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On fractional differentiation and integration on spaces of homogeneous type

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Abstract. In this paper we define derivatives of fractional order on spaces of homogeneous type by generalizing a classical formula for the fractional powers of the Laplacean [S1], [S2], [SZ] and introducing suitable quasidistances related to an approximation of the identity. We define integration of fractional order as in [GV] but using quasidistances related to the approximation of the identity mentioned before.

We show that these operators act on Lipschitz spaces as in the classical cases. We prove that the composition T_α of a fractional integral I_α and a fractional derivative D_α of the same order and its transpose (a fractional derivative composed with a fractional integral of the same order) are Calderón-Zygmund operators. We also prove that for small order α, T_α is an invertible operator in L^2. In order to prove that T_α is invertible we obtain Nahmod type representations for I_α and D_α and then we follow the method of her thesis [N1], [N2].

1. Definitions and statement of the main results.

In this paper (X, δ, μ) will be a space of homogeneous type which is normal and of order γ, $0 < \gamma \leq 1$, and such that $\mu(\{x\}) = 0$ for all x in X, and $\mu(X) = \infty$.

We recall that a *space of homogeneous type* consists of a set X, a

quasidistance δ, *i.e.* a function $\delta : X \times X \to [0, \infty)$ that satisfies

$$\delta(x,y) = 0 \qquad \text{if and only if} \qquad x = y\,, \tag{1.1}$$

$$\delta(x,y) = \delta(y,x)\,, \qquad \text{for every } x \text{ and } y \text{ in } X\,, \tag{1.2}$$

there is a positive constant κ such that

$$\delta(x,y) \leq \kappa\,(\delta(x,z) + \delta(z,y)) \tag{1.3}$$

for every x, y and z in X, and a measure μ defined on a σ-algebra of subsets of X which contains the open sets of X and the balls $B_r(x) = \{y : \delta(x,y) < r\}$ and satisfies the doubling condition: there exists a positive constant A such that for every x in X and every $r > 0$, $0 < \mu(B_{2r}(x)) \leq A\,\mu(B_r(x))$. If X has more than one element, as in this paper, the constant κ in (1.3) cannot be less than 1.

A space of homogeneous type is *normal* if there are positive constants A_1 and A_2 such that for all x in X

$$A_1\, r \leq \mu(B_r(x)) \leq A_2\, r\,, \qquad \text{for all } r > 0\,. \tag{1.4}$$

Two quasidistances δ and ρ are said to be *equivalent*, $\rho \cong \delta$, if there exist positive constants c_1 and c_2 such that for all x, y in X

$$c_1\,\delta(x,y) \leq \rho(x,y) \leq c_2\,\delta(x,y)\,. \tag{1.5}$$

It is easy to see that if (X, δ, μ) satisfies (1.4) then so does (X, ρ, μ).

A space of homogeneous type is of *order* γ, $0 < \gamma \leq 1$ if there is a positive constant M such that for all x, x', y in X

$$|\delta(x,y) - \delta(x',y)| \leq M\,\delta^{\gamma}(x,x')\,(\delta(x,y) + \delta(x',y))^{1-\gamma}\,. \tag{1.6}$$

It is shown in [MS] that in any space of homogeneous type there is a topologically equivalent quasidistance δ that satisfies (1.4) and (1.6).

For $0 < \beta \leq \gamma$, $\mathrm{Lip}\,(\beta)$ will denote the space of complex valued functions f such that for all x and y in X

$$|f(x) - f(y)| \leq C\,\delta^{\beta}(x,y) \tag{1.7}$$

holds with a constant C independent of x and y. The norm of an element f of $\mathrm{Lip}\,(\beta)$ is the infimum of the constants C in (1.7). Given a ball B, $C_0^{\beta}(B)$ will denote the space of functions f in $\mathrm{Lip}\,(\beta)$ with

compact support in B. We shall say that f belongs to C_0^β if f belongs to $C_0^\beta(B)$ for some B. The space C_0^β is the inductive limit of the Banach spaces $C_0^\beta(B)$ with the inductive limit topology and $(C_0^\beta)'$ will denote the space of all continuous linear functionals on C_0^β.

Let $s(x,y,t)$ be a symmetric approximation to the identity of the type introduced by Coifman, see Section 2. Let $-\infty < \alpha < 1$,we define $\delta_\alpha : X \times X \to [0,\infty)$ by

$$\delta_\alpha(x,y) = \Big(\int_0^\infty t^{\alpha-1}\, s(x,y,t)\, dt \Big)^{1/(\alpha-1)}, \qquad \text{for } x \neq y\,, \tag{1.8}$$

and

$$\delta_\alpha(x,y) = 0\,, \qquad \text{for } x = y\,.$$

We shall see in Section 2, Lemma 2.2, that for each α, δ_α is a quasidistance equivalent to δ, and it satisfies (1.6). Note that (X, δ_α, μ) is a normal space of order γ.

For $0 < \alpha < \gamma$ the fractional derivative of order α of f in $\mathrm{Lip}\,(\beta) \cap L^\infty$, $\alpha < \beta \leq \gamma$ is defined by

$$D_\alpha f(x) = \int_X \frac{f(y) - f(x)}{\delta_{-\alpha}^{\alpha+1}(x,y)}\, d\mu(y)\,. \tag{1.9}$$

The above definition extends the classical formula for functions on $\mathbb{R}^n$,

$$D_\alpha f(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{f(x+y) - f(x)}{|y|^{n+\alpha}}\, dy\,.$$

For f sufficiently restricted and $0 < \alpha < 2$, one has $D_\alpha f = c_\alpha\, (-\Delta)^{\alpha/2} f$, where Δ is the Laplacean [S1], [S2], [S3].

For $0 < \alpha < 1$, the fractional integral of order α of f in $\mathrm{Lip}\,(\beta) \cap L^1$ is defined by

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta_\alpha^{1-\alpha}(x,y)}\, d\mu(y)\,. \tag{1.10}$$

The definitions of D_α and I_α can be extended to $\mathrm{Lip}\,(\beta)$ for the same β as above. This requires the following modification similar to the one needed to define singular integrals on L^∞:

$$\widetilde{D}_\alpha f(x) = \int_X \Big(\frac{f(y) - f(x)}{\delta_{-\alpha}^{1+\alpha}(x,y)} - \frac{f(y) - f(x_0)}{\delta_{-\alpha}^{1+\alpha}(x_0,y)} \Big)\, d\mu(y)\,, \tag{1.11}$$

and

$$\widetilde{I}_\alpha f(x) = \int_X f(y) \left(\frac{1}{\delta_\alpha^{1-\alpha}(x,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(x_0,y)} \right) d\mu(y), \tag{1.12}$$

where x_0 is a fixed but arbitrary point of X. It will be shown in Theorem 1.1 and Theorem 1.2 that $\widetilde{D}_\alpha f(x)$ and $\widetilde{I}_\alpha f(x)$ converge absolutely for all x and therefore changing x_0 in the definitions above results in adding a constant. We show in Section 2, Lemma 2.4, that δ_α, for $0 < \alpha < \gamma$, has the cancellation property:

$$\int_X (\delta_\alpha^{\alpha-1}(x,y) - \delta_\alpha^{\alpha-1}(x',y))\, d\mu(y) = 0.$$

In [GV] it was shown that for fractional integrals defined with a quasi-distance which has the above properties the classical theorems on boundedness on L^p, BMO, Lip (β) and H^p hold. For the sake of completeness we prove the result for Lipschitz spaces in Theorem 1.1. See [GGW], [GV].

We recall the definition of a singular integral operator as given in [DJS] and [S3]. Let $\Omega = X \times X \setminus \Delta$ where Δ is the diagonal of $X \times X$. A continuous function $K : \Omega \to \mathbb{C}$ is a standard kernel if there exist a number η, $0 < \eta \leq 1$, and constants $\nu > 1$ and $c > 0$ such that

$$|K(x,y)| \leq \frac{c}{\delta(x,y)} \qquad \text{for } (x,y) \text{ in } \Omega, \tag{1.13}$$

and for $\nu\, \delta(x,y) < \delta(x,z)$ we have

$$|K(x,z) - K(y,z)| \leq c\, \frac{\delta^\eta(x,y)}{\delta^{1+\eta}(x,z)}, \tag{1.14}$$

and

$$|K(z,x) - K(z,y)| \leq c\, \frac{\delta^\eta(x,y)}{\delta^{1+\eta}(x,z)}. \tag{1.15}$$

A *singular integral operator* is a continuous linear operator $T : C_0^\beta \to (C_0^\beta)'$ associated with a standard kernel K in the following sense:

$$\langle Tf, g \rangle = \int_X \int_X K(x,y)\, g(x)\, f(y)\, d\mu(x)\, d\mu(y),$$

for all $f, g \in C_0^\beta$ with disjoint supports, and where $\langle Tf, g\rangle$ denotes the evaluation of Tf on g.

A singular integral operator is called a *Calderón-Zygmund operator* if it can be extended to a continuous operator from L^2 to L^2.

The *transpose* ${}^tT$ of a singular integral operator T is defined by

$$\langle {}^tTf, g\rangle = \langle Tg, f\rangle ,$$

for all $f, g \in C_0^\beta$, $0 < \beta \leq \gamma$.

The function $s(x, y, t)$ introduced before, is continuously differentiable in t. Let

$$q(x, y, t) = t\,\frac{\partial}{\partial t} s(x, y, t) \tag{1.16}$$

and set

$$-Q_t f(x) = \int_X q(x, y, t)\, f(y)\, d\mu(y) . \tag{1.17}$$

In this paper the letter c will denote a constant, not necessarily the same in different occurrences.

We can now state our main results.

Theorem 1.1. *Let* $0 < \alpha < \beta \leq \gamma$.

a) *If* $f \in \mathrm{Lip}(\beta) \cap L^1$ *then* $I_\alpha f(x)$ *converges absolutely for all* x *and there is a constant* c *independent of* f *such that*

$$\|I_\alpha f\|_{\mathrm{Lip}(\alpha+\beta)} \leq c\, \|f\|_{\mathrm{Lip}(\beta)} .$$

b) *If* $f \in \mathrm{Lip}(\beta)$, *then* $\widetilde{I}_\alpha f(x)$ *converges absolutely for all* x, *and there is a constant* c *independent of* f *such that*

$$\|\widetilde{I}_\alpha f\|_{\mathrm{Lip}(\alpha+\beta)} \leq c\, \|f\|_{\mathrm{Lip}(\beta)} .$$

c) *If* $f \in \mathrm{Lip}(\beta) \cap L^1$ *then* $\widetilde{I}_\alpha f$ *defines the same class as* $I_\alpha f$ *in* $\mathrm{Lip}(\alpha + \beta)$.

