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Self-similar solutions in weak L^p-spaces of the Navier-Stokes equations

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Abstract. The most important result stated in this paper is a theorem on the existence of global solutions for the Navier-Stokes equations in $\mathbb{R}^n$ when the initial velocity belongs to the space weak $L^n(\mathbb{R}^n)$ with a sufficiently small norm. Furthermore, this fact leads us to obtain self-similar solutions if the initial velocity is, besides, an homogeneous function of degree -1. Partial uniqueness is also discussed.

1. Introduction.

We start our exposition by setting the central problem and establishing the framework of the principal ideas. First of all we are going to try to give to the reader a modest approach to the physical meaning.

Let us consider the Navier-Stokes equations of a viscous incompressible fluid which fills an infinite cylinder of cross section Ω, an open subset of $\mathbb{R}^n$. These equations govern the flow of the fluid which moves parallel to the plane of Ω when an external force $f = (f_1, f_2, \dots, f_n)$ is present.

The vector $\mathbf{v}(t, x)$ represents the velocity field of a particle at spatial position x and time t, and the function $\mathrm{p}(t,x)$ the pressure at x and time t, respectively. They are the unknows of the Navier-Stokes system

$$\begin{cases} \rho\Big(\dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) - \mu\,\Delta\mathbf{v} + \nabla\mathrm{p} = f\,, \\ \nabla\cdot\mathbf{v} = 0\,, \end{cases} \tag{1}$$

where the constant $\rho > 0$ is the density of the fluid and $\mu > 0$ is the kinematic viscosity. As usual, $\nabla\mathrm{p} = (\partial_1\mathrm{p}, \partial_2\mathrm{p}, \dots, \partial_n\mathrm{p})$ denotes the gradient of p and $\nabla\cdot\mathbf{v} = \sum_{j=1}^{n}\partial_j\mathbf{v}_j$ the divergence of $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$.

The first equation of system (1) is the momentum conservation equation and the second one is the mass conservation equation (incompressibility condition) [T].

We shall limit ourselves to the study of the existence of solutions of the Cauchy problem for the equation (1) in the case that Ω is the whole space $\mathbb{R}^n$. Then, there will not be any external force ($f \equiv 0$). Moreover, we will assume that the density $\rho = 1$ and the viscosity $\mu = 1$. The initial data of the velocity is a vectorial field $\mathbf{v}_0$ satisfying the condition $\nabla\cdot\mathbf{v}_0 = 0$ in the distributional sense.

Therefore, the problem in which we are interested is

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathrm{p} = 0\,, \\ \nabla\cdot\mathbf{v} = 0\,, \\ \mathbf{v}(0,x) = \mathbf{v}_0(x)\,. \end{cases} \tag{2}$$

Of course, the pressure p will be disregarded. It will be automatically determined by mean of the first equation of (2) after computing the velocity $\mathbf{v}$, except for an additive function depending on time.

In his paper [K], Kato proved the existence and uniqueness of solutions for the problem (2) in the L^p-theory based on the technical details developed in [K-F]. In both works it was shown that such a global solution exists using the successive approximation method, applied to the integral equation formally equivalent to the initial value problem (2). This problem is recently solved by M. Cannone [C] for any abstract "adapted space" among the concrete applications we find the L^p spaces and some kind of Besov spaces. In Kozono-Yamazaki's paper [K-Y], the existence and the uniqueness of Cauchy problem (2) are treated too for a larger new function spaces constructed in the same way as the Besov ones, based on the Morrey spaces instead of the usual L^p spaces.

However, in the study of the attractors associated with the Navier-Stokes equations [T] it is necessary to be able to find "self-similar solutions". That is, solutions $\mathbf{v}(t,x)$ which satisfy $\mathbf{v}(t,x) = \lambda\mathbf{v}(\lambda^2 t, \lambda x)$

for all $x \in \mathbb{R}^n$, all $t > 0$ and all $\lambda > 0$. This kind of solutions are related to an asymptotic behaviour, for large time, of the global solutions of the Navier-Stokes equations. In the paper [G-M] the existence and uniqueness of self-similar mild-solutions are shown in the frame of the Morrey-type spaces of measures in $\mathbb{R}^3$, solving firstly the problem for the vorticity. In [F-M-T] the reader can find some implicit comments about the meaning of the self-similar solutions in the study of large time behaviour. If the functions $\mathbf{v}(t,x)$ and $\mathrm{p}(t,x)$ are global solutions of system (1), it is not difficult to show that, for each number $\lambda > 0$, the functions $\lambda \mathbf{v}(\lambda^2 t, \lambda x)$ and $\lambda^2 \mathrm{p}(\lambda^2 t, \lambda x)$ are solutions of the same system too.

Moreover, it is possible to characterize the self-similary condition in the following way. A vector field $\mathbf{v}(t,x)$ has the *self-similar property* if and only if there exists a vector field $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ such that $\mathbf{v}(t,x) = \mathbf{V}(x/\sqrt{t})/\sqrt{t}$, for all $x \in \mathbb{R}^n$ and all $t > 0$. In fact, when the field $\mathbf{V}$ exists, this last equality gives the definition for $\mathbf{v}(t,x)$, and it is straightforward to see that it is self-similar. Conversely, when the self-similar solution $\mathbf{v}(t,x)$ is given, we define $\mathbf{V}(x) = \mathbf{v}(1,x)$, for all $x \in \mathbb{R}^n$. Then, the self-similar condition on $\mathbf{v}$ turns out the expected equality between $\mathbf{v}$ and $\mathbf{V}$ choosing $\lambda = 1/\sqrt{t}$.

But, the problem of finding self-similar solutions is not evident. The initial velocity $\mathbf{v}_0(x)$ must be an homogeneous function of degree -1, if a self-similar solution exists. On $\mathbb{R}^3$, the typical elementary example of such functions is given by any linear combination of the vector fields

$$\Big(0, -\frac{x_3}{|x|^2}, \frac{x_2}{|x|^2}\Big), \ \Big(\frac{x_3}{|x|^2}, 0, -\frac{x_1}{|x|^2}\Big) \ \text{ and } \ \Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}, 0\Big)$$

which are homogeneous of degree -1 and verify $\nabla \cdot \mathbf{v} = 0$ (see [G-M], [C]). This fact produces a complicated situation. We would hope that the L^p-spaces might be the natural mathematical setting. However, it is trivial that there is not any homogeneous function of degree -1 (or any other degree) in L^p, for any power p. The existence and uniqueness of self-similar solutions for the problem (2) are proved in [C] for a certain family of Besov spaces.

Professor Y. Meyer suggested I should study the same problem in a much more natural frame what is the weak L^p spaces. So, our interest is particulary centered on the existence of global solutions $\mathbf{v}(t,x)$ of problem (2) in the space weak $L^n(\mathbb{R}^n)$, when the norm of the initial velocity is sufficiently small.

For each $0 < p < \infty$ the space weak $L^p(\mathbb{R}^n)$, denoted $L^{(p,\infty)}(\mathbb{R}^n)$ or shortly $L^{(p,\infty)}$, is the set of all the complex-valued Lebesgue measurable functions f defined on $\mathbb{R}^n$ such that exists a constant $A > 0$ satisfying

$$m\{x \in \mathbb{R}^n : \ |f(x)| > s\} \leq \frac{A}{s^p}\,, \tag{3}$$

for all $s > 0$. Here m denotes the Lebesgue measure on $\mathbb{R}^n$.

For $p = \infty$, the space weak $L^\infty(\mathbb{R}^n)$ will be $L^\infty(\mathbb{R}^n)$, and it will be denoted $L^{(\infty,\infty)}(\mathbb{R}^n)$, or simply $L^{(\infty,\infty)}$.

The space $L^{(n,\infty)}$ has the advantage to be one of the most natural extensions of $L^n(\mathbb{R}^n)$ that contains, besides, the homogeneous functions of degree -1. It is easy to see that the functions given above belong to $L^{(3,\infty)}(\mathbb{R}^3)$ but they are not included in $L^3(\mathbb{R}^3)$. This fact is the key that resolves the problem of finding self-similar solutions of the Navier-Stokes equations in a special sense to be explained later. This subject is one of the principal results presented here. All the main theorems will be stated in Section 2.

We shall need to make use of a class space which contains L^p and weak L^p spaces. That is, the Lorentz spaces $L^{(p,q)}$. In Section 3 we shall recall the definition, notation and some properties following the works [L], [H], [S-W] and [O]. At last, in Section 4, the proofs and some technical results will be given.

2. Our main theorems.

The aim of this section is to describe the principal results and their mathematical setting. First of all we need to specify the frame of problem (2), that is, the space of solutions of this system.

