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Generalized Fock spaces, interpolation, multipliers, circle geometry

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Abstract. By a (generalized) Fock space we understand a Hilbert space of entire analytic functions in the complex plane $\mathbb{C}$ which are square integrable with respect to a weight of the type $e^{-Q(z)}$, where $Q(z)$ is a quadratic form such that $\operatorname{tr} Q > 0$. Each such space is in a natural way associated with an (oriented) circle $\mathcal{C}$ in $\mathbb{C}$. We consider the problem of interpolation between two Fock spaces. If $\mathcal{C}_0$ and $\mathcal{C}_1$ are the corresponding circles, one is led to consider the pencil of circles generated by $\mathcal{C}_0$ and $\mathcal{C}_1$. If H is the one parameter Lie group of Moebius transformations leaving invariant the circles in the pencil, we consider its complexification H^c, which permutes these circles and with the aid of which we can construct the "Calderón curve" giving the complex interpolation. Similarly, real interpolation leads to a multiplier problem for the transformation that diagonalizes all the operators in H^c. It turns out that the result is rather sensitive to the nature of the pencil, and we obtain nearly complete results for elliptic and parabolic pencils only.

Introduction.

In this paper we shall understand by a *generalized Fock space* a Hilbert space of entire analytic functions in the complex plane $\mathbb{C}$ which are square integrable with respect to a weight of the type $e^{-Q(z)}$, where Q is a real quadratic form such that $\operatorname{tr} Q > 0$.

Such a quadratic form can be written as

$$Q(z) = k\,|z|^2 - \operatorname{Re}(l\,z^2)\,,$$

where k is a positive number ($k > 0$) and l is a complex number. Indeed, putting $z = x + iy$, we have

$$Q(z) = (k - \operatorname{Re} l)\,x^2 + (k + \operatorname{Re} l)\,y^2 + 2\,(\operatorname{Im} l)\,x\,y\,,$$

so that there are enough parameters to describe the most general real quadratic form. Moreover, we have

$$\operatorname{tr} Q = (k - \operatorname{Re} l) + (k + \operatorname{Re} l) = 2k > 0\,,$$

while

$$\det Q = (k - \operatorname{Re} l)(k + \operatorname{Re} l) - (\operatorname{Im} l)^2 = k^2 - |l|^2\,.$$

Thus, our spaces are labelled by pairs (k, l) and shall henceforth be denoted by $F_{(k,l)}$. If $f \in F_{(k,l)}$, its norm $\|f\|_{(k,l)}$ will be defined by

$$\|f\|^2_{(k,l)} = \frac{k^{1/2}}{\pi} \int_{\mathbb{C}} |f(z)|^2\, e^{-k|z|^2 + \operatorname{Re}(lz^2)}\, dm(z)\,,$$

where we have written $dm(z) = dx\,dy$ (Euclidean measure). The corresponding inner product will be written $\langle f, f_1\rangle_{(k,l)}$ if $f, f_1 \in F_{(k,l)}$.

More generally, for $0 < p \le \infty$ we let $F^p_{(k,l)}$ be the space of entire analytic functions f such that (with the usual interpretation as a supremum if $p = \infty$)

$$\|f\|^p_{(k,l);p} = \frac{k^{1/2}}{\pi} \int_{\mathbb{C}} \left(|f(z)|e^{-(k|z|^2 + \operatorname{Re}(lz^2))/2}\right)^p\, dm(z) < \infty\,;$$

for $p \ge 1$ the expression $\|f\|_{(k,l);p}$ is a norm and we have a Banach space; if $p < 1$ it is a quasi-norm and we have a quasi-Banach space. (In Section 4 we shall also briefly say a few words about (generalized)

Orlicz-Fock spaces $F^{\Phi}_{(k,l)}$.) Most of the time we shall however take $p = 2$.

EXAMPLE 1. In the paper [6] two special cases were considered:

$$F_{(k,0)} =: F_k \quad - \quad \text{weight } e^{-k|z|^2},$$
$$F_{(k,k)} =: G_k \quad - \quad \text{weight } e^{-2ky^2};$$

there the normalization was a slightly different one.

It will also be convenient to consider a certain limiting case of the spaces $F_{(k,l)}$, namely the case $l = ke^{i\theta}$, $k \to \infty$. To fix the ideas take first $\theta = 0$. Then formally

$$\frac{k^{1/2}}{\pi} \int_{\mathbb{C}} |f(z)|^2 \, e^{-2ky^2} \, dx \, dy \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|^2 \, dx \,.$$

This is because

$$\frac{k^{1/2}}{\pi} \, e^{-2ky^2} \, dy \to \frac{1}{\sqrt{2\pi}} \, \delta(y) \qquad \text{(Dirac measure)}.$$

Thus we are led to the space $L^2(\mathbb{R})$ of square integrable (non-analytic!) functions on the real line $\mathbb{R}$ equipped with the measure $dx/\sqrt{\pi}$. In the same way, for general θ we obtain the space $L^2(e^{-i\theta}\mathbb{R})$ of square integrable functions on the line $e^{-i\theta/2}\mathbb{R}$. We shall, alternatively, denote this space by S_θ (*Schrödinger space*). Its exact significance will be made more clear later on (see Example 1 in Section 1). We remark however right away that it should be viewed not primarily as a Lebesgue space, but as the completion in the metric in question of a space of certain analytic functions. The spaces S_θ have also a nice interpretation in terms of the heat equation (*cf.* [9]), but this point of view will not be pursued here.

The space $F := F_{(1,0)}$ will be called the *standard* Fock space and its norm will be written $\| \cdot \| = \| \cdot \|_{(1,0)}$.

In [6], among other things, the question of interpolation of the two scales of spaces F^p_k and G^p_k was raised.

1) Regarding complex interpolation the following result was established:

$$[F^{p_0}_{k_0}, F^{p_1}_{k_1}]_\theta = F^{p_\theta}_{k_\theta}\,, \qquad [G^{p_0}_{k_0}, G^{p_1}_{k_1}]_\theta = G^{p_\theta}_{k_\theta}\,,$$

where in both cases p_θ is given by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad 0 < \theta < 1,$$

while k_θ is in the former case a weighted geometric mean of k_0 and k_1:

$$k_\theta = k_0^{1-\theta} k_1^\theta$$

and in the latter case the corresponding weighted harmonic mean:

$$\frac{1}{k_\theta} = \frac{1-\theta}{k_0} + \frac{\theta}{k_1}.$$

2) What real interpolation concerns only a reduction to a multiplier problem was indicated in the case of the scale F_k^p (with p fixed, k variable).

This curious simultaneous occurence of both the geometric and the harmonic mean in essentialy the same context, already recorded in [6], has rised our curiosity. It is one of the objects of this paper to clarify this point and it is for this reason that it was decided that it is necessary to put oneself on the level of the generalized Fock spaces. At the same time we shall also settle the issue of real interpolation, at least in the two cases just indicated.[1]

It turns out that the subject is intimately connected with classical end 19th century higher geometry (German: "höhere Geometrie"), especially circle geometry. Namely, each space $F_{(k,l)}$ is in a natural way associated with a certain circle $C_{(k,l)}$ (or, perhaps rather, a disk $D_{(k,l)}$). And the problem of interpolation between two spaces $F_{(k_0,l_0)}$ and $F_{(k_1,l_1)}$ leads one to consider the pencil of circles generated by $C_{(k_0,l_0)}$ and $C_{(k_1,l_1)}$. There are basically three different types of pencils which we have decided to term elliptic, parabolic and hyperbolic. We have been able to settle most of our question in the elliptic and, to some extent, in the parabolic case but in the hyperbolic case some unexpected difficulties turn up so in this case our results are so far less complete.

Eventually we would like to extend the theory developed in the present paper to the case of several variables. We expect that the rôle

[1] As we shall see, the occurence of these special means is, however, a delusion to some extent!

played by the unit disk here will be taken over by a bounded symmetric domain of tube type, in the first instance one of type III. But this has to wait for the future ...

The plan of the paper is as follows. In Section 1 we set the foundations for the theory of generalized Fock spaces. In particular, we begin to uncover the geometric and the group theoretic aspects of the matter. In an appendix to Section 1 we discuss of the possiblity of assigning spaces not only to proper disks (not containing the point at infintity) but also to arbitrary disks on the Riemann sphere S^2. In the next two sections the interpolation theory of generalized Fock spaces will be developed, complex interpolation in Section 2 and real interpolation in Section 3. The short Section 4 contains some auxiliary results not directly related to the main theme of the paper. The theorems, lemmas etc. are numbered independently in each section.

1. Gauss-Weierstrass functions and Shale-Weil operators. Segal bundle.

We shall study our generalized Fock spaces $F_{(k,l)}$ with the aid of the family of functions e_{ac},

$$e_{ac}(z) = e^{(az^2+cz)/2},$$

where a and c are arbitrary complex numbers. In [9][2] these functions were referred to as *Gauss-Weierstrass functions*; other names current in the literature are: coherent states, Gabor wavelets etc. In our theory they serve as "atoms".

From [9] we take over the following formula:

$$\|e_{ac}\|^2 = \exp\left(\frac{\operatorname{Re} a\bar{c}^2 + |c|^2}{1-|a|^2}\right)(1-|a|^2)^{-1/2} \tag{1}$$

or in polarized form

$$\langle e_{ac}, e_{bd}\rangle = \exp\left(\frac{\dfrac{a\bar{d}^2 + \bar{b}c^2}{2} + c\bar{d}}{1-a\bar{b}}\right)(1-a\bar{b})^{-1/2}, \tag{2}$$

[2] We would like to turn the reader's attention to the circumstance that there is, regretfully, an abundance of misprints in [9]; this is most unfortunate as this reference was meant to be "a small compendium of useful formulae connected with ... Fock space".

the proper interpretation of these formulae being that $e_{ac} \in F$ if and only if $|a| < 1$. In particular, the system of functions $\{e_{ac}\}$, with $|a| < 1$, $c \in \mathbb{C}$, is total in F.

Next we perform a reduction to standard form. We rewrite the norm in our space $F_{(k,l)}$ as follows:

$$\|f\|_{(k,l)} = \|f e^{lz^2/2}\|_{(k,0)} = \left\| \frac{1}{k^{1/4}} f\left(\frac{z}{k^{1/2}}\right) e^{lz^2/2k} \right\| . \tag{3}$$

In other words, we have a unitary map

$$\begin{aligned} V : F_{(k,l)} &\longrightarrow F \\ f(z) &\mapsto \frac{1}{k^{1/4}} f\left(\frac{z}{k^{1/2}}\right) e^{lz^2/2k} , \end{aligned} \tag{4}$$

that is,

$$\|f\|_{(k,l)} = \|Vf\| , \quad \text{if } f \in F_{(k,l)} .$$

The inverse map reads

$$\begin{aligned} V^{-1} : F &\longrightarrow F_{(k,l)} \\ f(z) &\mapsto k^{1/4} f(k^{1/2} z)\, e^{-lz^2/2} . \end{aligned} \tag{5}$$

Using (3) in conjunction with (1) we can formally give an expression for the norm of a Gauss-Weierstrass function in the space $F_{(k,l)}$:

$$\|e_{ac}\|^2_{(k,l)} = k^{1/2} \exp\left(\frac{\operatorname{Re}\left(\dfrac{(a+l)\,\bar{c}^2}{k^2}\right) + \dfrac{|c|^2}{k}}{1 - \dfrac{|a+l|^2}{k^2}} \right) \left(1 - \frac{|a+l|^2}{k^2}\right)^{-1/2} . \tag{6}$$

Indeed, with the above notation we have

$$Ve_{ac} = \frac{1}{k^{1/4}}\, e_{(a+l)/k,\, ck^{-1/2}} , \qquad \|e_{ac}\|_{(k,l)} = \|Ve_{ac}\| ,$$

so, using (1) and (4), (6) readily follows. (The reader will have no difficulty in writing down the corresponding polarized identity.) The interpretation of (6) is the following:

$$e_{ac} \in F_{(k,l)} \quad \text{if and only if} \quad |a+l| < k .$$

Thus, *to the space* $F_{(k,l)}$ *there corresponds the disk* $D_{(k,l)}$ *with radius* k *and center at the point* $-l$, $D_{(k,l)} = \{a : |a+l| < k\}$. We denote by $\mathcal{C}_{(k,l)}$ the circle which constitutes the boundary of $D_{(k,l)}$, that is, $\mathcal{C}_{(k,l)} = \{a : |a+l| = k\}$. We put $D = D_{(1,0)}$ and $\mathcal{C} = \mathcal{C}_{(1,0)}$, unit disk and unit circle respectively. A total system of functions in this case is $\{e_{ac}\}$, with $a \in D_{(k,l)}$, $c \in \mathbb{C}$.

EXAMPLE 1. In [6] the following instances of this are found:

- to the space F_k there corresponds the disk $D_{k,0} = \{a : |a| < k\}$,
- to the space G_k there corresponds the disk $D_{k,k} = \{a : |a+k| < k\}$.

To this we may now add:

- to the space S_θ there corresponds the halfplane

$$P_\theta = \{a : \operatorname{Re} ae^{-i\theta} < 0\} \qquad \text{(a generalized disk)}.$$

We see that S_θ has the interpretation as the closure of the functions $\{e_{ac}\}$ with $a \in P_\theta$, $c \in \mathbb{C}$ in a suitable metric.

EXAMPLE 2. As another application of formula (6) let us record the following formula for the reproducing kernel in the space $F_{(k,l)}$:

$$K(z,w) = k^{1/2}\, e^{-l(z^2+\bar{w}^2)/2+kz\bar{w}}.$$

If $k = 1$, $l = 0$ it reduces, of course, to the well-known expression for the reproducing kernel in the standard Fock space F:

$$K(z,w) = e^{z\bar{w}};$$

see *e.g.* [6, formula (7.2)] with $\alpha = 0$ and $n = 1$. The reproducing kernel will not play any rôle in our discussion.