Theorem 1.2. *Let* $0 < \alpha < \beta \leq \gamma$.

a) *If* $f \in \mathrm{Lip}(\beta) \cap L^\infty$ *then* $D_\alpha f(x)$ *converges absolutely for all* x *and there is a constant* c *independent of* f *such that*

$$\|D_\alpha f\|_{\mathrm{Lip}(\beta-\alpha)} \leq c\, \|f\|_{\mathrm{Lip}(\beta)} .$$

b) *If* $f \in \mathrm{Lip}(\beta)$ *then* $\widetilde{D}_\alpha f(x)$ *converges absolutely for all* x *and there is a constant* c *independent of* f *such that*

$$\left\| \widetilde{D}_\alpha f \right\|_{\mathrm{Lip}(\beta-\alpha)} \leq c \, \|f\|_{\mathrm{Lip}(\beta)} \, .$$

c) *If* $f \in \mathrm{Lip}(\beta) \cap L^\infty$ *then* $\widetilde{D}_\alpha f$ *defines the same class as* $D_\alpha f$ *in* $\mathrm{Lip}(\beta - \alpha)$.

For similar classical results see [Z, Chapter XII].

Theorem 1.3. *Let* $0 < \alpha < \gamma$, *then* $T_\alpha = D_\alpha I_\alpha$ *is a singular integral operator with associated kernel*

$$K(x,y) = \int_X \frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)} \left(\frac{1}{\delta_\alpha^{1-\alpha}(y,t)} - \frac{1}{\delta_\alpha^{1-\alpha}(x,y)} \right) d\mu(t) \, . \tag{1.18}$$

Theorem 1.4. *Let* $0 < \alpha < \gamma$, *then* $T_\alpha = D_\alpha I_\alpha$ *is a Calderón-Zygmund operator.*

Theorem 1.5. *Let* $S_\alpha = I_\alpha D_\alpha$, *then* $S_\alpha f = {}^t T_\alpha f$ *for every* f *in* C_o^β, *with* $0 < \alpha < \beta \leq \gamma$, *and* S_α *is a Calderón-Zygmund operator.*

Theorem 1.6. *If* $Q_t(f)$ *is the operator defined by* (1.17) *then the following representation formulas hold pointwise everywhere and in the weak sense:*

$$\alpha \, I_\alpha f = \int_0^\infty t^\alpha \, Q_t(f) \, \frac{dt}{t} \, , \tag{1.19}$$

for f *in* $\mathrm{Lip}(\beta) \cap L^1$, $0 < \alpha$, $\alpha + \beta \leq \gamma$, *and*

$$-\alpha \, D_\alpha f = \int_0^\infty t^{-\alpha} \, Q_t(f) \, \frac{dt}{t} \, , \tag{1.20}$$

for f *in* $\mathrm{Lip}(\beta) \cap L^\infty$, $0 < \alpha < \beta \leq \gamma$.

The following theorem extends a result obtained by A. R. Nahmod in her Thesis [N2].

Theorem 1.7 *There exists* α_0, $0 < \alpha_0 < \gamma$, *such that for* $0 < \alpha < \alpha_0$ *the operator* T_α *as defined in Theorem* 1.3 *has a bounded inverse in* L^2.

2. Lemmas needed for the proofs of Theorems 1.1 through 1.5.

The first lemma states the properties of a Coifman type approximation to the identity. These properties are well known, see [DJS], and therefore the proofs will be omitted.

Let $h \geq 0$ be a C^∞ function on $[0,\infty)$ such that $h(r) = 1$ for $0 \leq r \leq 1/2$, and $h(r) = 0$ for $r \geq 2$. For $f \in L^1_{\text{loc}}(X)$ and $t > 0$ set

$$T_t f(x) = \frac{1}{t} \int_X h\Big(\frac{\delta(x,y)}{t}\Big) f(y)\, d\mu(y)\,,$$

$$M_t f(x) = \frac{1}{(T_t 1)(x)} f(x) = \varphi(x,t)\, f(x)\,,$$

$$V_t f(x) = \frac{1}{T_t\Big(\frac{1}{T_t 1}\Big)(x)} f(x) = \psi(x,t)\, f(x)\,.$$

Now define S_t by

$$S_t = M_t\, T_t\, V_t\, T_t\, M_t\,,$$

then

$$S_t f(x) = \int_X s(x,y,t)\, f(y)\, d\mu(y)\,,$$

where

$$s(x,y,t) = \frac{\varphi(x,t)\,\varphi(y,t)}{t^2} \int_X h\Big(\frac{\delta(x,u)}{t}\Big)\, h\Big(\frac{\delta(y,u)}{t}\Big)\, \psi(u,t)\, d\mu(u)\,.$$

Lemma 2.1. *There exist positive constants b_1, b_2, c_1, c_2, and c_3 independent of x, y, and t such that*

i) $s(x,y,t) = s(y,x,t)$ *for all x, y in X and $t > 0$,*

ii) $|s(x,y,t)| \leq c_1/t$ *for all x, y in X and $t > 0$, $s(x,y,t) = 0$ if $\delta(x,y) > b_1\, t$, and $c_2/t < s(x,y,t)$ if $\delta(x,y) < b_2\, t$,*

iii) $|s(x,y,t) - s(x',y,t)| < c_3\, \delta^\gamma(x,x')/t^{1+\gamma}$ *for all x, x' and y in X, and $t > 0$,*

iv) $\int s(x,y,t)\, d\mu(y) = 1$ *for all x in X, and $t > 0$,*

v) $s(x,y,t)$ *is continuously differentiable with respect to* t.

Lemma 2.2. *For each* α, $-\infty < \alpha < 1$, *the function* δ_α, *defined in* (1.8) *is a quasidistance equivalent to* δ *and it satisfies* (1.6).

PROOF. We shall prove first that there are positive constants c'_α and c''_α such that for all x, y in X

$$c'_\alpha\, \delta(x,y) \le \delta_\alpha(x,y) \le c''_\alpha\, \delta(x,y)\,.$$

Using the properties of $s(x,y,t)$ stated in Lemma 2.1, we have that $s(x,y,t) = 0$ if $\delta(x,y) > b_1\, t$. Then

$$\delta_\alpha^{\alpha-1}(x,y) = \int_{\delta(x,y)/b_1}^{\infty} t^{\alpha-1}\, s(x,y,t)\, dt\,.$$

On the other hand $\|s(\cdot,\cdot,t)\|_\infty \le c_1/t$, and therefore

$$\delta_\alpha^{\alpha-1}(x,y) \le c_1 \int_{\delta(x,y)/b_1}^{\infty} t^{\alpha-2}\, dt = \frac{c_1}{1-\alpha}\, b_1^{1-\alpha}\, \delta^{\alpha-1}(x,y)\,.$$

Raising this inequality to the power $1/(\alpha-1)$ we obtain the first inequality of (2.1).

To obtain the second inequality of (2.1) note that $s(x,y,t) \ge c_2/t$ if $\delta(x,y) < b_2\, t$, hence by (2.2)

$$\delta_\alpha^{\alpha-1}(x,y) \ge \int_{\delta(x,y)/b_2}^{\infty} t^{\alpha-1}\, \frac{c_2}{t}\, dt = \frac{c_2}{1-\alpha}\, b_2^{1-\alpha}\, \delta^{\alpha-1}(x,y)\,.$$

Raising this inequality to the power $1/(\alpha-1)$ we conclude the proof of (2.1).

The fact that $\delta_\alpha(x,y)$ is a quasidistance follows from the definition, property i) of $s(x,y,t)$ and (2.1). We will denote by κ_α the constant in the inequality (1.3) for δ_α.

We will show now that δ_α satisfies (1.6). If $\delta_\alpha(x,y) = 0$ then $x = y$ and $\delta_\alpha(x',y) = \delta_\alpha(x,x')$ and

$$|\delta_\alpha(x,y) - \delta_\alpha(x',y)| = \delta_\alpha(x,x') = \delta_\alpha^{\gamma}(x,x')\,(\delta_\alpha(x,y) + \delta_\alpha(x',y))^{1-\gamma}\,.$$

Similarly when $\delta_\alpha(x',y) = 0$ we get the estimate above.

Assume now that $\delta_\alpha(x,y) \neq 0$ and $\delta_\alpha(x',y) \neq 0$. Let

$$a = \frac{1}{b_1} \min\{\delta_\alpha(x,y), \delta_\alpha(x',y)\},$$

then by property ii) of Lemma 2.1

$$\begin{aligned}
&|\delta_\alpha(x,y) - \delta_\alpha(x',y)| \\
&= \left| \left(\int_a^\infty t^{\alpha-1}\, s(x,y,t)\, dt \right)^{1/(\alpha-1)} - \left(\int_a^\infty t^{\alpha-1}\, s(x',y,t)\, dt \right)^{1/(\alpha-1)} \right| \\
&\quad \leq \left(\int_a^\infty t^{\alpha-1}\, |s(x',y,t) + \theta(s(x,y,t) - s(x',y,t))|\, dt \right)^{(2-\alpha)/(\alpha-1)} \\
&\qquad \cdot \left(\int_a^\infty t^{\alpha-1}\, |s(x,y,t) - s(x',y,t)|\, dt \right),
\end{aligned}$$

with $0 < \theta < 1$. Using ii) and iii) of Lemma 2.1 we can majorize the last estimate by

$$\begin{aligned}
\left(c \int_a^\infty t^{\alpha-2}\, dt \right)^{(2-\alpha)/(\alpha-1)} \left(\int_a^\infty t^{\alpha-\gamma-2}\, c\, \delta_\alpha^\gamma(x,x')\, dt \right) & \\
\leq c\, \delta_\alpha^\gamma(x,x')\, a^{1-\gamma} & \\
\leq c\, \delta_\alpha^\gamma(x,x') \left(\delta_\alpha(x,y) + \delta_\alpha(x',y)\right)^{1-\gamma}. &
\end{aligned}$$

This concludes the proof of the lemma.

Lemma 2.3. *Let $\alpha < 1$ and $k_\alpha > \kappa_\alpha$. There exists a positive constant C_{k_α} such that*

$$|\delta_\alpha^{\alpha-1}(x,y) - \delta_\alpha^{\alpha-1}(x',y)| \leq C_{k_\alpha}\, \delta_\alpha^\gamma(x,x')^\gamma\, \delta_\alpha^{\alpha-1-\gamma}(x,y),$$

for all x, x', y in X such that

$$k_\alpha\, \delta_\alpha(x,x') \leq \delta_\alpha(x,y).$$

The exponent γ is the order of the space.

This result follows from property (1.6), it was proved in [GV] for $k_\alpha = 2\,\kappa_\alpha$, the proof for $k_\alpha > \kappa_\alpha$ is similar.