Definition. *Let $n > 1$ a positive integer number and let q be any fixed real number such that $n < q < \infty$. Let us define E the space of all the complex-valued functions $\mathbf{v}(t,x)$, with $t > 0$ and $x \in \mathbb{R}^n$, such that the following conditions are satisfied*

$$\mathbf{v}(t,x) \in C((0,\infty), L^{(n,\infty)})\,, \tag{4.1}$$

$$t^{(1-n/q)/2}\mathbf{v}(t,x) \in C((0,\infty), L^{(q,\infty)})\,, \tag{4.2}$$

(4.3) *the map $t \mapsto \mathbf{v}(t,\cdot)$ from $(0,\infty)$ to $L^{(n,\infty)}$ is continuous at the origin.*

Through this work C denotes the class of bounded and continuous function, and therefore the norm of $\mathbf{v}$ is naturally defined by

$$(5)\qquad \|\mathbf{v}\|_E := \sup_{t>0} t^{(1-n/q)/2}\, \|\mathbf{v}(t,x)\|_{(q,\infty)} + \sup_{t>0} \|\mathbf{v}(t,x)\|_{(n,\infty)}\,.$$

Once for all the space $L^{(n,\infty)}$ will be equipped with its canonical $\sigma(L^{(n,\infty)}, L^{(n',1)})$ topology and the symbol $\|\cdot\|_{(r,s)}$ denotes the norm on the Lorentz space $L^{(r,s)}(\mathbb{R}^n)$. (See Section 3.)

The expression (5) defines a norm on E, and the pair $(E, \|\cdot\|_E)$ is a Banach space. All this is a straightforward consequence of results of Section 3. In what follows we shall find global solutions of the problem (2) in E when the initial velocity $\mathbf{v}_0$ belongs to $L^{(n,\infty)}$ with sufficiently small norm, and verifies $\nabla\cdot\mathbf{v}_0 = 0$.

As usual, the problem (2) is written under the following integral form:

$$(6)\qquad \mathbf{v}(t,x) = S(t)\mathbf{v}_0(x) + B(\mathbf{v},\mathbf{v})(t,x)\,,$$

where

$$(7)\qquad S(t)\mathbf{v}_0(x) = \mathbb{P}e^{-t\Delta}\mathbf{v}_0(x)$$

and

$$(8)\qquad B(\mathbf{u},\mathbf{v})(t,x) = -\int_0^t \mathbb{P}S(t-s)(\mathbf{u}(s,x)\cdot\nabla)\mathbf{v}(s,x)\,ds\,.$$

Any vector field $\mathbf{v}(t,x)$ satisfying the equality (6) will be called a "mild-solution" of the initial value problem (2).

The bilinear map B is defined by a Bochner integral. $\mathbb{P}$ denotes the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the subspace $\mathbb{P}L^2(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) :\ \nabla\cdot f = 0\}$. As is well known (see, for instance [F-M]), $\mathbb{P}$ can be extended to a continuous operator on $L^p(\mathbb{R}^n)$ to $\mathbb{P}L^p(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) :\ \nabla\cdot f = 0\}$, for $1 < p < \infty$ (see [K]). Furthermore, it is trivial to extend $\mathbb{P}$ on $L^p(\mathbb{R}^n)$ to $\mathbb{P}L^{(p,\infty)}(\mathbb{R}^n) = \{f \in L^{(p,\infty)}(\mathbb{R}^n) :\ \nabla\cdot f = 0\}$ because $L^p(\mathbb{R}^n) = L^{(p,p)}(\mathbb{R}^n)$. (See Section 3.)

We notice that, since $\mathbb{P}$ commutes with the Laplacian Δ, the operator $S(t)$ agrees with the heat operator on functions in $\mathbb{P}L^{(p,\infty)}(\mathbb{R}^n)$.

We are now ready to state our main results.

Theorem 1. *Let* $\mathbf{v}_0$ *be any function in* $L^{(n,\infty)}(\mathbb{R}^n)$ *satisfying* $\nabla\cdot\mathbf{v}_0 = 0$ *in the distributional sense. Then, there exists a constant* $\delta > 0$ *such that if* $\|\mathbf{v}_0\|_{(n,\infty)} < \delta$ *the initial value problem* (2) *for the Navier-Stokes equations has, at least, a global mild-solution* $\mathbf{v}(t,x)$ *belonging to* E. *Furthermore, this solution satisfies* $\sup_{t>0} t^{(1-n/q)/2}\,\|\mathbf{v}(t,x)\|_{(q,\infty)} < \eta$ *where* $\eta = \eta(\delta) \to 0$ *with* δ.

Moreover, there exists a constant $\eta_0 > 0$ *such that if there is a solution* $\mathbf{v}(t,x)$ *satisfying* $\sup_{t>0} t^{(1-n/q)/2}\,\|\mathbf{v}(t,x)\|_{(q,\infty)} < \eta_0$, *then it is unique.*

Theorem 2. (Corollary). *Let* $\mathbf{v}_0 \in L^{(n,\infty)}(\mathbb{R}^n)$ *such that* $\nabla\cdot\mathbf{v}_0 = 0$ *in the distributional sense. It will be assumed that* $\mathbf{v}_0$ *is an homogeneous function of degree* -1, *that is,* $\mathbf{v}_0(\lambda x) = \lambda^{-1}\mathbf{v}_0(x)$ *for all* $x \in \mathbb{R}^n, x \neq 0$ *and all* $\lambda > 0$.

Then, there exists a constant $\delta > 0$ *such that if* $\|\mathbf{v}_0\|_{(n,\infty)} < \delta$ *a global mild-solution* $\mathbf{v}(t,x)$ *of the problem* (2) *of the Navier-Stokes equations in* E, *given by Theorem* 1, *satisfies*

$$\mathbf{v}(t,x) = \lambda\,\mathbf{v}(\lambda^2 t, \lambda x)\,, \tag{9}$$

for all $x \in \mathbb{R}^n$, *all* $t > 0$ *and all* $\lambda > 0$.

REMARK. We remind the reader that any global solution $\mathbf{v}(t,x)$ of the problem (2) having the property (9) is called "self-similar solution" of (2).

Theorem 3. (Corollary: Existence and uniqueness of self-similar solutions of Navier-Stokes equations in E). *Let* $\mathbf{v}_0$ *be any function defined on* $\mathbb{R}^n$ *which is homogeneous of degree* -1 *and satisfies* $\nabla\cdot\mathbf{v}_0 = 0$ *in the distributional sense. Furthermore, it will be supposed that the restriction of* $\mathbf{v}_0$ *to the unitary sphere* S^{n-1} *of* $\mathbb{R}^n$, *denoted* $\mathbf{v}_0|_{S^{n-1}}$, *belongs to* $L^n(S^{n-1})$.

Then, there exists a constant $\delta > 0$ *such that if* $\|\mathbf{v}_0|_{S^{n-1}}\|_{L^n(S^{n-1})} < \delta$, *the Cauchy problem* (2) *for the Navier-Stokes equations has, at least, a self-similar solution in* E.

As in Theorem 1, *this system admits a unique self-similar solution if such solution belongs to the ball*

$$\{\mathbf{v}(t,x) \in E :\ \sup_{t>0} t^{(1-n/q)/2}\,\|\mathbf{v}(t,x)\|_{(q,\infty)} < \eta_0\}\,,$$

for some $\eta_0 > 0$ *sufficiently small.*

3. Facts on Lorentz spaces.

In this section we shall present a brief summary of the definition of Lorentz spaces and the principal properties we are going to use.

Let us consider a non-atomic measure space $(X, \mathcal{M}, m)$. For each complex-valued, m-measurable function f defined on X, its distribution function is defined by

$$\lambda(s) = m\{x \in X : \, |f(x)| > s\}, \qquad \text{for } s > 0, \tag{10}$$

which is non-increasing and continuous from the right. This function has a "quasi-inverse", called the non-increasing rearrangement of f onto $(0, \infty)$, whose definition is

$$f^*(t) = \inf\{s > 0 : \lambda(s) \le t\}, \qquad \text{for } t > 0. \tag{11}$$

It should be noticed that $f^*(t)$ is the true inverse function of $\lambda(t)$ when this function is continuous and strictly decreasing. It results that $f^*(t)$ is also continuous from the right and has the same distribution function as f.

Thus, the Lorentz space $L^{(p,q)}$ on $(X, \mathcal{M}, m)$ will be the collection of all the complex-valued, m-measurable functions f defined on X such that $\|f\|^*_{(p,q)} < \infty$, where

$$\|f\|^*_{(p,q)} = \begin{cases} \left(\dfrac{p}{q}\displaystyle\int_0^\infty (t^{1/p} f^*(t))^q \,\frac{dt}{t}\right)^{1/q}, & \text{if } 0<p<\infty, 0<q<\infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), & \text{if } 0<p\le\infty, q=\infty. \end{cases} \tag{12}$$

The case $p = \infty$, $0 < q < \infty$ is out of interest since $\|f\|^*_{(\infty,q)} < \infty$ implies $f = 0$ almost everywhere.

