The spectrum of singularities
of Riemann's function

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Abstract. We determine the Hölder regularity of Riemann's function at each point; we deduce from this analysis its spectrum of singularities, thus showing its multifractal nature.

1. Introduction.

According to the tradition, Riemann would have proposed the function

$$\varphi(x) = \sum_1^\infty \frac{1}{n^2} \sin \pi n^2 x$$

as an example of continuous nowhere differentiable function. It turned out that, unlike lacunary series, the regularity of this function varies strongly from point to point. Let $x_0 \in \mathbb{R}$; by definition, a function f is $C^\alpha(x_0)$ if there exists a polynomial P of order at most α such that

$$(1) \qquad |f(x) - P(x - x_0)| \le C\,|x - x_0|^\alpha\,.$$

Let us recall the main steps of the determination of the Hölder regularity of φ at every point.

Hardy and Littlewood proved that φ is nowhere $C^{3/4}$ except perhaps at the rational points of the form $(2p+1)/(2q+1)$, $p, q \in \mathbb{Z}$ (see [9]). Their proof is interesting under many respects; for instance it

anticipates wavelet methods; they remark that the function

$$C(a,b) = \frac{a}{2}\left(\theta(b + ia) - 1\right)$$

(where θ is the Jacobi function $\theta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}$) is the convolution of φ with contractions (by a factor a) of

$$\psi(x) = \frac{1}{\pi(x - i)^2} \, .$$

Since ψ has a vanishing integral, an Abel type theorem (which we will state precisely in Proposition 1) shows that, if φ is smooth at x_0, $C(a,b)$ must have a certain decay when $a \to 0$ and $b \to x_0$. This method will thus yield upper bounds for the function

$$\alpha(x_0) = \sup\{\beta : \varphi \in C^\beta(x_0)\} \, .$$

This method actually yields the following more precise result which relates the pointwise behavior of φ at x_0 to the Diophantine approximation properties of x_0. Let $x_0 \notin \mathbb{Q}$, p_n/q_n be the sequence of its approximations by continued fractions and define

$$\tau(x_0) = \sup\left\{\tau : \left|x_0 - \frac{p_m}{q_m}\right| \le \frac{1}{q_m^\tau}\right\}$$

for infinitely many m's such that p_m and q_m are not both odd, then

$$\alpha(x_0) \le \frac{1}{2} + \frac{1}{2\,\tau(x_0)} \, .$$

a result which is actually stated by J. J. Duistermaat [4] where a more direct proof is given (this paper was actually one of the main motivations for writing the present one).

Converse results, which would yield an information about the pointwise regularity of φ from estimates on its convolutions are more difficult to obtain since they are of tauberian type, and were of course unavailable at the time of Hardy and Littlewood. This is why they had results concerning only the *irregularity* of Riemann's function and not its regularity. The Tauberian-type result we need is stated in Proposition 1.

Finally Gerver proved the differentiability at the rational points of the form $(2p + 1)/(2q + 1)$ [7] (where we now know that φ is exactly

$C^{3/2}$, see [13]). The analysis of the behavior of φ near such a rational point x_0 has been considerably sharpened since; Y. Meyer exhibited a complete "chirp" asymptotic expansion which describes the oscillations of φ:

$$\varphi(x) = u(x) + \sum_{n \geq 0} (x - x_0)^{n+3/2} \, v_+^n \Big(\frac{1}{x - x_0}\Big), \qquad \text{if } x \geq x_0 \,,$$

$$\varphi(x) = u(x) + \sum_{n \geq 0} |x - x_0|^{n+3/2} \, v_-^n \Big(\frac{1}{|x - x_0|}\Big), \qquad \text{if } x \leq x_0 \,,$$

where u is C^∞, the $v_\pm^n$ are 2π periodic, with a vanishing integral, and are $C^{n+1/2}$, see [13] (we will actually show that these points are the only ones where a chirp expansion exists).

The results of Hardy and Gerver left open the problem of the determination of the exact regularity of φ at irrational points; one of our purposes is to do this determination using the wavelet method we sketched.

Let the spectrum of singularities of φ be the function $d(\beta)$ which associates to each β the Hausdorff dimension of the set of points x where $\alpha(x) = \beta$ (conventionally the dimension of the empty set is $-\infty$). We will deduce form our study this spectrum which will be nonconstant on a whole interval. The Hölder singularities of φ are located on a whole collection of sets of different dimensions, so that φ is truly a "multifractal function". More precisely the determination of the spectrum of singularities is motivated by the following problem, referred to in the litterature as the "Multifractal Formalism for functions".

Let

$$L^{p,s} = \{f \in L^p : \ (-\Delta)^{s/2} f \in L^p\} \,.$$

If g is a one variable function, define

$$\eta(p) = \sup\{s : \ g \in L^{p,s/p}\} \,.$$

Frisch and Parisi in [6] conjectured that the spectrum of singularities of g is given by the following Legendre transform formula

$$(2) \qquad\qquad d(\alpha) = \inf_p \left(\alpha\,p - \eta(p) + 1\right).$$

Though one easily finds counterexamples, the exact mathematical range of validity of this formula is a fascinating problem (see [1], [3] and [12]).

