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Boundary Harnack Principle for separated semihyperbolic repellers, harmonic measure applications

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0. Introduction, notations and main results.

0.1. Boundary Harnack Principle.

There is an extensive literature on Boundary Harnack Principle (BHP). The results are of the following type: let Ω be a domain in $\mathbb{R}^n$ with a certain geometric property of $\partial\Omega$ and let u, v be positive harmonic functions on Ω vanishing on $V \cap \partial\Omega$ (V is an open set), then there exists a constant $C = C(\Omega, V, K)$ such that

$$\frac{u(x)/v(x)}{u(y)/v(y)} \le C\,, \qquad \text{for } x, y \in K \cap \Omega\,, \tag{0.1}$$

where K is a compact in V.

For domains with Lipschitz boundary this was proved independently by Ancona [A1] and Wu [W]. These results have been extended later by Bass and Burdzy [BB] to Hölder domains and L-harmonic functions, where L is a uniformly elliptic operator with bounded coefficients. Here we need a stronger version of BHP similar to the one in nontangentially accessible domains (NTA) which is due to Jerison and Kenig [JK]. Namely they proved a Hölder estimate:

$$\left|\frac{u(x)/v(x)}{u(y)/v(y)} - 1\right| \le C(\Omega, V, K)\,|x-y|^{\epsilon} \tag{0.2}$$

for $x, y \in K \cap \Omega$ if Ω is an NTA domain in $\mathbb{R}^n$.

A byproduct of their approach and important ingredient in proving (0.2) is a fundamental property of harmonic measure in NTA domains:

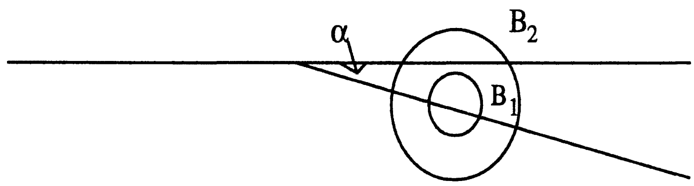
$$\omega(B(x, 2R)) \le C\,\omega(B(x, R)), \tag{0.3}$$

where $B(x, R)$ denotes *the euclidean ball* and $x \in \partial\Omega$. Inequality (0.3) is called *the doubling condition.*

Note that simply connected NTA domains in $\mathbb{R}^2$ are just quasidiscs and (0.2), (0.3) are standard.

We are going to extend these results to a wider class, namely the class of John domains. We send the reader to [Po] or [NV] for the detailed account on this class of domains. Also for terminology such as *John constant*, etc., see [Po], [NV].

Johnness is a very natural generalization of NTA property. However (0.3) and (0.2) are false even for simply connected John domains (*i.e. John discs*). Here is a simple example:



In this example Ω is the complement of three segments meeting at the origin. Then $\omega_\Omega(B_1) \sim r^{\pi/(\pi-\alpha)}$ and $\omega_\Omega(B_2) \sim r \gg r^{\pi/(\pi-\alpha)}$.

However, harmonic measure of John discs satisfies a certain doubling condition. One needs to replace *euclidean balls* by *the balls in internal metric* ρ ($\equiv$ *infimum diameters of curves*): for $Q \in \partial\Omega$, let

$$B_\rho(Q, r) = \{x \in \bar\Omega : \text{there exists } \gamma_{Q,x} \text{ such that } x, Q \in \gamma_{Q,x}\,, \\ \gamma_{Q,x} \setminus \{Q, x\} \subset \Omega\,, \text{ and } \operatorname{diam}\gamma_{Q,x} \le r\}\,.$$

Using the properties of the Riemann mapping (see [Po]) one can now prove:

Proposition 1 *Let Ω be a John disc. Then there exists $C_\Omega < \infty$ such that for all $Q \in \partial\Omega$, $r > 0$,*

$$\omega(\Omega, B_\rho(Q, 2r)) \leq C_\Omega\, \omega(\Omega, B_\rho(Q, r)) . \tag{0.4}$$

To formulate results of the first part of our paper let us introduce the following notion of uniformly John domain.

A domain Ω on the Riemann sphere is called *a uniformly John domain* if for any two points $x_1, x_2 \in \Omega$ there exists a curve $\gamma = \gamma_{x_1,x_2}$ connecting x_1 to x_2 and lying in Ω such that

i) for all $\xi \in \gamma$, $\operatorname{dist}(\xi, \partial\Omega) \geq c_1 \operatorname{dist}(\xi, \{x_1, x_2\})$;

ii) $\operatorname{diam} \gamma \leq C_2\, \rho(x_1, x_2)$.

In this definition "dist" and "diam" are understood in the spherical metric.

John discs are uniformly John domains by the results of Gehring, Hag and Martio [GHM] and Näkki, J. Väisälä [NV]. One can choose the hyperbolic geodesics to serve as γ_{x_1,x_2}.

Our main goal is to extend (0.2) and (0.3) to a broader class of domains. The class of John domains would be a good candidate if we replace the spherical metric by the internal metric. The change of metric is partially justified by the above example and proposition.

But it turns out that (0.2) and (0.3), even with the corresponding change of metric are generally false. This is shown by the example after Theorem 3.2. (Our example concerns (0.2) but may be used as well to disprove (0.3).)

To have the extension of (0.2) and (0.3) one wishes to have a certain approximate "self- similarity" of the boundary which is given, for example by uniform Johnness (see Proposition 2).

One can imagine a uniformly John domain as a domain which is uniformly thick at "every scale"(see Proposition 2 below). We also need the complement to be uniformly thick in the sense of potential theory. Here comes the widely used condition of uniformly perfectness of the boundary (UP). Let us recall that a set E is called *uniformly perfect with* UP *constant* $\alpha > 0$ if

$$\operatorname{cap}(B(x,r)\cap E)\geq \alpha \operatorname{cap}(B(x,r)), \quad \text{for all } x\in E, \text{ for all } r\leq \operatorname{diam} E.$$

Here are some of our results:

Theorem 3.1. *Let Ω be a uniformly John domain with uniformly perfect boundary. Then harmonic measure satisfies the doubling condition in the sense that* (0.4) *holds.*

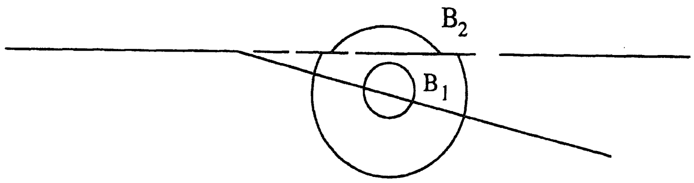
Theorem 3.2. *Let Ω be a uniformly John domain with uniformly perfect boundary. And let u, v be harmonic functions as in* (0.2). *Then for $x, y \in K$ we have*

$$\left|\frac{u(x)/v(x)}{u(y)/v(y)} - 1\right| \leq C(\Omega, V, K)\,\rho(x,y)^{\varepsilon}.$$

The above formulation of the result is stated in Section 3 as Corolary 3.3 and it follows immediately from Theorem 3.2.

Neither uniform Johnness nor UP is necessary for (0.4). It is not difficult to construct the corresponding examples. We do not know necessary and sufficient conditions neither for (0.4) nor for (0.5). On the other hand in our dynamical applications we always have the UP property as a natural feature. Uniform Johnness is also available for a wide class of holomorphic dynamical systems.

On the other hand, in a certain sense Theorems 3.1, 3.2 are sharp. At least there is no hope that they hold for an arbitrary John domain. The next example is an easy modification of the example above. We are grateful to Juha Heinonen for it -our initial one was more complicated. Domain Ω now is the complement of three segments united with $\cup(x_n^1, x_n^2)$, where $(x_n^2 - x_n^1)/x_n^1 \to 0$. It is a John domain and it is not uniformly John. The ball B_1 has harmonic measure of the order $r^{1+\varepsilon}$. As to B_2 it has a "mushroom" in its upper part. The size of this mushroom is $O(r)$. So the doubling condition (0.4) fails. By similar considerations the conclusion of Theorem 3.2 also fails for this domain. The conclusion: *Change of metric helps only if we can localize Ω in the sense which we are going to describe.*



The main difficulty of the proofs comes from the "generic" infinite connectivity of Ω. To overcome this, we use an important feature of uniformly John domains. That is: a geometric localization.

This property is similar to the one proven by P. Jones in [Jo] for NTA domains and it was used essentially by Jerison and Kenig in [JK] to prove (0.3). We would like to mention that there is also an abstract theory of BHP (on graphs, Riemann surfaces; see [A2]) which also suggests the use of localization.

To formulate this property, let Ω be a John domain and $Q \in \partial\Omega$, $r > 0$. We say that Ω admits (K, M, N)-localization in Q at the scale r if there exist John domains $\{\Omega_Q^\ell(r)\}_{\ell=1,\dots,N}$ such that

1) $\Omega_Q^\ell(r) \subset \Omega$, and John constants of $\Omega_Q^\ell(r)$ are bounded by K;

2) $\cup_\ell \Omega_Q^\ell(r) \supset B_\rho(Q, r)$;

3) $\operatorname{diam} \Omega_Q^\ell(r) \le M\, r$;

4) $\Omega_Q^i(r) \cap \Omega_Q^j(r) = \varnothing$, $i \neq j$.

The domain Ω is called *John localizable* if it is (K, M, N)-localizable for all $Q \in \partial\Omega$ and for all r, $0 < r < \operatorname{diam}\Omega$ with *uniform bounds* on K, M and N.

For details on this property we refer to [BV1] and [BV2], where the following assertion was proved:

Proposition 2. *A John domain is uniformly John if and only if it is John localizable.*

0.2. Applications to holomorphic dynamical systems.

In the second part of the paper we are going to apply our results to study the harmonic measure on Julia sets of a large class of dynamical systems.

Let us remind that a generalized polynomial-like map (GPL) is a triple (f, V, U) where U is a topological disc and V is the finite union of topological discs with disjoint closures and $f : V \to U$ is a branched (or regular) covering. The limit set ($\equiv$ *Julia set*) is $J_f = \partial K_f$, where $K_f = \bigcap_{n\geq 0} f^{-n}(U)$ is the filled Julia set. The dynamics is called *semihyperbolic* if

1) there are no parabolic points on J_f;

2) all critical points on J_f are separated from their orbit:

$$\text{dist}\,(c, orb(c)) \geq \Delta > 0\,, \qquad \text{for all } \ c \in \text{Crit}\,(f) \cap J_f\,.$$

The reader may find the detailed account on semihyperbolicity *e.g.* in [CJY], [Ma], [BV1], [BV2]. What we need is the important result of [CJY] which we formulate in a form convenient for us:

Theorem A ([CJY]). *For a generalized polynomial-like system (f,V,U), the domain $A_\infty(f) := \bar{\mathbb{C}} \setminus K_f)$ is a John domain if f is semihyperbolic.*

Another important fact is a recent result by Mañé and da Rocha [MR], Hinkkanen [H] and Eremenko [E].