We shall only recall the following two elementary properties for the Lorentz spaces.

a) For any $p > 0$ and any q and r such that $0 < q \le r \le \infty$ we have

$$\|f\|^*_{(p,r)} \le \|f\|^*_{(p,q)}, \tag{13}$$

and thus

$$L^{(p,q)} \subset L^{(p,r)}. \tag{14}$$

b) For any p such that $1 \leq p \leq \infty$ we have that $\|f\|^*_{(p,p)}$ is the usual L^p-norm, and then $L^{(p,p)} = L^p(X, \mathcal{M}, m)$.

Combining these two properties, we obtain in particular that for $0 < p \leq \infty$, $L^{(p,p)} = L^p \subset L^{(p,\infty)}$ with

$$\|f\|^*_{(p,\infty)} \leq \|f\|_p \,. \tag{15}$$

However, since the triangle inequality fails in general, $\|f\|^*_{(p,q)}$ is not a norm for $p < q$.

In view to build an "adequate" norm for the Lorentz spaces, it is necessary to define the function:

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(y)\, dy\,, \qquad \text{for} \quad t > 0\,. \tag{16}$$

This function is closely related with the Hardy-Littlewood maximal function of f because it can be expressed by ([H])

$$f^{**}(t) = \sup_{\substack{E \in \mathcal{M} \\ m(E) \geq t}} \frac{1}{m(E)} \int_E |f(x)|\, dx\,. \tag{17}$$

Hence, the norm $\|f\|_{(p,q)}$ is defined as:

$$\|f\|_{(p,q)} = \begin{cases} \left(\dfrac{p}{q} \displaystyle\int_0^\infty (t^{1/p} f^{**}(t))^q \, \frac{dt}{t} \right)^{1/q}, & \text{if } 1 < p < \infty, 1 \leq q < \infty, \\ \sup\limits_{t > 0} t^{1/p} f^{**}(t)\,, & \text{if } 1 < p \leq \infty\,,\ q = \infty\,. \end{cases} \tag{18}$$

These spaces $L^{(p,q)}$ with the norm $\|f\|_{(p,q)}$ are Banach spaces. In some sense $\|f\|_{(p,q)}$ is equivalent to $\|f\|^*_{(p,q)}$

$$\|f\|^*_{(p,q)} \leq \|f\|_{(p,q)} \leq \frac{p}{p-1} \|f\|^*_{(p,q)}\,, \tag{19}$$

for $1 < p < \infty$, $1 \leq q < \infty$. We observe that in definition (18) the case $p = 1$ has been excluded; although both expressions make sense, they do not define a norm. In this case property (19), or a similar one, is not valid. Besides, for $p = 1$ the second case in (18) would provide

$$\|f\|_{(1,\infty)} = \|f\|_1\,, \tag{20}$$

which is inadequate.

Since expression (19), it will be sufficient to estimate $\|f\|^*_{(p,q)}$ instead of $\|f\|_{(p,q)}$, which is more complicated to manipulate.

The reader who is interested in the Lorentz spaces $L^{(p,q)}$ and their properties is referred to [S-W, chapter V], and to [L] and [H] for more details.

In [O] estimates for the product operators and convolution operators are given. From there we extract the following definitions and results.

Given three measure spaces (X,μ), $(\bar{X},\bar{\mu})$ and (Y,ν), a bilinear operator T, which maps complex-valued measurable functions on X and $\bar{X}$ into complex-valued measurable functions on Y, is called a "convolution operator" if

$$\|T(f,g)\|_1 \le \|f\|_1\,\|g\|_1\,, \tag{21.a}$$

$$\|T(f,g)\|_\infty \le \|f\|_1\,\|g\|_\infty\,, \tag{21.b}$$

$$\|T(f,g)\|_\infty \le \|f\|_\infty\,\|g\|_1\,. \tag{21.c}$$

As usual, $\|\,.\,\|_1$ and $\|\,.\,\|_\infty$ are respectively the L^1-norm and the L^∞-norm on any of the three spaces X, $\bar{X}$, Y.

Proposition 1 ([O]) (Generalized Young's inequality). *Let $1 < p_1, p_2, r < \infty$. If T is a convolution operator and if $f \in L^{(p_1,q_1)}$, $g \in L^{(p_2,q_2)}$ where $1/p_1 + 1/p_2 > 1$, then $h = T(f,g)$ belongs to $L^{(r,s)}$ where $1/r = 1/p_1 + 1/p_2 - 1$, and $s \ge 1$ is any number such that $1/q_1 + 1/q_2 \ge 1/s$.*

Moreover,

$$\|h\|_{(r,s)} \le 3\,r\,\|f\|_{(p_1,q_1)}\,\|g\|_{(p_2,q_2)}\,. \tag{22}$$

Otherwise, a bilinear operator P, which maps complex-valued measurable functions on X and $\bar{X}$ into complex-valued measurable functions on Y, is called a "product operator" if

$$\|P(f,g)\|_\infty \le \|f\|_\infty\,\|g\|_\infty\,, \tag{23.a}$$

$$\|P(f,g)\|_1 \le \|f\|_1\,\|g\|_\infty\,, \tag{23.b}$$

$$\|P(f,g)\|_1 \le \|f\|_\infty\,\|g\|_1\,. \tag{23.c}$$

Proposition 2 ([O]) (Generalized Hölder's inequality). *If P is a product operator, $h = P(f,g)$, and if $f \in L^{(p_1,q_1)}$, $g \in L^{(p_2,q_2)}$ where $1/p_1 + 1/p_2 < 1$, then $h \in L^{(r,s)}$ where $1/r = 1/p_1 + 1/p_2$, and $s \geq 1$ is any number such that $1/q_1 + 1/q_2 \geq 1/s$.*

Moreover,

$$\|h\|_{(r,s)} \leq r' \, \|f\|_{(p_1,q_1)} \, \|g\|_{(p_2,q_2)} \,, \tag{24}$$

being r' the conjugate index of r.

Before leaving this section, we extract the following proposition from Hunt's paper [H] about the duality of Lorentz spaces.

Proposition 3 ([H]). *The conjugate space of $L^{(p,1)}$ is $L^{(p',\infty)}$, where $1/p + 1/p' = 1$.*

The conjugate space of $L^{(p,q)}, 1 < p < \infty, 1 < q < \infty$, is $L^{(p',q')}$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, and hence, these spaces are reflexive.

REMARK. In the same way that L^1 is not the conjugate space of L^∞, $L^{(p,1)}$ is not the conjugate space of $L^{(p',\infty)}$. ([H]).

4. Proofs of the Theorems.

In this last section we shall develope the proofs of the stated in Section 2, and some intermediate results.

First of all, we begin reminding that the proyection $\mathbb{P}$ commutes with the Laplacian Δ, and then the operator $S(t)$, defined by (7), is essentially the heat operator. This fact allows us to estimate $S(t)$ without taking into account the operator $\mathbb{P}$.

The Weierstrass kernel, or heat kernel, is given by

$$w_t(x) = \left(\frac{1}{4\pi t}\right)^{n/2} e^{-|x|^2/(4t)} \,, \tag{25}$$

for all $t > 0$ and all $x \in \mathbb{R}^n$.

Thus, we can write

$$S(t)\mathrm{v}_0(x) = \mathbb{P}(w_t * \mathrm{v}_0)(x) \,, \tag{26}$$

and

$$(27) \qquad B(\mathbf{u},\mathbf{v})(t,x) = -\int_0^t \mathbb{P}\big(w_{t-s} * ((\mathbf{u}(s)\cdot\nabla)\mathbf{v}(s))\big)(x)\,ds\,.$$

Here $*$ denotes the usual convolution of functions.

We begin giving a meaning to the right hand side of (26) because, in our case, $\mathbf{v}_0$ is a distribution in $L^{(n,\infty)}(\mathbb{R}^n)$ which is not reflexive.

The operator $S(t)(\varphi) = \mathbb{P}(w_t * \varphi)$ is well defined for all functions φ in $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the Schwartz class. These functions are dense in each Lorentz space $L^{(p,q)}$ with $1 < p < \infty$ and $1 \le q < \infty$. Then, we can extend the operator $S(t)$ to these Lorentz spaces and this extension will be denoted $S(t)$ too.

Let us remember that $L^{(p,\infty)} = (L^{(p',1)})^*$ for any $1 \le p < \infty$, where p' is the conjugate index of p (see Proposition 3). So, by duality, we can define the formal transposed operator of $S(t)$ on each $L^{(p,\infty)}$, which we shall also denote $S(t)$. We shall deduce some continuity properties of this operator.

Lemma 1. *For any $t > 0$, any r and any p such that $1 < p \le r < \infty$ the operator*

$$(28) \qquad S(t) : L^{(p,\infty)}(\mathbb{R}^n) \longrightarrow L^{(r,\infty)}(\mathbb{R}^n)$$

is weak-star continuous.