Returning to the general discusion, let us note that the intersection of two Fock spaces $F_{(k_0,l_0)}$ and $F_{(k_1,l_1)}$ is non-nil,

$$F_{(k_0,l_0)} \cap F_{(k_1,l_1)} \neq \{0\},$$

provided the corresponding disks have non-empty intersection,

$$D_{(k_0,l_0)} \cap D_{(k_1,l_1)} \neq \varnothing.$$

This is follows from the fact that $e_{ac} \in F_{(k_0,l_0)} \cap F_{(k_1,l_1)}$ if $a \in D_0 \cap D_1$, $c \in \mathbb{C}$.[3]

Next, we put into play the *Shale-Weil operators.* Let $G^c = \mathrm{Sp}(2,\mathbb{C})$ the group of complex 2×2 matrices $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ with $\alpha\delta - \beta\gamma = 1$. To such a matrix g we associate an integral operator

$$(7)\quad T_g f(z) = \frac{B^{1/2}}{\pi} \int_{\mathbb{C}} \exp\left(\frac{Az^2 + 2Bz\bar{w} + C\bar{w}^2}{2}\right) f(w)\, e^{-|w|^2}\, dm(w)\,,$$

where

$$A = \frac{\beta}{\delta}\,, \quad B = \frac{1}{\delta}\,, \quad C = -\frac{\gamma}{\delta}\,.$$

We proceed somewhat informally. We think of T_g as being defined on a suitable (preferably dense) subspace of our standard Fock space F and, for the time being (*cf.* Remark 1 below), we let T_g go undefined if $\delta = 0$. In addition, due to the ambiguity in the definition of the square root $B^{1/2}$, T_g is actually determined only up to sign $\pm$.

In [9] the following statement was proved:

T_g is unitary if and only if g is a pseudo-unitary matrix, i.e.
$$g \in G = \mathrm{SU}(1,1)\,;$$

we consider the previous group G^c as the complexification of the group G. It was also shown in [9] that the composition of two such operators T_{g_1} and T_{g_2}, if it makes sense, is again an operator of the same type; indeed, one has $T_{g_1}T_{g_2} = \pm T_{g_1 g_2}$. In other words, we have a unitary representation of a suitable double cover of $G = \mathrm{SU}(1,1) \approx \mathrm{Sp}(2,\mathbb{R})$ (the symplectic group), *viz.* the *metaplectic group* $\widetilde{G} = \mathrm{Mp}(2,\mathbb{R})$. It is the ambiguity in the definition of the square root that forces us to pass to a cover. A typical element of $\tilde{G}$ is given by a pair $\tilde{g}$, an element $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ of G *plus* a determination of the square root of δ, the composition being defined as follows: If we have a second element $\tilde{g}'$, then the composition $\tilde{g}'' = \tilde{g}'\tilde{g}$ is found exploiting the identity

$$\sqrt{\delta''} = \sqrt{1 + \frac{\gamma'}{\delta'}\, g0}\, \sqrt{\delta'}\, \sqrt{\delta}\,.$$

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[3] As the referee has pointed out to us, it is likely that, conversely, $F_{(k_0,l_0)} \cap F_{(k_1,l_1)} \neq \{0\}$ implies $D_{(k_0,l_0)} \cap D_{(k_1,l_1)} \neq \varnothing$, but we do not know how to prove such a result. We are indepted to him for several other precious remarks as well.

Added in proof (Nov. 95). This question has now been affirmatively settled by the authors.

We use the fact that, as $|\gamma'/\delta'| < 1$, one can define $a \mapsto (1+(\gamma'/\delta')a)^{1/2}$ as an analytic function in the unit disk $D(1,0)$ taking the value 1 at the origin.

This representation is known in the literature under various names: oscillator, harmonic, Bargmann-Segal, Shale-Weil, etc. representation. One speaks also, referring to the operators T_g, of the *oscillator group*. If one restricts attention to matrices g with the property that the corresponding Moebius transformation $a \mapsto ga = (\alpha a + \beta)/(\gamma a + \delta)$ maps the unit disk $D(0,1)$ into itself, not onto, then one obtains instead the *oscillator semi-group* (*cf.* [5], [10]).

Another formula, in [9] established for the group G, is

$$T_g e_{ac}(z) = \frac{1}{(\gamma a + \delta)^{1/2}} \exp\Big(- \frac{\gamma c^2}{2(\gamma a + \delta)}\Big) e_{ga, c/(\gamma a+\delta)}(z) , \tag{8}$$

which is easy to verify at least on the formal level.

REMARK 1. Note that, in contradistinction to (7) above, this formula (8) makes sense even if $\delta = 0$. Thus (8) may serve as a definition of T_g in this case; we view then T_g as a linear operator on the linear hull of the family of functions $\{e_{ac}\}$. We must only make sure that $ga \neq \infty$ or that $\gamma a + \delta \neq 0$.

Using (1) we find from this

$$\begin{aligned} \|T_g e_{ac}\|^2 &= \frac{1}{\pi} \frac{1}{|\gamma a + \delta|} \exp\Big(- \operatorname{Re} \frac{\gamma c^2}{\gamma a + \delta}\Big) \\ &\quad \cdot \exp\left(\frac{\operatorname{Re} ga \Big(\overline{\frac{c}{\gamma a + \delta}}\Big)^2 + \frac{|c|^2}{|\gamma a + \delta|^2}}{1 - |ga|^2} \right) (1 - |ga|^2)^{-1/2} . \end{aligned} \tag{9}$$

EXAMPLE 3. Let $g = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)$, so that $\delta = \alpha^{-1}$. Then $T_g f(z) = \alpha^{1/2} f(\alpha z) e^{i\beta\alpha z}$ for a general function f, while

$$T_g e_{ac}(z) = \alpha^{1/2} e_{(\alpha a + \beta)/\delta,\, c/\delta}(z) .$$

Let us write $\alpha = k^{-1/2}$, $\beta = k^{-1/2} l$, so that $\delta = k^{1/2}$. (The corresponding Moebius transformation is thus $a \mapsto (a + l)/k$.) Then we see that $V = T_g$ with $g = \left(\begin{smallmatrix} k^{-1/2} & k^{-1/2} l \\ 0 & k^{1/2} \end{smallmatrix}\right)$. That is, we have $\|f\|_{(k,l)} = \|T_g f\|$ for $f \in F_{(k,l)}$.

The above suggests to consider in general Hilbert spaces with a norm of the type $\|T_g f\|$ for some $g \in G^c$, where $\|\cdot\|$ stands for the standard Fock norm in our standard Fock space F.

In this direction we can establish the following basic result.

Theorem 1. *Let g be an element of G^c such that $g^{-1}(D) = D_{(k,l)}$ (in particular, one has $\infty \notin g^{-1}(D)$). Here $D = D_{(1,0)}$ is the unit disk. Then $T_g^{-1}(F) = F_{(k,l)}$. Moreover, we have $\|f\|_{(k,l)} = \|T_g f\|$ for $f \in F_{(k,l)}$.*

REMARK 2. The transformations g occurring in the statement give an element of $G^c \backslash G$, that is, a residue class modulo G in G^c.

The proof of this theorem which will be based on the following two lemmata may be of independent interest.

Lemma 1 (generalized Lagrange identity). *Let $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be a matrix in G^c such that the inverse image of the unit disk D is the disk $D_{(k,l)}$. Then*

$$1 - |ga|^2 = \frac{1}{|\gamma a + \delta|^2} \, \frac{k^2 - |a + l|^2}{k} \, . \tag{10}$$

Moreover, one has

$$k = \frac{1}{|\alpha|^2 - |\gamma|^2} \, , \qquad l = -\frac{\bar{\alpha}\beta - \bar{\gamma}\delta}{|\alpha|^2 - |\gamma|^2} \, . \tag{11}$$

Lemma 2. *Let the matrix g and the parameters k and l be as in Lemma 1. Then the following identity holds*

$$(\bar{\alpha}\bar{a} + \bar{\beta})\, k - (\gamma a + \delta)(\bar{a} + \bar{l}) = \gamma\,(k^2 - |a + l|^2). \tag{12}$$

REMARK 3. If $g(D) = D$, *i.e.* if $g \in G$, then we get from Lemma 1 the well-known identity

$$1 - |ga|^2 = \frac{1}{|\gamma a + \delta|^2}\,(1 - |a|^2)\,, \tag{13}$$

often used in function theory; in [9] it was called *Lagrange identity.* (The reason for this choice of name is the following: let us put ourselves in the case of the fundamental symmetry of D interchanging an arbitrary point $b \in D$ and the origin 0, that is, the mapping $a \mapsto (b-a)/(1-a\bar{b})$. Then (10) becomes

$$|1 - a\bar{b}|^2 - |b - a|^2 = (1 - |a|^2)(1 - |b|^2).$$

Introducing homogeneous coordinates (writing $a = a_1/a_0$, etc.), this gives

$$|\bar{a}_0 \bar{b}_0 - \bar{a}_1 \bar{b}_1|^2 = (|a_0|^2 - |a_1|^2)(|b_0|^2 - |b_1|^2).$$

This is the expression of the norm of a bivector in the pseudo-Hermitean metric $|a_0|^2 - |a_1|^2$.) In the same way from Lemma 2 we obtain the usual condition for a complex unimodular matrix to be in G, *viz.* $\bar{\alpha} = \delta$, $\bar{\beta} = \gamma$.

PROOF OF LEMMA 1. After having chased a denominator we perform the following chain of transformations:

$$\begin{aligned}|\gamma a + \delta|^2 - |\alpha a + \beta|^2 &= |\gamma|^2 |a|^2 + 2 \operatorname{Re} \gamma \bar{\delta} a + |\delta|^2 \\ &\quad - (|\alpha|^2 |a|^2 + 2 \operatorname{Re} \alpha \bar{\beta} a + |\beta|^2) \\ &= -(|\alpha|^2 - |\gamma|^2) \left| a + \frac{\bar{\alpha}\beta - \bar{\gamma}\delta}{|\alpha|^2 - |\gamma|^2} \right|^2 \\ &\quad + \left(|\delta|^2 - |\beta|^2 + \frac{|\bar{\alpha}\beta - \bar{\gamma}\delta|^2}{|\alpha|^2 - |\gamma|^2} \right).\end{aligned}$$

The last term in the last expression can be rewritten as

$$\begin{aligned}|\delta|^2 - |\beta|^2 + \frac{|\bar{\alpha}\beta - \bar{\gamma}\delta|^2}{|\alpha|^2 - |\gamma|^2} &= \frac{|\alpha|^2 |\delta|^2 + |\beta|^2 |\gamma|^2 - 2 \operatorname{Re} \alpha\delta\bar{\beta}\bar{\gamma}}{|\alpha|^2 - |\gamma|^2} \\ &= \frac{|\alpha\delta - \beta\gamma|^2}{|\alpha|^2 - |\gamma|^2} = \frac{1}{|\alpha|^2 - |\gamma|^2},\end{aligned}$$

where we in the last step used $\alpha\delta - \beta\gamma = 1$. Taking now (10) as definition of the numbers k and l, we formally arrive at formula (11). It is however readily seen from this equality that these parameters must have the desired significance.

PROOF OF LEMMA 2. *Step* 1. First we observe that if (12) holds for a matrix g, unimodular or not, then it holds for any multiple $t \cdot g$, where $t \in \mathbb{R}$.

Step 2. Reduction to the case $k = 1$, $l = 0$. Put $a_1 = (a + k)/k$. Then $|ga| < 1$ if and only if $|a_1| < 1$, while (12) can be written

$$\left(\bar{\alpha}\bar{a}_1 + \frac{\bar{\beta} - \bar{l}\bar{\alpha}}{k}\right) - \left(\gamma a_1 + \frac{\delta - l\gamma}{k}\right)\bar{a}_1 = \gamma(1 - |a_1|^2).$$

But this is nothing but (12) for $g_1 = \left(\begin{smallmatrix} \alpha & \beta_1 \\ \gamma & \delta_1 \end{smallmatrix}\right)$, where $\beta_1 = (\beta - l\alpha)/k$, $\delta_1 = (\delta - l\gamma)/k$, and this matrix represents the transformation $a_1 \mapsto b$, where $b = ga$. Clearly $\det g_1 = \alpha\delta_1 - \beta_1\gamma = 1/k \in \mathbb{R}$. By Step 1 the same equality holds then also for the corresponding unimodular matrix.

Step 3. The case $k = 1$, $l = 0$. In this case, as is we have already noted (see Remark 1) that in this case $\bar{\alpha} = \delta$, $\bar{\beta} = \gamma$. So then (12) is equivalent to the absolutely trivial relation

$$(\delta\bar{a} + \gamma) - (\gamma a + \delta)\,\bar{a} = \gamma(1 - |a|^2).$$

Next we proceed to the proof of the Theorem 1.

PROOF OF THEOREM 1. Let g be a matrix such that $g^{-1}(D) = D_{(k,l)}$. It suffices to show that

$$\|T_g e_{ac}\| = \|e_{ac}\|_{(k,l)}, \qquad \text{for } a \in D_{(k,l)},\ c \in \mathbb{C}. \tag{14}$$

For then we have by polarization

$$\langle T_g e_{ac}, T_g e_{bd}\rangle = \langle e_{ac}, e_{bd}\rangle_{(k,l)}, \qquad \text{for } a, b \in D_{(k,l)},\ c, d \in \mathbb{C},$$

whence, by considering linear combinations of Gauss-Weierstrass functions and applying a density argument (the Gauss-Weierstrass functions form a total set), it follows that $\|T_g f\| = \|f\|_{kl}$ for all $f \in F_{(k,l)}$.

Now we verify (14). To this end we must do some transformations in formula (9) showing that it reduces to (6).

