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Goldbach numbers represented by polynomials

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1. Introduction.

Let N be a large positive real number. It is well known that almost all even integers in the interval $[N, 2N]$ are Goldbach numbers, *i.e.* a sum of two primes. The same result also holds for short intervals of the form $[N, N+H]$, see Mikawa [4], Perelli-Pintz [7] and Kaczorowski-Perelli-Pintz [3] for the choice of admissible values of H and the size of the exceptional set in several problems in this direction.

One may ask if similar results hold for thinner sequences of integers in $[N, 2N]$, of cardinality smaller than the upper bound for the exceptional set in the above problems. In this paper we deal with the polynomial case.

Let $L = \log N$, $F \in \mathbb{Z}[x]$ with $\deg F = k \geq 1$ and with positive leading coefficient,

$$R(n) = \sum_{r+s=n} \Lambda(r)\, \Lambda(s)$$

and

$$\mathfrak{S}(n) = \prod_{p \nmid n} \Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{p \mid n} \Big(1 + \frac{1}{p-1}\Big) .$$

Our main result is the following

Theorem 1. *Let $k \in \mathbb{N}$, $A, \varepsilon > 0$ and $N^{1/(3k)+\varepsilon} \le H \le N^{1/k}$. Then*

$$\sum_{N^{1/k} \le n \le N^{1/k}+H} |R(F(n)) - F(n)\,\mathfrak{S}(F(n))|^2 \ll_{A,\varepsilon,F} H\,N^2\,L^{-A}.$$

For $d \in \mathbb{N}$ let

$$\varrho_F(d) = |\{m \,(\mathrm{mod}\ d) : \ F(m) \equiv 0 \,(\mathrm{mod}\ d)\}|.$$

If $\varrho_F(2) \neq 0$ then, writing

$$A_F(N,H) = \{N^{1/k} \le n \le N^{1/k} + H : \ F(n) \equiv 0 \,(\mathrm{mod}\ 2)\},$$

by standard techniques we have that

$$|A_F(N,H)| = \frac{\varrho_F(2)}{2}\,H + O(1).$$

Hence from Theorem 1 we easily obtain the following

Corollary 1. *Let $k \in \mathbb{N}$, $A, \varepsilon > 0$ and $N^{1/(3k)+\varepsilon} \le H \le N^{1/k}$. Then*

$$R(F(n)) = F(n)\,\mathfrak{S}(F(n)) + O_{A,\varepsilon,F}(N\,L^{-A}),$$

for all $n \in [N^{1/k}, N^{1/k} + H]$ but $O_{A,\varepsilon,F}(HL^{-A})$ exceptions. In particular, if $\varrho_F(2) \neq 0$, for all $n \in A_F(N,H)$ but $O_{A,\varepsilon,F}(HL^{-A})$ exceptions, $F(n)$ is a Goldbach number.

Since k can be chosen arbitrarily large, the above results provide examples of thin sequences in $[N, 2N]$ of cardinality $O(N^{\delta})$ with $\delta > 0$ arbitrarily small, having the property that almost all their elements are Goldbach numbers.

No attempt is made here to obtain results which are uniform in the coefficients of F. This problem would be of some interest, expecially in the case $k = 1$.

Theorem 1 is obtained by an extension of the techniques used in [7] and hence we shall refer to [7] at several places in the proof, to avoid merely repeating the arguments there. We also note that the technique in [7] of localizing the primes involved can be used in this paper too. This implies that the second statement of Corollary 1 remains valid for the shorter interval with $N^{7/(36k)+\varepsilon} \leq H \leq N^{1/k}$. Moreover, we can get stronger results under the assumption of the Generalized Riemann Hypothesis, by techniques similar to those in [3]. In particular we can obtain rather good uniformity in the coefficients in the case $k = 1$.