Theorem B. *Let (f, V, U) be a generalized polynomial-like map. Then J_f is uniformly perfect.*

These results were proven in the context of rational dynamics but the proofs can be carried over without major modifications. Or, one can use the fact that a GPL is quasiconformally (qc) conjugated to a polynomial. Johnness and the UP property are quasiconformally invariant.

We may ask when a GPL (f, U, V) gives rise to a uniformly John $A_\infty(f)$. For general semihyperbolic GPL this is not true (see [BV2]). On the other hand the uniform Johnness of $A_\infty(f)$ holds for the class of so called *separated semihyperbolic* GPL. We are going to recall this definition from [BV2].

Let us remind that K_f denotes the filled Julia set of f. For $x \in K_f$ we denote by K_x the component of connectivity of K_f containing x. We split the critical points in K_f in two parts:

$$C_1 = \{c_1 \in K_f : \ c_1 \text{ is critical point of } f, \ \ K_{c_1} = \{c_1\}\},$$
$$C_2 = \{c_2 \in K_f : \ c_2 \text{ is critical point of } f, \ \ K_{c_2} \neq \{c_2\}\}.$$

After this splitting we have:

Definition 1. *A semihyperbolic* GPL (f, U, V) *is called separated if there is* $\delta > 0$ *such that*

$$\text{dist}\,(\text{orb}\,(c_1), K_{c_2}) > \delta,$$

for any $c_1 \in C_1$ *and* $c_2 \in C_2$.

The following result from [BV2] shows that uniformly John domains appear naturally in dynamics:

Theorem C ([BV2]).

1. *If* J_f *is is totally disconnected, then the domain* $A_\infty(f) := \bar{\mathbb{C}} \backslash K_f$ *is a uniformly John domain if and only if* $A_\infty(f)$ *is a John domain.*

2. *Suppose that the generalized polynomial-like system* (f, V, U) *is semihyperbolic. The domain* $A_\infty(f)$ *is uniformly John if and only if* f *is separated semihyperbolic.*

So uniformly John domains can be met rather often. In the second part of this paper we are going to use Theorem 3.2 to prove a certain *rigidity result* of the harmonic measure on Julia sets of separated semi-hyperbolic GPL. To explain this result let us recall that a GPL (g, U, V) is called *maximal* if $\omega_g = m_g$, where ω_g is the harmonic measure on J_g e valuated at infinity and m_g denotes the measure of maximal entropy. We can now recall from [BPV]:

Definition 2. *The* GPL (f, U, V) *is called conformally maximal if* f *is conformally conjugated in a neighborhhod of* J_f *to a* GPL $(g, \tilde{U}, \tilde{V})$ *which is maximal* .

It is clear that for conformal maximality of (f, U, V) it is <u>necessary</u> to have $\omega_f \approx m_f$. By *rigidity* of harmonic measure, we understand the <u>sufficiency</u> of this condition. Our result in this direction is:

Theorem 5.5. *Let (f, V, U) be a generalized polynomial-like map which is separated semihyperbolic. Then, (f, U, V) is conformally maximal if and only if harmonic measure is absolutely continuous with respect to measure of maximal entropy.*

In particular if, in addition (f, V, U) is polynomial-like in the sense of Douady and Hubbard then f is conformally equivalent to a polynomial map if and only if harmonic measure is absolutely continuous with respect to measure of maximal entropy.

Note that infinite connectivity of $A_\infty(f)$ is a main difficulty again. Particular cases of this result can be found in [LyV] (hyperbolic case), [BPV] (J_f is a Cantor set). Our result is related to results of Lopes [Lo] and Mañe, Da Rocha [MR].

Commenting again the first part let us mention that throughout it we play with ideas virtually present in [A1], [W], [JK]. This is why Section 1 unites the results which have *exactly* the same proofs as corresponding results of [JK]. We give only the statements and references. Unfortunately two key results of [JK] cannot be imitated in our setting. The reason is that for NTA domains internal metric is equivalent to euclidean and for *uniformly John domains* this is no longer the case. Section 2 contains the proof of these stubborn results. In Section 3, Theorems 3.1 and 3.2 are proved.

In the second part of the paper we consider applications in dynamics. In Section 4 we apply Theorem 3.2 to construct a lifting of the harmonic measure to a one sided shift space.

In Section 5 we use this shift model of the harmonic measure to prove Theorem 5.5 .

Commenting on the assumptions of Theorem 5.5, we believe that the result is true for general f (without any condition of separated semihyperbolicity.)

The results of this article were partially announced in [BV3].

1. Estimates of harmonic measure and Green's function in John domains.

Throughout this section domains are John domains with John constants at most K and having uniformly perfect boundary with UP constant at least α.

Lemma 1.1. *There exists $\beta > 0$ and $r_0 > 0$ such that for all $Q \in \partial\Omega$ and $r < r_0$; and every positive harmonic function u in Ω, if u vanishes continuously on $B(Q,r) \cap \partial\Omega$, then there exists $M = M(\alpha)$ such that for $z \in \Omega \cap B(Q,r)$*

$$u(z) \le M\left(\frac{|z-Q|}{r}\right)^{\beta} \max_{\xi \in \Omega \cap \partial B(Q,r)} u(\xi). \tag{1.1}$$

The proof is exactly the same as for [JK, Lemma 4.1].

Lemma 1.2. *Let Ω be a John domain with John center A and diameter D. Let $Q \in \partial\Omega$, $d > 0$ and $\Omega' \subset \Omega$ be a domain with uniformly perfect $\partial\Omega'$ such that $\Omega \setminus \Omega' \subset B(Q, d/100)$. If u is positive harmonic in Ω' and u vanishes continuously on $B(Q,d) \cap \partial\Omega'$, then*

$$u(z) \le M'\, u(A) \tag{1.2}$$

for all $z \in B(Q, d/2) \cap \Omega'$. Here $M' = M'(d/D, K, \alpha)$.

The proof is exactly the same as for [JK, Lemma 4.4]. See also Carleson's article [C, p. 398].

In what follows $\delta(z) = \operatorname{dist}(z, \partial\Omega)$ and $G(\cdot, P)$ denotes Green's function for the domain Ω.

Lemma 1.3. *Let Ω be a John domain. Let $Q \in \partial\Omega$. Then there exists $M = M(K,\alpha)$ such that for every $r > 0$, $z \in B(Q, r/2)$ and $P \in \Omega \setminus B(z, \delta(z)/2)$*

$$G(z,P) \le M\, \omega(B_\rho(Q,r), P).$$

The proof is exactly the same as for [JK, (4.3)].

Lemma 1.4. *Let Ω be a John domain with John center A and diameter D. Let $d > 0, Q, Q' \in \partial\Omega, P \in \Omega \setminus (B_\rho(Q,2d) \cup B(Q',2d))$, $|P - A| \ge d$. Let Ω' be a subdomain of Ω with uniformly perfect $\partial\Omega'$ and such that $\Omega \setminus \Omega' \subset B(Q', d/100)$. Then there exists $M' = M'(D/d, K, \alpha) < \infty$ such that:*

$$G_{\Omega'}(A,P) \le M' \cdot \omega(\Omega', B_\rho(Q,d), P).$$

The independence of P is important here.

REMARK 1.5. The statement of the lemma stays true if we replace "B_ρ" by "B" everywhere.

2. Estimates of harmonic measure and Green's functions in uniformly John domains.

We first prove some assertions concerning arbitrary John domains. We have not been able to follow the lines of [JK], so the proofs are given. As before K is a John constant and $\alpha > 0$ is a UP constant of $\partial\Omega$.

Lemma 2.1. *Let Ω be a John domain with John center A and diameter D. Let $d > 0, Q, Q' \in \partial\Omega$, $P \in \Omega\setminus(B_\rho(Q, 20\,d) \cup B(Q', 20\,d))$, $|P - A| \geq 10\,d$. Let Ω' be a subdomain in Ω with uniformly perfect $\partial\Omega'$ and such that $\Omega\setminus\Omega' \subset B(Q', d/100)$. Then there exists $M' = M'(D/d, K, \alpha) < \infty$ independent of P such that:*

$$\omega(\Omega', B_\rho(Q, d), P) \leq M'\, G_{\Omega'}(A, P)\,.$$

PROOF. We consider only the case $\Omega = \Omega'$. The general case follows exactly the same lines but it is more tedious. Let $B(Q) = B(Q, 3d/2)$, $B(P) = B(P, d/4)$. Lemma 1.2 implies that (consider $\Omega\setminus B(P, \delta(P)/4)$):

$$G(z, P) \leq M_1'\, G(A, P)\,, \qquad z \in \Omega \setminus B(P)\,. \tag{2.1}$$

Points P and Q are $20\,d$-far apart in internal metric, but may not be even $2d$-far apart in euclidean metric. Still, if they are far enough in euclidean sense, namely, if

$$B(Q) \cap B(P) = \varnothing\,, \tag{2.2}$$

we carry out the proof of [JK, (4.6)] as follows. Let φ be a C^∞-function supported by $B(Q)$ such that $\varphi \equiv 1$ exactly on $B(Q, d)$. Then using (2.1) and the estimate $|\Delta\varphi| \leq C \cdot d^{-2}$ we get

$$\omega(B(Q, d), P) \leq \int \varphi\, d\omega(\cdot, P) = |\langle\varphi, \Delta G(\cdot, P)\rangle|$$

$$= \left| \int G(z,P)\,\Delta\varphi\,dA(z) \right|$$
$$\leq M'\,G(A,P) \int_{B(Q)} d^{-2}\,dA(z)$$
$$\leq M'\,G(A,P)\,,$$

which is even more than we wanted to prove: the left part is harmonic measure of $B(Q,d)$ and $B(Q,d) \supset B_\rho(Q,d)$.

Now assume the negation of (2.2):

$$B(Q) \cap B(P) \neq \varnothing\,. \tag{2.3}$$

We cannot modify the proof by modifying φ above. If we try $\varphi \equiv 1$ on $B_\rho(Q,d)$ and with support in $B_\rho(Q,3d/2)$ (which would be appropriate) we cannot claim that $|\Delta\varphi| \sim d^{-2}$. This might not be true.