First, we observe that the two exponent free factors combine in view of (10) in Lemma 1 to a factor

$$\frac{k^{1/2}}{\pi}\left(1 - \frac{|a + l|^2}{k^2}\right)^{-1/2}.$$

Next, we look at the exponential factors. Using Lemma 1 once more we see that the coefficient of $|c|^2$ in the exponent becomes

$$\frac{1}{1-\dfrac{|a+l|^2}{k^2}},$$

as it should. Similarly, formula (2) in Lemma 2 helps us to bring the c^2-term into the right shape. Indeed, we find (this is what stands after the sign Re after we have combined the two exponential factors)

$$\begin{aligned}
-\frac{\gamma c^2}{\gamma c+\delta}&+\frac{\overline{ga}}{1-|ga|^2}\,\frac{c^2}{(\gamma a+\delta)^2}\\
&=\Big(-\frac{\gamma c^2}{\gamma c+\delta}+\frac{\overline{\alpha a+\beta}}{(1-|ga|^2)\,|\gamma a+\delta|^2\,(\gamma a+\delta)}\Big)\,c^2\\
&=\frac{-(k^2-|a+l|^2)+\overline{\alpha a+\beta}k}{(\gamma a+\delta)\,(k^2-|a+l|^2)}\,c^2\\
&=\frac{a+l}{k^2-|a+l|^2}\,c^2,
\end{aligned}$$

which is precisely what is desired (see (6)). (In [9] the corresponding computations were done when $g\in G$.)

Let us also indicate an alternative less direct approach. Although it is apparently shorter than our previous proof, we prefer the form because of its constructive flavor involving also the beautiful identities in Lemma 1 and Lemma 2, which, as we have hinted at, may well be of independent interest.

ALTERNATIVE PROOF OF THEOREM 1. We begin by noting that the unitary map V in (4) obviously corresponds to the standard affine map $g_0 : D_{(k,l)} \to D$ given by $g_0 a = (a+l)/k$, *i.e.* $V = T_{g_0}$. It follows that if g is any element of G^c such that $g^{-1}(D) = D_{(k,l)}$ then we have $g = hg_0$ for some $h \in G$ (a pseudo-unitary matrix). But then $T_g = \pm T_h T_{g_0} = \pm T_h V$. As T_h is a unitary map on the Hilbert space F, this again implies that

$$\|T_g f\| = \|T_h V f\| = \|Vf\| = \|f\|_{(k,l)},$$

where we in the last step used (4).

Theorem 1 has an obvious generalization to the spaces F^p, $p \geq 1$.

Corollary. *Let g be as in Theorem 1. Let $p \geq 1$. Then again $T_g^{-1}(F^p) = F^p_{(k,l)}$. Moreover, we have the norm equivalence $\|f\|_{(k,l);p} \approx \|T_g f\|_p$ for $f \in F^p_{(k,l)}$.*

PROOF. In [9] it was shown that the group G^c acts on the spaces F^p (for the case $p < 1$ see Section 4). Therefore the previous (alternative) proof of Theorem 1 extends to the present situation without any changes.

Appendix to Section 1. Non-existence of a certain bundle.[4]

Now we have settled our main question (see the above Theorem 1) but only in the auxiliary assumption that the inverse image of the unit disk under g does not contain the point at infinity, ∞. It is a legitimate question whether it might be possible to free oneself of this assumption. In this appendix we give a brief discussion of this issue. However, it is mainly a negative experience.

First we recall that there are on the Riemann sphere S^2 three kinds of (generalized) disks:

1) proper disks;
2) halfplanes (limiting case of a disk);
3) exteriors of proper disk.

Alternatively, we could speak of oriented (generalized) circles: if an oriented circle is given, we pick up the disk that is to its "left". Thus there is a 1:1 correspondence

$$\text{disks} \longleftrightarrow \text{oriented circles.}$$

The question is thus whether it is possible to associate in a natural way to a generalized disk D on S^2 a "Fock space" $\mathfrak{F}_D$, extending the previous correspondence $D_{(k,l)} \mapsto F_{(k,l)}$. Introducing the notation $\mathfrak{M}$ for the manifold of disks (oriented circles) this would yield a bundle of Fock spaces $\mathfrak{F}$ over $\mathfrak{M}$, say. (Let us remark that in previous work

[4] A reader who is only interested in analysis questions (interpolation, multipliers) can safely omit this appendix. The senior author would like to thank Johan Råde for an illuminating discussion helping to clarifying some questions connected with the topology of the Lie groups G, $\tilde{G}$ and G^c.

one of us has already encountered several occurences of vector bundles (of infinite rank) over complex manifolds: the Fock bundle [11] and the Fischer bundle [12]; here we have a manifold $\mathfrak{M}$ which is not complex.) However, this seems to be a chimera: *the bundle* $\mathfrak{F}$ *does not exist*. Let us indicate why this is so.

Let us fix the disk D. Then D can be mapped conformally onto the unit disk $D(0,1)$ but not in a unique way. Any such map comes from a certain element g of the group G^c, for reference, let us call it a *frame*. It is natural to try to define the fiber $\mathfrak{F}_D$ as a kind of pullback of the standard Fock space $F = F_{(1,0)}$. More exactly, given any two frames coming from group elements g and g_1 we can write $g_1 = ug$ with $u \in G$ and one is then led to consider two functions f_1 and f in F as representatives of one and the same element of $\mathfrak{F}_D$ if $f_1 = T_u f$. However, by the above T_u is defined only up to sign $\pm$, which seems to be an unsurmountable difficulty and so our approach breaks down. It is only when we restrict ourselves to suitable open subsets of $\mathfrak{M}$ that we can make it work, for instance, when we consider the subset of all disks avoiding one point, say, the point at infinity, but then we are back in the situation considered already in Section 1.

A possible way out would be to count elements of F modulo sign but this would then essentially lead to a projective bundle, not a vector bundle, but this is not exactly what we desire.

One can give the above somewhat heuristic considerations also a somewhat more rigorous formulation using the language of principal bundles and their associated bundles, which we now indicate very quickly.[5] Let us denote by $\mathfrak{R}$ the manifold of all frames. Of course, we have the trivial identification $\mathfrak{R} \approx G^c$. Moreover, we can identify $\mathfrak{M}$ with a certain space of cosets of G^c, $\mathfrak{M} \approx G\backslash G^c$. It follows that $\mathfrak{R}$ can be viewed as a principal bundle over $\mathfrak{M}$ with G as structure group. If V is a any vector space on which G acts (a representation space), there is an associated vector bundle $\mathfrak{V}$ on which G acts. In our case we would like to take $V = F$ but the trouble is that its double cover the metaplectic $\tilde{G}$ acts on F, not G itself. There seems to be no way out of this dilemma. This is connected with the fact that while G admits a double cover, its complexification G^c does not. This again depends on the following facts: On the one hand, as G^c as a topological space

[5] We are now adressing ourselves to those readers who are familiar we the rudiments of this theory (see *e.g.* the book [8]). Notice that in the conventional treatment the structure group usually acts from the right, while in our formulation we have a group action (of the group G) from the left.

is simply connected its fundamental group is trivial, $\pi_1(G^c) = 1$, while, on the other hand, G is contractible to a circle S^1 and thus has the fundamental group $\pi_1(G) = \mathbb{Z}$.

2. Complex interpolation. Circle geometry.

Now we begin to interpolate. In this Section we shall deal with complex interpolation exclusively, thus relegating real interpolation to Section 3.

Our objective is to determine the complex interpolation spaces between two given Fock spaces $F_{(k_0,l_0)}$ and $F_{(k_1,l_1)}$. Assuming that their intersection is not nil, $F_{(k_0,l_0)} \cap F_{(k_1,l_1)} \neq \{0\}$, we shall show that the interpolation space $[F_{(k_0,l_0)}, F_{(k_1,l_1)}]_\theta$ again is a certain Fock space $F_{(k_\theta,l_\theta)}$. Indeed, if $C_{(k_0,l_0)}$ and $C_{(k_1,l_1)}$ are the circles corresponding to the spaces $F_{(k_0,l_0)}$ and $F_{(k_1,l_1)}$, then the circle $C_{(k_\theta,l_\theta)}$ corresponding to $F_{(k_\theta,l_\theta)}$ belongs to the pencil of circles generated by the two given circles $C_{(k_0,l_0)}$ and $C_{(k_1,l_1)}$.

First we recall some general facts about complex interpolation (for details, consult the excellent book Bergh-Löfström [2]).

Consider quite generally any Banach couple (A_0, A_1), *i.e.* A_0 and A_1 are two Banach spaces (over $\mathbb{C}$) both continuously imbedded in a Hausdorff topological vector space $\mathcal{A}$. An element a in the linear hull $A_0 + A_1$ of A_0 and A_1 in $\mathcal{A}$ is said to be in the complex interpolation space $[A_0, A_1]_\theta$, where $0 < \theta < 1$, if, informally speaking, there is a complex curve through a connecting A_0 and A_1. More exactly, we require that there exists a holomorphic function $f(\zeta)$, where $\zeta = \xi + i\eta$ is a complex variable, defined in the strip $0 < \operatorname{Re}\zeta < 1$ with values in $A_0 + A_1$ such that $a = f(\theta)$ and such that its boundary values satisfy $f(i\eta) \in A_0$, $f(1 + i\eta) \in A_1$. In addition, some growth conditions must be satisfied, and we have not told in what sense the boundary values are taken, but we shall not enter into such technicalities here.

Next, let us specialize to the case when

$$\begin{aligned} A_0 &= E = \text{a given Banach space,} \\ A_1 &= D(\Lambda) = \text{the domain of a closed} \\ &\qquad\qquad \text{unbounded operator } \Lambda \text{ acting in } E. \end{aligned}$$

Then one expects that, in suitable assumptions, one has $[E, D(\Lambda)]_\theta = D(\Lambda^\theta)$, where Λ^θ stands for the suitably defined θ-th power of Λ. For instance, it suffices that imaginary powers $\Lambda^{i\eta}$ make sense and satisfy a suitable growth estimate, *e.g.* $\|\Lambda^{i\eta}\| \leq C\,(1+|\eta|)^m$ or even $\|\Lambda^{i\eta}\| \leq Ce^{|\eta|^l}$, $l < 1$, will do and certainly $\|\Lambda^{i\eta}\| = 1$ (isometry). Then the canonical quasi-optimal choice of the function in the above construction is $f(\zeta) = \Lambda^{\theta-\zeta}e$, where e is an element of the space E. In particular, the following situation is allowed: E = a Hilbert space, Λ = a positive self-adjoint operator in E.

In the Fock case there is a natural choice for the operators Λ^ζ, namely $\Lambda^\zeta = T_{g_\zeta}$, where the transformations g_ζ form a certain complex one parameter subgroup of G^c leaving invariant the pencil generated by the given circles $C_{(k_0,l_0)}$ and $C_{(k_1,l_1)}$. Before making this more precise let us review some basic facts about circle geometry (classical references for "higher geometry" are Klein [7] and Blaschke [1][6]).

The equation of a (generalized) circle $\mathcal{C}$ on the Riemann sphere S^2 can be written

$$Aa\bar{a} + 2\,\mathrm{Re}\,B\bar{a} + C = 0 \tag{1}$$

(or equivalently $Aa\bar{a} + B\bar{a} + \bar{B}a + C = 0$), where A and C are real numbers, while B is a complex quantity. Thus (1) means one of the following: a genuine (real) circle; in a limiting case, a line (a circle through the point at infinity); a point circle; an imaginary circle. We see that each circle $\mathcal{C}$ gives a triple $\phi = (A, B, C)$ determined up to a non-zero real multiple. Note that such a triple consists of two real and one complex numbers; alternatively, splitting B into its real and imaginary parts, we could likewise have spoken of a quadruple of real numbers, thus a point in $\mathbb{R}^4$. (Sometimes it is also convenient to put $\phi = (A, B, \bar{B}, C)$.) A *pencil of circles* is a one parameter of family of circles of the form

$$(A_0 + tA_1)\,a\bar{a} + 2\,\mathrm{Re}(B_0 + tB_1)\,\bar{a} + (C_0 + tC_1) = 0\,, \qquad t \in \mathbb{R}\,.$$

We say that the pencil is generated by the circles $\mathcal{C}_0$ and $\mathcal{C}_1$ corresponding to the triples $\phi_0 = (A_0, B_0, C_0)$ and $\phi_1 = (A_1, B_1, C_1)$. It is the sign of the *discriminant* $D = AC - |B|^2$ that determines the geometric meaning of the equation (1): assuming that $A \neq 0$

if $D < 0$ it is a real circle;

if $D = 0$ it is a point circle;

---

6 The former book was actually edited by Blaschke.

if $D > 0$ it is an imaginary circle.

The assignment $\mathcal{C} \mapsto (A, B, C)$ thus defines a mapping from the space of all (generalized) circles to real projective space $\mathbb{PR}^3$ equipped with a distinguished quadric

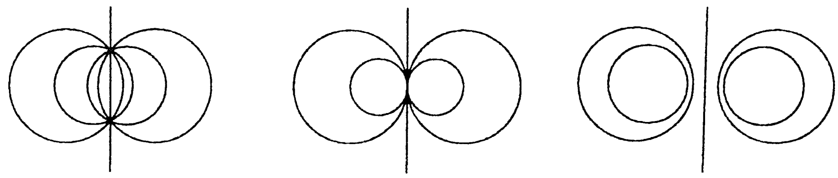
$$\mathcal{Q} : (\phi, \phi) = D = AC - |B|^2 = 0 ,$$

corresponding to a quadratic form in $\mathbb{R}^4$ of signature $+ - - -$ (or index of inertia $(1, 3)$).