Since φ has a nontrivial spectrum, it was natural to test this conjecture, and we will show that it is correct in this case. The interesting point here is that φ has a very different structure from the previous cases where the Multifractal Formalism was known to hold (see [3] and [12]).

Our main results are stated in the following theorem.

Theorem 1. *Let $x \notin \mathbb{Q}$ and let p_n/q_n be the sequence of its approximations by continued fractions. Let*

$$\tau(x) = \sup\left\{\tau : \left|x - \frac{p_m}{q_m}\right| \leq \frac{1}{q_m^\tau}\right\}$$

for infinitely many m's such that p_m and q_m are not both odd. Then

$$\alpha(x) = \frac{1}{2} + \frac{1}{2\tau(x)}.$$

The spectrum of singularities of φ is given by

$$d(\alpha) = \begin{cases} 4\alpha - 2, & \text{if } \alpha \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right], \\[2mm] 0, & \text{if } \alpha = \dfrac{3}{2}, \\[2mm] -\infty, & \text{else}; \end{cases}$$

and if $\alpha \leq 3/4$, $d(\alpha)$ satisfies (2).

The existence of this sets of smooth points is not only a consequence of the kind of lacunarity introduced by the frequencies n^2, but also of the very special coefficents that are chosen, which creates an exceptional behavior, as shown by the following remark: if the coefficients were multiplied by independant identically distributed Gaussians or Rademacher series (± 1), [14, Chapter 8, Theorem 4] shows that the corresponding random function would be almost surely nowhere $C^{1/2}$.

2. Pointwise regularity and wavelet transform.

Because of Hardy's result, we will only consider Hölder exponents smaller than $3/4$.

Suppose that a function ψ is nonvanishing and satisfies the following asumptions

$$(3) \quad |\psi(x)| + |\psi'(x)| \leq C\,(1+|x|)^{-2} \quad \text{and} \quad \int \psi(x)\,dx = 0\,,$$

and either

$$(4) \qquad \int_0^\infty |\hat\psi(\xi)|^2\,\frac{d\xi}{\xi} = \int_0^\infty |\hat\psi(-\xi)|^2\,\frac{d\xi}{\xi} = 1$$

or

$$(5) \qquad \hat\psi(\xi) = 0\,, \quad \text{if } \xi < 0 \quad \text{and} \quad \int_0^\infty |\hat\psi(\xi)|^2\,\frac{d\xi}{\xi} = 1\,.$$

In the last case the wavelet is said to be analytic. The wavelet transform of an L^∞ function f is defined by

$$C(a,b)(f) = \frac{1}{a}\int f(t)\,\bar\psi\big(\frac{t-b}{a}\big)\,dt\,.$$

We will consider the three following settings: In the first one the analyzed function f is real valued, and the wavelet satisfies (5); in the second one, f is complex valued and the wavelet satisfies (4); in the third one, $\hat f(\xi) = 0$ if $\xi < 0$ and the wavelet satisfies (5).

In each case, the following results concerning the relationships between the size of the wavelet transform and the regularity of the function hold.

Proposition 1. *Suppose that* $0 < \alpha < 1$. *Under the previous hypotheses if a function f is $C^\alpha(x_0)$,*

$$(6) \qquad |C(a,b)(f)| \leq C\,a^\alpha\left(1 + \frac{|b-x_0|}{a}\right)^\alpha\,.$$

Conversely, if

$$(7) \qquad |C(a,b)(f)| \leq C\,a^\alpha\left(1 + \frac{|b-x_0|}{a}\right)^{\alpha'} \quad \text{for an } \alpha' < \alpha\,,$$

and if $|x - x_0| \leq 1/2$, then

$$(8) \qquad |f(x) - f(x_0)| \leq C\,|x-x_0|^\alpha\,.$$

The first assertion was first stated in the wavelet terminology in [10] or [11], but it is only fair to say that it is at least implicitly contained in Hardy's paper [8]; and only the second assertion (see [11]) will have new implications on Riemann's function. Let us recall the proof of this proposition for the reader's convenience when $0 < \alpha < 1$ (the only case we will be interested in here).

PROOF OF PROPOSITION 1. If $f \in C^\alpha(x_0)$,

$$
|C(a,b)(f)| = \frac{1}{a} \left| \int f(x) \, \psi\left(\frac{x-b}{a}\right) dx \right|
$$

$$
= \frac{1}{a} \left| \int (f(x) - f(x_0)) \, \psi\left(\frac{x-b}{a}\right) dx \right|
$$

$$
\leq \frac{C}{a} \int |x - x_0|^\alpha \left(\frac{1}{1 + \left|\dfrac{x-b}{a}\right|} \right)^2 dx
$$

$$
\leq \frac{C}{a} \int \frac{|x - b|^\alpha}{\left(1 + \left|\dfrac{x-b}{a}\right|\right)^2} \, dx + |b - x_0|^\alpha \frac{C}{a} \int \frac{dx}{\left(1 + \left|\dfrac{x-b}{a}\right|\right)^2}
$$

$$
\leq C \, a^\alpha \left(1 + \left|\frac{b - x_0}{a}\right|\right)^\alpha .
$$