Given (2.3) let $\mathcal{O}(P)$ denote the component of $\Omega \cap B(P,5d)$ which contains P. Let $\Gamma(P)$ be the part of $\partial\mathcal{O}(P)$ which is not in $\partial\Omega$; $\Gamma(P)$ consists of parts of the circle $\partial B(P,5d)$ which separate P from A. By our assumptions

$$\mathcal{O}(P) \cap B_\rho(Q,d) = \varnothing. \tag{2.4}$$

Let P' denote an arbitrary point on $\Gamma(P)$. By (2.3)

$$B(P') := B(P',d/4) \cap B(Q) = \varnothing\,.$$

Thus, the first part of the proof shows that

$$\omega(B_\rho(Q,d),P') \leq M'\,G(A,P')\,. \tag{2.5}$$

Now the minimum principle is applicable to the harmonic function

$$M'\,G(A,z) - \omega(B_\rho(Q,d),z)\,, \qquad z \in \mathcal{O}(P)\,.$$

It is non-negative on $\partial\mathcal{O}(P)$ by (2.4) and (2.5). Evaluating it at $P \in \mathcal{O}(P)$ we finish the proof of Lemma 2.1.

REMARK 2.2. The statement of the lemma holds if we replace everywhere "B_ρ" by "B".

Lemma 2.3. *Let Ω be a John domain with center A and diameter D. Let $0 < d < D/10$, $Q \in \bar{\Omega}$, $|Q - A| \geq 2d$. Let Ω_1 be a John domain inside $\Omega \cap B_\rho(Q, d/4)$ with the same John and UP constants. We assume that*

$$\operatorname{dist}_\rho(Q, \Omega \cap \partial\Omega_1) \geq c_1\, d\,.$$

Let u, v be two positive harmonic functions in Ω continuous in $\bar{\Omega}$ and vanishing on $B(Q, d) \cap \partial\Omega$. Let $\Omega_1' := \Omega_1 \cap B_\rho(Q, c_1 d/(4K))$. We put

$$M = \max_{\Omega} \frac{u}{v}\,, \qquad m = \min_{\Omega} \frac{u}{v}\,, \qquad \ell = \frac{M}{m} - 1\,,$$
$$M_1 = \max_{\Omega_1'} \frac{u}{v}\,, \qquad m_1 = \min_{\Omega_1'} \frac{u}{v}\,, \qquad \ell_1 = \frac{M_1}{m_1} - 1\,.$$

There exists $q \in (0, 1)$, $q = q(d/D, K, \alpha, c_1)$, such that

$$\ell_1 \leq q\,\ell\,.$$

PROOF. Put $\Gamma = (\partial\Omega) \setminus B(Q, d)$. Without loss of generality we may assume that

$$v(\xi) \leq u(\xi) \leq (1 + \ell)\, v(\xi)\,, \qquad \xi \in \Gamma\,. \tag{2.6}$$

We are going to prove that for all $\xi \in \Omega_1'$ at once: either

$$v(\xi) \leq u(\xi) \leq (1 + q\ell)\, v(\xi)\,, \tag{2.7}$$

or

$$(1 + (1 - q)\ell)\, v(\xi) \leq u(\xi) \leq (1 + \ell)\, v(\xi)\,. \tag{2.8}$$

Any of these two relationships proves the lemma.

The left inequality (2.7) and the right one (2.8) are clear from (2.6) as

$$u = v = 0 \ \text{ on } \ (\partial\Omega) \cap B(Q, d) = (\partial\Omega) \setminus \Gamma\,.$$

Let us denote

$$\Gamma_1 = \left\{\xi \in \Gamma : \ u(\xi) \leq \left(1 + \frac{\ell}{2}\right) v(\xi)\right\}, \qquad \Gamma_2 = \Gamma \setminus \Gamma_1\,.$$

We put

$$v_i(z) = \int_{\Gamma_i} v(\xi)\, d\omega(\Omega, \xi, z)\,, \qquad i = 1, 2\,.$$

We first consider the case

$$v_1(A) \geq v_2(A)\,. \tag{2.9}$$

Starting from (2.9) we will prove the right inequality (2.7). The negation of (2.9) leads to the left inequality (2.8).

First step. We are going to construct a special disc on which (2.9) is nearly satisfied.

Let A_1 denote the John center of Ω_1. As by assumption $\operatorname{diam}\Omega_1 \geq c_1 d$ we get $\operatorname{dist}(A_1, \partial\Omega_1) \geq c_1 d/(2K)$. Let us consider a John arc in Ω connecting A and A_1. It intersects $\partial\Omega_1$ in, say, A_2. We have $|A_2 - A_1| \geq \operatorname{dist}(A_1, \partial\Omega_1) \geq c_1 d/(2K)$ and John property implies $\delta(A_2) := \operatorname{dist}(A_2, \partial\Omega) \geq c_1 d/(2K^2)$. Put

$$\beta = \min\left\{\frac{c_1 d}{4K^2}, \frac{c_1 d}{10}\right\}$$

and consider $B(A_2, \beta)$. By Harnack's inequality and (2.9)

$$v_1(\eta) \geq \gamma\, v_2(\eta)\,, \qquad \text{for all } \eta \in B(A_2, \beta)\,, \tag{2.10}$$

where $0 < \gamma = \gamma(d/D, K, c_1)$.

Second step. Let us prove that

$$u(\eta) \leq (1 + q'\ell)\, v(\eta)\,, \qquad \text{for all } \eta \in B(A_2, \beta)\,, \tag{2.11}$$

for a certain $q' \in (0, 1)$, $q' = q'(d/D, K, c_1)$.

Apply Poisson formula to the function u in Ω:

$$\begin{aligned} u(\eta) &= \int_\Gamma u(\xi)\, d\omega(\Omega, \xi, \eta) \\ &= \int_{\Gamma_1} + \int_{\Gamma_2} \\ &\leq \left(1 + \frac{\ell}{2}\right) \int_{\Gamma_1} v(\xi) + \cdots + (1 + \ell) \int_{\Gamma_2} v(\xi) \cdots \\ &:= \left(1 + \frac{\ell}{2}\right) v_1(\eta) + (1 + \ell)\, v_2(\eta)\,. \end{aligned}$$

Now $(v_1 + v_2)/v = 1$ and $v_1 \geq \gamma v_2$ on $B(A_2, \beta)$. Thus (2.11) follows.

If (2.11) were true in Ω_1' $(:= \Omega_1 \cap B_\rho(Q, c_1 d/4))$ we would be already done.

To push η in (2.11) from $B(A_2, \beta)$ to Ω_1' we need several steps more. Let $\partial_{\text{dist}} \Omega_1 = (\partial \Omega_1) \cap \Omega$.

Third step.

$$v(\eta) \geq \gamma' v(A), \qquad \eta \in B(A_2, \beta), \tag{2.12}$$

$$v(\eta) \leq M' v(A), \qquad \eta \in \bar{\Omega}_1 . \tag{2.13}$$

Here $0 < \gamma' = \gamma'(d/D, K, c_1)$, $M' = M'(d/D, K, \alpha) < \infty$. First inequality is just Harnack's inequality. The second one follows by Lemma 1.2.

Forth step. Let $\{x_i\}_{i=1}^N$ be a maximal $\beta/2$-net on $\partial_{\text{dist}} \Omega_1$ in the sense of metric $\rho_1 = \rho_{\Omega_1}$ on $\bar{\Omega}_1$. Clearly

$$N = N(K, c_1, d) < \infty . \tag{2.14}$$

Let $B_{\rho_1}^i$ denote $B_{\rho_1}(x_i, \beta/2)$.

We apply Lemmas 2.1 and 1.4 to Ω_1 and $\eta \in \Omega_1'$ playing the role of P, x_i playing the role of Q. Then

$$\omega(\Omega_1, B_{\rho_1}^i, \eta) \asymp G_{\Omega_1}(A_1, \eta), \qquad \eta \in \Omega_1' , \tag{2.15}$$

$$\omega(\Omega_1, B(A_2, \beta), \eta) \asymp G_{\Omega_1}(A_1, \eta), \qquad \eta \in \Omega_1' . \tag{2.16}$$

The constants implicitly involved here depend only on $d/D, K, \alpha$ and c_1.

Fifth step. We apply Poisson formula to u, (as in the second step) but now in Ω_1. For $\eta \in \Omega_1' (= \Omega_1 \cap B_\rho(Q, c_1 d/(4K)))$.

$$\begin{aligned} u(\eta) &= \int_{\partial_{\text{dist}} \Omega_1} u(\xi)\, d\omega(\Omega_1, \xi, \eta) \\ &= \int_{B(A_2,\beta) \cap \partial \Omega_1} u(\xi)\, d\omega(\Omega_1, \xi, \eta) + \int_{(\partial_{\text{dist}} \Omega_1) \setminus B(A_2,\beta)} u(\xi)\, d\omega(\Omega_1, \xi, \eta) \\ &\leq (1 + q'\ell) \int_{\dots} v(\xi) + \cdots + (1 + \ell) \int_{\dots} v(\xi) \cdots \end{aligned} \tag{2.17}$$

$$= (1 + q'\ell)\, v^1(\eta) + (1 + \ell)\, v^2(\eta)\,.$$

We are going to show that there exists $0 < \gamma'' = \gamma''(d/D, K, \alpha, c_1)$ such that

$$v^1(\eta) \geq \gamma''\, v^2(\eta)\,, \qquad \text{for all } \eta \in \Omega_1'\,. \tag{2.18}$$

To do this we proceed as follows. By (2.13)

$$\begin{aligned} v^2(\eta) &\leq \int_{\partial_{\text{dist}}\Omega_1} v(\xi)\, d\omega(\cdots) \\ &\leq \sum_{i=1}^{N} \int_{B_i} \cdots \\ &\leq M'\, v(A) \sum_{i=1}^{N} \omega(\Omega_1, B^i_{\rho_1}, \eta)\,. \end{aligned}$$

Now (2.14), (2.15) and (2.16) imply that for $M'' = M''(d/D, K, \alpha, c_1) < \infty$

$$v^2(\eta) \leq M''\, v(A)\, \omega(\Omega_1, B(A_2, \beta), \eta)\,. \tag{2.19}$$

On other hand by (2.12)

$$v^1(\eta) = \int\limits_{B(A_2,\beta)\cap\partial_{\text{dist}}\Omega_1} v(\xi) d\omega(\cdots) \geq \gamma'\, v(A)\, \omega(\Omega_1, B(A_2, \beta), \eta)\,.$$

Together with (2.19) this gives (2.18).

Taking into account that $(v^1 + v^2)/v = 1$ and (2.17), (2.18) we obtain the required inequality

$$u(\eta) \leq (1 + q\ell)\, v(\eta)\,, \qquad \eta \in \Omega_1'\,,$$

which is the right inequality in (2.7).