We can thus set up a small dictionary.

space of circles	projective space $\mathbb{PR}^3$
point circle	point on $\mathcal{Q}$
circle	point not on $\mathcal{Q}$
pencil of circles	line
the group G^c	the Lorentz group $\mathrm{SO}(1,3)$

We can apply the insights gained above to describe the structure of pencils of circles. There are essentially three cases depending on the mutual position of the corresponding line in $\mathbb{PR}^3$ and the quadric $\mathcal{Q}$. This is depicted in the figure on this page.



hyperbolic (time-like) parabolic (light-like) elliptic (space-like)

Thus in the former case the circles go through two real points, in the middle case they are tangent at a real point and in the last case they go through two imaginary points (and do not meet in the real).

It is clear that the spaces F_k correspond to an elliptic situation (the concentric circles $|a| < k$), while the spaces G_k correspond to a parabolic situation (the circles $|a + k| < k$ tangent to the imaginary axes at the origin). *This explains, in particular, their different interpolational behavior* (see Introduction).

REMARK 1. Note also that a pencil of circles contains in general a unique line called its *power line*. The exception is when we have a pencil of circles through the point at infinity. Then all elements of the pencil are lines, of course.

REMARK 2. Notice also that there is a duality for pencils of circles. The dual pencil consists of all circles orthogonal to the circles of the given pencil. This duality interchanges elliptic and hyperbolic, respects parabolic.

Now we discuss the subgroup of G^c which preserves a given pencil of circles. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ a pair of generating circles and denote by H the group of transformations leaving each of them invariant. Then we have the following lemma, which is the key to our discussion of complex interpolation of Fock spaces in general.

Lemma 1. *Let H^c be the complexification of the group H. Then H^c preserves the pencil (that is, the transformations in H^c map each circle in the pencil onto another circle of the same pencil -we say that they permute the circles in the pencil).*

PROOF. It can be shown (inspection!) that the group H is a one dimensional Lie group, hence commutative. So, using the exponential mapping, its element can be written in the form g_ξ, where ξ is a real parameter ($\xi \in \mathbb{R}$). Similarly the transformations in the complexification H^c will be written g_ζ, where ζ is a complex parameter ($\zeta = \xi + i\eta \in \mathbb{C}$). To fix the ideas, let us assume that we are in the hyperbolic case, denoting the points through which the circles go by p and q. (The other two cases are dealt with in a similar fashion.) Thus we have two equations of the type $g_\xi p = p$ and $g_\xi q = q$ ($\xi \in \mathbb{R}$). Then it is manifest that they remain true also after passing to the complexification (with ξ replaced by ζ). In other words, we have $g_\zeta p = p$ and $g_\zeta q = q$ ($\zeta = \xi + i\eta \in \mathbb{C}$). So if C is any circle passing through p and q, then its image $g_\zeta(C)$ under g_ζ is a circle which still passes through p and q and so belongs to the given pencil; but in general it is not the same circle ($g_\zeta(C) \neq C$).

Let us look at the three cases (hyperbolic, parabolic and elliptic) separately.

1. *Elliptic case.* Making a preliminary conformal transformation

we may pass to the normal form when it is question of concentric circles about the origin. The group H fixing any two of these circles, and thus all of them, consists of the maps $g_{e^{i\theta}} = e^{i\theta}a$ -rotation about the origin. Complexifying we get the transformations $g_\zeta a = \zeta a$, where we have put $\zeta = re^{i\theta}$ -rotations followed by dilation. (Note that here we made a passage from additive language to multiplicative language.)

2. *Parabolic case.* Now we may assume that we are dealing with straight lines parallel to the real axis -this is a pencil of degenerate circles. (In the case corresponding to the spaces G_k this can be achieved by applying the Bargmann transformation whereby Fock space G_k gets replaced by the Schrödinger space S_θ; see Introduction.) The maps in H consist of translations $a \mapsto a + \beta$ with β real. Complexifying yields the corresponding transformations with β complex. Note that in this limiting case the full group preserving the pencil is the 3-dimensional "$(\alpha a + \beta)$-group" with $\alpha \neq 0$ real, β complex.[8]

3. *Hyperbolic case.* As normal form we may use the straight lines through the origin. Then the transformations preserving the pencil are formally the same as in Case 1, $a \mapsto \zeta a$, the difference being that it is when we take the variable ζ real that we get the maps that leave invariant each element of the pencil (a degenerate circle).

REMARK 3. We note that H is compact precisely in the elliptic case.

Next we turn to the problem of the analytic description of the group H or H^c. Recall that if $\phi = (A, B, C)$ is the triple corresponding to a circle $\mathcal{C}$, we have already introduced the metric form

$$\langle \phi, \phi \rangle = AC - |B|^2 . \tag{2}$$

If we have one more circle $\mathcal{C}'$ corresponding to the triple $\phi' = (A', B', C')$, we obtain by polarization the inner product

$$\langle \phi, \phi' \rangle = \frac{1}{2}(AC' + CA') - \operatorname{Re} B\bar{B}' . \tag{2'}$$

EXAMPLE 1. A point circle can be identfied to the triple $\phi_a = (1, -a, |a|^2)$. Then (1) can, in view of (2′), be written as

$$\langle \phi, \phi_a \rangle = 0 .$$

[8] Since the letter a is occupied by the variable, we cannot speak of the $(ax+b)$-group!

We can also rewrite our previous formula for norm of the function e_{ac} (see Section 1) as

$$\|e_{ac}\|^2_{(k,l)} = \exp\left(\frac{\operatorname{Re}\left(\frac{\partial\langle\phi,\phi_a\rangle}{\partial\bar{a}}\,\bar{c}^2\right) - (-\langle\phi,\phi\rangle)^{1/2}\,|c|^2}{-\langle\phi,\phi_a\rangle}\right)\left(\frac{\langle\phi,\phi_a\rangle}{(-\langle\phi,\phi\rangle)}\right)^{-1/2},$$

which perhaps looks more convincing; here $\phi = (1, l, |l|^2, -k^2)$.

Let us now fix a pencil of circles. To determine the corresponding group H, the latter being one dimensional and hence commutative, it suffices to determine its infinitesimal generator X (forming a basis for the Lie algebra $\mathfrak{h}$ of H). This is essentially an exercise in linear algebra.

The image of the pencil under the circle-to-point map $\mathcal{C} \mapsto \phi = (A, B, C)$ is, by what we have said, a line L in $\mathbb{PR}^3$. Let $\phi_0 = (A_0, B_0, C_0)$ and $\phi_1 = (A_1, B_1, C_1)$ be on L. We seek a linear map $\hat{X}$ on $\mathbb{R}^4$ which vanishes on the span of the vectors ϕ_0 and ϕ_1, and is skew-Hermitean with respect to the metric $\langle\phi,\phi\rangle$. Clearly, X is the inverse image of $\hat{X}$.

It is easily seen that $\hat{X}$ is given by the the condition

$$i\begin{vmatrix} A' & B' & \bar{B}' & C' \\ A & B & \bar{B} & C \\ A_0 & B_0 & \bar{B}_0 & C_0 \\ A_1 & B_1 & \bar{B}_1 & C_1 \end{vmatrix} = \langle\phi', \hat{X}\phi\rangle\,, \quad \text{where } \phi' = (A', B', C') \text{ (and } i^2 = -1).$$

Expanding the determinant and comparing with (2′) shows that

$$\hat{X} : (A, B, C) \mapsto i\left(-\begin{vmatrix} A & B & \bar{B} \\ A_0 & B_0 & \bar{B}_0 \\ A_1 & B_1 & \bar{B}_1 \end{vmatrix}, \begin{vmatrix} A & B & C \\ A_0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \end{vmatrix}, \begin{vmatrix} B & \bar{B} & C \\ B_0 & \bar{B}_0 & C_0 \\ B_1 & \bar{B}_1 & C_1 \end{vmatrix}\right).$$

Putting $\phi^* = \hat{X}\phi = (A^*, B^*, C^*)$, we can write this, expanding the 3×3 determinants also, as

$$(3)\quad \begin{cases} A^* = i\left(-\begin{vmatrix} B_0 & \bar{B}_0 \\ B_1 & \bar{B}_1 \end{vmatrix} A + \begin{vmatrix} A_0 & \bar{B}_0 \\ A_1 & \bar{B}_1 \end{vmatrix} B - \begin{vmatrix} A_0 & B_0 \\ A_1 & B_1 \end{vmatrix} \bar{B} + 0\right), \\ B^* = i\left(-\begin{vmatrix} B_0 & C_0 \\ B_1 & C_1 \end{vmatrix} A - \begin{vmatrix} A_0 & C_0 \\ A_1 & C_1 \end{vmatrix} B + 0 + \begin{vmatrix} A_0 & B_0 \\ A_1 & B_1 \end{vmatrix} \bar{B}\right), \\ C^* = i\left(0 + \begin{vmatrix} \bar{B}_0 & C_0 \\ \bar{B}_1 & C_1 \end{vmatrix} B - \begin{vmatrix} B_0 & C_0 \\ B_1 & C_1 \end{vmatrix} \bar{B} + \begin{vmatrix} B_0 & \bar{B}_0 \\ B_1 & \bar{B}_1 \end{vmatrix} C\right). \end{cases}$$

That is, we have, in matrix form,

$$(4)\quad \hat{X} = i\begin{pmatrix} -\begin{vmatrix} B_0 & \bar{B}_0 \\ B_1 & \bar{B}_1 \end{vmatrix} & \begin{vmatrix} A_0 & \bar{B}_0 \\ A_1 & \bar{B}_1 \end{vmatrix} & -\begin{vmatrix} A_0 & B_0 \\ A_1 & B_1 \end{vmatrix} & 0 \\ -\begin{vmatrix} B_0 & C_0 \\ B_1 & C_1 \end{vmatrix} & -\begin{vmatrix} A_0 & C_0 \\ A_1 & C_1 \end{vmatrix} & 0 & \begin{vmatrix} A_0 & B_0 \\ A_1 & B_1 \end{vmatrix} \\ 0 & \begin{vmatrix} \bar{B}_0 & C_0 \\ \bar{B}_1 & C_1 \end{vmatrix} & -\begin{vmatrix} B_0 & C_0 \\ B_1 & C_1 \end{vmatrix} & \begin{vmatrix} B_0 & \bar{B}_0 \\ B_1 & \bar{B}_1 \end{vmatrix} \end{pmatrix}.$$

EXAMPLE 2 (The case of concentric circles). We can take $\phi_0 = (1,0,1)$, $\phi_1 = (1,0,2)$. Then $A^* = C^* = 0$, $B^* = B$. This corresponds to the circle transformations $B \mapsto e^{i\theta}B$, again induced by the point transformations $z \mapsto e^{i\theta}z$ (rotations about the origin). This we know, of course.

The map $\hat{X}$ is an element of the Lie algebra $\mathfrak{so}(1,3)$. Now we seek the corresponding element X in $\mathfrak{g}^c = \mathfrak{sl}(2,\mathbb{C})$.

First we work on the group level. Let $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be in G^c. Then a circle $\mathcal{C}$ corresponding to the quadruple $\phi = (A,B,\bar{B},C)$ is mapped into a circle $\mathcal{C}^*$ corresponding to the quadruple $\phi = (A^*,B^*,\bar{B}^*,C^*)$, where

$$\begin{aligned} A^* &= A\alpha\bar{\alpha} + B\gamma\bar{\alpha} + \bar{B}\alpha\bar{\gamma} + C\gamma\bar{\gamma}\,, \\ B^* &= A\beta\bar{\alpha} + B\delta\bar{\alpha} + \bar{B}\beta\bar{\gamma} + C\delta\bar{\gamma}\,, \\ C^* &= A\beta\bar{\beta} + B\delta\bar{\beta} + \bar{B}\beta\bar{\delta} + C\delta\bar{\delta}\,. \end{aligned}$$

Thus the point transformation g induces the circle transformation

$$\hat{g} = \begin{pmatrix} \alpha\bar{\alpha} & \gamma\bar{\alpha} & \alpha\bar{\gamma} & \gamma\bar{\gamma} \\ \beta\bar{\alpha} & \delta\bar{\alpha} & \beta\bar{\gamma} & \delta\bar{\gamma} \\ \alpha\bar{\beta} & \gamma\bar{\beta} & \alpha\bar{\delta} & \gamma\bar{\delta} \\ \beta\bar{\beta} & \delta\bar{\beta} & \beta\bar{\delta} & \delta\bar{\delta} \end{pmatrix}.$$

Passing to the infinitesimal (algebra) level we see that to the matrix

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$$

there corresponds the matrix

$$(5)\qquad \hat{X} = \begin{pmatrix} 2\,\mathrm{Re}\,\alpha & \gamma & \bar{\gamma} & 0 \\ \beta & -i\,\mathrm{Im}\,\alpha & 0 & \bar{\gamma} \\ \bar{\beta} & 0 & i\,\mathrm{Im}\,\alpha & \gamma \\ 0 & \bar{\beta} & \beta & -2\,\mathrm{Re}\,\alpha \end{pmatrix} \in \mathfrak{so}(1,3).$$

(Here we use $\alpha + \delta = 0$, corresponding to $\alpha\delta - \beta\gamma = 1$.)

Now we compare the general formula (4) to (3). This gives in our case

$$(6)\qquad X = i\begin{pmatrix} \begin{vmatrix} A_0 & C_0 \\ A_1 & C_1 \end{vmatrix} - \begin{vmatrix} B_0 & \bar{B}_0 \\ B_1 & \bar{B}_1 \end{vmatrix} & -\begin{vmatrix} B_0 & C_0 \\ B_1 & C_1 \end{vmatrix} \\ \begin{vmatrix} A_0 & \bar{B}_0 \\ A_1 & \bar{B}_1 \end{vmatrix} & -\begin{vmatrix} A_0 & C_0 \\ A_1 & C_1 \end{vmatrix} + \begin{vmatrix} B_0 & \bar{B}_0 \\ B_1 & \bar{B}_1 \end{vmatrix} \end{pmatrix}$$

This is the sought infinitesimal generator of the Lie algebra $\mathfrak{h}$.