Suppose now that we are in the first or the third case. In that case, f is reconstructed from its wavelet transform by

$$
f(x) = \iint C(a,b)(f) \, \psi\left(\frac{x-b}{a}\right) \frac{da \, db}{a^2} .
$$

Let

$$
\omega(a,x) = \int C(a,b)(f) \, \psi\left(\frac{x-b}{a}\right) \frac{db}{a} \; ;
$$

if (7) holds,

$$
|\omega(a,x)| \leq C \, a^\alpha \left(1 + \frac{|x - x_0|}{a}\right)^{\alpha'}
$$

and

$$
\left| \frac{\partial \omega(a,x)}{\partial x} \right| \leq C \, a^{\alpha - 1} \left(1 + \frac{|x - x_0|}{a}\right)^{\alpha'} .
$$

Using the second estimate (and the mean value theorem) for $a \geq |x - x_0|$ and the first estimate for $a \leq |x - x_0|$, we obtain

$$
|f(x) - f(x_0)| \leq \int_{a \geq |x - x_0|} C \, a^{\alpha - 1} \left(1 + \left|\frac{x - x_0}{a}\right|\right)^{\alpha'} |x - x_0| \frac{da}{a}
$$

$$+ \int\limits_{a \leq |x-x_0|} C \, a^\alpha \left(1 + \left|\frac{x - x_0}{a}\right|\right)^{\alpha'} \frac{da}{a}$$

$$\leq C \, |x - x_0|^\alpha .$$

This also implies the result in the second case by superposing reconstruction formulas for $\xi \geq 0$ and $\xi \leq 0$.

Using Cauchy's formula, we obtain that (using the wavelet $\psi(x) = (x - i)^{-2}$) the wavelet transform of $\varphi(x)$ is $2ia(\theta(b + ia) - 1)/2$. Since we want to determine Hölder exponents between $1/2$ and $3/4$, because of (7) we can add a term ia and the study of the pointwise regularity of φ reduces to obtaining estimates similar to (6) for the function

$$(9) \qquad\qquad C(a, b) = a \, \theta(b + ia) .$$

3. Theta Jacobi function and continued fractions.

The Theta modular group is obtained by composing the two transforms

$$x \longmapsto x + 2 \qquad \text{and} \qquad x \longmapsto -\frac{1}{x} .$$

It is composed of the fractional linear transformations

$$\gamma(x) = \frac{rx + s}{qx - p} ,$$

where $rp + sq = -1$, r, s, p, q are integers and the matrix

$$(10) \quad \begin{pmatrix} r & s \\ q & p \end{pmatrix} \text{ is of the form } \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \text{ or } \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} .$$

When γ belongs to the Theta modular group, θ is transformed following the formula (cf. [2] or [15])

$$(11) \qquad \theta(z) = \theta(\gamma(z)) \, e^{im\pi/4} \, q^{-1/2} \left(z - \frac{p}{q}\right)^{-1/2} ,$$

where m is an integer which depends on r, s, p, q.

Let $\rho \notin \mathbb{Q}$ and p_n/q_n the sequence of its approximations by continued fractions. The idea of the proof of Theorem 1 is to use (11),

which will allow us to deduce the behavior of $\theta(z)$ near p_n/q_n (hence near ρ) from its behavior near 0 or 1. Because of (10), we will have to separate two cases depending whether p_n and q_n are both odd or not; but let us first derive some straightforward estimates for θ near 0 and 1.

First remark that

$$(12) \qquad |\theta(z) - 1| \leq \frac{1}{2}, \qquad \text{if } \operatorname{Im} z \geq 1,$$

because in this case,

$$|\theta(z) - 1| \leq 2 \sum_{n \geq 1} e^{-\pi n^2 \operatorname{Im} z} \leq \frac{2\,e^{-\pi \operatorname{Im} z}}{1 - e^{-\pi \operatorname{Im} z}} \leq \frac{1}{2}\,.$$

We also have

$$(13) \qquad |\theta(z)| \leq C\,|\operatorname{Im} z|^{-1/2}, \qquad \text{if } \operatorname{Im} z \leq 1,$$

because $|\theta(z)| \leq \sum e^{-n^2 \operatorname{Im} z}$; the sum for $n \leq (\operatorname{Im} z)^{-1/2}$ is bounded trivially by $(\operatorname{Im} z)^{-1/2} + 1$ and the same bound holds for $n > (\operatorname{Im} z)^{-1/2}$ (by comparison with an integral).