Lemma 2.4. (Cancellation property of order $\alpha - 1$). *Let* $0 < \alpha < \gamma$, *then*

$$\int_X \left(\delta_\alpha^{\alpha-1}(x,y) - \delta_\alpha^{\alpha-1}(x',y)\right) d\mu(y) = 0\,,$$

for any x, x' *in* X.

PROOF. We show first that

$$\int_X \int_0^\infty t^{\alpha-1} \left|s(x,y,t) - s(x'y,t)\right| d\mu(y)\, dt < \infty\,.$$

We have

$$\int_X \int_0^1 t^{\alpha-1} \left|s(x,y,t) - s(x'y,t)\right| d\mu(y)\, dt \le 2 \int_0^1 t^{\alpha-1}\, dt < \infty\,.$$

To estimate $\int_X \int_1^\infty t^\alpha \left|s(x,y,t) - s(x',y,t)\right| d\mu(y)\, dt$, observe that the functions $s(x,\cdot,t)$ are supported in balls of radius $b_1\, t$, also by iii) of Lemma 2.1 we have

$$\left|s(x,y,t) - s(x',y,t)\right| \le c_3 \frac{\delta^\gamma(x,x')}{t^{1+\gamma}}\,.$$

Therefore using normality the double integral is majorized by

$$\int_1^\infty t^{\alpha-1} \frac{\delta^\gamma(x,x')\, c\, t}{t^{1+\gamma}}\, dt \le c\, \delta^\gamma(x,x') \int_1^\infty \frac{dt}{t^{1+\gamma-\alpha}} < \infty\,.$$

Since

$$\int_X \left(\delta_\alpha^{\alpha-1}(x,y) - \delta_\alpha^{\alpha-1}(x',y)\right) d\mu(y)$$
$$= \int_X \int_0^\infty t^{\alpha-1} \left(s(x,y,t) - s(x',y,t)\right) dt\, d\mu(y)\,,$$

by changing the order of integration and using v) of Lemma 2.1 we obtain that the integral is zero.

Lemma 2.5. *Let* $x \in X$ *and* $r > 0$. *Then*

$$\int\limits_{\delta(x,y)<r} \frac{1}{\delta^\lambda(x,y)}\, d\mu(y) \le c\, r^{-\lambda+1}\,, \qquad \text{for } \lambda < 1\,,$$

and

$$\int_{\delta(x,y)\geq r} \frac{1}{\delta^{\lambda}(x,y)}\, d\mu(y) \leq c\, r^{-\lambda+1}\,, \qquad \text{for } \lambda > 1\,,$$

where c is a constant independent of x.

Note that this lemma is valid for any quasidistance equivalent to δ. This lemma is well known. See for instance [GV].

3. Proofs of Theorems 1.1 through 1.5.

In the next proofs we will use without notice Lemma 2.2, Lemma 2.5 and normality.

PROOF OF THEOREM 1.1. To prove part a) observe that, since $f \in \mathrm{Lip}\,(\beta) \cap L^1$, the integral

$$I_\alpha f(x) = \int \frac{f(y)}{\delta_\alpha^{1-\alpha}(x,y)}\, d\mu(y)$$

converges absolutely for any x.

Now consider $x_1 \neq x_2$ and let $r = \delta_\alpha(x_1,x_2)$, $B = B_{2\kappa_\alpha r}(x_2)$ and B^c the complement of B. Since δ_α has the cancellation property stated in Lemma 2.4, we have

$$\begin{aligned} &I_\alpha f(x_2) - I_\alpha f(x_1) \\ &\qquad = \int_X (f(y) - f(x_2)) \left(\frac{1}{\delta_\alpha^{1-\alpha}(x_2,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(x_1,y)} \right) d\mu(y)\,. \end{aligned}$$

Then

$$\begin{aligned} &\left| I_\alpha f(x_2) - I_\alpha f(x_1) \right| \\ &\qquad \leq \int_B \frac{|f(y) - f(x_2)|}{\delta_\alpha^{1-\alpha}(x_2,y)}\, d\mu(y) + \int_B \frac{|f(y) - f(x_2)|}{\delta_\alpha^{1-\alpha}(x_1,y)}\, d\mu(y) \\ &\qquad\quad + \int_{B^c} |f(y) - f(x_2)| \left| \frac{1}{\delta_\alpha^{1-\alpha}(x_2,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(x_1,y)} \right| d\mu(y)\,. \end{aligned}$$

Since $|f(y) - f(x_2)| \leq c\, \|f\|_{\mathrm{Lip}(\beta)}\, r^{\beta}$ for $y \in B$, the first integral is less than or equal to $c\, \|f\|_{\mathrm{Lip}(\beta)}\, r^{\beta+\alpha}$.

To estimate the second integral observe that $B \subset B_{\kappa_\alpha(2\kappa_\alpha+1)r}(x_1)$ then using the previous argument and integrating over this ball we obtain that this integral is less than or equal to $c\,\|f\|_{\mathrm{Lip}(\beta)}\, r^{\beta+\alpha}$.

To estimate the third integral observe that for $y \in B^c$ we can apply Lemma 2.3, and using that $f \in \mathrm{Lip}\,(\beta)$ this integral is majorized by

$$c\,\|f\|_{\mathrm{Lip}(\beta)} r^{\gamma} \int_{B^c} \delta_\alpha^{\beta+\alpha-\gamma-1}(x_2, y)\, d\mu(y) \leq c\,\|f\|_{\mathrm{Lip}(\beta)}\, r^{\beta+\alpha}\,.$$

Since $r = \delta_\alpha(x_1, x_2)$ the proof of part a) is complete.

To prove part b) we show first that $\widetilde{I}_\alpha f(x)$ converges absolutely for every x. Since δ_α has the cancellation property stated in Lemma 2.4 we can write

$$\widetilde{I}_\alpha f(x) = \int (f(y) - f(x)) \Big(\frac{1}{\delta_\alpha^{1-\alpha}(x, y)} - \frac{1}{\delta_\alpha^{1-\alpha}(x_0, y)} \Big)\, d\mu(y)\,.$$

Now it is clear that the function inside this integral is integrable over the ball $B_{2\kappa_\alpha \delta_\alpha(x, x_o)}(x)$. To see that it is also integrable in the complement of this ball we apply Lemma 2.3 and use the fact that $f \in \mathrm{Lip}\,(\beta)$. The proof that $\|\widetilde{I}_\alpha f\|_{\mathrm{Lip}(\alpha+\beta)} \leq c\,\|f\|_{\mathrm{Lip}(\beta)}$ proceeds exactly as in part a).

Finally the fact that for $f \in \mathrm{Lip}\,(\beta) \cap L^1$, $\widetilde{I}_\alpha f$ coincides with $I_\alpha f$ as an element of $\mathrm{Lip}\,(\alpha+\beta)$, follows from the fact that for such a function

$$\widetilde{I}_\alpha f(x) = I_\alpha f(x) - I_\alpha f(x_0)\,.$$

PROOF OF THEOREM 1.2. Since $f \in \mathrm{Lip}\,(\beta) \cap L^\infty$ with $\alpha < \beta \leq \gamma$, the integral

$$D_\alpha f(x) = \int_X \frac{f(y) - f(x)}{\delta_{-\alpha}^{1+\alpha}(x, y)}\, d\mu(y)$$

converges absolutely for every x.

Now consider $x_1 \neq x_2$ and let $r = \delta_{-\alpha}(x_1, x_2), B = B_{2\kappa_{-\alpha} r}(x_2)$ and B^c the complement of B. We have

$$\begin{aligned} |D_\alpha f(x_2) - D_\alpha f(x_1)| \leq & \int_B \frac{|f(y) - f(x_2)|}{\delta_{-\alpha}^{1+\alpha}(x_2, y)}\, d\mu(y) \\ & + \int_B \frac{|f(y) - f(x_1)|}{\delta_{-\alpha}^{1+\alpha}(x_1, y)}\, d\mu(y) \end{aligned}$$

$$+\int_{B^c}\Big|\frac{f(y)-f(x_2)}{\delta_{-\alpha}^{1+\alpha}(x_2,y)}-\frac{f(y)-f(x_1)}{\delta_{-\alpha}^{1+\alpha}(x_1,y)}\Big|\,d\mu(y)\,.$$

Since $f\in \mathrm{Lip}\,(\beta)$ the first integral is majorized by $c\,\|f\|_{\mathrm{Lip}(\beta)}\,r^{\beta-\alpha}$. To estimate the second integral observe that $B\subset B_{\kappa_{-\alpha}(2\kappa_{-\alpha}+1)r}(x_1)$ then integrating over this ball and arguing as before this integral is majorized by $c\,\|f\|_{\mathrm{Lip}(\beta)}\,r^{\beta-\alpha}$.

To estimate the last integral we first rewrite the integrand as follows

$$\Big|\frac{f(x_1)-f(x_2)}{\delta_{-\alpha}^{1+\alpha}(x_2,y)}+(f(y)-f(x_1))\Big(\frac{1}{\delta_{-\alpha}^{1+\alpha}(x_2,y)}-\frac{1}{\delta_{-\alpha}^{1+\alpha}(x_1,y)}\Big)\Big|\,;$$

then this integral is less than or equal to

$$\int_{B^c}\frac{|f(x_1)-f(x_2)|}{\delta_{-\alpha}^{1+\alpha}(x_2,y)}\,d\mu(y)$$
$$+\int_{B^c}|f(y)-f(x_1)|\Big|\frac{1}{\delta_{-\alpha}^{1+\alpha}(x_2,y)}-\frac{1}{\delta_{-\alpha}^{1+\alpha}(x_1,y)}\Big|\,d\mu(y)\,.$$

The first term is majorized by $c\,\|f\|_{\mathrm{Lip}(\beta)}\,r^{\beta-\alpha}$. Using Lemma 2.3 we can majorize the second term by

$$\|f\|_{\mathrm{Lip}(\beta)}\,\delta_{-\alpha}^{\gamma}(x_1,x_2)\int_{B^c}\delta_{-\alpha}^{\beta-\alpha-1-\gamma}(x_2,y)\,d\mu(y)\le c\,\|f\|_{\mathrm{Lip}(\beta)}\,r^{\beta-\alpha}\,.$$

Since $r=\delta_{-\alpha}(x_1,x_2)$ the proof of part a) is complete.

To prove part b), we show first that $\widetilde{D}_\alpha f(x)$ converges absolutely for every x. For x fixed, since $f\in\mathrm{Lip}(\beta)$, the integral converges absolutely over the ball $B_{2\kappa_{-\alpha}\delta_{-\alpha}(x,x_0)}(x)$. To prove that it also converges absolutely in the complement of this ball rewrite the integrand as follows

$$\frac{f(x_0)-f(x)}{\delta_{-\alpha}^{1+\alpha}(x,y)}+(f(y)-f(x_0))\Big(\frac{1}{\delta_{-\alpha}^{1+\alpha}(x,y)}-\frac{1}{\delta_{-\alpha}^{1+\alpha}(x_0,y)}\Big)\,.$$

The first term is clearly integrable. The fact that the second term is integrable is a consequence of Lemma 2.3. The proof that $\|\widetilde{D}_\alpha f\|_{\mathrm{Lip}(\beta-\alpha)}$ $\le c\,\|f\|_{\mathrm{Lip}(\beta)}$ proceeds exactly as in part a).