We may summarize the preceding discussion as follows.

Lemma 2. *The Lie group H fixing the two circles C_0 and C_1 corresponding to the triple $\phi_0 = (A_0, B_0, C_0)$ and $\phi_1 = (A_1, B_1, C_1)$ is generated by the matrix given by formula* (6).

Let us give another example.

EXAMPLE 3. Consider the hyperbolic pencil of circles through the points 1 and -1. These circles correspond to the parameters $k = \sqrt{1+m^2}$, $l = im$ with m real. We may take $\phi_0 = (1, 0, -1)$ (unit circle), $\phi_1 = (0, i, 0)$ (real axis). Then (6) readily gives

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

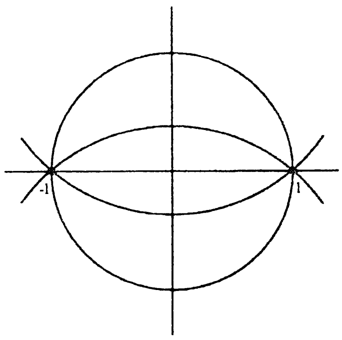
Thus integrating we get the transformations

$$g_\zeta = \begin{pmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{pmatrix}.$$

Each of the maps g_ζ preserves, if ζ is real, any of the circles of the pencil. If we let ζ assume complex values, we obtain transformations that permute the circles. If ζ is purely imaginary, $\zeta = i\eta$, then the image of the unit circle corresponds to

$$(7)\quad k = \frac{1}{\cos 2\eta} = \sec 2\eta\,, \quad l = i\,\frac{\sin 2\eta}{\cos 2\eta} = i\,\tan 2\eta\,, \qquad (m = \tan 2\eta)\,.$$

In particular, for $\eta = \pi/4$ the unit circle is mapped onto the real axis (Cayley transformation).



Now that we have a rather complete picture of the transformations permuting the circles of a given pencil, that is, of the complexification H^c of the group H of transformations fixing any two of them, it is possible also to answer the initial question of complex interpolation of two given spaces $F_{(k_0,l_0)}$ and $F_{(k_1,l_1)}$ corresponding to any two circles $\mathcal{C}_0 = C_{(k_0,l_0)}$ and $\mathcal{C}_1 = C_{(k_1,l_1)}$ in the pencil.

We let $D_0 = D_{(k_0,l_0)}$ and $D_0 = D_{(k_1,l_1)}$ be the corresponding disks (the disks bounding the circles and not containing the point at infinity). We assume that we have

$$D_0 \cap D_1 \neq \varnothing . \tag{8}$$

As was already recorded in Section 1, this implies that $F_{(k_0,l_0)} \cap F_{(k_1,l_1)} \neq \{0\}$. Changing somewhat the notation we may assume that the maps $g_{i\eta}$ ($\eta \in \mathbb{R}$) in H^c leave $\mathcal{C}_0$ and $\mathcal{C}_1$ invariant. We may also assume that $g_1(\mathcal{C}_1) = \mathcal{C}_0$. (This amounts to normalizing the group parameter.)

Lemma 3. *It is possible to choose g_1 such that $g_1(D_1) = D_0$.*

PROOF. By inspection. Expect in the hyperbolic case this is automatic. In the latter case we first choose g_1 to be <u>minimal</u>, that is, $g_1(\mathcal{C}_1) = \mathcal{C}_0$ but $g_\xi(\mathcal{C}_1) \neq \mathcal{C}_0$ for $0 < \xi < 1$. Then either $g_1(D_1) = D_0$ or else $g_1(D_1) = \tilde{D}_0$, where $\tilde{D}_0$ is the complementary disk $\tilde{D}_0 = S^2 \setminus \bar{D}_0$. In this case the group generated by g_1 must be compact. (On the other hand, the one generated by g_i equals H and is not compact *cf.* Remark 3.)

Therefore it must be periodic. If τ is the period, we can now achieve $g_1(D_1) = D_0$ replacing if necessary g_1 by $g_{1/2-\tau}$.

Corollary. *It follows that* $T_{g_1}(F_{(k_1,l_1)}) = F_{(k_0,l_0)}$ *and in particular that* $\|f\|_{(k_1,l_1)} = \|T_{g_1}f\|_{k_0}^{l_0}$ *for* $f \in F_{(k_1,l_1)}$.

For $\theta \in (0,1)$ let now the circle $\mathcal{C}_\theta$ be chosen in such a way that $g_\theta(\mathcal{C}_\theta) = \mathcal{C}_0$. We let further D_θ be the disk corresponding to the circle $\mathcal{C}_\theta$.

Lemma 4. *We have* $\infty \notin g_\theta(D_1)$, $\theta \in (0,1)$. *In particular, we have* $g_\theta(D_\theta) = D_0$.

PROOF. By a continuity argument. The elliptic and parabolic cases are quite obvious, because then the circles $\mathcal{C}_\theta$ lie all between $\mathcal{C}_0$ and $\mathcal{C}_1$. So let us again look at the hyperbolic case. In this case it is clear that the relation $g_\theta(D_\theta) = D_0$ holds true at least for θ close to 0. If the assertion were not true, then it is easy to see that for some particular value $\theta_0 \in (0,1)$ the corresponding circle $\mathcal{C}_{\theta_0}$ degenerates and becomes a line, the power line of our pencil (see Remark 3). But at that moment the corresponding disk degenerates into a halfplane. Continuing the parameter θ beyond the value θ_0 it is now easy to arrive at a contradiction, namely that $g_1(D_1) = \tilde{D}_0$, where again $\tilde{D}_0$ stands for the complementary disc.

We can now announce the following result.

Theorem 1. *Let* $F_{(k_0,l_0)}$ *and* $F_{(k_1,l_1)}$ *be the generalized Fock spaces corresponding to the circles* $C_{(k_0,l_0)}$ *and* $C_{(k_1,l_1)}$. *If* D_0 *and* D_1 *be the corresponding disks, we assume that* (8) *holds true. Let* g_ξ *be the one parameter group of conformal maps as defined above in the course of the discussion of Lemma 2 and 3. (In particular thus* $g_1(D_1) = D_0$.) *For* $0 < \theta < 1$ *define the circle* $C_{(k_\theta,l_\theta)}$ *by* $g_\theta(C_{(k_\theta,l_\theta)}) = C_{(k_0,l_0)}$. *Then we have the isometry*

$$[F_{(k_0,l_0)}, F_{(k_1,l_1)}]_\theta = F_{(k_\theta,l_\theta)}, \qquad 0 < \theta < 1. \tag{9}$$

PROOF. This follows from the general facts about complex interpolation which we recalled in the beginning of this section. In particular, the rôle of the operators Λ^ζ is now played by the maps T_{g_ζ}, as follows readily from Section 1, Theorem 1. The crucial thing is that for purely imaginary values of ζ these are unitary maps in $F_{(k_0,l_0)}$, $\|T_{g_{i\eta}}\| = 1$ for $\eta \in \mathbb{R}$. So there is really nothing to prove.

EXAMPLE 4. It is clear that Theorem 1 contains as special cases the results from [6] for $p = 2$ with the spaces F_k and G_k which were recalled in the Introduction. These are elliptic and parabolic cases respectively. A concrete example in a hyperbolic situation can easily be constructed at the hand of Example 3 *ultra*. Let us fix attention to the circles in the hyperbolic pencil there which lie in the upper halfplane, that is, if $k \geq 1$ is, as usual, the radius then the second parameter l is determined by $l = i\sqrt{1-k^2}$ (with the positive sign of the square root). We are thus lead to consider the family of spaces E_k of entire analytic functions f with the metric

$$\|f\|^2 = \frac{k^{1/2}}{\pi} \int_{\mathbb{C}} e^{-k|z|^2 - 2\sqrt{1-k^2}y^2} \, |f(z)|^2 \, dm(z) .$$

In agreement with our previous notation (*cf.* Introduction) we have in particular $E_1 = F_1 = F$ (our standard Foch space) and $E_0 = G_1$. Thus this connects the spaces F_1 and G_1. We conclude that we have the interpolation formula

$$[E_{k_0}, E_{k_1}]_\theta = E_{k_\theta} , \qquad 0 < \theta < 1 ,$$

where k_θ is obtained from k_0 and k_1 according to the following rule: if we write $k_0 = \sec 2\eta_0$ and $k_1 = \sec 2\eta_1$ then $k_\theta = \sec 2\eta$ with $\eta = (1-\theta)\eta_0 + \theta\eta_1$.

REMARK 5. The recepee for computing the "mean" of the parameters k_0 and k_1 is thus rather complicated in this case. That the rule has such a simple form in the case of the families F_k (geometric mean) and G_k (harmonic mean) is rather exceptional. In particular, the homogeneity is accounted for by the fact that the corresponding pencils are dilation invariant then. The phenomenon we initially set out to clarify in this paper has turned to be an exception!

So far we have only dealt with Hilbert spaces, that is, the problem of complex interpolation of the scale of generalized Fock spaces $F_{(k,l)}$. Now we pass to the corresponding problem for the Banach spaces $F^p_{(k,l)}$

($1 < p < \infty$). On a formal basis we expect that the obvious analogue of (9), *viz.* the interpolation formula,

$$[F^p_{(k_0,l_0)}, F^p_{(k_1,l_1)}]_\theta = F^p_{(k_\theta,l_\theta)}\,, \qquad 0 < \theta < 1\,, \tag{10}$$

to be true, perhaps not isometrically but at least up to an equivalence of norm. (For simplicity we keep the parameter p fixed taking $p_0 = p_1 = p$ and interpolate only k and l; how to treat the case $p_0 \neq p_1$ is indicated in [6].) The main difficulty is again to estimate the operator norm of $T_{g_{i\eta}}$, this time in the space $F^p_{(k_0,l_0)}$. It turns out that the different cases (elliptic, etc.) behave differently.

Let us first look at the elliptic case. Putting into play the map $V = V_0 : F^p_{(k_0,l_0)} \to F^p$, which by Section 1 is an isometry (only the case $p = 2$ was worked out there), we can reduce to the case when $\mathcal{C}_{(k_0,l_0)}$ is the unit circle $\mathcal{C}_{(1,0)}$. We recall from [9], Section 8, that the metaplectic group $\tilde{G}$ acts continuously on the spaces F^p, but this action is not isometric if $p \neq 2$. Indeed the operators T_g, $g \in \tilde{G}$, admit in F^p the following norm estimate:

$$\|T_g\|_p \approx |\delta|^{|1/p-1/2|}\,. \tag{11}$$

In the present elliptic case the one parameter group $g_{i\eta}$ is compact (cf. Remark 3). Therefore it follows from (11) that we have $\|T_{i\eta}\|_p \approx C$. So we are in business. We get thus back the result for the spaces F^p_k ([6, Theorem 9.3, the case $p_0 = p_1$]).

REMARK 6. In this situation we could have used instead of V_0 another more cleverly chosen Shale-Weil transformation reducing ourselves to the case when $\mathcal{C}_{(k_1,l_1)}$ is a concentric circle $\mathcal{C}_{(k,0)}$. Then we are back in the set up of [6].

Next, let us look look at the parabolic situation. It is then readily seen that the matrices $g_{i\eta}$ are conjugated to the matrices

$$\begin{pmatrix} 1 + i\frac{\eta}{2} & i\frac{\eta}{2} \\ -i\frac{\eta}{2} & 1 - i\frac{\eta}{2} \end{pmatrix}$$

by a fixed matrix. It follows then from (11) that now $\|T_{i\eta}\|_p \approx (1+|\eta|)^m$ for some number m, that is, we have power-like growth. We are again in business and have, in particular, essentially recovered the corresponding result for the spaces G_k^p ([6, formula 11.6, the case $p_0 = p_1$]).

Finally, we turn to the hyperbolic situation. From Example 3 it is seen that now the $g_{i\eta}$ are conjugated to the matrices

$$g_{i\eta} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} .$$

But according to (11) this gives exponential growth of the norm and we must conclude that the general theory is not applicable. The different cases thus behave essentially differently. We may summarize the preceding discussion as follows.

Theorem 2. *We return to the set up of Theorem* 1, *replacing everywhere the space* $F_{(k,l)}$ *by* $F^p_{(k,l)}$, $p \geq 1$. *Then we have the interpolation formula* (10)

$$(10) \qquad [F^p_{(k_0,l_0)}, F^p_{(k_1,l_1)}]_\theta = F^p_{(k_\theta,l_\theta)}, \qquad 0 < \theta < 1,$$

which is an isomorphism (*equality up to equivalence of norm*), *provided the pencil generated by the circles* $C_{(k_0,l_0)}$ *and* $C_{(k_1,l_1)}$ *is either elliptic or parabolic. If however this pencil is hyperbolic the natural approach fails and we do not know whether* (10) *is true or not.*

3. Real interpolation. Multipliers.

Now we turn to real interpolation. The problem is thus to describe the real interpolation spaces between two given space $F^p_{(k_0,l_0)}$ and $F^p_{(k_1,l_1)}$. First we recall some salient facts about real interpolation in general (and we refer again to [2] for details).

If (A_0, A_1) is any Banach couple, one begins by defining the K-functional: for any element a in the sum $A_0 + A_1$ and any t with $0 < t < \infty$ one puts[8]

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\},$$

[8] If X is any Banach space, we use a subsript X to designate the corresponding norm, thus writing $\|\cdot\|_X$.

where the infimum extends over all decompositions of a of the form $a = a_0 + a_1$ with $a_0 \in A_0$, $a_1 \in A_1$. One says that a belongs to the K-space $(A_0, A_1)_{\theta,q}$, where $0 < \theta < 1$ and $0 < q \leq \infty$, if and only if

$$\|a\|_{(A_0,A_1)_{\theta,q}} \stackrel{\text{def}}{=} \Big(\int_0^\infty (t^{-\theta} K(t,a))^q \,\frac{dt}{t} \Big)^{1/q} < \infty\,;$$

if $q = \infty$ the left hand side of the inequality is interpreted as a supremum.[9] In order to obtain a concrete representation of the K-spaces one has to compute the K-functional, at least approximately.