Completely similarly the negation of (2.9) leads to the proof of the left part of (2.8). Lemma 2.3 is completed.

From now on Ω is uniformly John, that is, by Proposition 2 it is localizable. Reminding that $\Omega^\ell_Q(r)$, $\ell = \overline{1, k(Q)}$, are local John domains with properties 1)-4) (see the introduction) we denote their John centers by $A^\ell_Q(r)$, and let K be their John constant. By G_Ω we denote Green's function with pole at A.

Clearly all $\partial\Omega^\ell_Q(r)$ are uniformly perfect (as $\partial\Omega$ was) and we denote by α, $\alpha > 0$, the least UP constant for them.

Lemma 2.4. *Let Ω be uniformly John with center A and let $0 < r \leq$ diam Ω, $Q \in \partial\Omega$. Then there exists $M = M(K,\alpha)$ such that*

$$\omega(\Omega, B_\rho(Q,r/2), A) \leq M \sum_{\ell=1}^{k(Q)} G_\Omega(A^\ell_Q(r)) .$$

PROOF. We may put $\operatorname{diam}\Omega = 1$. Let $k = k(Q)$. Fix $r > 0$ and let $\Gamma = B(Q,r) \cap \partial\Omega$. Let $\{Q_1, \ldots, Q_S\}$ be a maximal r-net of Γ in the metric ρ. Clearly $S = S(K) < \infty$ because Ω is a John domain.

In what follows $C_1, C_2, \ldots$ are large constants depending on K and α. Put $\Gamma_0 = \{z \in \Gamma : \rho(Q,z) < r/2\}$, $\tilde\Gamma = \{z \in \Gamma : \rho(Q,z) > C_1 r\}$. Let $\{Q_1, \ldots, Q_c\}$ be a subnet, $c \leq S$, consisting of points $Q_i \in \tilde\Gamma$.

Let us consider the finite family $\mathcal{F} = \{\Omega^\ell_{Q_i}(C_2 r)\}$, $i = \overline{1,C}$ $\ell = \overline{1, k(Q_i)}$. Let $\tilde\Omega$ be their union. We put $\tilde B = B(Q,r) \cap \tilde\Omega$. We have to delete somehow this set because Green's function on it cannot be controlled by $\sum_\ell G_\Omega(A^\ell_Q(r))$.

First step. Is to prove that

$$z \in B(Q,r) \text{ and } \rho(z,Q) > 2C_1 r \qquad \text{implies} \qquad z \in \tilde B . \tag{2.20}$$

In fact, let $\mathcal{O}$ be a component of $B(Q,r) \cap \Omega$ containing z and let $P \in \partial\Omega \cap \partial\mathcal{O}$. As $\rho(P,z) < 2r$ we see that $\rho(P,Q) > C_1 r$, that is $P \in \tilde\Gamma$. Let Q_i be a ρ-closest to P from our net. Then $\rho(z,Q_i) < r + 2r = 3r$ and $z \in \Omega^\ell_{Q_i}(3r)$. Thus $z \in \tilde\Omega$ and (2.20) is proved.

Second step. Let

$$\Omega_0 = \Omega, \quad \Omega_1 = \Omega_0 \setminus \overline{\bigcup_\ell \Omega^\ell_{Q_1}(C_2 r) \cap B(Q,r)}, \quad \ldots,$$
$$\Omega_k = \Omega_{k-1} \setminus \overline{\bigcup_\ell \Omega^\ell_{Q_{k-1}}(C_2 r) \cap B(Q,r)}, \quad \ldots, \quad \Omega_c = \Omega \setminus \overline{\tilde B} .$$

Denote by $G(z), G_1(z), \ldots, G_k(z), \ldots, G_c(z)$ Green's functions of corresponding domains with pole at A. They are ordered:

$$G_c(z) \leq \cdots \leq G_1(z) \leq G_1(z) \leq G(z) . \tag{2.21}$$

Harmonic functions $\omega_k(z) = \omega(\Omega_k, \Gamma_0, z) = \omega(\Omega_k, \Gamma_0 \cap \partial\Omega_k, z)$ are ordered in the same way

$$\omega_c(z) \leq \cdots \leq \omega_1(z) \leq \omega(z).$$

Let us prove first that

$$\omega_c(A) \leq M(K,\alpha) \sum_{\ell=1}^{k} G(A_Q^\ell(2r)). \tag{2.22}$$

To this end note that $B(Q,r) \cap \Omega_c \subset B_\rho(Q, 2C_1 r)$. This is just another way to state (2.20). Recall that $B_\rho(Q, 2C_1 r)$ is covered by

$$\bigcup_{\ell=1}^{k} \Omega_Q^\ell(2M_0\, C_1 r)$$

and apply Lemma 1.2 to $G(z)$ in each of this domains separately. Then

$$z \in B(Q,r) \cap \Omega_c \text{ implies } G(z) \leq M(K,\alpha) \sum_{\ell=1}^{k} G(A_Q^\ell(2M_0\, C_1 r)). \tag{2.23}$$

We may now go along the lines of [JK, (4.6)]. In fact, (2.21) and Harnack's inequality imply

$$G_c(z) \leq M \sum_{\ell=1}^{k} G(A_Q^\ell(r)), \qquad z \in B(Q,r) \cap \Omega_c. \tag{2.24}$$

Consider a C^∞-function φ supported by $B(Q,r)$ and such that $\varphi \equiv 1$ exactly on $B(Q, r/2)$. Then

$$\begin{aligned}
\omega_c(A) &= \omega(\Omega_c, B_\rho(Q, \tfrac{r}{2}), A) \\
&\leq \omega(\Omega_c, B(Q, \tfrac{r}{2}), A) \\
&\leq \int \varphi \, d\omega_c(\xi, A) \\
&= |\langle \varphi, \Delta G_c \rangle| \\
&= \Big| \int_{B(Q,r)} \Delta\varphi \, G_c(z,A)\, dA(z) \Big|
\end{aligned}$$

$$\leq M \sum_{\ell=1}^{k} G(A_Q^\ell(r))\, r^{-2} \int_{B(Q,r)} dA(z)$$
$$= M \sum_{\ell=1}^{k} G(A_Q^\ell(r))\,.$$

And (2.22) is proved. The key estimate was (2.23).

REMARK. We cannot hope to have (2.23) with Ω instead of $\Omega_{\text{(}}$

Just because $\Omega \cap B(Q,r)$ may contain the points which tremely far from Q in ρ-metric. For such points there is no estimate $G(z)$ by $G(A_Q^\ell(Cr)$ if r is small. This is the main c and the main difference from NTA domains. This is why we the procedure of excluding such points and this is why we nee(

Third step.

$$\omega_{k-1}(A) \leq M(K,\alpha)\,\omega_k(A)\,. \tag{2.25}$$

We denote $u_1 = \omega_k$, $u_2 = \omega_{k-1}$, thus having $u_1 \leq u_2$. Denote t matrix

$$\{\omega_{m,l}\}_{m,\ell=1}^{k(Q_k)} = \{\omega_k(\Omega_{Q_k}^m(C_2 r) \cap \partial B(Q,r)\,, \quad A_{Q_k}^\ell(C_2 r))\}_{m,}^{k((}$$

As C_2 grows the centers $A_{Q_k}^\ell(C_2 r)$ are getting more and more f from $\partial B(Q,r)$. In particular, the entries of this matrix can b(as small as we wish: C_2 rules that. To choose appropriate C notice that

$$\text{(2.26)}\quad \xi \in \partial B(Q,r) \cap \Omega_{Q_k}^\ell(C_2 r) \quad \text{implies} \quad u_2(\xi) \leq C_3\, u_2(A_{Q_k}^\ell($$

This is by Lemma 1.2 applied to $\Omega_{Q_k}^\ell(r)$ and $d \sim C_2 r$. Cons depends on John and UP constants of $\Omega_{Q_k}^\ell(C_2 r)$, that is inde of C_2. We choose C_2 so large that $\|(I - C_3\mathcal{U})^{-1}\| \leq 2$.

To compare $u_2(A_{Q_k}^\ell(C_2 r))$ and $u_1(A_{Q_k}^\ell(C_2 r))$ we write d Poisson formula for u_2 in Ω_k:

$$u_2(A_{Q_k}^\ell(C_2 r)) = u_1(A_{Q_k}^\ell(C_2 r)) + \int_{\Omega_{k-1} \cap \partial\Omega_k} u_2(\xi)\, d\omega_k(\xi, A_{Q_k}^\ell($$

We use (2.26) and the choice of $\mathcal{U}$ to conclude

$$u_2(A^\ell_{Q_k}(C_2 r)) \le 2\, u_1(A^\ell_{Q_k}(C_2 r)). \tag{2.27}$$

Put

$$\begin{aligned} 1 \ll C_2' \ll C_2\,, &\qquad 1 \ll C_2'' \ll C_2'\,, \\ \mathcal{O}' = \Big(\bigcup_\ell \Omega^\ell_{Q_k}(C_2' r)\Big) \bigcap \Omega_{k-1}, &\qquad \mathcal{O}'' = \Big(\bigcup_\ell \Omega^\ell_{Q_k}(C_2'' r)\Big) \bigcap \Omega_{k-1}\,, \\ T' = \partial \mathcal{O}' \cap \Omega_{k-1}\,, &\qquad T'' = \partial \mathcal{O}'' \cap \Omega_{k-1}\,. \end{aligned}$$

Clearly

$$\begin{cases} \operatorname{dist}(T', Q_k) \ge C_2' r\,, \\ C_2'' r \le \operatorname{dist}(T'', Q_k) \le \operatorname{dist}_\rho(T'', Q_k) \le \dfrac{1}{100}\, C_2' r\,. \end{cases} \tag{2.28}$$

Let $\{x_j\}_{j=1}^n$ be a maximal $C_2'' r$-net of T' in the metric of $\rho_{\mathcal{O}'}$. Clearly $n = n(K, C_2') < \infty$. Put $B_j = B_{\rho_{\mathcal{O}'}}(x_j, C_2'' r)$. Poisson formula for u_2 in $\mathcal{O}'$ gives that if $\xi \in T''$ then

$$\begin{aligned} u_2(\xi) &= \int_{T'} u_2(z)\, d\omega_{\mathcal{O}'}(z, \xi) \\ &= \sum_1^n \int_{B_j} \cdots \\ &\le C_3 \sum_\ell u_2(A^\ell_{Q_k}(C_2 r)) \sum_1^n \omega_{\mathcal{O}'}(B_j, \xi) \\ &\le 2\, C_3 \sum_\ell u_1(A^\ell_{Q_k}(C_2 r)) \sum_1^n \omega_{\mathcal{O}'}(B_j, \xi)\,. \end{aligned} \tag{2.29}$$