Finally, part c) follows from the fact that for $f\in\mathrm{Lip}(\beta)\cap L^\infty$,

$$\widetilde{D}_\alpha f(x)=D_\alpha f(x)-D_\alpha f(x_0)\,.$$

PROOF OF THEOREM 1.3. Let

$$\Psi(x,y) = \int_X \frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)} \left| \frac{1}{\delta_\alpha^{1-\alpha}(y,t)} - \frac{1}{\delta_\alpha^{1-\alpha}(x,y)} \right| d\mu(t)$$

for $x \neq y$. Observe that $|K(x,y)| \leq \Psi(x,y)$. We will show now that

$$\Psi(x,y) \leq \frac{c}{\delta(x,y)} \,. \tag{3.1}$$

For fixed $x \neq y$ we break up X into three regions:

$$D_1 = \{t : 2\,\kappa_\alpha\,\delta_\alpha(x,y) \leq \delta_\alpha(x,t)\}\,,$$

$$D_2 = \left\{t : \frac{1}{2\kappa_\alpha}\,\delta_\alpha(x,y) \leq \delta_\alpha(x,t) < 2\,\kappa_\alpha\,\delta_\alpha(x,y)\right\},$$

and

$$D_3 = \left\{t : \delta_\alpha(x,t) \leq \frac{1}{2\kappa_\alpha}\,\delta_\alpha(x,y)\right\}.$$

In D_1 we have that $\delta_\alpha(y,t) \geq \kappa_\alpha^{-1}\,\delta_\alpha(x,t) - \delta_\alpha(x,y) \geq \delta_\alpha(x,y)$, and therefore the integral over D_1 is less than or equal to

$$\frac{c}{\delta_{-\alpha}^{1-\alpha}(x,y)} \int_{D_1} \frac{1}{\delta_\alpha^{1+\alpha}(x,t)}\, d\mu(t) \leq c\,\frac{\delta_\alpha^{-\alpha}(x,y)}{\delta_\alpha^{1-\alpha}(x,y)} \leq \frac{c}{\delta(x,y)} \,.$$

The integral over D_2 is majorized by

$$\frac{c}{\delta_{-\alpha}^{1+\alpha}(x,y)} \Big(\int_{D_1} \frac{d\mu(t)}{\delta_\alpha^{1-\alpha}(y,t)} + \frac{1}{\delta_\alpha^{1-\alpha}(x,y)} \int\limits_{D_2} d\mu(t) \Big) \,.$$

For t in D_2 we have that

$$\delta_\alpha(y,t) \leq \kappa_\alpha\,(\delta_\alpha(t,x) + \delta_\alpha(x,y) \leq \kappa_\alpha\,(2\,\kappa_\alpha + 1)\,\delta_\alpha(x,y)\,,$$

Enlarging D_2 to the ball of center y and radius $\kappa_\alpha(2\,\kappa_\alpha + 1)\,\delta_\alpha(x,y)$ and integrating we have that the expression above is less than or equal to $c/\delta(x,y)$.

Since in D_3 , $2\,\kappa_\alpha\,\delta_\alpha(x,t) < \delta_\alpha(x,y)$, Lemma 2.3 can be applied, and the integral over D_3 is majorized by

$$c \int_{D_3} \frac{1}{\delta_\alpha^{1+\alpha}(x,t)}\, \delta_\alpha^{\gamma}(x,t)\, \delta_\alpha^{\alpha-1-\gamma}(x,y)\, d\mu(t)$$

$$\leq \delta_\alpha^{\alpha-1-\gamma}(x,y)\int_{D_3}\delta_\alpha^{\gamma-\alpha-1}(x,t)\,d\mu(t)$$
$$\leq c\,\delta_\alpha^{\alpha-1-\gamma}(x,y)\,\delta_\alpha^{\gamma-\alpha}(x,y)$$
$$\leq \frac{c}{\delta(x,y)}\,.$$

We are now going to show that T_α is associated with the kernel K, as defined in Section 1. Let f and g be in C_0^β, $0 < \alpha+\beta \leq \gamma$, with disjoint supports.

$$T_\alpha f(x) = D_\alpha I_\alpha f(x)$$
$$= \int_X \frac{I_\alpha f(t) - I_\alpha f(x)}{\delta_{-\alpha}^{1+\alpha}(x,t)}\,d\mu(t)$$
$$= \int_X \frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)}\left(\int_X\left(\frac{1}{\delta_\alpha^{1-\alpha}(t,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(x,y)}\right)f(y)\,d\mu(y)\right)d\mu(t)\,.$$

For $x \notin \operatorname{supp} f$ using the estimate obtained above for $\Psi(x,y)$ the last integral converges absolutely. Then changing the order of integration we have

$$T_\alpha f(x) = \int_X K(x,y)\,f(y)\,d\mu(y)\,,$$

where $K(x,y)$ is the kernel defined in (1.18). Furthermore for $x \in \operatorname{supp} g$, $\int_X |K(x,y)|\,|f(y)|\,d\mu(y)$ is bounded, and therefore

$$\langle T_\alpha f, g\rangle = \int_X T_\alpha f(x)\,g(x)\,d\mu(x) = \iint K(x,y)\,f(y)\,g(x)\,d\mu(x)\,d\mu(y)\,.$$

We will now prove condition (1.14). Note that $\delta_{-\alpha} \simeq \delta$ and therefore is suffices to prove (1.14) with $\delta_{-\alpha}$; *i.e.* we will show that there are positive constants ν, M, η, $1 < \nu, 0 < M$ and $0 < \eta \leq 1$ such that if

$$\nu\,\delta_{-\alpha}(x,y) < \delta_{-\alpha}(x,z) \tag{3.2}$$

then

$$|K(x,z) - K(y,z)| \leq M\,\frac{\delta_{-\alpha}^\eta(x,y)}{\delta_{-\alpha}^{1+\eta}(x,z)}\,.$$

Let $c_\alpha \geq 1$ be a constant such that

$$c_\alpha^{-1}\,\delta_\alpha(x,y) \leq \delta_{-\alpha}(x,y) \leq c_\alpha\,\delta_\alpha(x,y)\,.$$

Denote by κ_α and $\kappa_{-\alpha}$ the constants in the triangle inequalities for δ_α and $\delta_{-\alpha}$, respectively. Now we choose constants ν and k so that

$$c_\alpha^2\,\kappa_\alpha\,\kappa_{-\alpha}^2 < k < \frac{\nu}{\kappa_{-\alpha}\,c_\alpha^2\,\kappa_\alpha}\,. \tag{3.3}$$

Let x, y, z be fixed points satisfying (3.2). To estimate $|K(x,z)-K(y,z)|$ observe first that

$$\begin{aligned} &|K(x,z)-K(y,z)| \\ &\le \int_X \Big|\frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)}\Big(\frac{1}{\delta_\alpha^{1-\alpha}(t,z)}-\frac{1}{\delta_\alpha^{1-\alpha}(x,z)}\Big) \\ &\qquad -\frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)}\Big(\frac{1}{\delta_\alpha^{1-\alpha}(t,z)}-\frac{1}{\delta_\alpha^{1-\alpha}(y,z)}\Big)\Big|\,d\mu(t)\,. \end{aligned} \tag{3.4}$$

To estimate the integral in (3.4) we divide X into two regions:

$$A=\Big\{t:\ \frac{1}{k}\,\delta_{-\alpha}(x,z)<\delta_{-\alpha}(x,t)\Big\}$$

and its complement A^c.

To estimate the integral on A we rewrite the integrand as follows:

$$\begin{aligned} &\Big|\Big(\frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)}-\frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)}\Big)\frac{1}{\delta_\alpha^{1-\alpha}(t,z)} \\ &+\frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)}\Big(\frac{1}{\delta_\alpha^{1-\alpha}(y,z)}-\frac{1}{\delta_\alpha^{1-\alpha}(x,z)}\Big) \\ &+\frac{1}{\delta_\alpha^{1-\alpha}(y,z)}\Big(\frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)}-\frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)}\Big)\Big| \le |\mathrm{I}_1|+|\mathrm{I}_2|+|\mathrm{I}_3|\,. \end{aligned}$$

We first estimate $\int_A |\mathrm{I}_3|\,d\mu$. Observe that for t in A

$$\frac{\nu}{k}\,\delta_{-\alpha}(x,y)<\delta_{-\alpha}(x,t)\,.$$

On the other hand, by (3.3), $\nu/k > \kappa_{-\alpha}$ and therefore we can apply Lemma 2.3 to obtain

$$\begin{aligned} \int_A |\mathrm{I}_3|\,d\mu(t) &\le c\,\frac{\delta_{-\alpha}^\gamma(x,y)}{\delta_\alpha^{1-\alpha}(y,z)}\int_A \delta_{-\alpha}^{-\alpha-1-\gamma}(x,t)\,d\mu(t) \\ &\le c\,\frac{\delta_{-\alpha}^\gamma(x,y)}{\delta_{-\alpha}^{1-\alpha}(y,z)}\,\delta_{-\alpha}^{-\alpha-\gamma}(x,z)\,. \end{aligned}$$

Note that

$$\delta_{-\alpha}(x,z)\left(1-\frac{\kappa_{-\alpha}}{\nu}\right) \leq \kappa_{-\alpha}\,\delta_{-\alpha}(y,z)$$

and hence

$$\int_A |\mathrm{I}_3|\,d\mu(t) \leq c\,\frac{\delta^{\gamma}_{-\alpha}(x,y)}{\delta^{1+\gamma}_{-\alpha}(x,z)}\,.$$

Since

$$\frac{\nu}{c_\alpha^2}\,\delta_\alpha(x,y) \leq \delta_\alpha(x,z)$$

and since, by (3.3), $\nu/c_\alpha^2 > \kappa_\alpha$ we can apply Lemma 2.3 to I_2 to obtain

$$\int_A |\mathrm{I}_2|\,d\mu(t) \leq c\,\delta^{\gamma}_{\alpha}(x,y)\,\delta^{\alpha-1-\gamma}_{\alpha}(x,z)\int_A \frac{1}{\delta^{1+\alpha}_{-\alpha}(x,t)}\,d\mu(t)\,.$$

The last expression is majorized by

$$c\,\frac{\delta^{\gamma}_{-\alpha}(x,y)}{\delta^{1+\gamma}_{-\alpha}(x,z)}\,.$$