If we are in the situation of an "operator pair" $(E, D(\Lambda))$ (see Section 2), it is natural to try to exploit the functional or spectral calculus associated with the operator Λ in question. More specifically, in some situations one can prove that one has an estimate of the type

$$K(t,a) \approx \|\varphi(t\Lambda)a\|_E \tag{1}$$

with a suitable scalar function φ. (In particular, the couple $(E, D(\Lambda))$ is thus "quasi-linearizable" in a certain technical sense.) Then one has

$$a \in (E, D(\Lambda))_{\theta,q} \iff \Big(\int_0^\infty (t^{-\theta} \|\varphi(t\Lambda)a\|_E)^q \,\frac{dt}{t} \Big)^{1/q} < \infty\,. \tag{2}$$

Let us look at some special cases.

1) In the Hilbert case (*viz.* E = a Hilbert space, Λ a positive self-adjoint operator acting in E; *cf.* Section 2), there are plenty of such functions: any function φ defined on $(0, \infty)$ such that $\varphi(\lambda) \approx \min(1, \lambda)$ will do. With the aid of this one can prove that in case $q = 1/2$ indeed holds $(E, D(\Lambda))_{\theta,1/2} = D(\Lambda^\theta)$, up to equivalence of norm. (Indeed, there is now a canonical choice for the function φ: $\varphi(\lambda) = (1 + \lambda^{-2})^{-2}$; with this choice one has even an isometry, provide the K-functional is replaced by what is known as the K_2-functional.) Thus, in this case, and in general only in this case, the two approaches -real and complex interpolation- produce the same result.

2) In the general case, it is natural to try to exploit the resolvent $R(t) = (1 + t\Lambda)^{-1}$ $(t > 0)$. If one has the estimate $\|R(t)\| \leq C$, with C independent of t, it is not very hard to show that (1) is fulfilled with

[9] In the literature there is a J-functional and J-spaces, but these need not bother us here.

$\varphi(\lambda) = \lambda/(1+\lambda)$; operators Λ with this property are called *positive.*[10] Thus (2) in this case means that

$$(2') \qquad a \in (E, D(\Lambda))_{\theta,q} \Longleftrightarrow \Big(\int_0^\infty (t^{-\theta} \|R(t)\, t\Lambda a\|_E)^q \, \frac{dt}{t} \Big)^{1/q} < \infty .$$

3) If Λ is the generator of a semi-group of operators, it is again natural to use instead the semi-group of operators $G(t) = e^{t\Lambda}$. If one has $\|G(t)\| \le C$, with C independent of t, one speaks of a bounded semi-group and in this case one has the following result:

$$K(t,a) \approx \inf_{s \le t} \|G(s)a - a\|_E .$$

Although this is formally weaker than (1), it is nevertheless sufficient for establishing the desired analogue of (2), *viz.*

$$(2'') \quad a \in (E, D(\Lambda))_{\theta,q} \Longleftrightarrow \Big(\int_0^\infty (t^{-\theta} \|G(t)a - a\|_E)^q \, \frac{dt}{t} \Big)^{1/q} < \infty .$$

Let us mention that if Λ is the generator of a bounded semi-group of operators, then Λ is positive.

REMARK 1. Above we have summarized some classical results due to Lions, Grisvard, and others. Besides [2] we can also refer to the books Butzer-Behrens [3], Triebel [16]. Case 3) will not be used here, but we have now made mention of this case anyhow. (Perhaps somebody in the future might want to use semigroups in the Fock context ...)

After this long digression let us return to our generalized Fock spaces for good. More specifically, we are addressing ourselves to the problem of quasi-linearizability. The case $p = 2$ is essentially trivial, because as we have seen in 1) *ultra* in the Hilbert space case in general we can, in principle, even obtain an exact result. Therefore we proceed directly to the case of general p. (The problem of interpolation between two Fock spaces with different p's seems to be very hard; at least, it is not likely that one has a quasi-linearizable couple in that case.) So we are given two spaces $F^p_{(k_0,l_0)}$ and $F^p_{(k_1,l_1)}$ with $F^p_{(k_0,l_0)} \cap F^p_{(k_1,l_1)} \neq \{0\}$, assuming also that the corresponding disks have non-empty intersection, $D_{(k_0,l_0)} \cap D_{(k_1,l_1)} \neq \varnothing$. There is a natural candidate for the operator Λ,

10 For Hilbert spaces the two notions of positivity coincide.

namely the operator $T_1 = T_{g_1}$ constructed in Section 2. So the problem becomes to decide when operator T_1 is positive in the above sense, that is, when the operators $(1 + tT_1)^{-1}$ are uniformly bounded. Again this is basically a *multiplier problem* in the natural basis provided by the spectral theorem (if $p = 2$) for which T_1 comes in diagonal form. The nature of its solution depends on the type of the corresponding pencil of circle. Therefore we shall proceed by case by case study.

1. *Elliptic case.* Making a preliminary conformal mapping (see Section 1), one can put oneself in the situation of the two spaces $F^p = F_1^p = F_{(1,0)}^p$ and $F_k^p = F_{(k,0)}^p$, where we can assume, with no loss of generality, that $k > 1$. In this case we have $T_1 f(z) = f(k^{-1/2} z)$ (*cf.* Section 1, Corollary to Theorem 1). It will be expedient to write $\delta = k^{-1/2}$, so that $0 < \delta < 1$. In particular, we have then $T_1 : z^n \mapsto \delta^n z^n$, so a basis in which T_1 is in diagonal form is provided by the monomials $\{z^n\}$ $(n \neq 0)$. Moreover, we have as a consequence $R(t) : z^n \mapsto (1 + t\,\delta^n)^{-1} z^n$. This suggests to look quite generally at multiplier transforms on the space F^p.

Given a bounded function $\omega(n)$ defined on the set $\mathbb{N}$ of non-negative integers, $\mathbb{N} = \{0, 1, 2, \dots\}$, we define an operator R_ω by setting

$$R_\omega f(z) = \sum_{n=0}^{\infty} \omega(n)\, a_n z^n$$

whenever $f \in F^p$ has the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n\,.$$

It follows that R_ω maps each basis vector z^n into a multiple of itself, $R_\omega : z^n \mapsto \omega(n)\, z^n$. We are interested in the boundedness of R_ω on F^p. First we establish an easy transference result which reduces the study of R_ω to the study of Fourier multipliers.

Let us define the operator $\tilde{R}_\omega$ on $L^p(\mathbb{T})$, where $\mathbb{T}$ is the unit circle parametrized by arc length θ, by

$$\tilde{R}_\omega f(\theta) = \sum_{n=0}^{\infty} \omega(n) a_n e^{in\theta}$$

whenever $f \in L^p(\mathbb{T})$ has the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{in\theta}\,.$$

Proposition 1. *The operator R_ω is bounded on F^p whenever $\tilde{R}_\omega$ is bounded on $L^p = L^p(\mathbb{T})$.*

PROOF. We have

$$\int_{\mathbb{C}} |R_\omega f(z)|^p \, e^{-p|z|^2/2} \, dm(z) = \int_0^\infty \int_0^{2\pi} |R_\omega f(re^{i\theta})|^p \, d\theta \, e^{-pr^2/2} \, r \, dr$$
$$= \int_0^\infty \|\tilde{R}_\omega f_r(\,\cdot\,)\|_p^p \, e^{-pr^2/2} \, r \, dr \,,$$

where we have written $f_r(\theta) = f(re^{i\theta})$. If we assume that $\tilde{R}_\omega : L^p \to L^p$ is bounded, then

$$\|\tilde{R}_\omega f_r\|_p^p \le C \int_0^{2\pi} \Big| \sum_{n=0}^\infty a_n r^n e^{in\theta} \Big|^p \, d\theta \,.$$

It therefore follows that

$$\int_{\mathbb{C}} |R_\omega f(z)|^p \, e^{-p|z|^2/2} \, dm(z) \le C \int_{\mathbb{C}} |f(z)|^p \, e^{-p|z|^2/2} \, dm(z) \,.$$

We can now prove the following theorem; we assume now that ω is defined for all $\xi \ge 0$.

Proposition 2. *Let ω a bounded function on $(0, \infty)$ such that*

$$\int_0^\infty |\omega'(\xi)| \, d\xi < \infty \,.$$

Then for $1 < p < \infty$ the multiplier R_ω is bounded on F^p.

PROOF. In view of the previous Proposition 1 it is enough to show that $\tilde{R}_\omega$ is bounded on L^p, $1 < p < \infty$. Again, using another transference result between multipliers for the Fourier series and multipliers for the Fourier transform (see [14, Chapter VII, Theorem 3.8]) it is enough to show that the operator

$$S_\omega f(x) = \int_{-\infty}^\infty e^{ix\xi} \, \omega(\xi) \, \hat{f}(\xi) \, d\xi$$

is bounded on $L^p(\mathbb{R})$. But this follows from the Marcinkiewicz multiplier theorem (*cf.* [4, Proposition 4.1]) under the above hypothesis on ω.

REMARK 2. The proof of the Marcinkiewicz multiplier theorem actually gives a bound for the norm of R_ω on F^p. In fact,

$$\|R_\omega f\|_p \leq \left(\|\omega\|_\infty + \int_0^\infty |\omega'(\xi)|\, d\xi\right) \|f\|_p\,, \qquad 1 < p < \infty\,.$$

In applications one encounters multipliers of the form $\omega_t(\xi) = \varphi(t\,\psi(\xi))$, $t > 0$, where ψ is a positive monotone function. If we assume that φ is bounded and that $\int_0^\infty |\varphi'(\xi)|\, d\xi < \infty$, then the operator R_{ω_t} will be bounded on F^p with a bound independent of t, $t > 0$. Indeed, as $\omega_t'(\xi) = \varphi'(t\,\psi(\xi))\,t\,\psi'(\xi)$ we have $\|\omega\|_\infty + \int_0^\infty |\omega'(\xi)|\, d\xi \leq C$ with C independent of t, so that Proposition 2 is applicable (see Remark 1). In particular, taking $\psi(\xi) = \delta^\xi$, where $0 < \delta < 1$, we see that the operators

$$R_t f(z) = \sum_{n=0}^{\infty} a_n\,(1 + t\,\delta^n)^{-1}\, z^n \tag{3}$$

are uniformly bounded on F^p, $1 < p < \infty$. In view of the general remarks in the beginning of this Section we have thus established the following theorem, which thus in particular settles in part a question left over in [6].

Theorem 1. *Let $1 < p < \infty$, $0 < q \leq \infty$. Then the operators R_t as defined in* (3) (*with $\delta = k^{-1/2}$*) *are uniformly bounded in $F^p = F_1^p$ and for $f \in F^p = F^p + F_k^p$ we have*

$$f \in (F^p, F_k^p)_{\theta,q} \quad \Longleftrightarrow \quad \left(\int_0^\infty (t^{-\theta}\,\|R_t f\|_{F^p})^q\, \frac{dt}{t}\right)^{1/q} < \infty\,.$$

REMARK 3. Thus the pair (F^p, F_k^p) is quasi-linearizable in the technical sense.

There remain the cases $p = 1$, $p < 1$ and $p = \infty$. Here we shall only consider the former case. It is not hard to see that for each $t > 0$ the operator R_t is bounded on F^1. But unfortunately they are not uniformly bounded. This indicates that it is very unlikely that the Banach couple (F^1, F_k^1) is quasi-linearizable, again indicating that no result of the type of (2) can be true in this case. For reference, we state the result as a theorem.

Theorem 2. *The operators R_t are not uniformly bounded on F^1.*

The proof is by contradiction, but as it is rather long we prefer to split it up into several steps in the form Propositions 3-5 below. Let us begin by explaining the basic underlying idea.

Suppose that the operators R_t are uniformly bounded on F^1. Taking $f(z) = e_{0,c}(z) = e^{cz}$ $(c \in \mathbb{C})$ and noting that

$$\int_{\mathbb{C}} |e_{0,c}(z)|\, e^{-|z|^2/2}\, dm(z) = e^{|c|^2/2},$$

we see that one must have the estimate

$$\int_{\mathbb{C}} |R_t e_{0,c}(z)|\, e^{-|z|^2/2}\, dm(z) \leq C\, e^{|c|^2/2}. \tag{4}$$

(As the functions serve as "atoms" in the space F^1, one sees that, conversely, (4) implies uniform boundedness; *cf.* [6, Theorem 8.1.]) In what follows we shall show that (4) cannot hold true, proving the theorem. In doing this we may as well assume that c is a positive number.

Let us set

$$f(z,t,\delta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{1}{1 + t\,\delta^n}, \qquad t > 0, \quad 0 < \delta < 1. \tag{5}$$

We wish thus to test the hypothesis

$$\int_{\mathbb{C}} |f(cz,t,\delta)|\, e^{-|z|^2/2}\, dm(z) \leq C\, e^{c^2/2} \tag{6}$$

for $c > 0$ and C independent of t and c.

First we replace $f(z,t,\delta)$ by chain of simpler functions ending up with the function $j(z,t,\delta)$ in formula (13) below and then proceed to the study of that function. Our first intermediary result is thus the following.