In order to see this we must show the following continuity property of $S(t)$

$$(29) \qquad S(t) : L^{(r',1)} \longrightarrow L^{(p',1)}$$

is strong continuous for all $t > 0$. Moreover,

$$(30) \qquad \|S(t)\varphi\|_{(p',1)} \le c(n,p,r)\, t^{-(1/r'-1/p')n/2}\, \|\varphi\|_{(r',1)}\,.$$

Taking into account that $S(t)$ is essentially the convolution with w_t, the inequality (30), for the case $1 < p < r < \infty$, results directly from the generalized Young's inequality (22) and from the evident estimate

$$\|w_t\|_{(q_1,q_2)} \le c(n,q_1,q_2)\, t^{-(1-1/q_1)n/2} = c(n,p,r)\, t^{-(1/r'-1/p')n/2}\,,$$

where $q_1 > 1$ and $q_2 \geq 1$ must verify $1/p' = 1/r' + 1/q_1 - 1$ and $1 \leq 1 + 1/q_2$.

When $p = r > 1$ we obtain (30) from an elementary result over the homogeneous Banach space $L^{(p',1)}(\mathbb{R}^n)$ (see [Kn]).

Clearly, (30) implies (29). Then, a simple duality argument leads us to

$$\|S(t)f\|_{(r,\infty)} \leq c(n,p,r)\, \|f\|_{(p,\infty)}\, t^{-(1/p-1/r)n/2}\,.$$

completing the proof of Lemma 1.

We shall pass to study the behaviour of the operator $S(t)$ near the value $t = 0$. This fact will be closely related to the weak continuity at $t = 0$ of the mild-solution of the problem (2).

Once more we notice that the operator $S(t)$ is essentially the convolution with the heat kernel w_t so, taking into account that $w_t(x) \to 0$, when $t \to 0$, for all $x \in \mathbb{R}^n, x \neq 0$, one could hope that $S(t)$ tends to I, the identity operator. It is not difficult to obtain this convergence in the norm $\|\cdot\|_{(p,q)}$ when $S(t)$ acts on $L^{(p,q)}$ with $1 < p < \infty$, $1 < q < \infty$, that means, in the case that $L^{(p,q)}$ is a reflexive space (Proposition 3). But, unfortunately in our problem, the initial velocity $\mathbf{v}_0(x)$ belongs to $L^{(n,\infty)}$ which is not reflexive. However, for each $\varphi \in L^{(n',1)}$ we have by duality

$$|\langle S(t)\mathbf{v}_0 - \mathbf{v}_0, \varphi\rangle| = |\langle \mathbf{v}_0, S(t)\varphi - \varphi\rangle| \leq \|\mathbf{v}_0\|_{(n,\infty)}\, \|S(t)\varphi - \varphi\|_{(n',1)}$$

which tends to 0 with t. Hence, we have proved the following lemma.

Lemma 2. *If $\mathbf{v}_0 \in L^{(n,\infty)}$, then*

$$S(t)\mathbf{v}_0 \rightharpoonup \mathbf{v}_0\,, \quad \textit{when} \quad t \to 0^+\,.$$

In the next lemma our purpose will be the study of the continuity of the operator $S(t)$ as a function of t.

Lemma 3. *Let $f \in L^{(n,\infty)}(\mathbb{R}^n)$.*

a) *The function $t \mapsto S(t)f$ from $(0,\infty)$ to $L^{(n,\infty)}(\mathbb{R}^n)$ is continuous.*

b) *The function $t \mapsto t^{(1-n/q)/2}\, S(t)f$ from $(0,\infty)$ to $L^{(q,\infty)}(\mathbb{R}^n)$ is continuous.*

This result will be a consequence of the following estimate we shall suppose true for the moment. Hence we assume

$$\|\Delta S(t)f\|_{(r,\infty)} \le c(n,p,r)\,t^{-1-(1/p-1/r)n/2}\,\|f\|_{(p,\infty)}\,, \tag{31}$$

for $1 < p \le r < \infty$. Then, taking into account the mean-value theorem and

$$\frac{\partial}{\partial t}(S(t)f) = -\Delta S(t)f\,,$$

we obtain part a) from (31) giving, with $p = r = n$,

$$\left\|\frac{\partial}{\partial t}(S(t)f)\right\|_{(n,\infty)} \le \frac{c(n)}{t}\,\|f\|_{(n,\infty)}\,,$$

which is bounded in a neighbourhood of each $t > 0$.

Analogously, for part b) we apply again the mean-value theorem to evaluate the difference

$$\left\|t^{(1-n/q)/2}\,S(t)f - (t')^{(1-n/q)/2}\,S(t')f\right\|_{(q,\infty)}.$$

Thus it will be sufficient to estimate the derivative

$$\begin{aligned}\frac{\partial}{\partial t}\left(t^{(1-n/q)/2}\,S(t)f\right) &= \frac{1}{2}\left(1-\frac{n}{q}\right)t^{-(1+n/q)/2}S(t)f \\ &\quad + t^{(1-n/q)/2}\,\frac{\partial}{\partial t}\,(S(t)f)\,.\end{aligned} \tag{32}$$

Thanks to Lemma 1, the first term of the right hand side of (32) can be estimated by

$$\begin{aligned}&\left\|\frac{1}{2}\left(1-\frac{n}{q}\right)t^{-(1+n/q)/2}\,S(t)f\right\|_{(q,\infty)} \\ &\qquad \le c_1(n,q)\,t^{-(1+n/q)/2-(1-n/q)/2}\,\|f\|_{(n,\infty)} \\ &\qquad = c_1(n,q)\,t^{-1}\,\|f\|_{(n,\infty)}\,,\end{aligned} \tag{33}$$

where we have taken $p = n$ and $r = q$.

For the second term of the right hand side of (32) we use inequality (31) with $p = n$ and $r = q$ producing

$$\left\|t^{(1-n/q)/2}\,\frac{\partial}{\partial t}\,(S(t)f)\,\right\|_{(q,\infty)} \le c_2(n,q)\,t^{-1}\,\|f\|_{(n,\infty)}\,. \tag{34}$$

Finally, we have from (33), (34) and (32), that

$$\left\| \frac{\partial}{\partial t}\left(t^{(1-n/q)/2}\, S(t)f\right)\right\|_{(q,\infty)} \leq \frac{c(n,q)}{t}\, \|f\|_{(n,\infty)}\,,$$

completing part b).

It remains to prove inequality (31). In the same way as we have done in Lemma 1, we must show that

$$\|\Delta S(t)\varphi\|_{(p',1)} \leq c(n,p,r)\, t^{-1-(1/r'-1/p')n/2}\, \|\varphi\|_{(r',1)}\,, \tag{35}$$

for all $\varphi \in L^{(r',1)}$. It will be enough to see (35) for any testing function φ.

Then, it can be rapidly seen that

$$\Delta S(t)\varphi = u_t^{(1)} * \varphi + u_t^{(2)} * \varphi\,, \tag{36}$$

where

$$u_t^{(1)}(x) = -\frac{n}{2t}\, w_t(x) \qquad \text{and} \qquad u_t^{(2)}(x) = \frac{|x|^2}{4t^2}\, w_t(x)\,.$$

Both terms will be treated separately. The first one is just equal to $(c/t)\,(w_t * \varphi)(x)$, and then, as in Lemma 1, we have

$$\left\|u_t^{(1)} * \varphi\right\|_{(p',1)} \leq \tilde{c}(n,r,p)\, t^{-1-(n/p-n/r)/2}\, \|\varphi\|_{(r',1)}\,, \tag{37}$$

for $1 < p \leq r < \infty$.

For the second term $(u_t^{(2)} * \varphi)(x)$ we can follow the way sketched in Lemma 1. Thus, we obtain

$$\left\|u_t^{(2)} * \varphi\right\|_{(p',1)} \leq \tilde{c}(n,r,p)\, t^{-1-(n/r'-n/p')/2}\, \|\varphi\|_{(r',1)}\,. \tag{38}$$

Finally, applying triangular inequality in $L^{(p',1)}$ to (36) and taking into account estimates (37) and (38) we get (35). Then, (31) is deduced from (38) by duality and therefore Lemma 3 is proved.

We can now state a result concerning the continuity of the operator $S(t)$ which collects the properties given in Lemmas 1, 2 and 3.

Proposition 4. *Let $t > 0$. The operator $S(t) : L^{(n,\infty)}(\mathbb{R}^n) \longrightarrow E$ is continuous.*

Next, we go on with the bilinear operator B and their continuity properties. We need to define two auxiliar spaces F_1 and F_2. Let F_1 be the set of all the complex-valued functions $\mathbf{v}(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}^n$, such that $\mathbf{v}(t,\cdot) \in L^{(q,\infty)}(\mathbb{R}^n)$ and the number

$$\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{v}(t)\|_{(q,\infty)}$$

is finite.

Analogously, let F_2 be the set of all the complex-valued functions $\mathbf{v}(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}^n$, such that $\mathbf{v}(t,\cdot) \in L^{(n,\infty)}(\mathbb{R}^n)$ and the number $\sup_{t>0} \|\mathbf{v}(t,\cdot)\|_{(n,\infty)}$ is finite.

