the Fourier transform,

$$\hat{\varphi}(\xi) = m_0\big(\frac{\xi}{2}\big)\,\hat{\varphi}\big(\frac{\xi}{2}\big)\,, \tag{2}$$

with

$$m_0(\xi) = \sum_{k\in\mathbb{Z}} c_k\, e^{-ik\xi}\,. \tag{3}$$

This 2π-periodic function m_0 is called the *low-pass filter* associated with this MRA and satisfies the basic properties $m_0(0) = 1$ and

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1\,. \tag{4}$$

It is then easy to see that one can construct a 2π-periodic function m_1 such that

$$U(\xi) = \begin{pmatrix} m_0(\xi) & m_1(\xi) \\ m_0(\xi+\pi) & m_1(\xi+\pi) \end{pmatrix} \tag{5}$$

is an unitary matrix. The choice of m_1 is closely related to the construction of an orthonormal (or mother) wavelet: one can define $\psi \in L^2(\mathbb{R})$ by letting

$$\hat{\psi}(\xi) = m_1\big(\frac{\xi}{2}\big)\,\hat{\varphi}\big(\frac{\xi}{2}\big)\,, \tag{6}$$

in such a way that $\psi_{jk}(x) = 2^{j/2}\psi(2^jx+k)$ is an orthonormal basis of $L^2(\mathbb{R})$.

Let us remark that the choice of m_1 is not unique. The fact that $U(\xi)$ is unitary implies that any other $\tilde{m}_1$ is given by $\tilde{m}_1(\xi) = a(\xi)\,m_1(\xi)$, where a is π-periodic with values of modulus 1. For example, we can take $m_1(\xi) = e^{i\xi}\,\bar{m}_0(\xi+\pi)$.

The construction of an MRA, and the associated orthonormal wavelet basis, can also be done in terms of m_0 (if it satisfies appropriate properties). We define φ from m_0 by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi) . \tag{7}$$

I. Daubechies used this equality to construct her compactly supported wavelets. A characterization of the filters m_0 which generate an MRA has been obtained by A. Cohen in [1] (see Theorem 1.2 below). The basic question is: what property, in addition to (4), must m_0 satisfy to give us an MRA.

In order to understand such characterizations, we make the following remarks. Perhaps the "simplest" low-pass filter is the one associated with the Haar system: $m_0(\xi) = (1 + e^{i\xi})/2$. Clearly, $m_0(0) = 1$ and (4) is satisfied. Another simple function satisfying these properties is $m_0(\xi) = (1 + e^{i3\xi})/2$; but a simple calculation shows that (7), in this case, gives us, $\varphi = (1/3)\chi_{[-3,0]}$ for which $\{\varphi(x-k) : k \in \mathbb{Z}\}$ is not an orthonormal system. It is known, for example that if $m_0(\xi)$ is never 0 in $[-\pi/3, \pi/3]$, then (3) does give us a scaling function that generates a localized MRA. A. Cohen, in his thesis, shows that this is included in a characterization of these low-pass filter that we announce in condition 2.b) in Theorem 1.2 below. One of our aims is to show that if we rephrase this condition, then the set of functions m_0 may be seen as consisting of a "manifold".

More precisely, in this paper, we show that the set $\mathcal{E}$ of the C^∞ filters m_0, is a "connected manifold" in the Frechet space $C^\infty(\mathbb{T})$ of 2π-periodic functions, defined by the family of semi-norms $\|D^\alpha f\|_\infty$ ($\alpha = 0, 1, \dots$). In particular, we construct a continuous path in $\mathcal{E}$, connecting any element of $\mathcal{E}$ to the Haar filter $(1+e^{i\xi})/2$. This gives us a continuous "deformation" between any mother wavelet ψ with polynomial decay, and the Haar wavelet $h = \chi_{[0,1/2]} - \chi_{[1/2,1]}$. That is to say, we obtain a continuous function $t \mapsto \psi_t$, from $[0, 1]$ to $L^2((1 + |x|)^n dx)$ for any n, such that ψ_t is a wavelet, $\psi_0 = h$ and $\psi_1 = \psi$.

1. Characterization of the low-pass filter.

We start with the following definitions:

Definition 1.1. *We say that φ has polynomial decay in $L^2(\mathbb{R})$ if $|x|^N \varphi \in L^2(\mathbb{R})$ for all $N \in \mathbb{N}$, and that φ has exponential decay of order λ in $L^2(\mathbb{R})$ if there exists $\lambda > 0$ so that $e^{\lambda|x|}\varphi \in L^2(\mathbb{R})$.*

Our main interest is to study the existence of a scaling function φ, associated with a given low-pass filter m_0, that generates a multiresolution analysis (MRA). We quote a result in [1]:

Theorem 1.2. *Suppose $\varphi \in L^2(\mathbb{R})$ and m_0, 2π-periodic, are related as in equalities (2) and (7). Then the following properties are equivalent:*

1) *The function φ is the scaling function of an* MRA *and has polynomial decay in $L^2(\mathbb{R})$.*

2) *The function m_0 belongs to $C^\infty(\mathbb{T})$ and satisfies*

a) $m_0(0) = 1$ *and* $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$,

b) *There exists a finite union of closed bounded intervals K such that $0 \in K^\circ$, $\sum_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = 1$ almost everywhere and, for all $j \in \mathbb{N}$, $\xi \in K$, we have $m_0(2^{-j}\xi) \neq 0$.*