Proposition 3. *If inequality (6) holds with $f(cz,t,\delta)$, then it holds with $f(cz,t,\delta)$ replaced by $j(cz,t,\delta)$.*

PROOF. Performing a Mellin transformation with respect to t gives the representation

$$f(z,t,\delta) = \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{t^{-\lambda} e^{z\delta^{-\lambda}}}{\sin \pi\lambda}\, d\lambda \tag{7}$$

for $\lambda = \gamma + i\mu$, $0 < \gamma < 1$. It follows from (7) that

$$f(z,t,\delta) - f(z,t^{-1},\delta) = -\int_{-\infty}^{\infty} \frac{\sin(\mu \ln t)}{\sinh \pi\mu}\, e^{z\delta^{-i\mu}}\, d\mu\,, \tag{8}$$

where we also have moved the path of integration to the left so that it passes through the origin. (Note that the identity (8) is a special case of a more general formula stated in [13].) The essential feature of the right hand member of (8) is the factor $e^{z\delta^{-i\mu}}$ for which

$$\int_{\mathbb{C}} |e^{cz\delta^{-i\mu}}|\, e^{-|z|^2/2}\, dm(z) = 2\pi\, e^{c^2/2}\,. \tag{9}$$

Introducing the "c-norm" of an entire function f by

$$\|f\|_c = \int_{\mathbb{C}} |f(cz)|\, e^{|z|^2/2}\, dm(z)$$

(in the notation of the Introduction it is up to a factor just the norm in the space $F^1_{1/\sqrt{c}}$), we see that the right hand side of (8) is a superposition of functions all having the c-norm equal to $e^{c^2/2}$. Now, t is going to be large so the term $f(z,t^{-1},\delta)$ can easily be seen to satisfy (6). Thus the right hand member of (8) is essentially a representation of $f(z,t,\delta)$. Due to rapid convergence at infinity of the integral in (8) we can pass to the function

$$g(z,t,\delta) = \int_{-A}^{A} \frac{\sin(\mu \ln t)}{\sinh \pi\mu}\, e^{z\delta^{-i\mu}}\, d\mu \tag{10}$$

with A fixed > 0. This again may be replaced by

$$\begin{aligned} h(z,t,\delta) &= \int_{-A}^{A} \frac{\sin(\mu \ln t)}{\mu}\, e^{z\delta^{-i\mu}}\, d\mu \\ &= \int_{-A\ln(1/\delta)}^{A\ln(1/\delta)} \frac{\sin(\mu \ln t/\ln(1/\delta))}{\mu}\, e^{ze^{i\mu}}\, d\mu\,, \end{aligned} \tag{11}$$

where in the last equality we have made the change of variable $\mu \mapsto \mu/\ln(1/\delta)$. Choosing A so that $A\ln(1/\delta) = \pi$ we get

$$h(z,t,\delta) = \int_{-\pi}^{\pi} \frac{\sin(\mu \ln t/\ln(1/\delta))}{\mu}\, e^{ze^{i\mu}}\, d\mu\,. \tag{12}$$

Finally, we pass to the function

$$j(z,t,\delta) = \int_{-\pi}^{\pi} \frac{\sin(\mu \ln t/\ln(1/\delta))}{\sin(\mu/2)}\, e^{ze^{i\mu}}\, d\mu\,. \tag{13}$$

It is easy to estimate the error thereby committed but we shall not enter into details. Thus, *testing* (6) *for* $f(z,t,\delta)$ *is completely equivalent to testing it for* $j(z,t,\delta)$.

Next we establish the following result.

Proposition 4. *Let* φ *be a bounded radial function in* $\mathbb{C}$: $\varphi(z) = \sum_k \hat{\varphi}(k)\, e^{-ik\theta}$. *Then one has the identity*

$$\begin{aligned} &\int_{\mathbb{C}} j(cz,t_N,\delta)\,\varphi(z)\,e^{-|z|^2/2}\,dm(z) \\ &\qquad = (2\pi)^2 \sum_{n=0}^{N} \frac{c^n}{n!}\, 2^{n/2}\,\Gamma\Big(\frac{n}{2}+1\Big)\,\hat{\varphi}(-n)\,, \end{aligned} \tag{14}$$

where $\log t_N = (N+1/2)\log(1/\delta)$.

PROOF. For every bounded function φ we clearly have

$$\begin{aligned} &\int_{\mathbb{C}} j(z,t,\delta)\,\varphi(z)\,e^{-|z|^2/2}\,dm(z) \\ &= \Big(\int_{-\pi}^{\pi} \frac{\sin(\mu \ln t/\ln(1/\delta))}{\sin(\mu/2)}\,d\mu\Big)\Big(\int_{\mathbb{C}} e^{cze^{i\mu}}\,\varphi(z)\,e^{-|z|^2/2}\,dm(z)\Big). \end{aligned} \tag{15}$$

Setting $z = re^{i\theta}$, writing

$$\varphi(z) = \varphi(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n,r)\,e^{in\theta}\,,$$

(for our purposes we may without loss of generality assume that this is a finite sum) and using the expansion

$$e^{cze^{i\mu}} = \sum_{n=0}^{\infty} \frac{c^n}{n!} z^n e^{in\mu} = \sum_{n=0}^{\infty} \frac{c^n}{n!} r^n e^{in\theta} e^{in\mu},$$

we see that the inner integral in (15) equals

$$2\pi \sum_{n=0}^{\infty} \frac{c^n}{n!} e^{in\mu} \int_0^{\infty} \hat{\varphi}(-n, r)\, r^n\, e^{-r^2/2}\, r\, dr. \tag{16}$$

Now choose $t = t_N$ in (15) so that $\ln t / \ln(1/\delta) = N + 1/2$, where N is a positive integer at our disposal. Using the well-known identity

$$1 + 2\sum_{n=1}^{N} \cos n\mu = \frac{\sin(N+1/2)\mu}{\sin(\mu/2)},$$

we get inserting (16) into (15)

$$\begin{aligned} &\int_{\mathbb{C}} j(cz, t_N, \delta)\, \varphi(z)\, e^{-|z|^2/2}\, dm(z) \\ &\qquad = (2\pi)^2 \sum_{n=0}^{N} \frac{c^n}{n!} \int_0^{\infty} \hat{\varphi}(-n, r)\, r^n\, e^{-r^2/2}\, r\, dr. \end{aligned} \tag{17}$$

Specializing to the case $\varphi(e^{i\theta})$, that is, $\hat{\varphi}(n) = \hat{\varphi}(n, r)$ independent of r, we get

$$\begin{aligned} \int_0^{\infty} \hat{\varphi}(-n, r)\, r^n\, e^{-r^2/2}\, r\, dr &= \hat{\varphi}(-n) \int_0^{\infty} r^n\, e^{-r^2/2}\, r\, dr \\ &\underset{r^2=2s}{=} \hat{\varphi}(-n) \int_0^{\infty} (2s)^{n/2}\, e^{-s}\, ds \\ &= \hat{\varphi}(-n)\, 2^{n/2} \int_0^{\infty} s^{n/2}\, e^{-s}\, ds \\ &= \hat{\varphi}(-n)\, 2^{n/2}\, \Gamma\!\left(\frac{n}{2} + 1\right). \end{aligned}$$

Summing up we obtain in this case in view of (17) the desired formula, *viz.* (14).

Finally, we prove the following result.

Proposition 5. *There exist trigonometric polynomials $\varphi = \varphi_N$ of degree N uniformly bounded in z and N, such that*

$$\int_{\mathbb{C}} j(cz, t_N, \delta)\, \varphi_N(z)\, e^{-|z|^2/2}\, dm(z) \geq C\, e^{c^2/2} \log N\,, \tag{18}$$

where $C > 0$.

PROOF. We choose φ as a pure cosine series, *i.e.* $\hat{\varphi}(n) = \hat{\varphi}(-n)$:

$$\varphi = a_0 + 2 \sum_{n=1}^{\infty} c_n \cos n\theta\,. \tag{19}$$

Even more, we shall take φ to be the Fejér function

$$\begin{aligned} \varphi_N(z) = {} & \frac{\cos\theta}{2N-1} + \frac{\cos 2\theta}{2N-3} + \cdots + \frac{\cos N\theta}{1} \\ & - \frac{\cos(N+1)\theta}{1} - \frac{\cos(N+2)\theta}{2} - \cdots - \frac{\cos 2N\theta}{2N-1}\,; \end{aligned} \tag{20}$$

cf. [15, p. 416, 13.41], where it is proved that $\sup_{\mathbb{C}} |\varphi_N(z)| \leq C$. Comparing (19) and (20) we see that

$$\hat{\varphi}_N(0) = 0\,, \quad 2\,\hat{\varphi}_N(n) = \frac{1}{2N - (2n-1)}\,, \qquad (1 \leq n \leq N)\,,$$

i.e. (18) becomes

$$\begin{aligned} & \int_{\mathbb{C}} j(cz, t_N, \delta)\, \varphi_N(z)\, e^{-|z|^2/2}\, dm(z) \\ & \qquad = \frac{(2\pi)^2}{2} \sum_{n=1}^{N} \frac{c^n}{n!}\, \frac{\Gamma(\frac{n}{2}+1)\, 2^{n/2}}{2N - (2n-1)}\,. \end{aligned} \tag{21}$$

By Stirling's formula we have

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\, (1 + o(1))\,, \qquad \text{as } x \to \infty\,.$$

For large n we therefore get

$$\Gamma\big(\frac{n}{2}+1\big) = \Big(\frac{n}{2e}\Big)^{n/2}\sqrt{2\pi\,\frac{n}{2}}\,(1+o(1))\,;$$
$$n! = \Gamma(n+1) = \Big(\frac{n}{e}\Big)^{n}\sqrt{2\pi n}\,(1+o(1))\,,$$

so that

$$\frac{\Gamma(\frac{n}{2}+1)\,2^{n/2}}{n!} = \frac{\Big(\frac{n}{e}\Big)^{n/2}\sqrt{\pi n}}{\Big(\frac{n}{e}\Big)^{n}\sqrt{2\pi n}}\,(1+o(1)) = \Big(\frac{e}{n}\Big)^{n/2}\frac{1}{\sqrt{2}}\,(1+o(1))\,.$$

Hence, for n large,

$$\text{(22)}\qquad \frac{c^n}{n!}\,\Gamma\big(\frac{n}{2}+1\big)\,2^{n/2} = \Big(\frac{c^2e}{n}\Big)^{n/2}\frac{1}{\sqrt{2}}\,(1+o(1))\,.$$

For $N > M$ both large we now get by (21) and (22) (if we use $1+o(1)\geq 1/2$)

$$\text{(23)}\qquad \begin{aligned}&\int_{\mathbb{C}} j(cz, t_N, \delta)\,\varphi_N(z)\,e^{-|z|^2/2}\,dm(z)\\ &\qquad\geq (2\pi)^2\,\frac{1}{4\sqrt{2}}\sum_{n=M}^{N}\Big(\frac{c^2e}{n}\Big)^{n/2}\frac{1}{2N-(2n-1)}\,.\end{aligned}$$

Now, take c as a large positive integer and choose $N = c^2$, $M = c^2 - c$ in (23). Then the sum in the right hand side of (23) becomes

$$\begin{aligned}\sum_{n=M}^{N}\cdots &= \sum_{n=c^2-c}^{c^2}\Big(\frac{c^2e}{n}\Big)^{n/2}\frac{1}{2c^2-(2n-1)}\\ &\underset{\substack{n=c^2-j\\0\leq j\leq c}}{=}\sum_{j=0}^{c}\Big(\frac{c^2e}{c^2-j}\Big)^{(c^2-j)/2}\frac{1}{2j+1}\\ &= \sum_{j=0}^{c} e^{(c^2-j)/2}\underbrace{\Big(1+\frac{j}{c^2-j}\Big)^{(c^2-j)/2}}_{\approx e^{j/2}}\frac{1}{2j+1}\end{aligned}$$

$$\geq C\, e^{c^2/2} \sum_{j=0}^{c} \frac{1}{2j+1} \geq C_1\, e^{c^2/2} \log c\,,$$

proving (18).

It is clear that from Proposition 4 we get a contradiction to the hypothesis (6) (or (4)). Thereby we have proved also Theorem 3.

Next we treat the parabolic case. As this case is rather parallel to the elliptic one, we shall not be so detailed.

2. *Parabolic case.* We can again put ourselves in a model situation, namely, when we have the two spaces $G^p = G_1^p = F_{(1,1)}^p$ and $G_k^p = F_{(k,k)}^p$, where we without loss of generality can assume that $k < 1$ (or $k > 1$, whatever we like). Thus this situation corresponds to the pencil of circles tangent to the imaginary axis at the origin. We have to put the corresponding operators T_{g_ζ} on diagonal form. To this end we first take $p = 2$ so that we are dealing with Hilbert spaces. (As usual, we then drop the superscript p in the notation for the spaces.) Then we have the norms

$$\|f\|_G^2 = \int_{\mathbb{C}} |f(z)|^2\, e^{-2y^2}\, dm(z)\,,$$
$$\|f\|_{G_k}^2 = \int_{\mathbb{C}} |f(z)|^2\, e^{-2ky^2}\, dm(z)\,.$$

Introducing the Fourier transform

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx\,,$$

then the right hand sides of the previous formulae become

$$C \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2\, e^{\lambda^2/2}\, d\lambda\,,$$
$$C \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2\, e^{\lambda^2/2k}\, d\lambda\,.$$

From here it is readily seen that

$$\widehat{g_\zeta f}(\lambda) = \hat{f}(\lambda)\, e^{\lambda^2 \zeta (1/k-1)/4}\,. \tag{24}$$

In terms of Moebius transformations this corresponds to the following. First it is seen that to $f(z) \mapsto \hat{f}(\lambda)\, e^{\lambda^2/4}$ there corresponds the map

$$a \mapsto b = \frac{a+2}{2a} = \frac{1}{a} + \frac{1}{2}\,.$$

(It is the correspondence $a \mapsto 1/a$ that accounts for the Fourier transform; as before (see Section 1) the symbol a is used the designate a generic point of $\mathbb{C}$.) This gives the isomorphism $G \approx S_0$ (Schrödinger space). In the same way, to $f(z) \mapsto \hat{f}(\lambda)\, e^{\lambda^2/4k}$ there corresponds the map

$$a \mapsto b_1 = \frac{a+2k}{2ka} = \frac{1}{a} + \frac{1}{2k}\,.$$

Elimination of a between the two equations yields

$$\frac{1}{b_1} = \frac{1}{b} + \frac{1}{2}\Big(\frac{1}{k} - 1\Big)\,.$$

REMARK 3. Note that in particular this implies that for $\theta \in (0,1)$

$$\|g_\theta\|_G^2 = \int_{\mathbb{C}} |\hat{f}(k)|^2\, e^{-2ky^2}\, dm(z)\,,$$

where $1/k_\theta = 1-\theta+\theta/k$. This is the result from [JPR], which we wrote down already in the Introduction (see also Theorem 2 in Section 2).