Lemma 4. *The bilinear operator B is continuous from $F_1 \times F_1$ to F_1 and from $F_1 \times F_1$ to F_2 too. More precisely, there exists a constant $\mathcal{K} > 0$, depending only on n and q, such that for all pair of functions $\mathbf{u}$ and $\mathbf{v}$ in E the following estimates are satisfied:*

$$\begin{aligned} \sup_{t>0} \|B(\mathbf{u},\mathbf{v})(t)\|_{(n,\infty)} \le \frac{\mathcal{K}}{2} & \Big(\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{u}(t)\|_{(q,\infty)} \Big) \\ & \cdot \Big(\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{v}(t)\|_{(q,\infty)} \Big), \end{aligned} \tag{39}$$

and

$$\begin{aligned} \sup_{t>0} t^{(1-n/q)/2} \|B(\mathbf{u},\mathbf{v})(t)\|_{(q,\infty)} \le \frac{\mathcal{K}}{2} & \Big(\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{u}(t)\|_{(q,\infty)} \Big) \\ & \cdot \Big(\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{v}(t)\|_{(q,\infty)} \Big). \end{aligned} \tag{40}$$

We are going to deduce the estimates (39) and (40) on each component of the matrix operator $B(\mathbf{u},\mathbf{v})(t,x)$. When the fields $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are mild-solutions of (2), they must satisfy $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0$. Consequently,

$$\begin{aligned} (\mathbf{u} \cdot \nabla)\mathbf{v} &= \nabla(\mathbf{u} \otimes \mathbf{v}) \\ &= \Big(\sum_{j=1}^n \frac{\partial}{\partial x_j}(u_j v_1), \sum_{j=1}^n \frac{\partial}{\partial x_j}(u_j v_2), \dots, \sum_{j=1}^n \frac{\partial}{\partial x_j}(u_j v_n) \Big). \end{aligned}$$

Then, putting this in the definition (8) of B, and after an integration by parts, it is not difficult to see that each component of B, denoted $b(f,g)(t)$, can be expressed as

$$b(f,g)(t) = c_n \int_0^t (t-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big) * (fg)(s)\, ds\,, \tag{41}$$

being $\Theta(z)$ a smooth function which is $O(|z|^{-n-1})$ when $|z| \to \infty$, and c_n is a constant depending only on n. The symbol $*$ means the convolution in the variable x.

The totality of the Lemma 5 will be obtained from a unique estimate. Let $r \geq n$, frozen for the moment. According to Proposition 1, we infer from (41) that

$$\begin{aligned} &\|b(f,g)(t)\|_{(r,\infty)} \\ &\leq c_n \int_0^t (t-s)^{-(n+1)/2} \Big\|\Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big) * (fg)(s)\Big\|_{(r,\infty)} ds \\ &\leq c(n,r) \int_0^t (t-s)^{-(n+1)/2} \Big\|\Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big)\Big\|_p \|f(s)\, g(s)\|_{(q/2,\infty)}\, ds\,, \end{aligned} \tag{42}$$

with $1/r = 1/p + 2/q - 1$.

Now, due to Proposition 2, we write

$$\|f(s)\, g(s)\|_{(q/2,\infty)} \leq \frac{q}{q-2}\, \|f(s)\|_{(q,\infty)}\, \|g(s)\|_{(q,\infty)}\,, \tag{43}$$

where $q/(q-2)$ is the conjugate index of $q/2$.

Otherwise, after a simple change of variables, we have

$$\Big\|\Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big)\Big\|_p = c(p)\, \|\Theta\|_p\, \big(2\sqrt{t-s}\big)^{n/p}\,. \tag{44}$$

Replacing (43) and (44) in (42), we get

$$\begin{aligned} \|b(f,g)(t)\|_{(r,\infty)} \leq\; & c(n,r,p,q)\, \|\Theta\|_p \\ & \cdot \Big(\int_0^t (t-s)^{-(n+1)/2+n/(2p)}\, s^{-(1-n/q)}\, ds\Big) \\ & \cdot \Big(\sup_{0<t<T_0} t^{(1-n/q)/2}\, \|f(t)\|_{(q,\infty)}\Big) \\ & \cdot \Big(\sup_{0<t<T_0} t^{(1-n/q)/2}\, \|g(t)\|_{(q,\infty)}\Big)\,, \end{aligned} \tag{45}$$

for all $T_0 > t$, including $T_0 = \infty$. Notice that the integral on s in (45) is convergent since the numbers $q > n > 1$ are fixed parameters and the choice of p gives $1/p > (n-1)/n$, and then it is evident that the exponent $-(n+1)/2 + n/(2p) > -1$.

The very well known relation between Beta and Gamma functions yields

$$\int_0^t (t-s)^{-(n+1)/2+n/(2p)}\, s^{-(1-n/q)}\, ds = c(n,r,q)\, t^{-(1-n/r)/2}\,, \tag{46}$$

where

$$c(n,r,q) = \frac{\Gamma\Big(\frac{1}{2}+\frac{n}{2r}-\frac{n}{q}\Big)\,\Gamma\Big(\frac{n}{q}\Big)}{\Gamma\Big(\frac{1}{2}+\frac{n}{2r}\Big)}\,.$$

Case i): Taking $r = n$ and setting (46) in (45), it results

$$\begin{aligned}\|b(f,g)(t)\|_{(n,\infty)} \le c_1(n,q) &\Big(\sup_{0<t<T_0} t^{(1-n/q)/2}\, \|f(t)\|_{(q,\infty)}\Big) \\ &\cdot \Big(\sup_{0<t<T_0} t^{(1-n/q)/2}\, \|g(t)\|_{(q,\infty)}\Big)\,.\end{aligned} \tag{47}$$

In order to obtain (39), we take the supremum over all $t > 0$ in (47) and choose $T_0 = \infty$.

Case ii): Considering $r = q$, and following the same proceeding, we arrive to

$$\begin{aligned}\|b(f,g)(t)\|_{(q,\infty)} \le c_2(n,q)\, t^{-(1-n/q)/2} &\Big(\sup_{0<t<T_0} t^{(1-n/q)/2}\, \|f(t)\|_{(q,\infty)}\Big) \\ &\cdot \Big(\sup_{0<t<T_0} t^{(1-n/q)/2}\, \|g(t)\|_{(q,\infty)}\Big)\,.\end{aligned} \tag{48}$$

In the same way as before we get (40).

The continuity properties of B are direct consequences of the estimates (39) and (40), completing the proof of Lemma 4.

Another property on the continuity of the bilinear operator B is given in the following result.

Lemma 5. *Let* $\mathbf{u}(t,x)$ *and* $\mathbf{v}(t,x)$ *belong to* E. *Then,*

a) *The function* $t \mapsto B(\mathbf{u},\mathbf{v})(t,x)$ *from* $(0,\infty)$ *to* $L^{(n,\infty)}(\mathbb{R}^n)$ *is continuous.*

b) *The function* $t \mapsto B(\mathbf{u},\mathbf{v})(t,x)$ *from* $(0,\infty)$ *to* $L^{(q,\infty)}(\mathbb{R}^n)$ *is continuous.*

c) *The function* $t \mapsto t^{(1-n/q)/2} B(\mathbf{u},\mathbf{v})(t,x)$ *from* $(0,\infty)$ *to* $L^{(q,\infty)}(\mathbb{R}^n)$ *is continuous.*

We are going to show parts a) and b). The third one is a consequence of b).

Let $t > 0$ and h be sufficiently small (take $|h| < t/2$). We suppose $h > 0$. The case $h < 0$ can be analogously treated. We evaluate the difference of B between t and $t+h$. Then we write

$$\begin{aligned}
B(\mathbf{u},\mathbf{v})(t+h,x) &- B(\mathbf{u},\mathbf{v})(t,x)\\
&= \int_0^{t+h} S(t+h-s)\,\nabla(\mathbf{u}(s,x)\otimes\mathbf{v}(s,x))\,ds\\
&\quad - \int_0^{t} S(t-s)\,\nabla(\mathbf{u}(s,x)\otimes\mathbf{v}(s,x))\,ds\\
&= \int_t^{t+h} S(t+h-s)\,\nabla(\mathbf{u}(s,x)\otimes\mathbf{v}(s,x))\,ds\\
&\quad + \int_0^{t} (S(t+h-s)-S(t-s))\,\nabla(\mathbf{u}(s,x)\otimes\mathbf{v}(s,x))\,ds\\
&:= B^{(1)}(h) + B^{(2)}(h)\,.
\end{aligned}$$