Using (7), $m_0(0) = 1$ and $m_0 \in C^\infty(\mathbb{T})$, we observe that this last inequality is equivalent to $\hat{\varphi}(\xi) \neq 0$ for all $\xi \in K$. Thus we can restate this theorem by changing condition b) to

b') *There exists a finite union of closed bounded intervals K such that $0 \in K^\circ$, $\sum_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = 1$ almost everywhere and, for all $\xi \in K$, $\hat{\varphi}(\xi) \neq 0$.*

Moreover, we add two further conclusions:

Theorem 1.3. *With φ and m_0 related as above and satisfying one of the equivalent properties of Theorem 1.2, we have:*

i) *The support of φ is in $[0, N]$ if and only if m_0 is a trigonometric polynomial of degree $\leq N$.*

ii) *The function φ has exponential decay in $L^2(\mathbb{R})$ if and only if m_0, regarded as a function of $e^{i\xi}$ (on the unit circle), extends to a holomorphic function on an annulus (i.e. a region lying between two concentric circles centered at 0, including the unit circle).*

Property i) was proved by I. Daubechies in [2].

Property ii) is implicit in the theory of wavelets. We shall present an argument for completeness. Let us first assume that m_0 extends to a holomorphic function on an annulus. By the Cauchy formula, we have $\|m_0^{(l)}\|_\infty \le l!\,\tilde{M}^l$ for some constant $\tilde{M}$ depending on the size of the annulus (see [8]). From this we can deduce $\|\hat{\varphi}^{(n)}\|_2 \le n!\,M^n$. To see this, we write

$$\varphi^{(n)} = \sum_{J\in\mathbb{N}^n} 2^{-J} h_J ,$$

where $J = (j_1, \ldots, j_n)$, $2^{-J} = 2^{-j_1} \cdots 2^{-j_n}$ and the h_J are obtained by differenting the identity $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j)$ (and replacing $m_0(\xi/2^j)$ by $m_0^{(l)}(\xi/2^j)$ when j occurs l times in the sequence $j_1, \ldots, j_n$). Then, for $0 \le t < 1/\sqrt{M}$, we have

$$\begin{aligned} C\int \Big(\sum_{n=1}^{\infty} \frac{t^{2n}\,|x|^{2n}}{(2n)!}\Big)\, |\varphi(x)|^2\, dx &= \int \Big(\sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\Big)\, |\hat{\varphi}^{(n)}(\xi)|^2\, d\xi \\ &\le \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}\, t^{2n}\, M^n < \infty . \end{aligned}$$

From this it is then easy to check that

$$\begin{aligned} \int e^{t|x|}\, |\varphi(x)|^2\, dx &= \int \Big(\sum_{k=0}^{\infty} \frac{(t\,|x|)^k}{k!}\Big)\, |\varphi(x)|^2\, dx \\ &\le C\Big(\int \Big(\sum_{n=1}^{\infty} \frac{(t\,|x|)^{2n}}{(2n)!}\Big)\, |\varphi(x)|^2\, dx + \|\varphi\|_2^2\Big) < \infty . \end{aligned}$$

We thus obtain an exponential decay of order λ in $L^2(\mathbb{R})$, for φ, whenever $0 < \lambda < 1/\sqrt{M}$.

On the other hand, when φ has such an exponential decay, we have (from the definition of m_0), for $k \ge 0$

$$\begin{aligned} |e^{\lambda k}\, \hat{m}_0(k)| &= \Big| e^{\lambda k}\, \frac{1}{2} \int \varphi\big(\frac{x}{2}\big)\, \bar{\varphi}(x-k)\, dx \Big| \\ &= \Big| \frac{1}{2} \int e^{-\lambda x}\, \varphi\big(\frac{x}{2}\big)\, e^{\lambda(x+k)}\, \bar{\varphi}(x+k)\, dx \Big| , \end{aligned}$$

as well as a similar equality, for $k < 0$. Thus, the sequence $\hat{m}_0(k)$ has exponential decay $|e^{\lambda|k|}\, \hat{m}_0(k)| \le A < \infty$ since $\int e^{\lambda|x|}\, |\varphi(x)|^2\, dx < 0$.

This implies that m_0 is the restriction to the unit circle of a holomorphic function on an annulus about the origin.

2. A geometric interpretation.

We shall consider the Frechet space $C^\infty(\mathbb{T})$ of 2π-periodic functions endowed with the topology generated by the semi-norms $\|F^{(n)}\|_\infty$. We also consider the space $C^\infty(\mathbb{T}/2)$ of all π-periodic functions endowed with the same semi-norms. Let us define $\mathcal{E}$ by

$$\mathcal{E} = \{F \in C^\infty(\mathbb{T}) : \ F \text{ satisfies a) and b)}\} .$$

We shall show the following.

Theorem 2.1. *The set $\mathcal{E}$ is a Frechet manifold in the sense that each $m_0 \in \mathcal{E}$ has a neighborhood that is homeomorphic to a neighborhood of 0 in $C^\infty(\mathbb{T})$, where here 0 is the constant zero function.*

To prove the theorem we define the set

$$\mathcal{F} = \{F \in C^\infty(\mathbb{T}) : \ F \text{ satisfies a)}\} .$$

We shall show that $\mathcal{F}$ is a manifold in this sense and that $\mathcal{E}$ is an open set in $\mathcal{F}$.