Let us now obtain the behavior of θ near the point 1. Recall that θ satisfies (see [2])

$$(14) \qquad \theta(1 + z) = \sqrt{\frac{i}{z}} \left(\theta\left(-\frac{1}{4z}\right) - \theta\left(-\frac{1}{z}\right) \right),$$

so that

$$\theta(1 + z) = 2\sqrt{\frac{i}{z}} \left(A(4z) - A(z) \right),$$

where $A(z) = \sum_1^\infty e^{-i\pi n^2/z}$. If $\operatorname{Im}(-1/z) \geq 1$,

$$|A(z)| \leq 2 \exp\left(-\pi \operatorname{Im}\left(\frac{-1}{z}\right) \right),$$

so that in that case,

$$(15) \qquad |\theta(1 + z)| \leq C\,|z|^{-1/2} \exp\left(-\pi \operatorname{Im}\left(-\frac{1}{z}\right) \right).$$

Proposition 2. *Let* $\{p_n/q_n\}$ *be the sequence of approximations of* ρ *by continued fractions; let* τ_n *be defined by*

$$\text{(16)} \qquad \left| \rho - \frac{p_n}{q_n} \right| = \left(\frac{1}{q_n} \right)^{\tau_n}.$$

For each n, *if*

$$\text{(17)} \qquad 3 \left| \rho - \frac{p_n}{q_n} \right| \leq |b - \rho + ia| \leq 3 \left| \rho - \frac{p_{n-1}}{q_{n-1}} \right|$$

the following estimates hold:

If p_n *and* q_n *are not both odd but* p_{n-1} *and* q_{n-1} *are both odd, then*

$$\text{(18)} \qquad |C(a,b)| \leq C \, a^{(1+1/\tau_n)/2} \left(1 + \frac{|b-\rho|}{a} \right)^{1/2\tau_n}.$$

If p_n *and* q_n *are not both odd and* p_{n-1} *and* q_{n-1} *are not both odd, then*

$$\text{(19)} \qquad |C(a,b)| \leq C \, a^{(1+1/\tau_n)/2} \left(1 + \frac{|b-\rho|}{a} \right)^{1/2\tau_n}$$

or

$$\text{(20)} \qquad |C(a,b)| \leq C \, a^{(1+1/\tau_{n-1})/2} \left(1 + \frac{|b-\rho|}{a} \right)^{1/2\tau_{n-1}}.$$

If p_n *and* q_n *are both odd*

$$\text{(21)} \qquad |C(a,b)| \leq C \, a^{(1+1/\tau_{n-1})/2} \left(1 + \frac{|b-\rho|}{a} \right)^{1/2\tau_{n-1}}.$$

Furthermore, if p_n *and* q_n *are not both odd, these estimates are optimal, which means that there exists a point in the domain* (17) *where* (18) *or* (19) *are equalities.*

Remark that, since $\tau_n \geq 2$, this result together with Proposition 1 implies Hardy's result that $\varphi(x) - \varphi(x_0)$ is nowhere $o(|x - x_0|)^{3/4}$ except perhaps at the rational points quotient of two odd numbers. More precisely, we have

Corollary 1. *Let* $\rho \notin \mathbb{Q}$; *If there exists an infinity of integers* n *such that* p_n *and* q_n *are not both odd and* $\tau_n \geq \tau$, *then*

$$\varphi(x) - \varphi(x_0) \quad is \quad not \quad o(|x - x_0|)^{(1+1/\tau)/2}$$

but if there exists N such that for any $n \geq N$ (verifying p_n and q_n are not both odd), $\tau_n \leq \tau$, then

$$\varphi(x) - \varphi(x_0) = O(|x - x_0|)^{(1+\tau)/2}.$$

Define $\Gamma^\alpha(x_0)$ as the set of functions f such that

$$\begin{cases} \text{for all } \beta > \alpha, & f \notin C^\beta(x_0), \\ \text{for all } \beta < \alpha, & f \in C^\beta(x_0). \end{cases}$$

If $\eta(\rho) = \limsup \tau_n(\rho)$, where the lim sup bears only on the n's such that p_n and q_n are not both odd, this result implies that $\varphi \in \Gamma^{(1+1/\eta(\rho))/2}(\rho)$. We will prove this proposition in Sections 5 and 6, and in the next section, we will show how to derive the spectrum of singularities from this result.

Let us now recall a few properties of approximations by continued fractions.

Since

$$(22) \qquad p_n\, q_{n-1} - p_{n-1}\, q_n = (-1)^{n-1}$$

thus

$$\frac{1}{q_n\, q_{n+1}} = \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| \geq \left|\rho - \frac{p_n}{q_n}\right|,$$

because (see [10]) p_n/q_n and p_{n+1}/q_{n+1} are not on the same side of ρ; and

$$\frac{1}{q_n\, q_{n+1}} = \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| \leq 2\left|\rho - \frac{p_n}{q_n}\right|,$$

so that

$$(23) \qquad \left(\frac{1}{q_n}\right)^{\tau_n-1} \leq \frac{1}{q_{n+1}} \leq 2\left(\frac{1}{q_n}\right)^{\tau_n-1}.$$

4. Spectrum of singularities and Multifractal Formalism.

Recall that if

$$E_\tau = \left\{\rho : \left|\rho - \frac{p_n}{q_n}\right| \leq \frac{C}{q_n^\tau} \text{ for infinitely many } n\text{'s}\right\},$$

the Hausdorff dimension of E_τ is $2/\tau$ and the $\mathcal{H}^{2/\tau}$-measure of E_τ is positive (a direct consequence of [5, Propositions 10.4 and 8.5]).