We have to see

$$\|B^{(1)}(h)\|_{(n,\infty)} \to 0\,, \qquad \text{when } h \to 0\,, \tag{49}$$

$$\|B^{(1)}(h)\|_{(q,\infty)} \to 0\,, \qquad \text{when } h \to 0\,, \tag{50}$$

$$\|B^{(2)}(h)\|_{(n,\infty)} \to 0\,, \qquad \text{when } h \to 0\,, \tag{51}$$

$$\|B^{(2)}(h)\|_{(q,\infty)} \to 0\,, \qquad \text{when } h \to 0\,. \tag{52}$$

Let $r \geq n$. As in Lemma 4, denoting $b^{(1)}(f,g)(h)$ the entries of $B^{(1)}(h)$ and after applying generalized Young and Hölder inequalities, we arrive to

$$
(53)\qquad
\begin{aligned}
&\|b^{(1)}(f,g)(h)\|_{(r,\infty)}\\
&\leq c(n,r,p,q)\int_t^{t+h}(t+h-s)^{-(n+1)/2+n/(2p)}\,s^{-(1-n/q)}\,ds\\
&\quad\cdot\Big(\sup_{t>0}t^{(1-n/q)/2}\,\|f\|_{(q,\infty)}\Big)\\
&\quad\cdot\Big(\sup_{t>0}t^{(1-n/q)/2}\,\|g\|_{(q,\infty)}\Big),
\end{aligned}
$$

where $1/p = 1/r + 1 - 2/q$. A simple computation (by homogeneity) gives

$$
(54)\qquad
\begin{aligned}
&\int_t^{t+h}(t+h-s)^{-(n+1)/2+n/(2p)}\,s^{-(1-n/q)}\,ds\\
&\qquad\qquad= c_1(n,q,r)\,t^{-(1-n/q)}\,h^{n/(2r)-n/q+1/2},
\end{aligned}
$$

for all $r \geq n$ such that $n/(2r) - n/q + 1/2 > 0$. As it was noticed in the proof of Lemma 4, the last integral is finite. If we take $r = n$, we have from (53) and (54)

$$
\begin{aligned}
\|b^{(1)}(f,g)(h)\|_{(n,\infty)} \leq \tilde{c}(n,q)\,t^{-(1-n/q)}\,h^{1-n/q}
&\cdot\Big(\sup_{t>0}t^{(1-n/q)/2}\,\|f\|_{(q,\infty)}\Big)\\
&\cdot\Big(\sup_{t>0}t^{(1-n/q)/2}\,\|g\|_{(q,\infty)}\Big),
\end{aligned}
$$

and then $\|B^{(1)}(h)\|_{(n,\infty)} \to 0$ with h.

Otherwise, choosing $r = q$ we obtain from (53) and (54)

$$
\begin{aligned}
\|b^{(1)}(f,g)(h)\|_{(q,\infty)} \leq \tilde{\tilde{c}}(n,q)\,t^{-(1-n/q)}\,h^{(1-n/q)/2}
&\cdot\Big(\sup_{t>0}t^{(1-n/q)/2}\,\|f\|_{(q,\infty)}\Big)\\
&\cdot\Big(\sup_{t>0}t^{(1-n/q)/2}\,\|g\|_{(q,\infty)}\Big),
\end{aligned}
$$

yielding $\|B^{(1)}(h)\|_{(q,\infty)} \to 0$ with h. Thus, we have just proved (49) and (50).

In order to see (51) and (52) we take again the entries of the operator $B^{(2)}(h)$ which we denote by $b^{(2)}(f,g)(h)$. Hence, using the same notation as in Lemma 5 we have

$$
\begin{aligned}
b^{(2)}(f,g)(h) = c_n \int_0^t \Big((t+h-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t+h-s}}\Big) \\
- (t-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big)\Big) * (fg)(s)\, ds\,.
\end{aligned}
\tag{55}
$$

Let $r \geq n$. We are interested on $r = n$ and $r = q$. It is easy to show that the expression under the integral sign in (55) has bounded $L^{(r,\infty)}$-norm. Indeed, if $1/r = 1/p + 2/q - 1$, after applying generalized Young and Hölder inequalities we obtain

$$
\begin{aligned}
&\Big\| \Big((t+h-s)^{-(n+1)/2} \Theta\Big(\frac{\cdot}{2\sqrt{t+h-s}}\Big) \\
&\quad - (t-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big)\Big) * (fg)(s) \Big\|_{(r,\infty)} \\
&\qquad \leq c(n,r,q) \Big\| \Big((t+h-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t+h-s}}\Big) \\
&\qquad\qquad - (t-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big)\Big)\Big\|_p \|(fg)(s)\|_{(q/2,\infty)} \\
&\qquad \leq \tilde{c}(n,r,q) \Big\| \Big((t+h-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t+h-s}}\Big) \\
&\qquad\qquad - (t-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big)\Big)\Big\|_p \\
&\qquad\quad \cdot \|f(s)\|_{(q,\infty)}\, \|g(s)\|_{(q,\infty)}\,.
\end{aligned}
\tag{56}
$$

Taking into account (44) and reminding that $h > 0$, we can write

$$
\begin{aligned}
&\Big\| \Big((t+h-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t+h-s}}\Big) \\
&\quad - (t-s)^{-(n+1)/2}\, \Theta\Big(\frac{\cdot}{2\sqrt{t-s}}\Big)\Big) * (fg)(s) \Big\|_{(r,\infty)} \\
&\qquad \leq c'(n,r,q)\Big((t+h-s)^{-(n+1)/2+n/(2p)} + (t-s)^{-(n+1)/2+n/(2p)} \Big) \\
&\qquad\quad \cdot \|\Theta\|_p\, \|f(s)\|_{(q,\infty)}\, \|g(s)\|_{(q,\infty)}
\end{aligned}
$$

$$\leq c''(n,r,q)\,(t-s)^{-(n+1)/2+n/(2p)}\,s^{-(1-n/q)}\,\|\Theta\|_p \cdot \Big(\sup_{t>0} t^{(1-n/q)/2}\,\|f(s)\|_{(q,\infty)}\Big) \cdot \Big(\sup_{t>0} t^{(1-n/q)/2}\,\|g(s)\|_{(q,\infty)}\Big),$$

which, due to $-(n+1)/2+n/(2p)>-1$ once more, belongs to $L^1([0,t],ds)$ and it is independent of h. It remains to prove that the function under the integral sign in (55) tends to zero with h in the $L^{(r,\infty)}$-norm. But from (56) this is evident since the continuity on t of the function $(t-s)^{-(n+1)/2}\,\Theta(\,\cdot\,/2\sqrt{t-s})$ from $(0,\infty)$ to L^p and the behaviour of $\Theta(z)$ for $|z|\to+\infty$. Thus, thanks to the Lebesgue dominated convergence theorem we get $\lim_{h\to 0^+}\|b^{(2)}(f,g)(h)\|_{(r,\infty)}=0$. In particular we have shown (51) and (52). Then, the proof of Lemma 5 is complete.

Lemma 6. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *belong to* E*, then*

$$B(\mathbf{u},\mathbf{v})(t,x)\rightharpoonup 0\,,\qquad \text{when } t\to 0^+\,. \tag{57}$$

As it was noticed in Lemma 4, it is not difficult to see that if $u_j(s)$ and $v_k(s)$ denote respectively the j-th and the k-th components of the vectors $\mathbf{u}(s)$ and $\mathbf{v}(s)$, then $B_k(\mathbf{u},\mathbf{v})(t,x)$, the k-th component of $B(\mathbf{u},\mathbf{v})(t,x)$, is written as

$$B_k(\mathbf{u},\mathbf{v})(t,x)=\sum_{j=1}^{n}\int_0^t Q_{jk}(t-s)\,(u_j(s)v_k(s))(x)\,ds\,, \tag{58}$$

where $Q_{jk}(t-s)$ are pseudodifferential operators with symbols

$$\sigma(Q_{jk}(t-s))(\xi)=i\,\xi_j\left(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\right)e^{-(t-s)|\xi|^2} \tag{59}$$

for $j,k=1,2,\ldots,n$. Here, δ_{jk} denotes the Kronecker's delta function and $i^2=-1$.

More generally, taking into account (58) and (59), we are going to consider a pseudodifferential operator $Q(t-s)$ on $\mathbb{R}^n$ whose symbol has the form $\sigma(Q(t-s))(\xi)=q_1(\xi)q_2(\xi)e^{-(t-s)|\xi|^2}$, where $q_1(\xi)$ and $q_2(\xi)$ are homogeneous functions on $|\xi|>0$ of degree $1/2$. Note that in (59) we can take $q_1(\xi)=\xi_j/|\xi|^{1/2}$ and $q_2(\xi)=i\,(\delta_{jk}\,|\xi|^2-\xi_j\,\xi_k)/|\xi|^{3/2}$.

We define $T^{(1)}(\tau)$ and $T^{(2)}(\tau)$ as the pseudodifferential operators whose symbols are

$$\sigma(T^{(1)}(\tau))(\xi) = q_1(\xi)\, e^{-\tau|\xi|^2/2}$$

and

$$\sigma(T^{(2)}(\tau))(\xi) = q_2(\xi)\, e^{-\tau|\xi|^2/2},$$

and such that $Q(t-s) = T^{(1)}(t-s)\, T^{(2)}(t-s)$.