Let $m_0 \in \mathcal{E}$, and let $m_1 \in C^\infty(\mathbb{T})$ be such that

$$U(\xi) = \begin{pmatrix} m_0(\xi) & m_1(\xi) \\ m_0(\xi+\pi) & m_1(\xi+\pi) \end{pmatrix}$$

is a unitary matrix. We shall use the elementary lemma:

Lemma 2.2. *Any C^∞ and 2π-periodic function F may be written uniquely*

$$F = G\, m_0 + H\, m_1 ,$$

where G and H are C^∞ and π-periodic. Moreover $F \mapsto (G, H)$ is an isomorphism between $C^\infty(\mathbb{T})$ and $C^\infty(\mathbb{T}/2) \times C^\infty(\mathbb{T}/2)$.

To prove the lemma, it is sufficient to remark that G and H are given by

$$\begin{pmatrix} G(\xi) \\ H(\xi) \end{pmatrix} = U(\xi)^{-1} \begin{pmatrix} F(\xi) \\ F(\xi+\pi) \end{pmatrix} , \tag{8}$$

since U^{-1} has C^∞ coefficients.

Moreover, U being unitary,

$$|F(\xi)|^2 + |F(\xi+\pi)|^2 = |G(\xi)|^2 + |H(\xi)|^2 ,$$

so that condition a) for F becomes

a') $G(0) = 1$, $H(0) = 0$, $|G|^2 + |H|^2 = 1$.

From (4), we see that

$$\begin{pmatrix} G(\xi) - 1 \\ H(\xi) - 0 \end{pmatrix} = U(\xi)^{-1} \begin{pmatrix} F(\xi) - m_0(\xi) \\ F(\xi+\pi) - m_0(\xi+\pi) \end{pmatrix} .$$

Hence, if F is close to m_0 in $C^\infty(\mathbb{T})$, then G is close to 1 and H is close to 0 in the topology we introduced. Then, to show that $\mathcal{F}$ is a manifold, it is sufficient to show that a neighborhood of $(1,0)$ in $\mathcal{F}' = \{(G,H) \in C^\infty(\mathbb{T}/2) \times C^\infty(\mathbb{T}/2) : (G,H) \text{ satisfies a')}\}$ is homeomorphic to a neighborhood of $(0,0)$ in $C^\infty(\mathbb{T}/2) \times C^\infty(\mathbb{T}/2)$ (that is homeomorphic to a neighborhood of 0 in $C^\infty(\mathbb{T})$, by Lemma 2.2. We can take the neighborhood given by $\|H\|_\infty < 1/2$ and $\|1-G\|_\infty \le 1/2$. Then, clearly, $G = e^{iA}\sqrt{1-|H|^2}$, where A is C^∞, π-periodic with values in $[-\pi/2, \pi/2]$, $A(0) = 0$, and the application $(F,G) \mapsto (A,H)$ is a continuous bijection. This shows that $\mathcal{F}$ is a manifold in the sense we described above.

Let us prove that $\mathcal{E}$ is an open set in $\mathcal{F}$. We have to prove that if $m_0 \in \mathcal{E}$ then for F close to m_0, the scaling function $\hat{\varphi}_F$ which corresponds to F satisfies a condition equivalent to b).

We shall use the following lemma that will be proved later.

Lemma 2.3. *If F tends to m_0 in $\mathcal{F}$, then $\hat{\varphi}_F$ tends to $\hat{\varphi}$ uniformly on compacts.*

From b'), we have $|\hat{\varphi}(\xi)| \ge C > 0$ when $\xi \in K$, since the latter is compact and $\hat{\varphi}$ is continuous. The fact that φ_F tends to φ uniformly on K permits us to conclude that $|\hat{\varphi}_F(\xi)| \ge C/2$ on the same compact set, for F close enough to m_0. This shows that $\mathcal{E}$ is open in $\mathcal{F}$.

Let us prove the last lemma now. In fact, we are going to prove a more powerful property that we shall use later.

Proposition 2.4. *If F tends to m_0 in $\mathcal{F}$, then $\hat{\varphi}_F$ and its derivatives tend uniformly to $\hat{\varphi}$ and its derivatives on compacts. Moreover if $m_0 \in \mathcal{E}$, the convergence of F towards m_0 in $\mathcal{E}$ is equivalent to the convergence of $x^n\varphi_F$ toward $x^n\varphi$ in $L^2(\mathbb{R})$ for each n.*

A version of this proposition has been obtained, by a different method, by P.G. Lemarié-Rieusset.

First, to prove the uniform convergence on compacts, it is sufficient to prove it on an interval $[-a,a]$ on which $|m_0(\xi)-1| < 1/2$. We see this since it gives us the convergence on $[-2^N a, 2^N a]$ for each N, by using $\hat{\varphi}_F(\xi) = \prod_{j=1}^N F(\xi/2^j)\,\hat{\varphi}_F(\xi/2^N)$. We can also assume that $\|F-m_0\|_\infty < 1/4$, so that the logarithms in the sequel are well defined and belong to C^∞.

Let us first prove that $\hat{\varphi}_F$ tends to $\hat{\varphi}$ uniformly on $[-a,a]$. We prove this by showing $\hat{\varphi}_F/\hat{\varphi} \to 1$ or $\log(\hat{\varphi}_F/\hat{\varphi}) \to 0$ uniformly on $[-a,a]$; that is, using (7), we show

$$\log \frac{\hat{\varphi}_F(\xi)}{\hat{\varphi}(\xi)} = \sum_{j=1}^{\infty} \log \frac{F(\xi/2^j)}{m_0(\xi/2^j)} \to 0$$

as $F \to m_0$ when $\xi \in [-a,a]$. But, by mean value theorem,

$$\left|\log \frac{F(\xi/2^j)}{m_0(\xi/2^j)}\right| \le \frac{|\xi|}{2^j} \sup_{[-a,a]} \left|\frac{F'}{F} - \frac{m_0'}{m_0}\right|.$$

Hence, we have

$$\left|\sum_{j=1}^{\infty} \log \frac{F(\xi/2^j)}{m_0(\xi/2^j)}\right| \le a \sup_{[-a,a]} \left|\frac{F'}{F} - \frac{m_0'}{m_0}\right|,$$

which tends to zero as $\|F-m_0\|_\infty$ and $\|F'-m_0'\|_\infty$ tend to zero.