To estimate $\int_A |\mathrm{I}_1|\,d\mu$ we will further subdivide A into

$$D_1 = \{t:\ \delta_{-\alpha}(x,t) > k\,c_\alpha\,\delta_{-\alpha}(x,z)\}$$

and

$$D_2 = \left\{t:\ \frac{1}{k}\,\delta_{-\alpha}(x,z) < \delta_{-\alpha}(x,t) \leq k\,c_\alpha\,\delta_{-\alpha}(x,z)\right\}.$$

For t in D_1 we have

$$\delta_{-\alpha}(x,t) > c_\alpha\,k\,\delta_{-\alpha}(x,z) > c_\alpha\,k\,\nu\,\delta_{-\alpha}(x,y)$$

and, by (3.3), $c_\alpha k\,\nu > \kappa_{-\alpha}$ and therefore we can apply Lemma 2.3 to obtain

$$\int_{D_1} |\mathrm{I}_1|\,d\mu(t) \leq c\,\delta^{\gamma}_{-\alpha}(x,y)\int_{D_1} \delta^{-\alpha-\gamma-1}_{-\alpha}(x,t)\,\delta^{\alpha-1}_{\alpha}(t,z)\,d\mu(t)\,.$$

Now note that for t in D_1

$$\left(1-\frac{\kappa_{-\alpha}}{c_\alpha\,k}\right)\delta_{-\alpha}(x,t) \leq c_\alpha\,\kappa_{-\alpha}\,\delta_\alpha(t,z)\,. \tag{3.5}$$

By (3.3), $1 - \nu_{-\alpha}/c_\alpha k > 0$, and hence by (3.5) we obtain

$$\int_{D_1} |\mathrm{I}_1|\, d\mu(t) \le c\, \frac{\delta^\gamma_{-\alpha}(x,y)}{\delta^{1+\gamma}_{-\alpha}(x,z)}\,.$$

To estimate $\int_{D_2} |\mathrm{I}_1|\, d\mu(t)$ observe that for t in D_2, by (3.2),

$$\nu\, \delta_{-\alpha}(x,y) < \delta_{-\alpha}(x,z) < k\, \delta_{-\alpha}(x,t)\,.$$

By (3.3) $\kappa_{-\alpha} < \nu/k$, and therefore we can apply Lemma 2.3 to the integral to get

$$|\mathrm{I}_1| \le c\, \frac{\delta^\gamma_{-\alpha}(x,y)}{\delta^{1+\alpha+\gamma}_{-\alpha}(x,t)}\, \frac{1}{\delta^{1-\alpha}_{-\alpha}(t,z)} \le c\, \frac{\delta^\gamma_{-\alpha}(x,y)}{\delta^{1+\alpha+\gamma}_{-\alpha}(x,z)}\, \frac{1}{\delta^{1-\alpha}_{-\alpha}(t,z)}\,.$$

Since D_2 is contained in the ball

$$B = \{t:\ \delta_{-\alpha}(t,z) \le (\kappa_{-\alpha}\, c_\alpha\, k + \kappa_{-\alpha})\, \delta_{-\alpha}(x,z)\}$$

we get

$$\int_{D_2} |\mathrm{I}_1|\, d\mu(t) \le \int_B |\mathrm{I}_1|\, d\mu(t) \le c\, \frac{\delta^\gamma_{-\alpha}(x,y)}{\delta^{1+\gamma}_{-\alpha}(x,z)}\,.$$

Now we estimate integral in (3.4) on $A^c = \{t:\, k^{-1}\, \delta_{-\alpha}(x,z) \ge \delta_{-\alpha}(x,t)\}$. We divide this region into two subregions:

$$B_1 = \{t:\ \delta_{-\alpha}(x,t) < k\, \delta_{-\alpha}(x,y)\}\,,$$
$$B_2 = A^c \backslash B_1 = \left\{t:\ k\, \delta_{-\alpha}(x,y) \le \delta_{-\alpha}(x,t) \le \frac{1}{k}\, \delta_{-\alpha}(x,z)\right\}.$$

We estimate first on B_1 :

$$\begin{aligned}
\int_{B_1} \Big| &\frac{1}{\delta^{1+\alpha}_{-\alpha}(x,t)} \Big(\frac{1}{\delta^{1-\alpha}_{\alpha}(t,z)} - \frac{1}{\delta^{1-\alpha}_{\alpha}(x,z)}\Big) \\
&- \frac{1}{\delta^{1+\alpha}_{-\alpha}(y,t)} \Big(\frac{1}{\delta^{1-\alpha}_{\alpha}(t,z)} - \frac{1}{\delta^{1-\alpha}_{\alpha}(y,z)}\Big)\Big|\, d\mu(t) \\
&\le \int_{B_1} \frac{1}{\delta^{1+\alpha}_{-\alpha}(x,t)} \Big|\frac{1}{\delta^{1-\alpha}_{\alpha}(t,z)} - \frac{1}{\delta^{1-\alpha}_{\alpha}(x,z)}\Big|\, d\mu(t) \\
&\quad + \int_{B_1} \frac{1}{\delta^{1+\alpha}_{-\alpha}(y,t)} \Big|\frac{1}{\delta^{1-\alpha}_{\alpha}(t,z)} - \frac{1}{\delta^{1-\alpha}_{\alpha}(y,z)}\Big|\, d\mu(t)\,.
\end{aligned}$$

Now observe that for t in A^c by (3.3) there are constants $c, c' > \kappa_\alpha$,

$$c = \frac{k}{c_\alpha^2}, \qquad c' = c_\alpha^2 \kappa_{-\alpha}^2 \frac{\frac{1}{k} + \frac{1}{\nu}}{1 - \frac{\kappa_{-\alpha}}{\nu}},$$

such that $c\,\delta_\alpha(x,t) \le \delta_\alpha(x,z)$ and $c'\,\delta_\alpha(y,t) \le \delta_\alpha(y,z)$; therefore we can apply Lemma 2.3 to the integrands of both terms to obtain, for the first term:

$$\begin{aligned}\int_{B_1} \frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)} \frac{c\,\delta_\alpha^\gamma(t,x)}{\delta_\alpha^{1-\alpha+\gamma}(x,z)}\, d\mu(t) &\\ &\le \frac{c}{\delta_\alpha^{1-\alpha+\gamma}(x,z)} \int_{B_1} \frac{1}{\delta_{-\alpha}^{1-\gamma+\alpha}(x,t)}\, d\mu(t) \\ &\le c\,\frac{\delta_{-\alpha}^{\gamma-\alpha}(x,y)}{\delta_{-\alpha}^{1+\gamma-\alpha}(x,z)},\end{aligned}$$

and for the second term:

$$\begin{aligned}\int_{B_1} \frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)} \frac{c\,\delta_\alpha^\gamma(t,y)}{\delta_\alpha^{1-\alpha+\gamma}(y,z)}\, d\mu(t) &\\ &\le \frac{1}{\delta_{-\alpha}^{1+\gamma-\alpha}(y,z)} \int_{B_1^*} \frac{1}{\delta_{-\alpha}^{1-\gamma+\alpha}(t,y)}\, d\mu(t),\end{aligned}$$

where $B_1^* = \{t : \delta_{-\alpha}(y,t) \le \kappa_{-\alpha}(k+1)\,\delta_{-\alpha}(x,y)\}$.

Integrating and using the fact that

$$\frac{1}{\kappa_{-\alpha}} \left(1 - \frac{\kappa_{-\alpha}}{\nu}\right) \delta_{-\alpha}(x,z) \le \delta_{-\alpha}(y,z),$$

the last integral is majorized by

$$c\,\frac{\delta_{-\alpha}^{\gamma-\alpha}(x,y)}{\delta_{-\alpha}^{1+\gamma-\alpha}(x,z)}.$$

Now we estimate

$$\int_{B_2} \left| \frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)} \left(\frac{1}{\delta_\alpha^{1-\alpha}(t,z)} - \frac{1}{\delta_\alpha^{1-\alpha}(x,z)} \right) \right.$$

$$
\begin{aligned}
&- \frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)}\Big(\frac{1}{\delta_{\alpha}^{1-\alpha}(t,z)} - \frac{1}{\delta_{\alpha}^{1-\alpha}(y,z)}\Big)\Big|\, d\mu(t) \\
&\quad \le \int_{B_2} \Big|\frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)} - \frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)}\Big| \\
&\qquad \cdot \Big|\frac{1}{\delta_{\alpha}^{1-\alpha}(t,z)} - \frac{1}{\delta_{\alpha}^{1-\alpha}(x,z)}\Big|\, d\mu(t) \\
&\qquad + \int_{B_2} \frac{1}{\delta_{-\alpha}^{1+\alpha}(y,t)} \Big|\frac{1}{\delta_{\alpha}^{1-\alpha}(y,z)} - \frac{1}{\delta_{\alpha}^{1-\alpha}(x,z)}\Big|\, d\mu(t) \\
&\quad = \mathrm{J}_1 + \mathrm{J}_2 \,.
\end{aligned}
$$

To estimate J_1 observe that $k\,\delta_{-\alpha}(x,y) \le \delta_{-\alpha}(x,t)$ and that

$$
\frac{k}{c_\alpha^2}\,\delta_\alpha(x,t) \le \delta_\alpha(x,z)\,,
$$

therefore we apply Lemma 2.3 to both brackets in absolute value to obtain

$$
\begin{aligned}
\mathrm{J}_1 &\le \int_{B_2} \frac{c\,\delta_{-\alpha}^{\gamma}(x,y)}{\delta_{-\alpha}^{1+\alpha+\gamma}(x,t)}\, \frac{\delta_{\alpha}^{\gamma}(x,t)}{\delta_{\alpha}^{1-\alpha+\gamma}(x,z)}\, d\mu(t) \\
&\le \frac{c\,\delta_{-\alpha}^{\gamma}(x,y)}{\delta_{\alpha}^{1+\gamma-\alpha}(x,z)} \int_{B_2} \frac{1}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t)\,.
\end{aligned}
$$

Note that $B_2 \subset \{t:\ k\,\delta_{-\alpha}(x,y) \le \delta_{-\alpha}(x,t)\}$ then integrating over this set we have that the last expression is majorized by

$$
c\,\frac{\delta_{-\alpha}^{\gamma-\alpha}(x,y)}{\delta_{-\alpha}^{1+\gamma-\alpha}(x,z)}\,.
$$

This concludes the proof of (1.14) with exponent $\eta = \gamma - \alpha$.