Theorem 1 deals with a short interval mean square estimate of the error term for the number of Goldbach representations of $F(n)$. By similar, but simpler, techniques we can also obtain the asymptotic formula for the average of $R(F(n))$ over shorter intervals. Writing $F(x) = a_k x^k + \cdots + a_0$ with $a_k > 0$,

$$C(F) = a_k \frac{\varrho_F(2)}{2} \mathfrak{S} \prod_{p>2} \left(1 + \frac{\varrho_F(p)}{p(p-2)}\right)$$

and

$$\mathfrak{S} = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

we have

Theorem 2. *Let $k \in \mathbb{N}$, $A, \varepsilon > 0$ and $N^{1/(6k)+\varepsilon} \leq H \leq N^{1/k-\varepsilon}$. Then*

$$\sum_{N^{1/k} \leq n \leq N^{1/k}+H} R(F(n)) = C(F)\, H\, N + O_{A,\varepsilon,F}(H\, N\, L^{-A}).$$

An easy consequence of Theorem 2 is

Corollary 2. *Let $k \in \mathbb{N}$, $\varepsilon > 0$, $N^{1/(6k)+\varepsilon} \leq H \leq N^{1/k-\varepsilon}$ and $\varrho_F(2) \neq 0$. Then there exists $n \in [N^{1/k}, N^{1/k} + H]$ such that $F(n)$ is a Goldbach number.*

The above remarks concerning uniformity in the coefficients, localization of the prime summands and conditional results also apply, in an appropriate form, to Theorem 2 and Corollary 2. We finally note that the N^{ε} in the above results may be replaced by a suitable power of L.

We wish to thank the referee for having pointed out several inaccuracies in the paper.

2. Proof of Theorem 1.

We first note that we may assume that $H = N^{1/(3k)+\varepsilon}$, $\varepsilon > 0$ is sufficiently small, $A > 0$ is a sufficiently large and $N \geq N_0(A,\varepsilon)$, a large constant.

Let $P = L^B$, where $B > 0$ is a suitable constant which will be chosen later on in terms of A and F, $\overline{Q} = H^k L^{-B/4}$ and $Q = \overline{Q}^{1/2}/2$. Denote by $\mathfrak{M}(q,a)$ and $\overline{\mathfrak{M}}(q,a)$ the Farey arc with centre at a/q of the Farey dissections of order Q and $\overline{Q}$ respectively, and let

$$\mathfrak{M}'(q,a) = \left\{\alpha : \left|\alpha - \frac{a}{q}\right| \leq \frac{L^B}{N}\right\},$$

$$\mathfrak{M} = \bigcup_{q\leq P} \bigcup_{a=1}^{q}{}^{*} \mathfrak{M}'(q,a), \qquad \overline{\mathfrak{M}} = \bigcup_{q\leq L^{B/4}} \bigcup_{a=1}^{q}{}^{*} \overline{\mathfrak{M}}(q,a),$$

$$\mathfrak{m} = [0,1] \setminus \mathfrak{M} (\bmod 1) \qquad \text{and} \qquad \overline{\mathfrak{m}} = [0,1] \setminus \overline{\mathfrak{M}} (\bmod 1),$$

where * means that $(a,q) = 1$.

Writing

$$S(\alpha) = \sum_{n\leq c_1 N} \Lambda(n)\, e(n\alpha)$$

and

$$K(F,\alpha) = \sum_{N^{1/k}\leq n\leq N^{1/k}+H} e(F(n)\,\alpha),$$

where $c_1 = c_1(F) > 0$ is a suitable constant, we have

$$\begin{aligned} &\sum_{N^{1/k}\leq n\leq N^{1/k}+H} |R(F(n)) - F(n)\,\mathfrak{S}(F(n))|^2 \\ &\ll \sum_{N^{1/k}\leq n\leq N^{1/k}+H} \left|\int_{\mathfrak{M}} S(\alpha)^2\, e(-F(n)\,\alpha)\, d\alpha - F(n)\,\mathfrak{S}(F(n))\right|^2 \\ &\qquad + \sum_{N^{1/k}\leq n\leq N^{1/k}+H} \left|\int_{\mathfrak{m}} S(\alpha)^2\, e(-F(n)\,\alpha)\, d\alpha\right|^2 \\ &:= \sum_{\mathfrak{M}} + \sum_{\mathfrak{m}}, \end{aligned} \tag{1}$$

say.