Let us now prove that $\hat{\varphi}_F^{(n)} \to \hat{\varphi}^{(n)}$ uniformly on $[-a,a]$, for $n>0$. It is equivalent to show that $(d/d\xi)^n(\hat{\varphi}_F/\hat{\varphi})$ tends uniformly to 0 on $[-a,a]$ since $\hat{\varphi}$ is bounded away from zero on $[-a,a]$. Thus, we consider

$$\sum_{j=1}^{\infty} \frac{1}{2^{nj}} \left((\log F)^{(n)}\left(\frac{\xi}{2^j}\right) - (\log m_0)^{(n)}\left(\frac{\xi}{2^j}\right)\right);$$

and, it is elementary to deduce that $(\log F)^{(n)} - (\log m_0)^{(n)}$ is uniformly small on $[-a,a]$ when $\|F^{(k)} - m_0^{(k)}\|_\infty$ is small for each $k \ge 0$, since our assumptions imply that the values assumed by F and m_0, and their derivatives, are appropriately restricted.

Let us show that if $m_0 \in \mathcal{E}$ then the convergence of φ_F to φ is in $L^2(\mathbb{R})$ (remember that $\int |\hat{\varphi}_F|^2 = \int |\hat{\varphi}|^2 = 2\pi$). Let K be a compact such that $\int_K |\hat{\varphi}|^2/2\pi > (1-\varepsilon)^2$. Then, if F is close enough to m_0, so that $\int_K |\hat{\varphi}_F - \hat{\varphi}|^2 < 2\pi\varepsilon^2$, then $\int_K |\hat{\varphi}_F|^2 > 2\pi(1-2\,\varepsilon)^2$, and $\int_{CK} |\hat{\varphi}_F|^2 < 2\pi(2\,\varepsilon)^2$. Finally, $\int |\hat{\varphi}_F - \hat{\varphi}|^2 < 2\pi(4\,\varepsilon)^2$.

Let us now prove the convergence in L^2 of the derivatives. We claim that if $m_0 \in \mathcal{F}$, then $\hat{\varphi}_F^{(n)} \to \hat{\varphi}^{(n)}$ weakly (in $L^2(\mathbb{R})$) for any n. Observe that it follows from the proof of Theorem 1.3.ii) that $\|\hat{\varphi}_F^{(n)}\|_2$ is bounded uniformly when F lies in a neighborhood of m_0. Hence, there exists a subsequence that converges weakly. Clearly the only possible limit is $\hat{\varphi}^{(n)}$. Since any sequence of F's converging to m_0 has a subsequence such that $\hat{\varphi}_F^{(n)}$ converge weakly to $\hat{\varphi}^{(n)}$, our claim follows.

As $\hat{\varphi}_F^{(2n)} \to \hat{\varphi}^{(2n)}$ weakly and $\hat{\varphi}_F \to \hat{\varphi}$ in $L^2(\mathbb{R})$, we have

$$\int \hat{\varphi}_F^{(2n)}\, \bar{\hat{\varphi}}_F \to \int \hat{\varphi}^{(2n)}\, \bar{\hat{\varphi}}\,.$$

Thus,

$$\int x^{2n}\, |\varphi_F(x)|^2\, dx \to \int x^{2n}\, |\varphi(x)|^2\, dx\,,$$

and

$$\int |\hat{\varphi}_F^{(n)}|^2 \to \int |\hat{\varphi}^{(n)}|^2\,.$$

Then let J be a compact set such that

$$\int_J |\hat{\varphi}^{(n)}|^2 > (1-\varepsilon)^2\, \|\hat{\varphi}^{(n)}\|^2\,.$$

If F is close enough to m_0 so that

$$\int_J |\hat{\varphi}_F^{(n)} - \hat{\varphi}^{(n)}|^2 < \varepsilon^2\, \|\hat{\varphi}^{(n)}\|_2$$

(since $\hat{\varphi}_F^{(n)}$ converges uniformly to $\hat{\varphi}^{(n)}$ on the compact set J) and

$$\big|\|\hat{\varphi}_F^{(n)}\|^2 - \|\hat{\varphi}^{(n)}\|^2\big| < \varepsilon^2 \|\hat{\varphi}^{(n)}\|^2\,,$$

then

$$\int_J |\hat{\varphi}_F^{(n)}|^2 \ge (1-2\,\varepsilon)^2 \|\hat{\varphi}^{(n)}\|^2\,, \qquad \int_{J^c} |\hat{\varphi}_F^{(n)}|^2 \le (3\,\varepsilon)^2\, \|\hat{\varphi}^{(n)}\|^2\,,$$

and

$$\int |\hat{\varphi}_F^{(n)} - \hat{\varphi}^{(n)}|^2 \le (5\,\varepsilon)^2\, \|\hat{\varphi}^{(n)}\|^2 .$$