We will now prove the condition (1.15). Since $\delta \approx \delta_\alpha$ it suffices to prove condition (1.15) for δ_α, we will show that there are positive constants ν', M', η', $1 < \nu'$, $0 < M'$ and $0 < \eta' \le 1$, such that if

$$
\nu'\,\delta_\alpha(x,y) < \delta_\alpha(x,z) \tag{3.6}
$$

then

$$
|K(z,x) - K(z,y)| \le M'\,\frac{\delta_\alpha^{\eta'}(x,y)}{\delta_\alpha^{1+\eta'}(x,z)}\,.
$$

We choose $\kappa_\alpha < \nu'$ and $\kappa_\alpha < k'$ such that

$$\frac{\kappa_\alpha^2}{\nu'} < 1 - \frac{\kappa_\alpha}{k'} . \tag{3.7}$$

Let x, y, z be fixed points satisfying (3.6). To estimate $|K(z,x) - K(z,y)|$ observe first that

$$|K(z,x) - K(z,y)| \le \int_X \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \Big| \frac{1}{\delta_\alpha^{1-\alpha}(t,x)} - \frac{1}{\delta_\alpha^{1-\alpha}(z,x)} - \frac{1}{\delta_\alpha^{1-\alpha}(t,y)} + \frac{1}{\delta_\alpha^{1-\alpha}(z,y)} \Big| \, d\mu(t) . \tag{3.8}$$

To estimate we divide X into three regions:

$$A = \Big\{ t : \ \delta_\alpha(z,t) < \frac{1}{k'} \min\{\delta_\alpha(y,z), \delta_\alpha(x,z)\} \Big\} ,$$

$$B = \Big\{ t : \ \frac{1}{k'} \min\{\delta_\alpha(y,z), \delta_\alpha(x,z)\} \le \delta_\alpha(z,t) < k' \, \delta_\alpha(x,z) \Big\} ,$$

and

$$C = \{ t : \ k' \, \delta_\alpha(x,z) \le \delta_\alpha(z,t) \} .$$

To estimate the integral on A we further subdivide A into two subregions:

$$A_1 = \{ t : \delta_\alpha(z,t) \le k' \, \delta_\alpha(x,y) \} , \qquad \text{and} \qquad A_2 = A \backslash A_1 .$$

The integral over A_1 is less than or equal to

$$\int_{A_1} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \Big| \frac{1}{\delta_\alpha^{1-\alpha}(t,x)} - \frac{1}{\delta_\alpha^{1-\alpha}(z,x)} \Big| \, d\mu(t) + \int_{A_1} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \Big| \frac{1}{\delta_\alpha^{1-\alpha}(z,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(t,y)} \Big| \, d\mu(t) .$$

Note that for t in A, $k' \, \delta_\alpha(z,t) < \delta_\alpha(x,z)$ and $k' \, \delta_\alpha(z,t) < \delta_\alpha(y,z)$ by (3.7), $\kappa_\alpha < k'$. Therefore we can apply Lemma 2.3 in both integrands. The first term is majorized by

$$\frac{c}{\delta_\alpha^{1-\alpha+\gamma}(z,x)} \int_{A_1} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \, \delta_\alpha^\gamma(t,z) \, d\mu(t) \le c \, \frac{\delta_\alpha^{\gamma-\alpha}(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(z,x)} .$$

The second term is majorized by

$$\frac{c}{\delta_\alpha^{1-\alpha+\gamma}(z,y)} \int_{A_1} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)}\, \delta_\alpha^\gamma(z,t)\, d\mu(t) \le c\, \frac{\delta_\alpha^{\gamma-\alpha}(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(z,y)}\,,$$

since by (3.6)

$$\frac{1}{\kappa_\alpha}\left(1-\frac{\kappa_\alpha}{\nu'}\right)\delta_\alpha(x,z) \le \delta_\alpha(y,z)\,,$$

the last expression is less than or equal to

$$c\,\frac{\delta_\alpha^{\gamma-\alpha}(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(z,x)}\,.$$

The integral over A_2 is less than or equal to

$$\int_{A_2} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \left|\frac{1}{\delta_\alpha^{1-\alpha}(t,x)} - \frac{1}{\delta_\alpha^{1-\alpha}(t,y)}\right| d\mu(t) + \int_{A_2} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \left|\frac{1}{\delta_\alpha^{1-\alpha}(z,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(z,x)}\right| d\mu(t)$$

and for t in A_2, by (3.7) there is $a > \kappa_\alpha$,

$$\nu'\left(1-\frac{\kappa_\alpha}{k'}\right)\frac{1}{\kappa_\alpha} = c\,,$$

such that $a\,\delta_\alpha(x,y) < \delta_\alpha(x,t)$ also by (3.6) $\nu'\,\delta_\alpha(x,y) < \delta_\alpha(x,z)$ and by (3.7), $\kappa_\alpha < \nu'$ therefore we can apply Lemma 2.3 in both terms.

The first term is less than or equal to

$$\delta_\alpha^\gamma(x,y) \int_{A_2} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)}\, \frac{1}{\delta_\alpha^{1-\alpha+\gamma}(x,t)}\, d\mu(t) \le c\,\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(x,z)} \int_{A_2} \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)}\, d\mu(t)$$

because for t in A_2,

$$\delta_\alpha(x,z)\left(1-\frac{\kappa_\alpha}{k'}\right) \le \kappa_\alpha\,\delta_\alpha(t,x)\,.$$

Now integrating over $\{t : \delta_\alpha(z,t) > k'\,\delta_\alpha(x,y)\}$ we get that the last expression is majorized by

$$c\,\frac{\delta_\alpha^{\gamma-\alpha}(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(x,z)}\,.$$

The second term is less than or equal to

$$\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(x,z)} \int_{A_2} \frac{1}{\delta_\alpha^{1+\alpha}(z,t)}\, d\mu(t)\,.$$

Integrating as before over $\{t : \delta_\alpha(z,t) > k'\,\delta_\alpha(x,y)\}$ the last expression is majorized by

$$c\,\frac{\delta_\alpha^{\gamma-\alpha}(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(x,z)}\,.$$

The integral over B is less than or equal to

$$\int_B \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \left|\frac{1}{\delta_\alpha^{1-\alpha}(z,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(z,x)}\right| d\mu(t) + \int_B \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)} \left|\frac{1}{\delta_\alpha^{1-\alpha}(t,x)} - \frac{1}{\delta_\alpha^{1-\alpha}(t,y)}\right| d\mu(t)\,.$$

To estimate the first integral observe that by (3.6) and (3.7) we can apply Lemma 2.3 to the integrand, and majorize this integral by

$$\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1-\alpha+\gamma}(x,z)} \int_B \frac{d\mu(t)}{\delta_{-\alpha}^{1+\alpha}(z,t)}\,.$$

Now integrating over

$$\left\{t :\ \delta_\alpha(z,t) > \frac{1}{k'}\,\min\{\delta_\alpha(y,z), \delta_\alpha(x,z)\right\}$$

and using

$$\frac{1}{\kappa_\alpha}\left(1 - \frac{\kappa_\alpha}{\nu'}\right)\delta_\alpha(x,z) \le \delta_\alpha(y,z)$$

we get that the last expression is majorized by

$$c\,\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\gamma}(x,z)}\,.$$

To estimate the second term we consider two regions

$$D_1 = B \cap \{t :\ \delta_\alpha(t,x) < k'\,\delta_\alpha(x,y)\}\,,$$

and

$$D_2 = B \cap \{t :\ \delta_\alpha(t,x) \ge k'\,\delta_\alpha(x,y)\}\,.$$

By (3.6) and (3.7) there are constants a_1 and a_2 such that

$$a_1\,\delta_\alpha(z,y) \le \delta_\alpha(z,x) \le a_2\,\delta_\alpha(z,y)\,,$$

therefore the integral over D_1 is less than or equal to

$$\frac{c}{\delta_\alpha^{1+\alpha}(z,x)}\int_{D_1}\frac{d\mu(t)}{\delta_\alpha^{1-\alpha}(t,x)} + \frac{c}{\delta_\alpha^{1+\alpha}(z,x)}\int_{D_1}\frac{d\mu(t)}{\delta_\alpha^{1-\alpha}(t,y)}\,.$$

Since $D_1 \subset \{t : \delta_\alpha(t,x) < k'\,\delta_\alpha(x,y)\}$ integrating over this ball we get that the first term is majorized by $c\,\delta_\alpha^\alpha(x,y)/\delta_\alpha^{1+\alpha}(x,z)$. On the other hand

$$D_1 \subset \{t :\ \delta_\alpha(t,y) \le \kappa_\alpha\,(k'+1)\,\delta_\alpha(x,y)\}\,.$$

Therefore integrating over this ball the second integral is majorized by $c\,\delta_\alpha^\alpha(x,y)/\delta_\alpha^{1+\alpha}(x,z)$. For t in D_2, $\delta_\alpha(t,x) \ge k'\,\delta_\alpha(x,y)$, and therefore we can apply Lemma 2.3 to the integrand, and majorize the integral by

$$c\,\delta_\alpha^\gamma(x,y)\int_{D_2}\frac{d\mu(t)}{\delta_\alpha^{1+\alpha}(z,t)\,\delta_\alpha(x,t)^{1+\gamma-\alpha}}\,.$$

On the other hand there is $a_3 > 0$ such that $a_3\,\delta_\alpha(x,z) \le \delta_\alpha(z,t)$, and therefore the last expression is less than or equal to

$$c\,\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\alpha}(z,x)}\int_{D_2}\frac{d\mu(t)}{\delta_\alpha^{1+\gamma-\alpha}(x,t)}\,.$$

Now integrating over $\{t : \delta_\alpha(t,x) \ge k'\,\delta(x,y)\}$ the last expression is majorized by

$$c\,\frac{\delta_\alpha^\alpha(x,y)}{\delta_\alpha^{1+\alpha}(x,z)}\,.$$

Finally, we will estimate the integral over C. This integral is less than or equal to

$$\int_C \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)}\left|\frac{1}{\delta_\alpha^{1-\alpha}(t,x)} - \frac{1}{\delta_\alpha^{1-\alpha}(t,y)}\right| d\mu(t) + \int_C \frac{1}{\delta_{-\alpha}^{1+\alpha}(z,t)}\left|\frac{1}{\delta_\alpha^{1-\alpha}(z,y)} - \frac{1}{\delta_\alpha^{1-\alpha}(z,x)}\right| d\mu(t)\,.$$

For t in C by (3.6) and (3.7) there are constants a_4, a_5

$$a_4 = \frac{k'-\kappa}{\kappa}\,\nu'\,, \qquad a_5 = \frac{\kappa_\alpha}{k'\left(1-\dfrac{\kappa_\alpha}{k'}\right)}\,,$$

such that $a_4\,\delta_\alpha(x,y) \le \delta_\alpha(x,t)$ and $\delta_\alpha(x,z) \le a_5\,\delta_\alpha(x,t)$. Therefore we can apply Lemma 2.3 to the integrand of the first term, and majorize the integral by

$$c\,\delta_\alpha^\gamma(x,y)\int_C \frac{d\mu(t)}{\delta_\alpha^{1+\alpha}(t,z)\,\delta_\alpha^{1+\gamma-\alpha}(x,t)} \le c\,\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(x,z)}\int_C \frac{d\mu(t)}{\delta_\alpha^{1+\alpha}(t,z)} \le c\,\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\gamma}(x,z)}\,.$$

By (3.6), $\nu'\,\delta_\alpha(x,y) \le \delta_\alpha(x,z)$, and by (3.7), $\kappa_\alpha < \nu'$, therefore we can apply Lemma 2.3 to the integrand of the second term and majorize the integral by

$$c\,\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\gamma-\alpha}(x,z)}\int_C \frac{d\mu(t)}{\delta_\alpha^{1+\alpha}(t,z)} \le c\,\frac{\delta_\alpha^\gamma(x,y)}{\delta_\alpha^{1+\gamma}(x,z)}\,.$$

To conclude the proof choose η' to be α.