We used (2.26), (2.27) and Lemma 1.2 with $d \sim C_2 r$. Let a^ℓ_k be a point of intersection of T' with the John arc connecting $A^\ell_{Q_k}(C_2 r)$ with Q_k. Let S^ℓ_k denote a disc from the family $\{B_j\}$ containing a^ℓ_k, and $\tilde{S}^\ell_k$ denote $B(a^\ell_k, \delta(a^\ell_k)/2)$. Notice that

$$\delta(a^\ell_k) \ge C(K)\, C_2' r \ge 10\, C_2'' r$$

if C_2' is large enough in comparison to C_2''. In particular $S_k^\ell \subset \tilde{S}_k^\ell$. Clearly (using the notations $A_k^\ell := A_{Q_k}^\ell(C_2 r)$) if $\xi \in T''$, then

$$\begin{aligned} u_1(\xi) &= \int u_1(z)\, d\omega_{\mathcal{O}'\cap\Omega_k}(z,\xi) \\ &\geq C(K)\sum_\ell u_1(a_k^\ell)\,\omega_{\mathcal{O}'\cap\Omega_k}(\tilde{S}_k^\ell,\xi) \\ &\geq C(K, C_2/C_2')\sum_\ell u_1(A_k^\ell)\,\omega_{\mathcal{O}'\cap\Omega_k}(\tilde{S}_k^\ell,\xi)\,. \end{aligned} \tag{2.30}$$

Now we apply Lemmas 1.4 and 2.1 to the domain $\mathcal{O}'$ and the fact that $d \sim C_2'' r$ to obtain

$$\omega_{\mathcal{O}'}(B_s,\xi) \leq C(C_2'/C_2'', K, \alpha)\,\omega_{\mathcal{O}'}(B_t,\xi)\,,$$

for all $\xi \in T''$, and for all s, t.

Thus, if $\xi \in T''$ then

$$u_2(\xi) \leq C(C_2'/C_2'', K, \alpha)\, n \sum_\ell u_1(A_k^\ell)\,\omega_{\mathcal{O}'}(\tilde{S}_k^\ell,\xi)\,. \tag{2.31}$$

If we could prove that

$$\omega_{\mathcal{O}'}(\tilde{S}_k^\ell,\xi) \leq 2\,\omega_{\mathcal{O}'\cap\Omega_k}(\tilde{S}_k^\ell,\xi), \qquad \xi \in T'', \tag{2.32}$$

then (2.30)-(2.31)-(2.32) combined would imply

$$u_2(\xi) \leq C(C_2/C_2'', K, \alpha)\, u_1(\xi)\,, \qquad \text{if } \xi \in T''\,.$$

Applying this and the maximum principle to the component of $\Omega \setminus T''$ which contains A we would get (2.25).

So we are left to prove (2.32).

Let us put

$$w_2(\xi) = \omega_{\mathcal{O}'}(\tilde{S}_k^\ell,\xi)\,, \qquad w_1(\xi) = \omega_{\mathcal{O}'\cap\Omega_k}(\tilde{S}_k^\ell,\xi)\,.$$

Then $w_1 \leq w_2$. To prove the inverse inequality let us consider both functions in $\mathcal{O}' \cap \Omega_k$ and let $\gamma_k = \mathcal{O}' \cap (\partial(\mathcal{O}' \cap \Omega_k))$. From the construction we conclude that $\gamma_k \subset \partial B(Q,r)$ and

$$\operatorname{dist}_{\rho_{\mathcal{O}'}}(\gamma_k, Q_k) \leq 2r\,.$$

Clearly

$$w_2(\xi) \le w_1(\xi) + \omega_{\mathcal{O}' \cap \Omega_k}(\gamma_k, \xi)\, m \,, \tag{2.33}$$

where $m = \max\{w_2(z) : \ z \in \gamma_k\}$.

Introduce $v(z) = \omega(\mathcal{O}', B_{\rho_{\mathcal{O}'}}(Q_k, C(K)C_2'' r), z)$. On γ_k we have by α-uniform perfectness that if $z \in \gamma_k$

$$v(z) \ge \gamma \ge \gamma\, \omega_{\mathcal{O}' \cap \Omega_k}(\gamma_k, z) \,,$$

where $\gamma = \gamma(\alpha) > 0$. On the rest of $\partial(\mathcal{O}' \cap \Omega_k)$ both functions vanish. Thus

$$\xi \in \mathcal{O}' \cap \Omega_k \quad \text{implies} \quad v(\xi) \ge \gamma\, \omega_{\mathcal{O}' \cap \Omega_k}(\gamma_k, \xi) \,. \tag{2.34}$$

Applying Lemmas 1.4 and 2.1 to $\mathcal{O}'$ and the fact that $d \sim C_2'' r$ and taking into account that the radius of $\tilde{S}_k^\ell$ is at least $C(K)\, C_2'\, r$ we get that if $\xi \in T''$, then

$$\begin{aligned} v(\xi) &\le M'(C_2'/C_2'', K, \alpha)\, \omega_{\mathcal{O}'}(\tilde{S}_k^\ell, \xi) \\ &= M'(C_2'/C_2'', K, \alpha)\, w_2(\xi) \,. \end{aligned} \tag{2.35}$$

Uniting (2.33)-(2.35) we obtain that if $\xi \in T''$, then

$$w_2(\xi) \le w_1(\xi) + \gamma^{-1} M'(C_2'/C_2'', K, \alpha)\, m\, w_2(\xi) \,.$$

By an obvious extremal length estimate

$$m = \max\{w_2(z) : \ z \in \gamma_k\}$$

can be made as small as we wish by taking C_2' very large. We keep the ratio C_2'/C_2'' bounded as we make C_2' so large that

$$m\, \gamma^{-1} M'(C_2'/C_2'', K, \alpha) < \frac{1}{2} \,.$$

Then, (2.32) follows and the lemma is completed.

3. Doubling condition and BHP for uniformly.

3.1. John domains.

As always ρ denotes the internal metric.

Theorem 3.1. *Let Ω be a uniformly John domain with constant K and center A, and let $\partial\Omega$ be uniformly perfect with constant α. Then there exists $M = M(K, \alpha)$ such that*

$$\omega(\Omega, B_\rho(Q, 2r), A) \leq M(K, \alpha)\, \omega(\Omega, B_\rho(Q, r), A)\,.$$

PROOF. This follows from Lemma 2.4 if we apply the classical Harnack inequality to $G_\Omega(z) := G_\Omega(z, A)$ for passing from $G_\Omega(A_Q^\ell(4r))$ to $G_\Omega(A_Q^\ell(2r))$, $\ell = 1, \ldots, k(Q)$. We just need to apply Lemma 1.3 to finish the estimate.

Recall that in the introduction we showed that it was essential to replace euclidean balls by B_ρ. The second essential thing was to consider uniformly John property (equivalent to John localizability). The examples given in the introduction show that for general John domains the doubling property in ρ-metric (and in euclidian metric as well) fails to be true.

Theorem 3.2. *Let Ω be a uniformly John domain with constant K and let $\partial\Omega$ be uniformly perfect with constant α. Let $Q \in \partial\Omega$, $R > 0$, and let u, v be two positive harmonic functions in Ω vanishing continuously on $B(Q, 4R) \cap \partial\Omega$. Then there exist $M = M(K, \alpha)$ and $\varepsilon = \varepsilon(K, \alpha) > 0$ such that*

$$\Big|\frac{u(\xi)/v(\xi)}{u(\eta)/v(\eta)} - 1\Big| \leq M\Big(\frac{\rho(\xi, \eta)}{R}\Big)^{\varepsilon} \max_{x, y \in B(Q, 3R)} \Big(\Big|\frac{u(x)/v(x)}{u(y)/v(y)}\Big| + 1\Big)\,,$$

for all $\xi, \eta \in B(Q, R)$.

PROOF. We are going to use Proposition 2. from the introduction and Lemma 2.3.

Let $Q_1 \in \partial\Omega$ be a closest point to ξ. Then $Q_1 \in B(Q, 2R)$. Let $r' = |Q_1 - \xi|$. If $\rho(\xi, \eta) \leq r'/2$, put $r = 2r'$; if $\rho(\xi, \eta) > r'/2$, put $r = 4\,\rho(\xi, \eta)$. Consider the local John domain $\Omega_1 := \Omega_{Q_1}^\ell(r)$ containing

ξ, η; $\ell \in \overline{1, k(Q_1)}$. There exists $M = M(K)$ (this is exactly M from (3) in the introduction) such that $\operatorname{diam} \Omega_1 \leq Mr$, that is $\Omega_1 \subset B_\rho(Q_1, Mr)$. Choose a local John domain $\Omega_2 := \Omega^\ell_{Q_1}(M^2 r)$ containing Ω_1. Again, $\operatorname{diam} \Omega_2 \leq M^3 r$, that is $\Omega_2 \subset B_\rho(Q_1, M^3 r)$. Choose $\Omega_3 := \Omega^\ell_{Q_1}(M^4 r)$ containing Ω_2, etc. We stop when we choose Ω_n such that

$$n = \max\{k : \; M^{2k-1}\, r \leq R\}\,,$$

that is

$$n \geq \frac{1}{2}\, \frac{\log(R/r)}{\log M}\,.$$

Now we are left to apply Lemma 2.3 $n-1$ times to pairs of nested John domains $\Omega_n \supset \Omega_{n-1} \supset \cdots \supset \Omega_2 \supset \Omega_1$. For example, considering the pair $\Omega_k \supset \Omega_{k-1}$ we apply the lemma with $M^{2k-2} r \leq D \leq M^{2k-1} r$, $d = M^{2k-3} r$, $c_1 = 1/M$. If we denote by ℓ_k the quantity

$$\max_{\xi,\eta \in \Omega_k} \left| \frac{u(\xi)/u(\eta)}{v(\xi)/v(\eta)} - 1 \right|,$$

we obtain

$$\ell_{k-2} \leq q\,(d/D, K, \alpha, c_1)\,\ell_k = q(M,\alpha)\,\ell_k\,, qquadq = q(M,\alpha) \in (0,1)\,.$$

Finally

$$\ell_1 \leq q^{n/2-1}\,\ell_n\,,$$

which results in the estimate

$$\max_{\xi,\eta \in \Omega_1} \left| \frac{u(\xi)/v(\xi)}{u(\eta)/v(\eta)} - 1 \right| \leq \left(\frac{r}{R}\right)^{\varepsilon} \max_{x,y \in B(Q,3R)} \left(\left| \frac{u(x)v(x)}{u(y)/v(y)} \right| + 1 \right).$$

If r was chosen to be $4\,\rho(\xi,\eta)$ this is what we want. Otherwise $\rho(\xi,\eta) \leq r/4 = |Q_1 - \xi|/2 = \operatorname{dist}(\xi, \partial\Omega)/2$ and we apply the usual Harnack's inequality in the disc $B(\xi, 3|Q_1 - \xi|/4)$ lying entirely in Ω_1. Then

$$\left| \frac{u(\xi)/v(\xi)}{u(\eta)/v(\eta)} - 1 \right| \leq C\, \frac{\rho(\xi,\eta)}{r} \max_{B(\xi,3|Q_1-\xi|/4)} \left| \frac{u(x)/v(x)}{u(y)/v(y)} - 1 \right|,$$

where C is absolute.