Let

$$F_\tau = \left\{ \rho : \ \left| \rho - \frac{p_n}{q_n} \right| \leq \frac{C}{q_n^\tau} \right\},$$

for infinitely many n's such that p_n and q_n are not both odd, and let

$$G_\tau = \left\{ \rho : \ \left| \rho - \frac{p_n}{q_n} \right| \leq \frac{C}{q_n^\tau} \right\},$$

for infinitely many n's such that p_n and q_n are both odd. Of course, because of the best approximation properties of continued fractions, we have

$$E_\tau = F_\tau \cup G_\tau .$$

We will need the following lemma proved in [17].

Lemma 1. *Let $\rho \in \mathbb{R}$; If p and q have no common factor and*

$$|q\,\alpha - p| < \frac{1}{2q} ,$$

then p/q is a continued fraction approximation of ρ.

This lemma implies that if p/q is a continued fraction which approximates ρ and such that p and q are odd and $|\rho - p/q| \leq q^{-\tau}$ with $\tau > 2$, $p/(2q)$ is a continued fraction which approximates $\rho/2$.

Let us prove that the $\mathcal{H}^{2/\tau}$-measure of F_τ is positive. If the $\mathcal{H}^{2/\tau}$-measure of G_τ vanishes, we have nothing to prove. Else, the remark we just made shows that if $\tau > 2$ and $x \in G_\tau$ then $x/2 \in F_\tau$; thus, if the $\mathcal{H}^{2/\tau}$-measure of G_τ is positive, the $\mathcal{H}^{2/\tau}$-measure of F_τ is also positive (if $\tau = 2$, $F_\tau = \mathbb{R}$).

Consider the set

$$F_\tau \setminus \bigcup_{\tau' > \tau} E_{\tau'} .$$

The $\mathcal{H}^{2/\tau}$-measure of $\cup_{\tau' > \tau} E_{\tau'}$ vanishes; since F_τ has a positive $\mathcal{H}^{2/\tau}$-measure, $F_\tau \setminus \cup_{\tau' > \tau} E_{\tau'}$ has dimension $2/\tau$.

If $\rho \in F_\tau \setminus \cup_{\tau' > \tau} E_{\tau'}$, since $\rho \in F_\tau$, Proposition 2 implies that φ is not smoother than $(1 + 1/\tau)/2$ at ρ and since $\rho \notin \cup_{\tau' > \tau} E_{\tau'}$, φ is $C^{(1+1/\tau)/2-\varepsilon}(\rho)$ for all $\varepsilon > 0$; thus $\varphi \in \Gamma^{(1+1/\tau)/2}(\rho)$ and the dimension of $\{\rho : \ \varphi \in \Gamma^{(1+1/\tau)/2}(\rho)\}$ is at least $2/\tau$.

Suppose that ρ is such that $\varphi \in \Gamma^{(1+1/\tau)/2}(\rho)$; then φ is $C^{(1+1/\tau)/2-\varepsilon}(\rho)$ for all $\varepsilon > 0$ and thus $\rho \in E_{\tau'}$ for all $\tau' < \tau$; thus

$$\{\rho : \ \varphi \in \Gamma^{(1+1/\tau)/2}(\rho)\} \subset \bigcup_{\tau' > \tau} E_{\tau'}$$

and the dimension of $\{\rho : \ \varphi \in \Gamma^{(1+1/\tau)/2}(\rho)\}$ is bounded by $2/\tau$, hence the second part of Theorem 1 follows.

Let us now check that the Multifractal Formalism is true for Riemann's function. Let

$$S_n(x) = \sum_{m=1}^{n} e^{im^2 \pi x} .$$

In [18], Z. Zalcwasser proves that

$$\int_0^1 |S_n(x)|^p \, dx \sim \begin{cases} n^{p/2} , & \text{if } 0 < p < 4 , \\ n^2 \log(n+1) , & \text{if } p = 4 , \\ n^{p-2} , & \text{if } p > 4 . \end{cases}$$

Thus, taking D-adic blocs (for a D large enough),

$$\left\| \sum_{D^j \le m^2 < D^{j+1}} e^{im^2 \pi x} \right\|_{L^p} \sim \begin{cases} D^{j/4} , & \text{if } 0 < p < 4 , \\ D^{j(p-2)/(2p)} , & \text{if } p > 4 . \end{cases}$$