The quantity $\sum_{\mathfrak{M}}$ can be estimated by standard methods. Using the arguments of Vaughan [10, Chapter 3] we obtain that

$$\begin{aligned}(2)\qquad &\int_{\mathfrak{M}} S(\alpha)^2\, e(-F(n)\,\alpha)\, d\alpha - F(n)\,\mathfrak{S}(F(n)) \\ &\ll N\Big|\sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2}\, c_q(-F(n))\Big| + N\,L^{-B+1} + N\,L^{-A/2}\,,\end{aligned}$$

where $c_q(m)$ is the Ramanujan sum. By the well known formula

$$c_q(m) = \varphi(q)\, \frac{\mu(q/(q,m))}{\varphi(q/(q,m))}$$

we have that

$$\begin{aligned}(3)\qquad \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2}\, c_q(-F(n)) &\ll \sum_{q>P} \varphi(q)^{-1}\, \varphi\Big(\frac{q}{(q,F(n))}\Big)^{-1} \\ &\ll \sum_{d|F(n)} \varphi(d)^{-1} \sum_{r>P/d} \varphi(r)^{-2} \\ &\ll P^{-1} \sum_{d|F(n)} \frac{d}{\varphi(d)} \\ &\ll \frac{F(n)\,\tau(F(n))}{P\,\varphi(F(n))}\,,\end{aligned}$$

where τ is the divisor function. Hence from (2), (3) and the Theorem of Nair [6] we get

$$(4)\qquad \sum_{\mathfrak{M}} \ll H\,N^2\,L^{-2B+c_2} + H\,N^2\,L^{-A}\,,$$

where $c_2 = c_2(F) > 0$ is suitable constant.

From Parseval's identity we have that

$$\begin{aligned}(5)\qquad \sum_{\mathfrak{m}} &= \int_{\mathfrak{m}} S(\xi)^2 \int_{\mathfrak{m}} \overline{S(\alpha)}^2\, K(F,\alpha-\xi)\, d\alpha\, d\xi \\ &\ll N\,L \sup_{\xi\in\mathfrak{m}} \int_{\mathfrak{m}} |S(\alpha)|^2\, |K(F,\alpha-\xi)|\, d\alpha\,.\end{aligned}$$

We need the following slight variant of Weyl's inequality.

Lemma. *Let $|\alpha - a/q| \le 1/q^2$ and $(a,q)=1$. Then for any $D>0$ we have that*

$$K(F,\alpha) \ll_{F,D} H\left(\frac{1}{q}+\frac{1}{H}+\frac{q}{H^k}+L^{-2D+k^2-2}\right)^{1/K} L^{(D+1)/K},$$

where $K = 2^{k-1}$.

PROOF. Arguing as in Lemma 2.4 of [10] we get

$$|K(F,\alpha)|^K \ll H^{K-k}\Big(H^{k-1} + \sum_{h=1}^{c_3 H^{k-1}} \tau_k(h)\Big|\sum_{n\in I} e(n\alpha h)\Big|\Big), \tag{6}$$

where τ_k is the k-th divisor function, $I \subset [N^{1/k}, N^{1/k}+H]$ is a suitable interval and $c_3 = c_3(F) > 0$ is a suitable constant. The contribution of the h with $\tau_k(h) > L^D$ is

$$\ll H^{K-k+1}L^{-D}\sum_{h=1}^{c_3 H^{k-1}} \tau_k(h)^2 \ll H^K L^{-D+k^2-1}, \tag{7}$$

by the well known inequality

$$\sum_{n\le x} \tau_k(n)^2 \le x(\log ex)^{k^2-1},$$

see, *e.g.*, [10, p. 120]. The contribution of the h with $\tau_k(h) \le L^D$ is, by [10, Lemma 2.2],

$$\ll H^K\Big(\frac{1}{q}+\frac{1}{H}+\frac{q}{H^k}\Big)L^{D+1}, \tag{8}$$

and the Lemma follows from (6)-(8).