PROOF OF THEOREM 1.4. To prove Theorem 1.4 we will use the "$T1$ theorem" (see [Ch]), *i.e.* "A singular integral operator T is a Calderón-Zygmund operator if and only if

1) T is weakly bounded,

2) $T1 \in$ BMO,

3) ${}^tT1 \in$ BMO."

We recall that an operator $T : C_0^\eta \to (C_0^\eta)'$ is *weakly bounded* if there exists a constant c such that

$$|\langle Tf, g\rangle| \le c\,\mu(B)^{1+2\eta}\,\|f\|_\eta\,\|g\|_\eta \tag{3.9}$$

for every f, g in $C_0^\eta(B)$ and for every ball B.

We will show that

i) T_α is weakly bounded,

ii) $T_\alpha 1 = 0$,

iii) ${}^tT_\alpha 1 = 0$,

To prove i) we will show first the following estimate for $f \in C_0^\eta(B)$, $0 < \eta + \alpha < \gamma$,

$$\|T_\alpha f\|_\infty \leq c\,\mu(B)^\eta \|f\|_\eta \,. \tag{3.10}$$

Consider $f \in C_0^\eta(B)$, $B = B_r(x_0)$. Observe that

$$|I_\alpha f(x)| \leq \int_B \frac{|f(y)|}{\delta_\alpha^{1-\alpha}(x,y)}\, d\mu(y) \leq c\,\|f\|_\infty\,(\mu(B))^\alpha \,.$$

Now

$$\begin{aligned} |T_\alpha f(x)| &\leq \int \frac{|(I_\alpha f)(t) - (I_\alpha f)(x)|}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t) \\ &\leq \int\limits_{\delta(x,t)<r} \frac{|I_\alpha f(t) - I_\alpha f(x)|}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t) \\ &\quad + \int\limits_{\delta(x,t)\geq r} \frac{|I_\alpha f(t) - I_\alpha f(x)|}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t) \,. \end{aligned}$$

To estimate the first integral we use the fact proved in Theorem 1.1 that $|I_\alpha f(t) - I_\alpha f(x)| \leq c\,\|f\|_\eta\, \delta_{-\alpha}^{\eta+\alpha}(t,x)$ then integrating this integral we see that it is less than or equal to $c\,\|f\|_\eta (\mu(B))^\eta$. For the second integral we use the estimate for $I_\alpha f$ obtained above and integrating we obtain that this integral is less than or equal $c\,\|f\|_\infty$.

Note that for $f \in C_0^\eta(B)$, $\|f\|_\infty \leq c\,\|f\|_\eta\,\mu(B)^\eta$; this concludes the proof of (3.10). Let f and g be in $C_0^\eta(B)$, then

$$\begin{aligned} |\langle T_\alpha f, g\rangle| &\leq \int_B |T_\alpha f(x)|\; |g(x)|\, d\mu(x) \\ &\leq \|T_\alpha f\|_\infty\; \|g\|_\infty\; \mu(B) \\ &\leq c\,\mu(B)^{1+2\eta}\; \|f\|_\eta\; \|g\|_\eta \,. \end{aligned}$$

The last inequality follows from (3.10) and the fact that $g \in C_0^\eta(B)$.

To prove ii) we observe that the extension of T_α to $L^\infty \cap \mathrm{Lip}\,(\eta)$ coincides with the operator $\widetilde{T}_\alpha = \widetilde{D}_\alpha \widetilde{I}_\alpha$. Since, by Lemma 2.4, $\widetilde{I}_\alpha 1 = 0$ we have $T_\alpha 1 = \widetilde{T}_\alpha 1 = 0$.

To prove iii) we use the following fact

$${}^tT_\alpha = I_\alpha D_\alpha \,, \tag{3.11}$$

which will be proved in Theorem 1.5. Now, since $D_\alpha 1 = 0$ we have ${}^tT_\alpha 1 = 0$. This concludes the proof of the Theorem.

PROOF OF THEOREM 1.5. Let $S_\alpha = I_\alpha D_\alpha$ and consider f and g in C_o^β, $0 < \alpha + \beta \leq \gamma$. We want to show that

$$\langle T_\alpha f, g\rangle = \langle f, S_\alpha g\rangle \ . \tag{3.12}$$

We will show first that for $f \in L^\infty \cap \mathrm{Lip}\,(\eta)$, $\alpha < \eta \leq \gamma$ and $g \in C_0^\beta$,

$$\langle D_\alpha f, g\rangle = \langle f, D_\alpha g\rangle \ . \tag{3.13}$$

For every $f \in L^\infty \cap \mathrm{Lip}\,(\eta)$, note that

$$\int \frac{|f(t) - f(x)|}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t)$$

is bounded as a function of x and therefore

$$\langle D_\alpha f, g\rangle = \iint \frac{f(t) - f(x)}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, g(x)\, d\mu(t)\, d\mu(x)$$

because the double integral converges absolutely. Now rewrite this integral as follows

$$\iint \frac{f(t)\, g(x) - f(x)\, g(t)}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t)\, d\mu(x) + \iint \frac{f(x)\, g(t) - f(x)\, g(x)}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t)\, d\mu(x) \,.$$

The second integral converges absolutely since for $g \in C_0^\beta(B_r(x_0))$

$$\int \frac{|g(t) - g(x)|}{\delta_{-\alpha}^{1+\alpha}(x,t)}\, d\mu(t) \leq \frac{c}{1 + \delta_{-\alpha}^{1+\alpha}(x_0, x)} \ ,$$

and it is equal to $\langle f, D_\alpha g\rangle$. Since the second integral is absolutely convergent so is the first one, and it is equal to zero because its integrand, $h(x,t)$, satisfies $h(x,t) = -h(t,x)$.

Now consider f and g in C_o^β. It was shown before (see Theorem 1.1 and (3.10)) that $I_\alpha f \in L^\infty \cap \mathrm{Lip}\,(\eta)$, therefore

$$\langle D_\alpha I_\alpha f, g\rangle = \langle I_\alpha f, D_\alpha g\rangle = \iint D_\alpha g(x)\, \frac{f(t)}{\delta_\alpha^{1-\alpha}(x,t)}\, d\mu(t)\, d\mu(x) \,.$$

Since $I_\alpha|f| \in L^\infty$ and

$$|D_\alpha g(x)| \leq \frac{c}{1+\delta_{-\alpha}^{1+\alpha}(x,x_0)}\,,$$

the double integral converges absolutely and by Fubini's theorem is equal to $\langle f, I_\alpha D_\alpha g\rangle$.

The fact that $S_\alpha = {}^tT_\alpha$ is a Calderón-Zygmund operator follows from the fact that T_α is a Calderón-Zygmund operator.

4. Lemmas needed for the proofs of Theorem 1.6 and Theorem 1.7

Lemma 4.1. *The kernel $q(x,y,t)$ defined in* (1.16) *has the following properties*

i) $q(x,y,t) = q(y,x,t)$ *for all* x,y *in* X, *and* $t>0$,

ii) $q(x,y,t) = 0$ *if* $\delta(x,y) > b_1\, t$,

iii) $|q(x,y,t)| \leq c_4/t$ *for all* x,y *in* X *and* $t>0$,

iv) $|q(x,y,t) - q(x',y,t)| \leq c_5 \dfrac{\delta^\gamma(x,x')}{t^{1+\gamma}}$ *for all* x,x',y *in* X *and* $t>0$,

v) $\displaystyle\int q(x,y,t)\,d\mu(y) = 0$ *for all* x *in* X *and* $t>0$.

Lemma 4.1 is the continuous version of known results [N2], [DJS].

Lemma 4.2. *The integral operator Q_t introduced in* (1.17) *satisfy the following estimates*

$$\|Q_tQ_s\| \leq c \begin{cases} \left(\dfrac{s}{t}\right)^\gamma, & \text{if } 0<s\leq t\,, \\ \left(\dfrac{t}{s}\right)^\gamma, & \text{if } 0<t\leq s\,, \end{cases}$$

where the norm is the operator norm in L^2, and c is a constant independent of s and t.

From this result one obtains the next lemma.

Lemma 4.3. *For positive r, s, t define a function $h_t(s,r)$ as follows*

$$h_t(s,r) = \begin{cases} \min\left\{t^\gamma, \left(\frac{s}{r}\right)^{\gamma/2}\right\}, & \text{if } \frac{s}{r} \le 1 \text{ and } 0 < t \le 1, \\ \min\left\{t^\gamma, \left(\frac{s}{r}\right)^{-\gamma/2}\right\}, & \text{if } \frac{s}{r} > 1 \text{ and } 0 < t \le 1, \end{cases}$$

$$h_t(s,r) = \begin{cases} \min\left\{t^{-\gamma}, \left(\frac{s}{r}\right)^{\gamma/2}\right\}, & \text{if } \frac{s}{r} \le 1 \text{ and } t > 1, \\ \min\left\{t^{-\gamma}, \left(\frac{s}{r}\right)^{-\gamma/2}\right\}, & \text{if } \frac{s}{r} > 1 \text{ and } t > 1. \end{cases}$$

Then the operators Q_t introduced in (1.17) satisfy

$$\|Q_s Q_{ts}(Q_r Q_{tr})^*\| \le h_t^2(s,r)$$

and

$$\|(Q_s Q_{ts})^* Q_r Q_{tr}\| \le h_t^2(s,r).$$

Moreover, setting

$$c(t) = \sup_s \int_0^\infty h_t(s,r)\,\frac{dr}{r}$$

one has the estimates

$$c(t) \le c \begin{cases} t^\gamma + t^\gamma \log\frac{1}{t}, & \text{for } 0 < t \le 1, \\ t^{-\gamma} + t^{-\gamma}\log t, & \text{for } t > 1. \end{cases} \tag{4.1}$$

Lemmas 4.2 and 4.3 are continuous versions of known results, see *e.g.* [DJS] and [N2].