Let $\Phi = \sum e^{in^2 \pi x} / n^2$; we have $\Phi' \in B_p^{-1/4,\infty}$ if $p < 4$ and $\Phi' \in B_p^{-1/2+1/p,\infty}$ if $p > 4$. So that $\Phi \in B_p^{3/4,\infty}$ if $p < 4$ and $\Phi \in B_p^{1/2+1/p,\infty}$ if $p > 4$, and these estimates are optimal. Because of the continuity of the Hilbert transform on Besov spaces, the same result holds for φ. Since $\eta(p)$ can also be defined by

$$\eta(p) = \sup\{s : \ \varphi \in B_p^{s/p,\infty}\} ,$$

we have

$$\eta(p) = \begin{cases} 3p/4 , & \text{if } 0 < p \le 4 , \\ 1 + p/2 , & \text{if } p \ge 4 . \end{cases}$$

If $\alpha < 1/2$, $\inf_p(\alpha p - \eta(p) + 1) = -\infty$ and if $1/2 \le \alpha \le 3/4$, $\inf_p(\alpha p - \eta(p) + 1) = 4\alpha - 2$; we recover thus the increasing part of the spectrum, thus showing the validity of the Multifractal Formalim in that case.

Remark that Propositions 1 and 2 imply that if $x_0 \in F_\tau$, φ is $C^{(1+1/\tau)/2}(x_0)$.

We now prove Proposition 2.

5. The case when p_n and q_n are not both odd.

We first determine $\gamma_n = (r_n x + s_n)/(q_n x - p_n)$ in the theta modular group such that the pole of γ_n is p_n/q_n. Because of (22) if p_{n-1} and q_{n-1} are not both odd, we can choose

$$r_n = (-1)^n q_{n-1}, \qquad s_n = (-1)^{n+1} p_{n-1} \ ;$$

the corresponding transform satisfies (10) and thus belongs to the theta modular group; and if p_{n-1} and q_{n-1} are both odd, we can choose

$$r_n = (-1)^n q_{n-1} + q_n, \qquad s_n = (-1)^{n+1} p_{n-1} - p_n \ .$$

Since

$$\gamma_n\left(\frac{p_n}{q_n} + z\right) = \frac{r_n}{q_n} - \frac{1}{q_n^2 z} \ ,$$

applying (11) to $p_n/q_n + z$ and γ_n we obtain

$$(24) \qquad \left|\theta\left(\frac{p_n}{q_n} + z\right)\right| = \left|\theta\left(\frac{r_n}{q_n} - \frac{1}{q_n^2 z}\right)\right| \frac{1}{\sqrt{q_n|z|}} \ .$$

Since $\mathrm{Im}\,(-1/q_n^2 z) = \mathrm{Im}\,(z)/q_n^2|z|^2$, we consider the two following cases.

First case: $\mathrm{Im}\,(z)/q_n^2|z|^2 \geq 1$; then (24) and (12) imply that

$$\left|\theta\left(\frac{p_n}{q_n} + z\right)\right| \sim \frac{1}{\sqrt{q_n|z|}}$$

so that

$$|C(a,b)| \sim \frac{C\,a}{\sqrt{q_n(a + |b - \rho|)}}$$

(note that here and hereafter, $\sim$ means that the two quantities are equivalent, the constants in the equivalence being independant of n). Because of (17),

$$a + |b - \rho| \geq \frac{1}{q_n^{\tau_n}} \ ,$$

so that

$$|C(a,b)| \leq C \, a^{(1+1/\tau_n)/2} \left(1 + \frac{|b-\rho|}{a}\right)^{(1/\tau_n-1)/2};$$

and because of (12) this upper bound becomes an equality if we choose $a = 1/q_n^{\tau_n}$, $b = 0$, hence (18) and (19) in this case, and we also have proved their optimality.

Second case: $\mathrm{Im}\,(z)/q_n^2|z|^2 \leq 1$; we separate this case into two subcases:

First subcase: p_{n-1} and q_{n-1} are not both odd; then

$$(25) \qquad \left|\theta\left(\frac{p_n}{q_n} + z\right)\right| \leq \frac{1}{\sqrt{q_n|z|}} \left(\frac{\mathrm{Im}\,(z)}{q_n^2\,|z|^2}\right)^{-1/2} = \sqrt{\frac{q_n|z|}{Im(z)}},$$

so that, since $|z| \geq 2\,|\rho - p_n/q_n|$,

$$(26) \qquad |C(a,b)| \leq 2\,\sqrt{a\,q_n(a + |b-\rho|)}\,.$$

Because of (17),

$$a + |b - \rho| \leq 6\left|\rho - \frac{p_{n-1}}{q_{n-1}}\right| \leq 6\left(\frac{1}{q_{n-1}}\right)^{\tau_{n-1}} \leq 6\left(\frac{1}{q_n}\right)^{\tau_{n-1}/(\tau_{n-1}-1)};$$

thus

$$(27) \qquad \begin{aligned} |C(a,b)| &\leq C\,a\,q_n^{1/2}\left(1 + \frac{|b-\rho|}{a}\right)^{1/2} \\ &\leq C\,a\left(\frac{1}{a+|b-\rho|}\right)^{(\tau_{n-1}-1)/(2\tau_{n-1})}\left(1 + \frac{|b-\rho|}{a}\right)^{1/2} \\ &\leq C\,a^{1/2+1/(2\tau_{n-1})}\left(1 + \frac{|b-\rho|}{a}\right)^{1/(2\tau_{n-1})}; \end{aligned}$$

hence (20).