Conversely, we now show that $m_0 \in \mathcal{E}$ and $x^n \varphi_F \to x^n \varphi$ in $L^2(\mathbb{R})$, for each n, implies $F \to m_0$ in C^∞. We have

$$\hat{m}_0(k) = \frac{1}{2} \int \varphi\big(\frac{x}{2}\big)\, \bar{\varphi}(x+k)\, dx . \tag{9}$$

Since $\hat{\varphi}_F$ tends to $\hat{\varphi}$ uniformly on compacts, we can assume that $\hat{\varphi}_F$ also satisfies condition b'). Thus, by Theorem 1.2, equality (9) is also true for F, and we have

$$\begin{aligned}|k|^n(\hat{m}_0(k) - \hat{F}(k)) &= \frac{|k|^n}{2} \int \Big(\varphi\big(\frac{x}{2}\big)\, \bar{\varphi}(x+k) - \varphi_F\big(\frac{x}{2}\big)\, \bar{\varphi}_F(x+k)\Big)\, dx \\ &= \frac{|k|^n}{2} \int \bar{\varphi}(x+k)\, \Big(\varphi\big(\frac{x}{2}\big) - \varphi_F\big(\frac{x}{2}\big)\Big)\, dx \\ &\quad + \frac{|k|^n}{2} \int \varphi_F\big(\frac{x}{2}\big)\, \big(\bar{\varphi}(x+k) - \bar{\varphi}_F(x+k)\big)\, dx .\end{aligned}$$

Let us majorize

$$\frac{|k|^n}{2} \int \varphi_F\big(\frac{x}{2}\big)\, \big(\bar{\varphi}(x+k) - \bar{\varphi}_F(x+k)\big)\, dx .$$

For any $x \in \mathbb{R}$,

$$|k|^n \le 2^n\, (|x|^n + |k+x|^n) .$$

But

$$\int |x|^{2n}\, \big|\varphi_F\big(\frac{x}{2}\big)\big|^2\, dx \le C < \infty$$

and

$$\int |\varphi(x+k) - \varphi_F(x+k)|^2\, dx \longrightarrow 0 .$$

Thus,

$$\int |x|^n\, \varphi_F\big(\frac{x}{2}\big)\, (\bar{\varphi}(x+k) - \bar{\varphi}_F(x+k))\, dx \longrightarrow 0 .$$

Moreover,

$$\int \big|\varphi_F\big(\frac{x}{2}\big)\big|^2\, dx \le C < \infty$$

and

$$\int |\varphi(x) - \varphi_F(x)|^2 \, |x|^{2n} \, dx \longrightarrow 0 .$$

Thus,

$$\int |k + x|^n \, \varphi_F\big(\frac{x}{2}\big) \, (\bar{\varphi}(x + k) - \bar{\varphi}_F(x + k)) \, dx \longrightarrow 0 .$$

We majorize

$$\frac{|k|^n}{2} \int \bar{\varphi}(x + k) \left(\varphi_F\big(\frac{x}{2}\big) - \varphi\big(\frac{x}{2}\big)\right) dx$$

in the same way and we obtain

$$\| \, |k|^n \hat{m}_0(k) - |k|^n \hat{F}(k) \|_\infty \longrightarrow 0 .$$

In particular, if $n \geq 2$, we obtain

$$| \, |k|^{n-2} \hat{m}_0(k) - |k|^{n-2} \hat{F}(k) \, | \leq |k|^{-2} \, o(1) .$$

Therefore,

$$\|m_0^{(n-2)} - F^{(n-2)}\|_\infty \leq \Big(\sum_{k \in \mathbb{Z}} |k|^{-2} \Big) \, o(1) .$$

EXAMPLE. let $\mathcal{F}_N$ be the set of polynomials with real coefficients of degree N that belong to $\mathcal{F}$; that is, $m \in \mathcal{F}_N$ if and only if $m(\xi) = a_0 + a_1 e^{i\xi} + \cdots + a_N e^{iN\xi}$, the a_j's are real and m satisfies a). Let $\mathcal{E}_N = \mathcal{E} \cap \mathcal{F}_N$.

Let us examine the example given in the end of the introduction in these terms. For $N = 3$, $\mathcal{F}_N$ consists of those m satisfying

$$m(\xi) = \frac{1 + e^{i\xi}}{2} \, (a + b \, e^{i\xi} + c \, e^{2i\xi}) ,$$

with $b = 1 - a - c$, and $a^2 + c^2 = a + c$. $\mathcal{E}_3$ corresponds to the circle

$$\{(a, c) : \; a^2 + c^2 = a + c\} \setminus \{(1, 1)\} .$$

If $(1, 1)$ were a point of this circle, the corresponding filter would be $m(\xi) = (1 + e^{3i\xi})/2$ and the corresponding scaling function would be $\varphi = (1/3) \, \chi_{[-3,0]}$. but the latter has L^2-norm $1/\sqrt{3}$ ($\neq 1$), thus, as observed before, we would not obtain an MRA.

3. The deformation of wavelets associated with the class $\mathcal{E}$.

Let us introduce a dense subset in $\mathcal{F}$ that will be useful to us: let $\mathcal{F}_{\exp}$ (respectively $\mathcal{E}_{\exp}$) be the set for which $\hat{m}_0(k)$ has exponential decay. Then we have the following:

Proposition 3.1. *$\mathcal{F}_{\text{exp}}$ is dense in $\mathcal{F}$.*

To prove the proposition, we take m_0 in $\mathcal{F}$ and select a sequence of trigonometric polynomials P_n which tends to m_0 in $C^\infty(\mathbb{T})$.