Combining two inequalities above we finish the proof in this case too.

Corollary 3.3. *Let Ω be a domain containing ∞. Let $\mathcal{U}$ be an open set containing $\mathbb{C}\setminus\Omega$ and u, v be two continuous functions in $\mathcal{U}$; positive, harmonic in $\mathcal{U}\cap\Omega$ and vanishing on $\mathbb{C}\setminus\Omega$. Suppose also that Ω is uniformly John with constant K and $\partial\Omega$ is uniformly perfect with constant α. Then if C is a compact subset of $\mathcal{U}$ we have for all $\xi, \eta \in C$ that*

$$\left|\frac{u(\xi)/v(\xi)}{u(\eta)/v(\eta)} - 1\right| \leq M(C, K, \alpha)\,(\rho(\xi, \eta))^{\varepsilon}, \tag{3.1}$$

where $\varepsilon = \varepsilon(C, K, \alpha) > 0$.

4. Generalized polynomial-like maps. Geometric coding tree. Shift model for harmonic measure.

Throughout this section (f, V, U) is a generalized polynomial-like system (see the introduction), $J = J_f$. Let us remind an important notion from [PS]. One constructs the geometric coding tree as follows. Let $z_0 \in U \setminus J$. Let $z^1, \ldots, z^d$ be its preimages. Let γ^i be curves joining z_0 to z^j, $j = 1, \ldots, d$, such that

$$\overline{\operatorname{orb}(c)} \cap \bigcup_{j=1}^{d} \gamma^j = \varnothing$$

for any $c \in \operatorname{Crit}(f)$.

Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$ be the one sided shift space with σ denoting the shift to the left, and ρ_Σ be the standard metric on Σ:

$$\rho_\Sigma(\alpha, \beta) = e^{-k(\alpha,\beta)},$$

where $k(\alpha, \beta)$ is the least integer n for which $\alpha_n \neq \beta_n$. For each sequence α we put $\gamma^1(\alpha) = \gamma^{\alpha_1}$. Suppose that for every m, $1 \leq m \leq n$ and all $\alpha \in \sum$ the curves $\gamma^m(\alpha) : [0, 1] \to U$ are already defined in such a way that $f(\gamma^m(\alpha)) = \gamma^{m-1}(\sigma(\alpha))$ and $\gamma^m(\alpha)(0) = \gamma^{m-1}(\alpha)(1)$. Define $\gamma^{n+1}(\alpha)$ by taking respective preimages of $\gamma^n(\sigma(\alpha))$. Put $z_n(\alpha) = \gamma^n(\alpha)(1)$. The graph $T = T(z_0, \gamma^1, \ldots, \gamma^d)$ with vertices z_0, $z_n(\alpha)$ and edges $\gamma^n(\alpha)$ is called a *geometric coding tree with root at* z_0. Given $\alpha \in \Sigma$, the subgraph composed by $z_0, z_n(\alpha), \gamma^n(\alpha)$ is called *α-branch* and is denoted by $b(\alpha)$. The branch $b(\alpha)$ is called *strongly convergent* if $\{\gamma^n(\alpha)\}$ converges to a point as $n \to \infty$.

It is easily seen from [CJY, Theorem 2.1] or from [BV1], [BV2] that the following simple proposition holds

Lemma 4.1. *Let f be semihyperbolic, then there exist $C_1 < \infty$, $0 < \theta_1 < 1$, such that*

$$\operatorname{diam} \gamma^n(\alpha) \le C_1\, \theta_1^n\,, \qquad \alpha \in \Sigma\,. \tag{4.1}$$

In particular each $b(\alpha)$ strongly converges to a point of J. (For a general f strong convergence of $b(\alpha)$ holds for all α except a set of zero Hausdorff dimension in (Σ, ρ_Σ).) Let $\pi(\alpha) = \lim z_n(\alpha)$ be a point of convergence of $b(\alpha)$. Let $r_n(\alpha) := |z_n(\alpha) - \pi(\alpha)|$.

Lemma 4.2. *Let f be semihyperbolic. Then*

$$\#\{z_n(\beta) :\ \beta \in \Sigma\,,\ |z_n(\beta) - \pi(\alpha)| \le k\, r_n(\alpha)\} \le C(k, f)\,, \tag{4.2}$$

for any $\alpha \in \Sigma$ and any n.

PROOF. Put $x = \pi(\alpha)$. Choose M_0 in such a way that

$$\frac{C_1\, \theta_1^{M_0}}{1 - \theta_1} < \frac{\varepsilon}{2}\,, \tag{4.3}$$

where ε is from [CJY, Theorem 2.1] (Theorem B of [BV1], [BV2]).

Denote $\beta = \sigma^{n-M_0}(\alpha)$. Then $\pi(\beta) = f^{n-M_0}(x)$. Let $b_k(\alpha) = \cup_{i \ge k} \gamma^i(\alpha)$. Using (4.3) we have that $\operatorname{diam} b_{M_0}(\beta) \le C_1 \theta_1^{M_0}/(1 - \theta_1) \le \varepsilon/2$. Therefore $b_{M_0}(\beta) \subseteq U_0 := B(f^{n-M_0}(x), \varepsilon/2)$. Denote by W_{n-M_0} the component of $f^{-(n-M_0)}(B(f^{n-M_0}(x), \varepsilon/2))$ which contains x. We conclude that $b_n(\alpha) \subseteq W_{n-M_0}$, thus $z_n(\alpha) \in W_{n-M_0}$, and so

$$\operatorname{diam} W_{n-M_0} \ge r_n(\alpha)\,.$$

Also $f^{n-M_0} : W_{n-M_0} \longrightarrow U_0$ is a branched covering of degree at most D (see [CJY, Theorem 2.1]) and the same is true with $2U_0$ if we replace W_{n-M_0} by a corresponding component of $f^{-(n-M_0)}(2U_0)$. Applying Lemma C from [BV1], [BV2] we see that W_{n-M_0} is τ-thick at x, that is $W_{n-M_0} \supset B(x, \tau\, r_n(\alpha))$.

Our purpose now is to enlarge W_{n-M_0} to the size of $B(x, k\, r_n(\alpha))$. To do this consider $U_1 = B(f^{n-M_0-M}(x), \varepsilon/2)$ for a certain M to be chosen later. Let U_2 be a component of $f^{-M}(U_0)$ containing $f^{n-M_0-M}(x)$. Denote by W_{n-M_0-M} the component of $f^{-(n-M_0-M)}(U_1)$ which contains x. We have the covering $f^{n-M_0-M} : W_{n-M_0-M} \longrightarrow U_1$ which sends W_{n-M_0} to U_2. This is the covering of degree at most D (see [CJY, Theorem 2.1]) and the same is true for $2U_1$ if W_{n-M_0-M} is replaced by a corresponding component of $f^{-(n-M_0-M)}(2U_1)$. Applying Lemma C from [BV1], [BV2] or using [HR] we see that with constants independent of n we have

$$\frac{\operatorname{diam} W_{n-M_0-M}}{\operatorname{diam} W_{n-M_0}} \sim \frac{\operatorname{diam} U_1}{\operatorname{diam} U_2}\,.$$

But from [CJY, Theorem 2.1] it follows that we can make $\operatorname{diam} U_2$ as small as we wish by choosing M large. Since W_{n-M_0-M} is τ-thick at x given k we can choose $M = M(k, f)$ so large that

$$B(x, k\, r_n(\alpha)) \subset W_{n-M_0-M}\,.$$

The degree of the map $f^{n-M_0-M} : W_{n-M_0-M} \longrightarrow U_1$ is bounded by D independent of n. So

$$\begin{aligned}\#\{z_n(\beta) : z_n(\beta) \in B(x, k\, r_n(\alpha))\} \\ &\le D\,\#\{z_{M+M_0}(\beta) :\ z_{M+M_0}(\beta) \in U_1\} \\ &\le D\, d^{M+M_0} = C(k, f)\,.\end{aligned}$$

Now we are in a position to prove

Theorem 4.3. *Let f be semihyperbolic. Then $\pi : \Sigma \longrightarrow J$ has the following properties:*

1) *π is Hölder continuous,*

2) *π is onto,*

3) *$\#\{\pi^{-1}(x)\} \le K(f)$ for any $x \in J$.*

PROOF. Hölder continuity is obvious from Lemma 4.1. Also 3) follows immediately from Lemma 4.2. "Onto" part is also easy.

One just applies the criterion of accessibility obtained by Przytycki in [P]. Or one can proceed as follows. Given $Q \in J$ let $z_{n(k)}(\alpha_k) \to Q$. We may assume that $\alpha_k \to \alpha$ in Σ. By Lemma 4.1 $|z_{n(k)}(\alpha_k) - \pi(\alpha_k)| \le C_1\, \theta_1^{n(k)}$. Thus $|Q - \pi(\alpha)| \le |Q - z_{n(k)}(\alpha_k)| + |z_{n(k)}(\alpha_k) - \pi(\alpha_k)| + |\pi(\alpha_k) - \pi(\alpha)| \to 0$, when $k \to \infty$. So $Q = \pi(\alpha)$ and the proof is completed.

REMARK. In fact, more is true. If we assume that f is separated semihyperbolic we can claim that the branches $b(\alpha)$ approximate the endpoint $\pi(\alpha) = Q$ by exhausting all the prime ends with impression Q.

PROOF. To see this we are going to prove that for any local domain $\Omega^i_Q(r)$ there is a point $z_n(\alpha) \in \Omega^i_Q(r)$. Because of the qc conjugacy it is enough to prove this for polynomials.

Suppose the contrary: there exists a local domain, say $\Omega^i_Q(r)$ free from preimages $z_n(\alpha)$. Consider the function u, which is harmonic on $\bar{\mathbb{C}} \setminus K_f$ and such that $u(z) \to 0$ if $z \to K_f$, $z \in \bar{\mathbb{C}} \setminus \Omega^i_Q(r)$ and $u(z) \to 1$ if $z \to K_f$, $z \in \Omega^i_Q(r)$. We define a sequence of functions $(u_n)_n$ by the formula:

$$u_n(z) = \sum_{y \in f^{-n} z} \frac{1}{d^n}\, u(y)\,,$$

where d is the degree of the polynomial f.