Second Subcase: p_{n-1} and q_{n-1} are both odd; then

$$r_n = (-1)^n q_{n-1} + q_n\,, \qquad s_n = (-1)^{n+1} p_{n-1} - p_n\,.$$

We now want to estimate θ near the points p_n/q_n where p_n and q_n are both odd; we will deduce this estimate from (15).

We see that (24) becomes

$$\left|\theta\left(\frac{p_n}{q_n}+z\right)\right| = \left|\theta\left(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z} + 1\right)\right| \frac{1}{\sqrt{q_n|z|}}$$

let

$$g_n(z) = \frac{(-1)^n q_n q_{n-1} z - 1}{q_n^2 z}$$

From (14), we get

$$\left|\theta\left(\frac{p_n}{q_n}+z\right)\right| = \frac{1}{\sqrt{q_n|z|\,|g_n(z)|}} \left|\theta\left(\frac{-1}{4\,g_n(z)}\right) - \theta\left(\frac{-1}{g_n(z)}\right)\right|.$$

Remark that

$$\text{Im}\left(\frac{-1}{g_n(z)}\right) = \frac{\text{Im}\,(g_n(z))}{|g_n(z)|^2} = \frac{\text{Im}\,(z)}{q_n^2\,|z|^2\,|g_n(z)|^2}.$$

If $\text{Im}\,(-1/g_n(z)) \geq 1$, using (15),

$$\left|\theta\left(\frac{p_n}{q_n}+z\right)\right| \leq \frac{1}{\sqrt{q_n\,|z|\,|g_n(z)|}} \exp\left(-\pi\,\frac{\text{Im}\,(z)}{q_n^2\,|z|^2\,|g_n(z)|^2}\right)$$

$$\leq \frac{1}{\sqrt{q_n\,|z|\,|g_n(z)|}} \left(\frac{q_n^2\,|z|^2\,|g_n(z)|^2}{\text{Im}\,(z)}\right)^{1/4}$$

$$\leq \left(\frac{1}{\text{Im}\,(z)}\right)^{1/4}.$$

so that $|C(a,b)| \leq a^{3/4}$; hence (18) in that case.

Suppose now that $\text{Im}\,(-1/g_n(z)) \leq 1$. Then

$$\left|\theta\left(\frac{p_n}{q_n}+z\right)\right| \leq \frac{1}{\sqrt{q_n\,|z|\,|g_n(z)|}} \left(\text{Im}\left(\frac{-1}{g_n(z)}\right)\right)^{-1/2}$$

$$\leq \frac{1}{\sqrt{q_n\,|z|\,|g_n(z)|}} \frac{q_n\,|z|\,|g_n(z)|}{\sqrt{\text{Im}\,(z)}}$$

$$\leq \sqrt{\frac{q_n\,|z|\,|g_n(z)|}{\text{Im}\,(z)}}.$$

Because of (17), $|z| \leq 6/(q_n q_{n-1})$, so that $|g_n(z)| \leq 7/(q_n^2|z|)$ and

$$\left|\theta\left(\frac{p_n}{q_n}+z\right)\right| \leq \sqrt{\frac{q_n\,|z|}{q_n^2\,|z|\,\text{Im}\,(z)}} \leq \frac{1}{\sqrt{q_n\,\text{Im}\,(z)}}.$$

Thus

$$|C(a,b)| \leq \frac{\sqrt{a}}{\sqrt{q_n}} \, .$$

From (17), we have $(1/q_n)^{r_n} \leq (a + |b - \rho|)$ so that

$$|C(a,b)| \leq a^{1/2 + 1/(2r_n)} \left(1 + \frac{|b - \rho|}{a}\right)^{1/(2r_n)} ;$$

Hence (18) in that case.

6. The case when p_n and q_n are both odd.

Following the same procedure as in the previous section, we first determine $\gamma_n = (ax + b)/(cx + d)$ such that $\gamma_n(p_n/q_n) = 1$. We choose either $r_n = q_{n+1}$, $s_n = p_{n+1}$ or $r_n = -q_{n+1}$, $s_n = -p_{n+1}$ such that

$$p_n r_n - s_n q_n = 1$$

(which is possible because of (22)). Now γ_n is defined by the coefficients

$$a = q_n + r_n , \quad b = -p_n - s_n , \quad c = r_n , \quad d = -s_n .$$

One easily checks that (10) holds and thus γ_n belongs to the Theta modular group;

$$\gamma_n\left(\frac{p_n}{q_n} + z\right) = \frac{1 + q_n (r_n + q_n) z}{1 + r_n q_n z} = 1 + f_n(z)$$

with

$$f_n(z) = \frac{q_n^2 z}{1 + r_n q_n z}$$

so that, because of (11),

$$\left|\theta\left(\frac{p_n}{q_n} + z\right)\right| = |\theta(1 + f_n(z))| \frac{1}{r_n^{1/2}\left(z + \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right)^{1/2}}$$

but (17) implies

$$|z| \geq 3\left|\rho - \frac{p_n}{q_n}\right| \geq \frac{3}{2}\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right|,$$

so that

$$\left|\theta\left(\frac{p_n}{q_n} + z\right)\right| \leq \frac{|\theta(1 + f_n(z))|}{r_n^{1/2} |z|^{1/2}} .$$

Remark that the condition $\mathrm{Im}\,(-1/f_n(z)) \geq 1$ is equivalent to $\mathrm{Im}\,(z) \geq q_n^2 |z|^2$. Thus we now consider the two following cases.