We can assume that, for all integer n, $P_n(0) = 1$ and $P_n(\pi) = 0$. Indeed, let $\tilde{m}_0(\xi) = ((1 + e^{i\xi})/2)^{-1} m_0(\xi)$. Since $m_0(\pi) = 0$, $\tilde{m}_0$ is well defined and C^∞. Build a sequence of trigonometric polynomials $\tilde{P}_n$ which tends to $\tilde{m}_0$. The sequence $P_n(\xi) = (1+e^{i\xi})(\tilde{P}_n(\xi)+1-\tilde{P}_n(0))/2$ tends to m_0 and satisfies the required properties.

Now $|P_n(\xi)|^2 + |P_n(\xi + \pi)|^2$ tends to 1 in $C^\infty(\mathbb{T})$. So $(|P_n(\xi)|^2 + |P_n(\xi + \pi)|^2)^{-1/2}$ is well defined for n big enough, and tends to 1 in $C^\infty(\mathbb{T})$. Finally we take

$$F_n(\xi) = \frac{P_n(\xi)}{(|P_n(\xi)|^2 + |P_n(\xi + \pi)|^2)^{1/2}} ,$$

which belongs to $\mathcal{F}$, and tends to m_0.

The only thing to prove is that $F_n \in \mathcal{F}_{\text{exp}}$. By the argument that establishes Theorem 1.3, part ii), it suffices to show that $(|P_n(\xi)|^2 + |P_n(\xi+\pi)|^2)^{-1/2}$ extends to a holomorphic function on an annulus. But $|P_n(\xi)|^2 + |P_n(\xi + \pi)|^2$ is a trigonometric polynomial. There exists an integer m such that $e^{im\xi}(|P_n(\xi)|^2 + |P_n(\xi + \pi)|^2)$ is a polynomial in $e^{i\xi}$ which extends to a holomorphic function on $\mathbb{C}$. We can then extend $|P_n(\xi)|^2 + |P_n(\xi + \pi)|^2$ to a holomorphic function f_n on $\mathbb{C} \setminus \{0\}$. And, since there exists a neighborhood of the unit circle on which $\operatorname{Re} f_n(z) > 1/2$ for n big enough, $f_n^{-1/2}$ is also holomorphic on that neighborhood.

Finally, let us prove the following:

Theorem 3.2. *$\mathcal{F}$ and $\mathcal{E}$ are connected.*

We shall prove that any m_0 in $\mathcal{F}$ can be joined to $(1 + e^{i\xi})/2$ by a continuous path, and that this path can be chosen within $\mathcal{E}$ if m_0 is in $\mathcal{E}$.

Let us first remark that, if we go back to the example discussed in Section 2, we can clearly see that there exists such a continuous path between $e^{i\xi}(1 + e^{i\xi})/2$ and $(1 + e^{i\xi})/2$ (just follow the circle indicated). Then, for any integer k, there exists a continuous path between $e^{i(k+1)\xi}(1 + e^{i\xi})/2$ and $e^{ik\xi}(1 + e^{i\xi})/2$, and consequently, between $e^{ik\xi}(1 + e^{i\xi})/2$ and $(1 + e^{i\xi})/2$. Thus, it suffices to join m_0 to $e^{ik\xi}(1 + e^{i\xi})/2$ for an appropriate integer k.

We begin by constructing a path made up of trigonometric polynomials (not necessarily in $\mathcal{F}$).

Lemma 3.3. *Let F be a trigonometric polynomial of degree less than or equal to N such that $F(0) = 1$ and $F(\pi) = 0$. There exists a continuous map $t \mapsto F_t$ from $[0,1]$ to the space of trigonometric polynomials of degree $\leq N$ such that*

1) $F_t(0) = 1,\ F_t(\pi) = 0,$ *if* $0 \leq t \leq 1$,

2) $F_1(\xi) = F(\xi)$,

3) $|F_t(\xi)|^2 = (1-t)\left(\frac{1+e^{i\xi}}{2}\right)\left(\frac{1+e^{-i\xi}}{2}\right) + t\,|F(\xi)|^2$.

Moreover, $F_0(\xi) = e^{ik\xi}\,(1+e^{i\xi})/2$ for an integer k.

We postpone the proof of this lemma and, using it, pass to the proof of Theorem 3.2. In the general case, since $\mathcal{F}$ is a manifold, one can join m_0 to a neighboring m belonging to the dense subset $\mathcal{F}_{\exp} \subset \mathcal{F}$; thus, we can assume that $m \in \mathcal{F}_{\exp}$. If we examine the proof of Proposition 3.1, in fact, we observe that we can assume

$$m(\xi) = \frac{F(\xi)}{(|F(\xi)|^2 + |F(\xi+\pi)|^2)^{1/2}},$$

where F is a trigonometric polynomial which satisfies $F(0) = 1$, $F(\pi) = 0$ and $|1 - |F(\xi)|^2 - |F(\xi+\pi)|^2| < 1/2$. We can then apply Lemma 3.3 to F. We obtain a continuous function $t \mapsto F_t$ with $F_0(\xi) = e^{ik\xi}(1+e^{i\xi})/2$ and $F_1(\xi) = m_0(\xi)$.

So let

$$G_t(\xi) = |F_t(\xi)|^2 + |F_t(\xi+\pi)|^2.$$

We also have $G_t(\xi) = 1 - t + t\,G(\xi)$, where $G(\xi) = |F(\xi)|^2 + |F(\xi+\pi)|^2$.