It is easy to see that u_n are harmonic in $\bar{\mathbb{C}} \setminus K_f$. Furthermore if $z \neq \infty$, $u_n(z) \to 0$. On the other hand, since ∞ is superattracting fixed point we have $u_n(\infty) = u(\infty) > 0$.

This observation leads to a contradiction proving the assertion.

Lemma 4.4. *If f is semihyperbolic then*

$$\frac{\operatorname{dist}(z_n(\alpha), J)}{r_n(\alpha)} \sim 1\,,$$

where constants depend only on f.

This is another standard application of [CJY, Theorem 2.1] and Lemma C of [BV1], [BV2] (see also [HR]).

Let us now consider (f, V, U), $\Omega = \bar{\mathbb{C}} \setminus K_f$. Our main assumption is that f is separated semihyperbolic (see the introduction for definitions).

Then Theorem C claims that Ω is uniformly John. We are going to apply Corollary 3.3 to

$$u = G \circ f\,, \qquad v = G\,,$$

where G is Green's function of Ω with pole at ∞. Immediately we obtain

Theorem 4.5. *In the setting above for each $\alpha \in \Sigma$ there exists the limit*

$$\varphi(\alpha) = \lim_{n\to\infty} -\log \frac{(G\circ f)(z_n(\alpha))}{G(z_n(\alpha))} \tag{4.4}$$

and moreover φ is a Hölder continuous function on (Σ, ρ_Σ).

Let h_ν denote the entropy of the invariant measure ν on Σ (see *e.g.* [Wa]). Applying Sinai-Bowen-Ruelle thermodynamical formalism we construct the measure μ as in

Theorem D. *Let φ be a Hölder continuous function on (Σ, σ). Then there exists the limit (independent of $\beta \in \Sigma$)*

$$P = P(\varphi, \sigma) = \lim \frac{1}{n} \log\Big(\sum_{\alpha:\sigma^n\alpha=\beta} E_n(\alpha)\Big), \tag{4.5}$$

where $E_n(\alpha) = \exp(\varphi(\alpha) + \varphi(\sigma\alpha) + \cdots + \varphi(\sigma^{n-1}\alpha))$. Furthermore, there exists an invariant, ergodic probability measure μ on Σ such that for any $\beta \in \Sigma$

$$C_1\, e^{-nP} \le \frac{\mu\{\alpha:\ \alpha_1 = \beta_1, \dots, \alpha_n = \beta_n\}}{\exp(\varphi(\beta) + \cdots + \varphi(\sigma^{n-1}\beta))} \le C_2\, e^{-nP}. \tag{4.6}$$

The measure μ is unique with this property and it is equilibrium measure for φ which means that μ maximizes the functional $h_\nu + \int \varphi\, d\nu$. Moreover,we have:

$$h_\mu + \int \varphi\, d\mu = P.$$

Definition. *Measure μ is called Gibbs measure with potential φ.*

Theorem E. *If φ, ψ are two Hölder continuous function on $\sum$ then their Gibbs measures are either singular or the homologous equation*

$$\gamma\circ\sigma - \gamma = \varphi - \psi - P_\varphi + P_\psi$$

has a solution γ among Hölder continuous functions.

See [Bo] for Theorems D, E. In what follows φ is the function constructed in Theorem 4.5.

Lemma 4.6. $P(\varphi, \sigma) \leq 0$.

PROOF. Boundary Harnack principle of Corollary 3.3 implies more than the existence of the limit in (4.5). Actually

$$\left|\varphi(\alpha) - \log \frac{G(z_n(\alpha))}{G(f(z_n(\alpha)))}\right| \leq C\, q^n, \tag{4.8}$$

uniformly in α and n. In particular

$$\frac{\sum_{\sigma^n \alpha = \beta} E_n(\alpha)}{\sum_{\sigma^n \alpha = \beta} G(z_n(\alpha))} \asymp 1 .$$

Now we apply Lemma 1.3 (even with B_ρ replaced by B):

$$G(z_n(\alpha)) \leq M(K, \alpha)\, \omega(B(z_n(\alpha), 4\, r_n(\alpha))) .$$

Denote $B_{n,\alpha} = B(z_n(\alpha), 4\, r_n(\alpha))$. Discs $\{B_{n,\alpha}\}_{\alpha \in \sum}$ form the covering of J. Lemma 4.2 implies readily that this covering has a finite multiplicity independent of n.

Combining these facts we get

$$\sum E_n(\alpha) \leq C(K, \alpha)$$

and the lemma follows.

REMARK. One can prove that $P(\varphi, \sigma) = 0$, see [B]. But this requires more careful estimates. Moreover it is proved in [B] that $\pi^* \mu$ is mutually absolutely continuous with respect to harmonic measure ω.

Definition. *We call $\pi^* \mu$ the shift model measure for ω. It is clearly f-invariant.*

5. $\omega \approx m \Leftrightarrow$ conformal maximality.

We recall the definition from [BPV]. The system (f, V, U) is called *conformally maximal* if it is conformally equivalent to (g, V_g, U_g) for which harmonic measure ω_g of $\mathbb{C} \setminus K_g$ equals measure of maximal entropy of g on J_g. Measure of maximal entropy will be denoted by m (or m_g, m_f, m_σ to highlight the dynamical system if we need this).

Lemma 5.1. $\pi^* m_\sigma = m_f$.

PROOF. This is clear from [LW] and Theorem 4.3.

From Lemma 4.2 it is easy to deduce the following natural assertion.

Lemma 5.2. *Let F be a Borel subset of Σ such that $m_\sigma(F) = 0$. Then $m_\sigma(\pi^{-1}\pi F) = 0$.*

Now we are going to consider two cases:

First case: $\mu \perp m_\sigma$. Then Theorem D and Lemma 4.6 give

$$\log d + \int_\Sigma \varphi \, dm_\sigma < P(\varphi) \le 0 .$$

Second case: $\mu = m_\sigma$. Then Theorem E gives

$$\varphi + \log d = \gamma \circ \sigma - \gamma + P(\varphi) .$$

Anyway, either we have (for a certain positive ε)

$$\log d + \int_\Sigma \varphi \, dm_\sigma = -2\varepsilon < 0 \tag{5.1}$$

or

$$\varphi + \log d = \gamma \circ \sigma - \gamma , \qquad \gamma \in \text{Höld}(\Sigma) . \tag{5.2}$$

The last possibility occurs only if $\mu = m_\sigma$ and $P(\varphi) = 0$.

Lemma 5.3. *If* (5.1) *happens then* $\omega \perp m$.

PROOF. By Birkhoff's ergodic theorem:

$$\frac{1}{n}(\varphi(\alpha) + \cdots + \varphi(\sigma^{n-1}\alpha)) \to \int_\Sigma \varphi \, dm_\sigma ,$$

for almost every α with respect to m_σ. Combining this with (5.1) we see that

$$\frac{1}{n}(\varphi(\alpha) + \cdots + \varphi(\sigma^{n-1}\alpha)) \le -\varepsilon + \log\frac{1}{d} , \qquad n \ge n(\alpha) ,$$

for almost every α with respect to m_σ. Combining this with (4.7) we get the estimate of Green's function

$$G(z_n(\alpha)) \le C\, e^{-\varepsilon n}\, d^{-n}\,, \qquad n \ge n(\alpha)\,, \tag{5.3}$$

for almost every α with respect to m_σ. Let E be the set of α for which (5.3) holds.

By Lemma 5.2 we may assume that

$$E = \pi^{-1} e\,, \qquad e \subset J\,, \quad m(J \setminus e) = 0\,. \tag{5.4}$$

We are going to show that

$$\omega(e) = 0. \tag{5.5}$$

Clearly (5.4), (5.5) finishes the proof. To prove (5.5) we need Lemma 2.4. Put $n(Q) = \max_{\alpha \in \pi^{-1}(Q)} n(\alpha)$, where $n(\alpha)$ is taken from (5.3). Let N be fixed and put

$$r(Q) = \frac{1}{100} \min_{\alpha \in \pi^{-1}(Q)} |z_{\max(N, n(Q))}(\alpha) - Q|\,.$$

"Discs" $\{\bar{B}_\rho(Q, r(Q))\}_{Q \in e}$ cover e and let B_ρ^i, $B_\rho^i := \bar{B}_\rho(Q_i, r(Q_i))$, be a disjoint family such that $e \subset \bigcup_{i \ge 1} 5\, B_\rho^i$. This family exists by Vitali's lemma. Lemma 2.4 claims that

$$\omega(5\, B_\rho^i) \le M_1 \sum_\ell G(A_{Q_i}^\ell(10\, r(Q_i))). \tag{5.6}$$

Remind that $b_k(\alpha) = \cup_{i \ge k} \gamma^j(\alpha)$. Let m_i be the first index greater than $\max(N, n(Q_i))$ such that $\alpha \in \pi^{-1}(Q_i)$ implies

$$z_{m_i}(\alpha) \quad \text{and} \quad b_{m_i}(\alpha) \quad \text{lie inside} \quad \bigcup_\ell \Omega_{Q_i}^\ell \Big(\frac{r(Q_i)}{M_2} \Big)\,. \tag{5.7}$$

By the Remark after Theorem 4.3 we see that in each local domain there is at least one point $z_{m_i}(\alpha)$.