Lemma 4.4. *Let Q_t be the operators defined by* (1.17), *and let $f \in L^2(X)$, then for every set E of finite measure of the measure space $([0,\infty), dt/t)$ one has*

$$\left\| \int_E Q_s Q_{st} f\,\frac{ds}{s} \right\|_2 \le c(t)\,\|f\|_2\,,$$

where $c(t)$ is the quantity introduced in Lemma 4.3. *Furthermore*

$$W_t f = \int_0^\infty Q_s Q_{ts} f\,\frac{ds}{s}$$

exists in the weak L^2 sense and satisfies

$$\|W_t f\|_2 \leq c(t)\,\|f\|_2\,. \tag{4.2}$$

This result follows from Lemma 4.3 and the continuous version of the Cotlar-Knapp-Stein Lemma, see [CV] and [F].

5. Proofs of Theorems 1.6 and 1.7.

PROOF OF THEOREM 1.6. To prove (1.19) observe that for $f \in \mathrm{Lip}\,(\beta) \cap L^1$ the integral (1.10) converges absolutely for every x, therefore using (1.8) we have

$$\alpha\, I_\alpha f(x) = \alpha \int_X \int_0^\infty t^{\alpha-1}\, s(x,y,t)\, dt\; f(y)\, d\mu(y)$$

and the double integral converges absolutely for every x. Then by changing the order of integration we obtain

$$\alpha\, I_\alpha f(x) = \alpha \int_0^\infty t^{\alpha-1}\, u(x,t)\, dt\,, \tag{5.1}$$

where

$$u(x,t) = \int_X s(x,y,t)\, f(y)\, d\mu(y)\,. \tag{5.2}$$

Since

$$\frac{\partial}{\partial t}\, s(x,y,t) = \frac{1}{t}$$

and $q(x,y,t)$ has the properties ii) and iii) of Lemma 4.1, we can differentiate with respect to t under the integral sign of (5.2) to get

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{t}\, v(x,t)\,, \tag{5.3}$$

where

$$v(x,t) = \int_X q(x,y,t)\, f(y)\, d\mu(y) = -Q_t f(x)\,. \tag{5.4}$$

Now integrating the integral in (5.1) by parts, using (5.3) and the fact that $f \in \mathrm{Lip}(\beta) \cap L^1$ we obtain

$$\begin{aligned} \alpha\, I_\alpha f(x) &= \lim_{\substack{a\to 0\\ b\to\infty}} \alpha \int_a^b t^{\alpha-1}\, u(x,t)\, dt \\ &= \lim_{\substack{a\to 0\\ b\to\infty}} (t^\alpha\, u(x,t)|_a^b) - \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b t^\alpha\, v(x,t)\, \frac{dt}{t} \\ &= \int_0^\infty t^\alpha\, Q_t(f)(x)\, \frac{dt}{t}\,. \end{aligned}$$

To prove (1.20) observe that for $f \in \mathrm{Lip}(\beta) \cap L^\infty$ the integral (1.9) converges absolutely for every x, therefore using (1.8) we have

$$-\alpha\, D_\alpha f(x) = -\alpha \int_X \int_0^\infty t^{-\alpha-1}\, s(x,y,t)\, dt\, (f(y) - f(x))\, d\mu(y)$$

and the double integral converges absolutely. Then by changing the order of integration we have

$$-\alpha\, D_\alpha f(x) = -\alpha \int_0^\infty t^{-\alpha-1}\, (u(x,t) - f(x))\, dt\,. \tag{5.5}$$

Now integrating the integral in (5.5) by parts, using (5.3), and the fact that $f \in \mathrm{Lip}(\beta) \cap L^\infty$, $\alpha < \beta \le \gamma$, we obtain

$$\begin{aligned} -\alpha\, D_\alpha f(x) &= \lim_{\substack{a\to 0\\ b\to\infty}} -\alpha \int_a^b t^{-\alpha-1}\, (u(x,t) - f(x))\, dt \\ &= \lim_{\substack{a\to 0\\ b\to\infty}} (t^{-\alpha}\, (u(x,t) - f(x))|_a^b) - \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b t^{-\alpha}\, v(x,t)\, \frac{dt}{t} \\ &= \int_0^\infty t^{-\alpha}\, Q_t(f)(x)\, \frac{dt}{t}\,. \end{aligned}$$

The fact that representation formulas hold in the weak sense, *i.e.* that

$$\lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b t^\alpha\, \langle Q_t f, \varphi\rangle\, \frac{dt}{t} = \langle \alpha\, I_\alpha f, \varphi\rangle$$

and

$$\lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b t^{-\alpha}\, \langle Q_t f, \varphi\rangle\, \frac{dt}{t} = -\langle \alpha\, D_\alpha f, \varphi\rangle$$

for all $\varphi \in C_0^\mu$, $0 < \mu < \gamma$, follows from observing that the double integrals

$$\int_0^\infty \int_X t^\alpha \, Q_t f(x) \, \varphi(x) \, d\mu(x) \, \frac{dt}{t}$$

and

$$\int_0^\infty \int_X t^{-\alpha} \, Q_t f(x) \, \varphi(x) \, d\mu(x) \, , \frac{dt}{t}$$

are absolutely convergent.

This concludes the proof of Theorem 1.6.

PROOF OF THEOREM 1.7. The proof of Theorem 1.7 is a continuous version of the method of Nahmod [N2].

We first show that for $0 \leq \alpha < \gamma$ the integral

$$\int_0^\infty \int_0^\infty s^{-\alpha} \, t^\alpha \, \langle Q_s Q_t f, g \rangle \, \frac{ds}{s} \, \frac{dt}{t} \tag{5.6}$$

converges absolutely for f and g in $L^2(X)$. We make the following change of variables in (5.6)

$$s = u \, , \qquad t = u \, v \, ,$$

and obtain

$$\int_0^\infty \int_0^\infty v^\alpha \, \langle Q_u Q_{uv} f, g \rangle \, \frac{du}{u} \, \frac{dv}{v} \, . \tag{5.7}$$

By Cotlar's lemma (Lemma 4.3) it can be seen that

$$\int_0^\infty |\langle Q_u Q_{uv} f, g \rangle| \, \frac{du}{u} \leq 4 \, c(v) \, \|f\|_2 \, \|g\|_2 \, ,$$

where $c(v)$ is the constant of Lemma 8. Using the estimates (4.1) for $c(v)$ one sees that (5.7), and therefore (5.6) are absolutely convergent.

We will show next that for $f \in C_0^\beta$, $g \in C_0^\mu$, where $0 < \alpha + \beta < \gamma$, and $0 < \mu < \gamma$, (5.6) is equal to $-\alpha^2 \, \langle T_\alpha f, g \rangle$ for $0 < \alpha < \gamma$, and equal to $\langle f, g \rangle$ for $\alpha = 0$.

In other words

$$f = \int_0^\infty \int_0^\infty Q_t Q_s f \, \frac{ds}{s} \, \frac{dt}{t} \, , \tag{5.8}$$

and

$$-\alpha^2 T_\alpha f = \int_0^\infty \int_0^\infty s^{-\alpha} t^\alpha Q_s Q_t f \frac{ds}{s}\frac{dt}{t}\,, \tag{5.9}$$

where the integrals are in the weak L^2 sense. The equality (5.8) is a well known formula of Coifman, see [C]. Let, then, $f \in C_0^\beta$, $0 < \alpha + \beta < \gamma$ and $g \in C_0^\mu$, $0 < \mu < \gamma$. Since (5.6) is absolutely convergent it can be written as an iterated integral

$$\begin{aligned}
&\int_0^\infty \int_0^\infty s^{-\alpha} t^\alpha \langle Q_s Q_t f, g\rangle \frac{ds}{s}\frac{dt}{t} \\
&\qquad = \int_0^\infty s^{-\alpha} \Big(\int_0^\infty t^\alpha \langle Q_s Q_t f, g\rangle \frac{dt}{t}\Big) \frac{ds}{s} \\
&\qquad = \int_0^\infty s^{-\alpha} \Big(\int_0^\infty t^\alpha \langle Q_t f, Q_s g\rangle \frac{dt}{t}\Big) \frac{ds}{s} \\
&\qquad = \int_0^\infty s^{-\alpha} \Big\langle \int_0^\infty t^\alpha Q_t f \frac{dt}{t}, Q_s g\Big\rangle \frac{ds}{s} \\
&\qquad = \int_0^\infty s^{-\alpha} \langle \alpha I_\alpha f, Q_s g\rangle \frac{ds}{s} \\
&\qquad = \alpha \int_0^\infty s^{-\alpha} \langle Q_s (I_\alpha f), g\rangle \frac{ds}{s} \\
&\qquad = \alpha \Big\langle \int_0^\infty s^{-\alpha} Q_s (I_\alpha f) \frac{ds}{s}, g\Big\rangle \\
&\qquad = -\alpha^2 \langle D_\alpha I_\alpha f, g\rangle \\
&\qquad = -\alpha^2 \langle T_\alpha f, g\rangle\,.
\end{aligned}$$

The above chain of equalities is easily justified by using the properties of the kernel $q(x, y, t)$ and Theorem 1.6. For $\alpha = 0$ the calculation is quite similar except for the fact that instead of Theorem 1.6 one uses the known identity

$$\int_0^\infty Q_t f \frac{dt}{t} = f\,.$$

Let now $0 < \alpha$, $f \in C_0^\beta$, $\alpha + \beta < \gamma$. Using that (5.6) and (5.7) are the same we can write

$$(I + \alpha^2 T_\alpha) f = \int_0^\infty (1 - v^\alpha) W_v f \frac{dv}{v}\,,$$

where W_v is the operator defined in Lemma 4.4. Applying the estimate (4.2) we obtain

$$\|(I+\alpha^2 T_\alpha)f\|_2 \leq \int_0^\infty |1-v^\alpha|\, c(v)\,\frac{dv}{v}\,\|f\|_2 \,.$$

To estimate the last integral we write it as the sum

$$\int_0^{1/N} |1-v^\alpha|\, c(v)\,\frac{dv}{v} + \int_{1/N}^{N} |1-v^\alpha|\, c(v)\,\frac{dv}{v} + \int_N^\infty |1-v^\alpha|\, c(v)\,\frac{dv}{v} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 \,.$$

Using the estimate (4.1) for $c(v)$ we can find $N = N_0$ sufficiently large so that I_1 and I_3 are less than $1/4$ uniformly with respect to α with α in $(0, \gamma']$ for a fixed γ' less than γ. Having chosen N_0 we can find an α_0 such that for $0 < \alpha < \alpha_0$, I_2 is less than $1/2$. Therefore $\|I+\alpha^2 T_\alpha\| < 1$, and hence $-\alpha^2 T_\alpha$ is invertible and therefore so is T_α.

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