The path will join $e^{ik\xi}(1+e^{i\xi})/2$ to m_0, via

$$t \mapsto F_t(\xi)\,(G_t(\xi))^{-1/2} = \Phi_t(\xi).$$

It remains for us to show that $t \mapsto \Phi_t$ is continuous from $[0,1]$ to $C^\infty(\mathbb{T})$, and that, for all t, Φ_t belongs to $\mathcal{F}$, or to $\mathcal{E}$ if $m_0 \in \mathcal{E}$.

It is clear that Φ_t is well defined and satisfies a) (by the arguments in the proof of Proposition 3.1, we see that Φ_t is the restriction to the unit circle of a holomorphic function on an annulus). Since $t \mapsto G_t^{-1/2}$

is continuous from $[0,1]$ to the space of holomorphic functions on the annulus (in the L^∞ norm), it is continuous from $[0,1]$ to $C^\infty(\mathbb{T})$. So $t \mapsto \Phi_t$ is continuous from $[0,1]$ to $C^\infty(\mathbb{T})$ (and even maps into the class of functions which extend to a holomorphic function on an annulus).

Finally, Φ_t belongs to $\mathcal{E}_{\exp}$ for all $0 \leq t < 1$ as Φ_t has no other zero on the unit disc than the one at -1. So the path is in $\mathcal{E}$, except, perhaps, for its endpoint Φ_1 which is m_0.

We have also proved

Proposition 3.4. *Let ψ a wavelet that arises from an* MRA *with a scaling function that has polynomial decay. Then there exists a continuous family of such wavelets, $t \mapsto \psi_t$, $t \in [0,1]$, such that $\psi_0 = h$, $\psi_1 = \psi$, where h is the Haar wavelet.*

Proposition 3.5. (P.G. Lemarié-Rieusset and G. Malgouyres [6]). *Let ψ a compactly supported wavelet that arises from an* MRA. *Then there exists a continuous family of such wavelets, $t \mapsto \psi_t$, $t \in [0,1]$, such that $\psi_0 = h$, $\psi_1 = \psi$, where h is the Haar wavelet.*

Observe that if the scaling function has polynomial decay, so does the wavelet Φ. "Continuous" means that $t \mapsto \psi_t$ is continuous from $[0,1]$ to $L^2((1+|x|)^n\,dx)$ for any n.

Finally, let us prove Lemma 3.3. In fact, we are going to show a version of the lemma, where trigonometric polynomials have been replaced by polynomials in z. It will be clear that Lemma 3.3 follows from Lemma 3.6 using $F_t(\xi) = e^{-il\xi}(1+e^{i\xi})P_t(e^{i\xi})/2$ for an appropriate positive integer l.

Lemma 3.6. *Let P be a polynomial in $z \in \mathbb{C}$ of degree less than or equal to N, such that $P(2) = 1$. There exists a continuous map $t \to P_t$, from $[0,1]$ to the space of polynomials of degree $\leq N$ such that*

1) $P_t(2) = 1$, *if* $0 \leq t \leq 1$,

2) $P_1(z) = P(z)$,

3) $|P_t(z)|^2 = (1-t) + t\,|P(z)|^2$, *if* $|z| = 1$.

Moreover, there exists an integer k such that $P_0(z) = z^k$.

We can assume that $P(0) \neq 0$, otherwise $P(z) = z^j \tilde{P}(z)$ with $\tilde{P}(0) \neq 0$, and we can take $P_t(z) = z^j \tilde{P}_t(z)$. Let

$$Q(z) = z^N P(z) \bar{P}(\frac{1}{z}),$$

and

$$Q_t(z) = (1-t) z^N + t Q(z).$$

The map $t \mapsto Q_t$ is obviously continuous, $Q_t(2) = 1$, $Q_1 = Q$ and $Q_0(z) = z^N$. We shall introduce polynomials P_t such that $Q_t(z) = z^N P_t(z) \bar{P}_t(1/z)$. These polynomials are constructed with the aid of the zeros of the polynomials $Q_t(z)$ by an argument very similar to that used to establish the Lemma of Fejér-Riesz (see [3, p. 117]).

Lemma 3.7. *Let $z_1, z_2, \dots, z_N$ be the zeros of P (possibly repeated with their multiplicity) chosen so that $z_1, \dots, z_k$ are the only zeros inside the unit disc. Let $z_j = 1/\bar{z}_{j-N}$ for $j = N+1, \dots, 2N$. Then there exist $2N$ continuous functions on $(0,1]$ such that $z_1(t), z_2(t), \dots, z_{2N}(t)$ are the $2N$ zeros of Q_t, $z_j(t) = 1/\bar{z}_{j-N}(t)$ for $j = N+1, \dots, 2N$, and $z_1(t), \dots, z_k(t)$ are inside the unit disc while $z_{k+1}(t), \dots, z_N(t)$ are outside the unit disc.*

Let us remark that, since $Q_t(2) = 1$ we must have $z_j(t) \neq 1$. Assuming that Lemma 3.7 is proved, we can then define

$$P_t(z) = \prod_{j=1}^{N} \frac{z - z_j(t)}{1 - z_j(t)}, \qquad 0 < t \leq 1.$$

It is clear that $Q_t(z) = z^N P_t(z) \bar{P}_t(1/z)$ and $t \mapsto P_t$ is continuous. In order to obtain Lemma 3.6, it suffices to prove that $P_t \to z^k$ when $t \to 0$. But $Q_t \to z^N$ as $t \to 0$; then, for $t < \eta$, N of the $z_j(t)$'s are inside a small disc $\{|z| < \varepsilon\}$,and the other N (which are their reciprocals) are outside the disc $\{|z| < 1/\varepsilon\}$. That is to say, $z_1(t), \dots, z_k(t)$ tend to 0 while $|z_{k+1}(t)|, \dots, |z_N(t)|$ tend to ∞. Thus the polynomials $P_t(z)$ (each of degree $\leq N$) tend uniformly with the unit circle to z^k. This shows Lemma 3.6.