We consider too two functions f and g satisfying $\|f\|_E < \infty$ and $\|g\|_E < \infty$. In fact we shall only need that $\sup_{t>0} \|f(t)\|_{(n,\infty)} < \infty$ and $\sup_{t>0} \|g(t)\|_{(n,\infty)} < \infty$. Then, we shall prove (57) for the operator $\int_0^t Q(t-s)\,(f(s)g(s))(x)\,ds$.

Let φ be a testing function ($\varphi \in \mathcal{S}$). Hence we have

$$\begin{aligned} \Big|\Big\langle \int_0^t Q(t-s)\,(f(s)g(s))\,ds, \varphi\Big\rangle\Big| &= \Big|\int_0^t \langle Q(t-s)\,(f(s)g(s)), \varphi\rangle\, ds\Big| \\ &\le \int_0^t \big|\langle T^{(1)}(t-s)\, T^{(2)}(t-s)\,(f(s)g(s)), \varphi\rangle\big|\, ds \\ &= \int_0^t \big|\langle T^{(2)}(t-s)\,(f(s)g(s)), T^{(1)}(t-s)\,\varphi\rangle\big|\, ds \\ &\le \int_0^t \big\|T^{(2)}(t-s)\,(f(s)g(s))\big\|_{(n,\infty)} \big\|T^{(1)}(t-s)\,\varphi\big\|_{(n',1)}\, ds\,. \end{aligned} \tag{60}$$

Thanks to Propositions 1 and 2 we get

$$\begin{aligned} &\|T^{(2)}(t-s)\,(f(s)g(s))\|_{(n,\infty)} \\ &\qquad \le c(n)\, \|k_2(t-s)\|_{p_2}\, \|f(s)\|_{(n,\infty)}\, \|g(s)\|_{(n,\infty)} \end{aligned} \tag{61}$$

with $1/n = 1/p_2 + 2/n - 1$ and the function $k_2(t-s) \in \mathcal{S}$ verifies

$$\widehat{k_2(t-s)}(\xi) = q_2(\xi)\, e^{-(t-s)|\xi|^2/2}\,.$$

It is clear that $p_2 = n'$. As usual, we denote by $\widehat{h}(\xi)$ the Fourier transformation of a function h at the point ξ.

Analogously,

$$\big\|T^{(1)}(t-s)\varphi\big\|_{(n',1)} \le c(n,r)\, \|k_1(t-s)\|_{p_1}\, \|\varphi\|_{(r,1)}\,, \tag{62}$$

where $1/n' = 1/p_1 + 1/r - 1$, $p_1 > 1$, $r > 1$, and the function $k_1(t-s) \in \mathcal{S}$ satisfies

$$\widehat{k_1(t-s)}(\xi) = q_1(\xi)\, e^{-(t-s)|\xi|^2/2}\,.$$

Then, for each function $k_j(t-s)$, $j = 1, 2$, we can rapidly see that

$$\begin{aligned}
\|k_j(t-s)\|_{p_j} &= \Big\|(t-s)^{-1/4-n/2}\, \tilde{k}_j\big(\frac{\cdot}{\sqrt{t-s}}\big)\Big\|_{p_j} \\
&= (t-s)^{-1/4-n/2+n/(2p_j)}\, \|\tilde{k}_j\|_{p_j} \\
&= c(n,p_j)\,(t-s)^{-1/4-n/2+n/(2p_j)}\,,
\end{aligned} \tag{63}$$

with $\widehat{\tilde{k}_j}(\xi) = q_j(\xi)\, e^{-|\xi|^2/2}$.

Due to (63) and (62) we obtain

$$\begin{aligned}
\|T^{(1)}(t-s)\varphi\|_{(n',1)} &\le \tilde{c}(n,r)\,(t-s)^{-1/4-n/2+n/(2p_1)}\, \|\varphi\|_{(r,1)} \\
&= \tilde{c}(n,r)\,(t-s)^{-3/4+n/2-n/(2r)}\, \|\varphi\|_{(r,1)}\,.
\end{aligned} \tag{64}$$

Besides, from (63) and (61) we have

$$\begin{aligned}
&\|T^{(2)}(t-s)\,(f(s)g(s))\|_{(n,\infty)} \\
&\qquad \le \tilde{c}(n)\,(t-s)^{-3/4}\, \|f(s)\|_{(n,\infty)}\, \|g(s)\|_{(n,\infty)} \\
&\qquad \le \tilde{c}(n)\,(t-s)^{-3/4}\, \sup_{\tau>0} \|f(\tau)\|_{(n,\infty)} \\
&\qquad\quad \cdot \sup_{\tau>0} \|g(\tau)\|_{(n,\infty)}\,.
\end{aligned} \tag{65}$$

Therefore, replacing (64) and (65) in (60) we get

$$\begin{aligned}
&\Big|\Big\langle \int_0^t Q(s-t)\,(f(s)g(s))\,ds, \varphi\Big\rangle\Big| \\
&\qquad \le \tilde{\tilde{c}}(n,r)\, \|\varphi\|_{(r,1)} \int_0^t (t-s)^{-3/2+(n-n/r)/2}\, ds \\
&\qquad\quad \cdot \sup_{\tau>0} \|f(\tau)\|_{(n,\infty)}\, \sup_{\tau>0} \|g(\tau)\|_{(n,\infty)}\,.
\end{aligned}$$

For instance, choosing $r = 3$, it results that $r > n'$ since $n \ge 2$ and then the s-integral is convergent and it is like $t^{-1/2+(n-n/r)/2}$ which tends to zero with t. Finally we have that

$$\lim_{t\to 0} \Big|\Big\langle \int_0^t Q(s-t)\,(f(s)g(s))\,ds, \varphi\Big\rangle\Big| = 0\,.$$

This proves Lemma 6.

The conclusions of Lemmas 4, 5 and 6 can be briefly summarized in the following terms.

Proposition 5. *The bilinear operator* $B : E \times E \longrightarrow E$ *is continuous.*

Now, we are ready to prove Theorem 1. The method we shall apply takes the approach on succesive approximations (called Picard's method) developed in [K-F], [F-K], [K] and [C]. Picard's sequence is defined as follows.

$$\begin{aligned} \mathbf{v}_1(t,x) &= (S(t)\mathbf{v}_0)(x)\,, \\ \mathbf{v}_{m+1}(t,x) &= \mathbf{v}_1(t,x) + B(\mathbf{v}_m,\mathbf{v}_m)(t,x)\,, \end{aligned} \tag{66}$$

for $m = 1, 2, 3, \ldots$ Most of the estimates given here are the weak versions of those presented in [K]. In [K-F] and [F-K] some techniques based on the fractional powers of the Laplace operator are used. We are not going to employ them here.

Previously to explain the details of the proof, we need the following abstract result extracted from [C].

Lemma 7 ([C]). *Let* X *be a Banach space with norm* $\|\cdot\|_X$, *and* $B : X\times X \longrightarrow X$ *a continuous bilinear map. That is, there exists a constant* $\mathcal{K} > 0$ *such that for all* x_1 *and* x_2 *in* X *we have*

$$\|B(x_1,x_2)\|_X \le \mathcal{K}\,\|x_1\|_X\,\|x_2\|_X\,.$$

Then, for any vector $y \in X$, $y \neq 0$, *such that* $4\mathcal{K}\,\|y\|_X < 1$, *there exists a solution* $x \in X$ *for the equation* $x = y + B(x,x)$. *Moreover, this solution* x *satisfies* $\|x\|_X \le 2\,\|y\|_X$.

This Lemma will be proved in the same way as in [C]. Let

$$R = \frac{1 - \sqrt{1 - 4\mathcal{K}\,\|y\|_X}}{2\mathcal{K}}\,,$$

which is the smallest solution of the equation $\|y\|_X + \mathcal{K}\,R^2 = R$. We can easily observe that $R \le 2\,\|y\|_X$.