From (24) we see now that our question is about the multiplier

$$\frac{1}{1+t\, e^{(1/k-1)\lambda^2/2}}\,.$$

Imitating what we have done already in the elliptic case (this Section *infra*, we associate, quite generally, with any suitable locally integrable function $\Omega(\lambda)$ on the real line $\mathbb{R}$ a multiplier transform P_Ω (it is the analogue of the previous R_ω) defined on the space G^p by the formula

$$\widehat{P_\Omega f}(\lambda) = \Omega(\lambda)\, \hat{f}(\lambda)\,.$$

We denote by $\tilde{P}_\Omega$ the same transformation when consider on the Lebesgue space $L_p(\mathbb{R})$. Then we have the following analogue of Proposition 1.

Proposition 6. *The operator P_Ω is bounded on G^p whenever $\tilde{P}_\Omega$ is bounded on $L^p(\mathbb{R})$.*

PROOF. The proof parallels the proof of Proposition 1. Assuming that $\tilde{P}_\Omega$ is bounded on $L^p(\mathbb{R})$ we obtain

$$\begin{aligned}\int_{\mathbb{C}} |P_\Omega f(z)|^p \, e^{-2y^2} \, dm(z) &= \int_{-\infty}^{\infty} e^{-2y^2} \left(\int_{-\infty}^{\infty} |P_\Omega f_y(x)|^p \, dx \right) dy \\ &\le C \int_{-\infty}^{\infty} e^{-2y^2} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \right) dy \\ &= C \int_{\mathbb{C}} |f(z)|^p \, e^{-2y^2} \, dm(z).\end{aligned}$$

Thus P_Ω is bounded on G^p.

Similarly, it is easy to carry over Proposition 2 and from there one derives the expected analogue of Theorem 1 (in the statement of the theorem replace everywhere F^p and F_p^k by G^p and G_p^k respectively), but according to our above promise we omit the details.

There remains the hyperbolic case. But there is a difficulty hidden which we have not been able to overcome ...

3. *Hyperbolic case.* In a model situation the pencil might consist of lines through a point, say, the origin -the configuration that perhaps first comes to our mind. So if $p = 2$ we are in a situation when the spaces to be interpolated are Schrödinger spaces S_θ. The trouble is that in the general case $p \neq 2$ we do not possess a workable analogue of these spaces; in particular, we know of no counterpart of the Corollary to Theorem 1 in Section 1.

So let us instead take as model the case of the spaces F^p and $F^p_{(k,l)}$ with $k = \sec\theta$, $l = i\tan\theta$. The pencil then consists of the circles through the points 0 and $-2 - 2i\tan\theta$. (It is easy to see that this is basically the set up of Section 2, Example 3 shifted the amount 2 to the left.)

It is convenient to introduce, quite generally, the notation $H_k^p = F^p_{(k,l)}$ if k and l are related by the previous relation $k = \sec\theta$, $l = i\tan\theta$. The corresponding circle with center at the point $-1 - i\tan\theta$ may be written C_θ. We are going to follow our usual abuses putting $H = H_1$ and also dropping the superscript p if $p = 2$.

So we want to interpolate between H^p and a fixed space H_k^p. The first thing is to determine the corresponding diagonalizing map (taking temporarily $p = 2$). In the case of H we have $\theta = 0$ and the circle C_0 is the unit circle so transforming the space H into a Schrödinger space S_0 goes as before (in the parabolic case) via the Moebius map

$$a \mapsto b = \frac{a+2}{2a} = \frac{1}{a} + \frac{1}{2} \, .$$

In the case of H_k and the circle C_θ we first observe that the parameter θ has a geometric meaning: it is the angle between this circle and C_0. Hence the previous map must essentially be composed by a rotation by an angle θ. This leads to the map

$$a \mapsto b' = \frac{e^{i\theta} a + 2 \sec\theta}{2\, e^{i\theta} \sec\theta a} = \frac{1}{a} + \frac{1}{2} \, .$$

Elimination between the last two identities, as in the parabolic case, yields

$$b' = \frac{1}{2}\sin\theta + \left(b - \frac{1}{2}\right) e^{-i\theta} = \frac{e^{-i\theta}}{a} + \frac{1}{2}\, i \sin\theta \, ,$$

where we have used Euler's formula $e^{-i\theta} = \cos\theta - i\sin\theta$. From this we get the map

$$f(z) \mapsto \hat{f}(e^{i\theta}\lambda)\, e^{i \sin\theta \lambda^2/4} \, . \tag{25}$$

(This would correspond to $\hat{f}(\lambda) \mapsto \hat{f}(\lambda)\, e^{\zeta(1/k-1)\lambda^2/4}$ in the parabolic case; see formula (24).) But, in view of the appearence, of the factor $e^{i\theta}$ in front of the variable λ in the second half of (25), the map given by this formula does not give a simultanous diagonalization of the group operators $\zeta \mapsto g_\zeta$. To get a *bona fide* diagonalization we must first apply a Mellin type transformation to the Fourier transform $\hat{f}(\lambda)$. But then we do not have anymore any such simple transfer results as the above Propositions 1 and 6. Therefore we stop here hoping to be able to resume this thread on a future occasion. Concluding let us only remark (as a conjecture!) that perhaps it is the case that the presence of a hyperbolic pencil does not imply quasi-linearizability.

4. Concluding remarks.

In this section we consider some left-overs from the previous sections, also complementing some points in [9]. We begin by some easy observations on the Orlicz case.

4.1. On Orlicz-Fock spaces.

Recall that, generally speaking, a measurable function f on some measure space X endowed with a measure μ is said to belong to the Orlicz space $L^\Phi = L^\Phi(X,\mu)$, where Φ is an Orlicz function (in particular, increasing), if

$$\int_X \Phi\left(\frac{|f(x)|}{\alpha}\right) d\mu(x) < \infty \tag{1}$$

for some number $\alpha > 0$. It is well-known that L^Φ is a (quasi-)Banach space with the (quasi-)norm of f defined by

$$\|f\|_{L^\Phi} = \inf \alpha\,, \tag{2}$$

where α ranges over all numbers satisfying (1). If Φ is convex, we can drop the affix "quasi" everywhere. If $\Phi(u) = u^p$, $p > 0$, then we get back the Lebesgue space L^p.

This suggests (*cf.* Section 1) to introduce in our case the Orlicz-Fock spaces $F^\Phi_{(k,l)}$ as the space of entire analytic functions f in $\mathbb{C}$ such that the function $|f(z)|\, e^{-(k|z|^2-\mathrm{Re}(lz^2))/2}$ belongs to L^Φ when $X = \mathbb{C}$ and $\mu = m$ (Euclidean measure). Again it is clear that if $\Phi(u) = u^p$, $p > 0$, they reduce to the spaces $F^p_{(k,l)}$. Also in the general case they should have similar properties as the spaces $F^\Phi_{(k,l)}$. The norm of f in $F^\Phi_{(k,l)}$ is the induced norm and will be written $\|f\|_{(k,l);\Phi}$. (In the special case $k = 1$, $l = 0$ we allow us to drop these indices in the notation.)

We shall limit ourselves to calculating the norm of the corresponding Gauss-Weierstrass functions e_{ac} in one simple case, namely when Φ is of the form

$$\Phi(u) = \sum_{p=1}^{\infty} A_p u^p \qquad \text{with} \ \ A_p \geq 0\,,$$

for simplicity's sake also taking $k = 1$, $l = 0$.

REMARK 1. More generally we could have allowed functions admitting an integral representation (Mellin transform):

$$\Phi(u) = \int_0^\infty A(p)\, u^p\, dp \qquad \text{with } A(p) \geq 0\,.$$

Using a formula established in [9, Section 2] for the norm of e_{ac} in the space F^p, we then find

$$(3)\qquad \begin{aligned} &\int_{\mathbb{C}} \Phi\left(\frac{|e_{ac}(z)|\, e^{-|z|^2/2}}{\alpha}\right)^p dm(z) \\ &= \sum_{p=1}^\infty A_p \int_{\mathbb{C}} \frac{(|e_{a,c}(z)|\, |e^{-|z|^2/2})^p}{\alpha^p}\, dm(z) \\ &= \sum_{p=1}^\infty A_p \left(\frac{p}{2}\right)^{-1} \frac{\exp\left(\frac{p}{2}\,\frac{\operatorname{Re} a\bar{c}^2 + |c|^2}{1-|a|^2}\right)}{\alpha^p}\, \frac{1}{(1-|a|^2)^{1/2}} \\ &= \Phi_1\left(\frac{\exp\left(\frac{1}{2}\,\frac{\operatorname{Re} a\bar{c}^2 + |c|^2}{1-|a|^2}\right)}{\alpha^p}\right) \frac{1}{(1-|a|^2)^{1/2}}\,, \end{aligned}$$

where we in the last step have introduced the notation

$$\Phi_1(u) = \sum_{p=1}^\infty \left(\frac{p}{2}\right)^{-1} A_p u^p\,.$$

We now take account of (2) letting α tend to $\|e_{ac}\|_\Phi$. Then we end up with the formula

$$(4)\qquad \|e_{ac}\|_\Phi = \exp\left(\frac{1}{2}\,\frac{\operatorname{Re} a\bar{c}^2 + |c|^2}{1-|a|^2}\right) \frac{1}{\Phi_1^{-1}((1-|a|^2)^{1/2})}$$

In particular, we draw the conclusion that $e_{a,c} \in F^\Phi_{(k,l)}$ if and only if $|a| < 1$. Obviously, a similar result must hold true for general k and l as well.

4.2. Fock space in the case $0 < p < 1$.

Now we return to the case of Lebesgue spaces L^p but for a change take $p < 1$. (Thus we are leaving the realm of Banach spaces.) Also we take again $k = 1$, $l = 0$. In this situation we have the following obvious generalizing of a corresponding result in [9, Section 8] for $p \geq 1$.[11]

Theorem 1. *The metaplectic group $\tilde{G} = \mathrm{Mp}(2, \mathbb{R})$ acts on the space F^p, $0 < p < 1$. Indeed, if $g \in \tilde{G}$ then we have the following estimate for the opearator norm of T_g in F^p:*

$$\|T_g\|_p \approx \frac{1}{(1 - |g0|^2)^{(1/p - 1/2)/2}} \,. \tag{5}$$

PROOF. Let f be in F^p. Let us write $\|f\|_p = \|f\|_{F_p}$ for the (quasi-) norm in F_p; we use this notation even for $p \geq 1$. Let us further put $e_c(z) = e_{0,c}(z) = e^{cz}$ (exponential function). In view of Wallstén's theorem [17] (atomic decomposition in F_p, $0 < p < 1$; generalization of the corresponding result for F_1 in [6, Theorem 8.1]) we can write

$$f = \sum_k \lambda_k \frac{e_{c_k}}{\|e_{c_k}\|_2} \qquad \text{with} \quad \sum_k |\lambda_k|^p < \infty \,,$$

for some sequence of complex numbers $\{c_k\}$. For $g \in \tilde{G}$ we now obtain

$$T_g f = \sum_k \lambda_k \frac{T_g e_{c_k}}{\|e_{c_k}\|_2} \,.$$

For each index k we have ([9], Section 3)

$$T_g e_{c_k} = e_{g0, c_k/\delta}\, \delta^{-1/2}\, e^{-\gamma c_k^2/2\delta} \,.$$

In particular, we have

$$\begin{aligned} \|T_g e_{c_k}\|_p &= \|e_{g0,c_k/\delta}\|_p\, |\delta|^{-1/2} \exp\left(- \mathrm{Re}\, \frac{\gamma c_k^2}{2\delta} \right) \\ &= \left(\frac{p}{2}\right)^{-1/p} \exp\left(\frac{1}{2} \frac{\mathrm{Re}\, \dfrac{\beta\, \bar{c}_k^2}{\delta\, |\delta|^2} + \dfrac{|c_k|^2}{\delta^2}}{1 - \dfrac{|\beta|^2}{|\delta|^2}} \right) \end{aligned}$$

---

[11] This theorem again was used already in Section 2.

$$\cdot\left(1-\frac{|\beta|^2}{|\delta|^2}\right)^{1/2p}|\delta|^{-1/2}\exp\left(-\frac{1}{2}\operatorname{Re}\frac{\gamma c_k^2}{\delta}\right)$$
$$=\left(\frac{p}{2}\right)^{-1/p}|\delta|^{1/p-1/2}e^{|c_k|^2/2},$$

while

$$\|e_{c_k}\|_2=e^{|c_k|^2/2}.$$

It follows that

$$\|T_gf\|_p^p\le 2\sum_k|\lambda_k|^p\left(\frac{\|T_ge_{c_k}\|_p}{\|e_{c_k}\|_2}\right)^p\le 2\left(\frac{p}{2}\right)^{-1}|\delta|^{1-p/2}\sum_k|\lambda_k|^p<\infty,$$

whence $T_gf\in F_p$. As

$$g0=\frac{\beta}{\delta},\qquad 1-|g0|^2=1-\frac{|\beta|^2}{|\delta|^2}=\frac{1}{|\delta|^2},$$

it likewise follows that inequality (5) is true.

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