First Case: $\mathrm{Im}\,(z) \geq q_n^2 |z|^2$. In that case, because of (15),

$$\left|\theta\left(\frac{p_n}{q_n} + z\right)\right| \leq C \, \exp\left(\frac{-\pi \mathrm{Im}\,(z)}{q_n^2 |z|^2}\right) \left(\frac{1}{|f_n(z) r_n z|}\right)^{1/2}$$

$$\leq C \, \exp\left(\frac{-\pi \mathrm{Im}\,(z)}{q_n^2 |z|^2}\right) \frac{|1 + r_n q_n z|^{1/2}}{(|r_n z|)^{1/2} |q_n^2 z|^{1/2}} .$$

Because of (16),

$$1/|r_n q_n| \leq 2/q_n^{r_n} ,,$$

and thus

$$|r_n q_n z| \geq \frac{3}{2} .$$

Thus

$$\left|\theta\left(\frac{p_n}{q_n} + z\right)\right| \leq C \, \exp\left(\frac{-\pi \mathrm{Im}\,(z)}{q_n^2 |z|^2}\right) \frac{1}{|z|^{1/2} q_n^{1/2}}$$

$$\leq C \left(\frac{q_n^2 |z|^2}{\mathrm{Im}\,(z)}\right)^{1/4} \frac{1}{|z|^{1/2} q_n^{1/2}} = \frac{C}{\mathrm{Im}\,(z)^{1/4}}$$

so that

$$|C(a,b)| \leq a^{3/4} ;$$

hence (21) in that case.

Second Case: $\mathrm{Im}\,(z) \leq q_n^2 |z|^2$. In that case, from (13), (14) and (28), we obtain

$$\left|\theta\left(\frac{p_n}{q_n} + z\right)\right| \leq \frac{1}{\sqrt{|f_n(z) r_n z|}} \left|\theta\left(\frac{-1}{4 f_n(z)}\right) - \theta\left(\frac{-1}{f_n(z)}\right)\right|$$

$$\leq C \, \frac{q_n |z|^{1/2}}{\sqrt{r_n |f_n(z)| \mathrm{Im}\,(z)}}$$

$$\leq C \, \frac{|1 + r_n q_n z|^{1/2}}{\sqrt{r_n \mathrm{Im}\,(z)}}$$

$$\leq C \, \frac{|q_n z|^{1/2}}{\sqrt{\mathrm{Im}\,(z)}}$$

as above $|r_n \, q_n \, z| \geq 3/2$, so that

$$|C(a,b)| \leq C a^{1/2} \sqrt{q_n(a + |b - \rho|)} \,.$$

Because of (17),

$$a + |b - \rho| \leq 3 \, (1/q_n)^{\tau_{n-1}/(\tau_{n-1}-1)} \,,$$

so that

$$|C(a,b)| \leq C \, a^{1/2+1/(2\tau_{n-1})} \left(1 + \frac{|b - \rho|}{a}\right)^{1/(2\tau_{n-1})} \,.$$

Thus (21) holds in this case and Proposition 3 is proved.

The fact that (18) and (21) cannot be improved in a cone

$$\operatorname{Im}(z - \rho) \geq C \operatorname{Re}(z - \rho)$$

yields a slightly more precise information than the fact that φ is not smoother than $1/2 + 1/(2\eta(\rho))$ at ρ because it shows that φ has no chirp expansion at an irrational point ρ (see [9]); thus the only points where φ has a chirp expansion are the rationals of the form odd / odd. It also shows that fractional integrals of φ of order s will be exactly $\Gamma^{s+1/2+1/(2\eta(\rho))}(\rho)$. Actually, from Proposition 2, and the chirp characterization given in [13], one easily obtains the following corollary.

Corollary 2. *Let*

$$\varphi_s(x) = \sum_1^\infty \frac{1}{n^{2+2s}} \sin \pi n^2 x \,.$$

If $s \in (-1/2, +\infty)$ and if ρ is not a rational quotient of two odd numbers, $\varphi_s \in \Gamma^{s+1/2+1/(2\eta(\rho))}(\rho)$ and the spectrum of singularities of φ_s is given by

$$d(\alpha) = \begin{cases} 4(\alpha - s) - 2, & \text{if } \alpha \in \left[s + \dfrac{1}{2}, s + \dfrac{3}{4}\right], \\ 0, & \text{if } \alpha = 2s + \dfrac{3}{2}, \\ -\infty, & \text{else;} \,. \end{cases}$$

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