In X, we consider the closed ball $B_R = \{x \in X : \|x\|_X \le R\}$. Let us define $F : X \longrightarrow X$ by $F(x) = y + B(x,x)$. First, we note that for all $x \in B_R$,

$$\|F(x)\|_X \le \|y\|_X + \mathcal{K}\,\|x\|_X^2 \le \|y\|_X + \mathcal{K}\,R^2 = R\,.$$

That is, F maps B_R into B_R. Besides, for all x and x' in B_R we have

$$\begin{aligned} \|F(x) - F(x')\|_X &= \|B(x,x) - B(x',x')\|_X \\ &\leq \|B(x-x',x)\|_X + \|B(x',x-x')\|_X \\ &\leq 2\mathcal{K}R\,\|x-x'\|_X\,. \end{aligned}$$

From the definition of R, it becomes

$$0 < 2\mathcal{K}R = 1 - \sqrt{1 - 4\mathcal{K}\,\|y\|_X} < 1\,.$$

Hence, $\|F(x) - F(x')\|_X \leq c\,\|x - x'\|_X$, with $0 < c < 1$. Thus, the map $F : B_R \longrightarrow B_R$ is a contraction. Consequently, from the Picard contractions Theorem applied to the sequence

$$\begin{aligned} x_1 &= y\,,, \\ x_{m+1} &= F(x_m)\,, \qquad \text{for } m = 1, 2, \dots, \end{aligned}$$

we obtain a unique solution x in B_R, but perhaps not unique in X. The last estimates is trivial, since as it was observed

$$\|x\|_X \leq R \leq 2\,\|y\|_X\,. \tag{67}$$

PROOF OF THEOREM 1. For the existence of mild-solutions of problem (2) we want to apply Lemma 7 to the integral equation (6) with the Banach space $X = E$ and the vector $y = S(t)\mathbf{v}_0$. This fact leads us to check the condition

$$4\mathcal{K}\,\|S(t)\mathbf{v}_0\|_E < 1\,, \tag{68}$$

being $\mathcal{K}$ the constant given in Lemma 4. Besides, from Proposition 4 there exists a constant $c > 0$ such that

$$\|S(t)\mathbf{v}_0\|_E \leq c\,\|\mathbf{v}_0\|_{(n,\infty)}\,. \tag{69}$$

Choosing $0 < \delta < (4\mathcal{K}c)^{-1}$ and making use of (69), inequality (68) is satisfied provided that $\|\mathbf{v}_0\|_{(n,\infty)} < \delta$. Thus, Lemma 7 guarantees the existence of a global mild-solution $\mathbf{v}(t,x)$ in E. Moreover, we know from (67) that $\|\mathbf{v}\|_E \leq 2\,\|S(t)\mathbf{v}_0\|_E \leq 2c\delta$, which goes to 0 with δ.

Now, we shall pass to consider the uniqueness. Let us suppose that $\mathbf{u}$ and $\mathbf{v}$ are two mild-solutions of the Navier-Stokes equations in E with the same initial data $\mathbf{v}_0$, and such that there exists $0 < \eta_0 < 1/\mathcal{K}$ for which the following inequalities

$$\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{u}(t)\|_{(q,\infty)} < \eta_0$$

and

$$\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{v}(t)\|_{(q,\infty)} < \eta_0$$

hold. Thanks to the estimate (48) of the proof of Lemma 4, we note that the difference

$$\begin{aligned}\mathbf{w}(t,x) &:= \mathbf{u}(t,x) - \mathbf{v}(t,x)\\ &= B(\mathbf{u},\mathbf{u})(t,x) - B(\mathbf{v},\mathbf{v})(t,x)\\ &= B(\mathbf{w},\mathbf{u})(t,x) - B(\mathbf{v},\mathbf{w})(t,x)\end{aligned}$$

satisfies (with $T_0 = \infty$)

$$\begin{aligned}&\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{w}(t)\|_{(q,\infty)}\\ &\quad \le \frac{\mathcal{K}}{2}\Big(\sup_{t>0} t^{(1-n/q)/2} \|\mathbf{u}(t)\|_{(q,\infty)} + \sup_{t>0} t^{(1-n/q)/2} \|\mathbf{v}(t)\|_{(q,\infty)}\Big)\\ &\qquad \cdot \sup_{t>0} t^{(1-n/q)/2} \|\mathbf{w}(t)\|_{(q,\infty)}\\ &\quad < \mathcal{K}\,\eta_0 \sup_{t>0} t^{(1-n/q)/2} \|\mathbf{w}(t)\|_{(q,\infty)}\\ &\quad < \sup_{t>0} t^{(1-n/q)/2} \|\mathbf{w}(t)\|_{(q,\infty)} ,\end{aligned}$$

since for $\eta_0 > 0$ it was supposed $\mathcal{K}\eta_0 < 1$. This fact implies that $\mathbf{w}(t) = 0$ for all $t > 0$, and then, $\mathbf{v}(t) = \mathbf{u}(t)$. This concludes the proof.

PROOF OF THEOREM 2. From Theorem 1 we know that the system (2) with initial data $\mathbf{v}_0(x)$ admits, at least, a global mild-solution $\mathbf{v}(t,x) \in E$. In fact, this solution was founded as the limit of the sequence $\{\mathbf{v}_m(t,x)\}$ defined by (66). It is straightforward to verify that $\mathbf{v}_1(t,x) = (S(t)\mathbf{v}_0)(x)$ satisfies $\lambda\,\mathbf{v}_1(\lambda^2 t, \lambda x) = \mathbf{v}_1(t,x)$ by a change of variables. Also, it is evident to see by induction on m that all the functions $\mathbf{v}_m(t,x)$ have this property. Then, it is clear that the limit $\mathbf{v}(t,x)$ must verify

$$\lambda\,\mathbf{v}(\lambda^2 t, \lambda x) = \mathbf{v}(t,x)$$

for all $\lambda > 0$, all $t > 0$ and all $x \in \mathbb{R}^n$. Theorem 2 is proved.

PROOF OF THEOREM 3. Theorem 3 follows directly from Theorem 2 and the following characterization.

Lemma 8 (A characterization of homogeneous functions). *Let $0 < d < n$ and let f be a complex-valued function defined on $\mathbb{R}^n$. Suppose that f is homogeneous of degree $-d$. The function $f \in L^{(n/d,\infty)}(\mathbb{R}^n)$ if and only if its restriction $f|_{S^{n-1}}$ to the sphere S^{n-1} of $\mathbb{R}^n$ belongs to $L^{n/d}(S^{n-1})$.*

The key of this Lemma is centered in the following computation. Let $s > 0$. Recall that m denotes the Lebesgue measure in $\mathbb{R}^n$.

Therefore, taking polar coordinates and reminding the homogeneity assumption on f, we can write

$$\begin{aligned} m\{x \in \mathbb{R}^n : \ |f(x)| > s\} &= \int_{|f(x)|>s} 1\, dx \\ &= \int_0^\infty \int_{\{\xi \in S^{n-1} : |f(\xi)| > r^d s\}} r^{n-1}\, d\sigma_\xi\, dr\,. \end{aligned}$$

Here, $d\sigma_\xi$ denotes the differential area over the sphere. Applying now Fubini's Theorem we conclude

$$\begin{aligned} m\{x \in \mathbb{R}^n : \ |f(x)| > s\} &= \int_{S^{n-1}} \int_0^{(|f(\xi)|/s)^{1/d}} r^{n-1}\, dr\, d\sigma_\xi \\ &= \int_{S^{n-1}} \frac{1}{n} \frac{|f(\xi)|^{n/d}}{s^{n/d}}\, d\sigma_\xi\,. \end{aligned}$$

Finally, we get

$$\begin{aligned} \|f\|^*_{(n/d,\infty)} &= \sup_{s>0} s\, m\{x \in \mathbb{R}^n : \ |f(x)| > s\}^{d/n} \\ &= \frac{1}{n^{d/n}} \|f|_{S^{n-1}}\|_{L^{n/d}(S^{n-1})}\,. \end{aligned}$$

Then, one of both norms is finite if and only if it is finite the other. It is necessary to take into account the "equivalence" (19) and the hypothesis $0 < d < n$ to complete the proof of Lemma 8.

REMARK. It is easy to observe after Fubini's Theorem and a standard change of variables that there is not any homogeneous function defined on $\mathbb{R}^n$ belonging to $L^p(\mathbb{R}^n)$, for any $p > 0$. This fact leads us to try to find " adequate" extensions of $L^p(\mathbb{R}^n)$ as , for instance, the $L^{(p,\infty)}(\mathbb{R}^n)$ spaces.

FINAL REMARK. The results shown throughout the present paper can be obtained as a consequence of the general theory exposed by M. Cannone in [C], for the case of spatial dimension $n = 3$. It is not true that the Lorentz space $L^{(3,\infty)}(\mathbb{R}^3)$ is a subset of the Besov space $\dot{B}_3^{0,\infty}$ but, on the other hand, we have that

$$L^{(3,\infty)}(\mathbb{R}^3) \subset \dot{B}_q^{-\alpha,\infty}\,, \tag{70}$$

for $q > 3$ and $\alpha = 1 - 3/q$. So, the Besov space $\dot{B}_q^{-\alpha,\infty}$ can be useful as the auxiliary space to build the artificial norm in Kato's theory (see, for instance, [F-K], [K-F], [K]).

From [C] we learn that the bilinear operator B defined by (8) is continuous from $\dot{B}_q^{-\alpha,\infty} \times \dot{B}_q^{-\alpha,\infty}$ to $L^3(\mathbb{R}^3)$. Therefore, taking into account (70), if the initial condition v_0 satisfies $\|v_0\|_{(3,\infty)} < \alpha$, for some $\alpha > 0$ small enough, we have $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < c\,\alpha$, and then, thanks to Cannone's results [C] once more, we know that the solution

$$\mathbf{v}(t,x) = S(t)\mathbf{v}_0(x) + B(\mathbf{v},\mathbf{v})(t,x)\,,$$

where $\|B(\mathbf{v},\mathbf{v})(t)\|_3 \le c_1$. On the other hand $L^{(3,\infty)}$ is a translation invariant space which yields $S(t)\mathbf{v}_0 \in L^{(3,\infty)}$ uniformly on t.

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