We can apply now Harnack's inequality:

$$\sum_\ell G(A_{Q_i}^\ell(10\, r(Q_i))) \le M_3 \sum_{\alpha \in \pi^{-1}(Q_i)} G(z_{m_i}(\alpha)), \tag{5.8}$$

where M_3 depends only on M_2 and John constant of John arcs $b(\alpha)$ (see Lemma 4.4 which claims that $b(\alpha)$ are John arcs with uniform John constant). Now if M is the constant from (3) of the introduction and $M_2 \geq M$ then

$$\Omega^{\ell}_{Q_i}\left(\frac{r(Q_i)}{M_2}\right) \subset B_\rho(Q_i, r(Q_i)) = B^i_\rho\,.$$

In particular from (5.7) we see that

$$\text{(5.9)}\quad \alpha \in \pi^{-1}(Q_i) \quad \text{implies} \quad z_{m_i}(\alpha) \quad \text{and} \quad b_{m_i}(\alpha) \text{ lie inside } B^i_\rho\,.$$

Combine (5.8), (5.6) and (5.3) to write

$$\text{(5.10)}\qquad \begin{aligned}\sum_{i\geq 1}\omega(5B^i_\rho) &\leq C\, M_1\, M_3\, e^{-\varepsilon N}\sum_{i\geq 1}\#\pi^{-1}(Q_i)\, d^{-m_i}\\ &\leq M_4\, e^{-\varepsilon N}\sum_{i\geq 1} d^{-m_i}\,.\end{aligned}$$

On the other hand we have that if $Q_i \neq Q_j$ then $z_{m_i}(\alpha) \notin b_{m_j}(\beta)$ for any pair α, β, $\alpha \in \pi^{-1}(Q_i)$, $\beta \in \pi^{-1}(Q_j)$. This is just (5.9) combined with the fact that B^i_ρ and B^j_ρ are disjoint. If we denote $C_i := \{x \in \Sigma : x_k = \alpha_k$, for some $\alpha \in \pi^{-1}(Q_i)$, and all k, $1 \leq k \leq m_i\}$ we conclude that

$$\text{(5.11)}\qquad Q_i \neq Q_j \quad \text{implies} \quad C_i \cap C_j = \varnothing.$$

On the other hand $m_\sigma(C_i) \geq d^{-m_i}$. This, (5.10) and (5.11) now imply

$$\omega(e) \leq M_4\, e^{-\varepsilon N}\, m_\sigma(\cup C_i) \leq M_4\, e^{-\varepsilon N}\,.$$

We are done because N is arbitrary.

In what follows an important part is played by the so called automorphic harmonic function. Given a generalized polynomial-like system (f, V, U) (see the introduction) we say that the function τ on U is an automorphic harmonic function (for f) if

1) τ is nonnegative subharmonic function on U,
2) τ is positive and harmonic in $U \setminus K_f$,
3) $\tau|_{K_f} \equiv 0$,

and

$$(Aut) \qquad \tau(fz) = d\,\tau(z)\,.$$

Lemma 5.4. *Assume that* (5.2) *holds. Then there exists an automorphic harmonic function for* f.

PROOF. To avoid unimportant technicalities we are going to present the proof only in the case of totally disconnected J. This gives the advantage of only one local domain for given Q, r. The general case follows along the same lines. The procedure in the "boundedly many local domains" case differs only technically from the "only one local domain" case.

Applying Corollary 3.3 to $u = G\circ f$, $v = G$ and taking into account (5.2) we write

$$(5.12) \qquad \left|\log\frac{G(fz)}{G(z)} - \log d - (\gamma(\sigma\alpha) - \gamma(\alpha))\right| \leq C\,\rho(z,\pi\alpha)^{\varepsilon}\,.$$

Let $\alpha_i = i,\dots,i,\dots, i = 1,\dots,d$, and let $q_i = \pi(\alpha_i)$.

Let $B = B_{q_i}$ to be such a small disc centered at q_i that all components W_n of $f^{-n}(B)$ containing q_i are free from critical points of f.

Thus a univalent branch $g = g_{q_i}$ of f^{-1} is defined in B and $g^n : B \to W_n$ is univalent. Putting $\alpha = \alpha_i$ into (5.12) we define a positive finite limit

$$U_{q_i}(z) = \lim_{n\to\infty} d^n\,G(g^n z)\,, \qquad z\in B\,,$$

and $U_{q_i}(z) \leq C\,G(z)$, $z\in B$. Let $V_{q_i} = U_{q_i}\,e^{\gamma(q_i)}$. Thus V_{q_i} is positive harmonic on $B\backslash K_f$, nonnegative subharmonic on B and by construction

$$\lim_{z\to\pi\alpha}\frac{G(z)}{U_{q_i}(z)} = e^{\gamma(\alpha)-\gamma(q_i)}\,,$$

that is

$$(5.13) \qquad \lim_{z\to\pi\alpha}\frac{G(z)}{V_{q_i}(z)} = e^{\gamma(\alpha)}\,.$$

Now let I be the set of periodic points of Σ, $q_\chi = \pi(\chi)$, $\chi\in I$. We can repeat the construction above and obtain $U_\chi, V_\chi = U_\chi\,e^{\gamma(\chi)}$

defined in B_{q_χ}. Let $\mathrm{L} = \cup_{n\geq 0}\sigma^{-n}I$. Fix $l \in \mathrm{L}$ and define V_l in a small neighborhood of B_l of $\pi(l)$ by the relation

$$V_l(z) = d^{-n}V_\chi(f^n z)$$

if $\chi = \sigma^n l$. We clearly can use (5.12) to extend (5.13) on V_l literally. Let us try to prove that

$$V_{l_1} = V_{l_2} \text{ on } B_{l_1} \cap B_{l_2}\,. \tag{5.14}$$

Actually we are going to prove that either (5.14) holds, or a certain symmetrized version (see (5.16) below) holds.

We use (5.13) for V_{l_1}, V_{l_2} to conclude that

$$|V_{l_1}(z) - V_{l_2}(z)| = o(G(z))\,, \qquad z \in B_{l_1} \cap B_{l_2}\,. \tag{5.15}$$

Let us remind

Lemma F (A.F. Grishin [Gr]). *Let w_1, w_2 be two nonnegative subharmonic functions on an open set $\mathcal{O}$ and $w_1 \geq w_2$. Let*

$$J = \left\{z \in \mathcal{O}:\ (w_1 - w_2)(z) = \liminf_{r\to 0} \int_0^{2\pi} (w_1 - w_2)(z + re^{i\theta})\, d\theta = 0\right\}.$$

Then

$$\Delta w_1 \geq \Delta w_2 \text{ on } J\,.$$

Lemma F and (5.15) imply that

$$\Delta V_{l_1} = \Delta V_{l_2} \text{ on } B_{l_1} \cap B_{l_2}\,.$$

Thus the difference $V_{l_1} - V_{l_2}$ is harmonic in $B_{l_1} \cap B_{l_2}$. Remind that it vanishes on $K_f \cap B_{l_1} \cap B_{l_2}$.

We conclude that either (5.14) holds or $K_f \cap B_{l_1} \cap B_{l_2}$ is covered by finitely many real analytic curves. In the latter case $K_f = J_f$ is covered by finitely many real analytic curves. As in [LyV] we conclude that these curves are disjoint. Now let $*$ be a holomorphic symmetry with respect to these curves.

Consider

$$\tilde{V}_l(z) := \frac{V_l(z) + V_l(z^*)}{2}\,.$$

Now $\tilde{V}_{l_1} - \tilde{V}_{l_2}$ vanishes on these curves together with its normal derivative. That means $\tilde{V}_{l_1} = \tilde{V}_{l_2}$ in $B_{l_1} \cap B_{l_2}$. In this case we have a family of functions $\tilde{V}_l$ and small neighborhoods B_l of l, $l \in \mathrm{L}$ with properties

$$\begin{aligned} \tilde{V}_{l_1} &= \tilde{V}_{l_2} \text{ on } B_{l_1} \cap B_{l_2}\,, \\ \tilde{V}_{\sigma l}(fz) &= d\,\tilde{V}_l(z)\,. \end{aligned} \tag{5.16}$$

Let $\tilde{V}_l$ denote V_l if (5.14) holds and denote $\tilde{V}_l$ if (5.16) holds. The occurance of these alternatives depends on whether J is contained in a finite union of analytic curves or not.

Put $\tau|_{B_l} = \tilde{V}_l|_{B_l}$. So we choose a neighborhood $\mathcal{O}$ of L, such that $f^{-1}(\mathcal{O}) \subset \mathcal{O}$ and a subharmonic nonnegative function τ on $\mathcal{O}$ such that

$$\tau(z) = d\,\tau(f^{-1}z)\,, \qquad z \in \mathcal{O}\,. \tag{5.17}$$

The function τ is positive and harmonic in $\mathcal{O} \setminus K_f$. It may happen that $\mathcal{O}$ does not contain the whole J_f inside. So we are going to extend τ as follows: Let ε be a number such that diameters of components of $f^{-n}(B(x,\varepsilon))$ are at most $C\,\theta^n$ (see [CJY, Theorem 2.1]). We are going to extend τ to $B(q_\chi,\varepsilon)$, $\chi \in I$ ($\equiv$ periodic points). Let N be so large that the component W_N of $f^{-N}(B(q_\chi,\varepsilon))$ which contains q_χ is contained in $\mathcal{O}$. Thus $f^N : W_N \to B(q_\chi,\varepsilon)$ is a branched covering. Let $z \in B(q_\chi,\varepsilon)$ be a critical value of this covering and let Γ be a curve not meeting the critical values of f^N and lying in $B(q_\chi,\varepsilon)$. We choose Γ to connect z with a certain $\xi \in W_N$.

Let x, y be any two f^N-preimages of z lying in W_N. Let γ_x, γ_y be two liftings of Γ by f^N into W_N, starting at x, y respectively. In a small neighborhood U of Γ we can define τ^x and τ^y by

$$\tau^x(t) = d^N\,\tau(f_x^{-N}t)\,, \qquad \tau^y(t) = d^N\,\tau(f_y^{-N}t)\,.$$

Here f_x^{-N}, f_y^{-N} are branches of f^{-N} on U mapping U into neighborhoods of γ_x, γ_y respectively. Note that τ^x, τ^y coincide near ξ because of (5.17). So as harmonic function they should coincide in the whole U.

Clearly we have for this extension

$$\lim_{\substack{z \to \pi\alpha \\ z \in B(q_\chi,\varepsilon)}} \frac{G(z)}{\tau(z)} = e^{\gamma(\alpha)}\,.$$

As before we can see that on $\Omega = \cup_{\chi \in I} B(q_\chi,\varepsilon)$ a subharmonic function τ is defined such that

$$\tau(fz) = d\,\tau(z)\,, \qquad \text{if } z,\ f(z) \in \Omega\,.$$

The advantage of Ω in comparison to $\mathcal{O}$ is that $\Omega \supset J_f$. Let m be so large that $f^{-m}(U) \setminus K_f \subset \Omega$. Define

$$\tau(z) = d^m \, \tau(f^{-m} z)\,, \qquad z \in U \setminus K_f\,. \tag{5.18}$$

Repeating the lifting argument above we see that (5.18) gives a single valued function. Clearly τ is an automorphic harmonic function we were looking for and Lemma 5.4 is proved.

Now let us use the following

Theorem G ([BPV]). *Let (f, V, U) be a generalized polynomial-like system. Then it is conformally maximal if and only if there exists a harmonic automorphic function for f.*

Uniting Lemmas 5.3, 5.4 with Theorem G we obtain the following criterion of *conformal maximality.*

Theorem 5.5. *Let (f, V, U) be a system with separated semihyperbolic f. Then it is conformally maximal if and only if harmonic measure on J_f is not singular with respect to measure of maximal entropy on J_f.*

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