Hence, we just have to prove Lemma 3.7. Let us start by defining $z_j(t)$ for t close to 1. Let z_0 be a zero of P with multiplicity k_0.

First case: $|z_0| < 1$. We can assume that $z_1 = \cdots = z_{k_0} = z_0$. $1/\bar{z}_0$ may or may not be a zero of P. In the first case, let $k_0' \geq 1$ be its multiplicity, and let $z_{k+1} = \cdots = z_{k+k'_0} = 1/\bar{z}_0$. We shall define $z_j(t)$ for $j \in J_0 = \{1, \dots, k_0, N+k+1, \dots, N+k+k'_0\}$. We know that z_0 is a zero of Q with multiplicity $k_0 + k'_0$. In a neighborhood of z_0, $Q(z)/z^N$ can also be written $(z - z_0)^{k_0+k'_0} F(z)$, with $F(z_0) \neq 0$, and $Q_t(z)$ has $k_0 + k'_0$ distinct zeros which are solutions of

$$\frac{Q(z)}{z^N} = -\frac{1-t}{t} = (z - z_0)^{k_0+k'_0} F(z) . \tag{10}$$

From this, we obtain

$$e^{2\pi i l(j)/(k_0+k'_0)} s = (z - z_0)\, \alpha(z)^{-1} ,$$

where $\alpha(z)^{k_0+k'_0} = (-F(z))^{-1}$, $s = ((1-t)/t)^{1/(k_0+k'_0)}$ and $j \mapsto l(j)$ is a bijection between J_0 and $\{1, \dots, k_0 + k'_0\}$. It is then easy to see that we can define $z_j(t)$ so that $t \mapsto z_j(t)$ is tangent at z_0 to the half-lines $s \mapsto z_0 + \alpha\, e^{2\pi i l(j)/(k_0+k'_0)} s$, where $\alpha = \alpha(z_0)$.

When $1/\bar{z}_0$ is not a zero of P, we obtain the same result with k_0 instead of $k_0 + k'_0$.

Second case: $|z_0| > 1$. The previous reasoning applies and allows us to define $z_j(t)$ for t close to 1 and j such that $|z_j| \neq 1$.

Third case: $|z_0| = 1$. This time, z_0 is a zero of Q with multiplicity $2k_0$, and we can assume that $z_0 = z_{k+1} = \cdots = z_{k+k_0} = z_{N+k+1} = \cdots = z_{N+k+k_0}$. Once again, the question is to define $z_j(t)$ for $j \in J_0 = \{k+1, \dots, k+k_0, N+k+1, \dots, N+k+k_0\}$ so that $|z_j(t)| > 1$ for $j \leq N$ and $|z_j(t)| < 1$ for $j \geq N$. Again, the z_j's can be chosen tangent to the half-lines $s \mapsto z_0 + \alpha\, e^{2\pi i l(j)/2k_0} s$, where $\alpha^{2k_0} = -1/F(z_0)$ and l is a bijection between J_0 and $\{1, \dots, 2k_0\}$.

But the positivity of $Q(z)/z^N$ on the unit circle implies that $(-1)^{k_0}$ $z_0^{-2k_0} F(z_0)$ is positive; hence, we can take $\alpha = z_0\, \beta$, where $\beta > 0$, if k_0 is odd, and $\alpha = z_0\, e^{i\pi/2k_0} \beta$, where $\beta > 0$, if k_0 is even. In both cases half of the half-lines are outside the unit circle, the rest are inside. We choose $l(j)$ so that the half-line lies outside the unit circle if $j \leq N$, while, for $j > N$, the half-line crosses the circle.

We can now finish the proof of Lemma 3.7. By continuity, $z_j(t)$ is well defined as long as it is distinct from $z_l(t)$ for $l \neq j$. Otherwise let ε be such that $z_{j_1}(t) \to z_0$, $z_{j_2}(t) \to z_0$, …, $z_{j_l}(t) \to z_0$ when $t \to \varepsilon$, $\varepsilon > 0$, while the other zeros stay outside a neighborhood of z_0. That is to say, Q_ε has at z_0 a zero with multiplicity l. Moreover $z_0 \neq 0$ since $Q_\varepsilon(0) = \varepsilon Q(0)$ of z_0, we have, once again, $Q_\varepsilon(z)/z^N = (z - z_0)^l F(z)$, $F(z_0) \neq 0$; thus, for $t < \varepsilon$, $Q_t(z) = 0$ if and only if $(z - z_0)^l F(z) =$

$-(\varepsilon-t)/t$. As before we obtain continuous functions $z_{j_1}(t)(t),\dots,z_{j_l}(t)$ defined for $t \le \varepsilon$, t close to ε, and equal to z_0 at ε.

REMARK. In the third case, we could just as well have chosen to reverse the property of the half-lines. That is to say that, among $z_1(t),\dots,z_N(t)$, we can choose l of them in the unit circle, with $k \le l \le k'$, where k is the number of zeros of P in the open unit circle, and k' the number of zeros of P in the closed unit circle. This is a way of interchanging z^k and z^l.

REMARK. In this paper, we considered only the case of 0-regular MRA's as they have been defined by Y. Meyer in [7]. The question whether r-regular MRA's have the same property remains open.

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