If $\alpha - \xi \in \overline{\mathfrak{m}}$, choosing $D = B/8 + k^2/2 - 1$, from the Lemma we have that

$$K(F,\alpha-\xi) \ll HL^{-(B-4k^2)/(8K)}, \tag{9}$$

hence from (5) and (9) we get

$$\sum_{\mathfrak{m}} \ll HNL \sup_{\xi\in\mathfrak{m}} \int_{\mathfrak{m}\cap(\xi+\overline{\mathfrak{M}})} |S(\alpha)|^2\, d\alpha + HN^2L^{-(B-4k^2-16K)/(8K)}, \tag{10}$$

where $\xi + \overline{\mathfrak{M}}$ denotes the set $\overline{\mathfrak{M}}$ shifted by ξ.

Since $\overline{\mathfrak{M}}$ is the union of at most $L^{B/2}$ Farey arcs, from (10) we have that

$$\sum_{\mathfrak{m}} \ll HNL^{B/2+1} \sup_{\xi\in\mathfrak{m}} \max_{\substack{\overline{q}\le L^{B/4} \\ (\overline{a},\overline{q})=1}} \int_{\mathfrak{m}\cap(\xi+\overline{\mathfrak{M}}(\overline{q},\overline{a}))} |S(\alpha)|^2\, d\alpha + HN^2L^{-(B-4k^2-16K)/(8K)}\,. \tag{11}$$

From the definition of $\overline{\mathfrak{M}}(\overline{q},\overline{a})$ we have

$$\max_{\substack{\overline{q}\le L^{B/4} \\ (\overline{a},\overline{q})=1}} |\overline{\mathfrak{M}}(\overline{q},\overline{a})| \le \frac{1}{\overline{Q}}\,. \tag{12}$$

Since for $a/q \ne a'/q'$ and $q, q' \le Q$ we have

$$\left|\frac{a}{q} - \frac{a'}{q'}\right| \ge \frac{1}{Q^2} = \frac{4}{\overline{Q}}\,, \tag{13}$$

from (12) and (13) we see that there are at most two punctured arcs $\mathfrak{M}''(q,a)$, where

$$\mathfrak{M}''(q,a) = \begin{cases} \mathfrak{M}(q,a) \setminus \mathfrak{M}'(q,a)\,, & \text{if } q \le P\,, \\ \mathfrak{M}(q,a)\,, & \text{if } P < q \le Q\,, \end{cases}$$

with $q \le Q$ and $(a,q) = 1$, which intersect any of the $\xi + \overline{\mathfrak{M}}(\overline{q},\overline{a})$. Hence

$$\sup_{\xi\in\mathfrak{m}} \max_{\substack{\overline{q}\le L^{B/4} \\ (\overline{a},\overline{q})=1}} \int_{\mathfrak{m}\cap(\xi+\overline{\mathfrak{M}}(\overline{q},\overline{a}))} |S(\alpha)|^2\, d\alpha \ll \max_{\substack{q\le Q \\ (a,q)=1}} \int_{\mathfrak{M}''(q,a)} |S(\alpha)|^2\, d\alpha\,. \tag{14}$$

Writing

$$S\Big(\frac{a}{q} + \eta\Big) = \frac{\mu(q)}{\varphi(q)}\, T(\eta) + R(\eta, q, a)\,, \tag{15}$$

where

$$T(\eta) = \sum_{n\le c_1N} e(n\eta)\,,$$

$$R(\eta,q,a)=\frac{1}{\varphi(q)}\sum_{\chi}\chi(a)\,\tau(\overline{\chi})\,W(\chi,\eta)+O(L^2)\,,$$
$$W(\chi,\eta)=\sum_{n\le c_1N}\Lambda(n)\,\chi(n)\,e(n\eta)-\delta_\chi\,T(\eta)\,,$$
$$\delta_\chi=\begin{cases}1\,, & \text{if } \chi=\chi_0\,,\\ 0\,, & \text{if } \chi\neq\chi_0\,,\end{cases}$$

and $\tau(\chi)$ is the Gauss sum, we have that

$$\int_{\mathfrak{M}''(q,a)}|S(\alpha)|^2\,d\alpha\ll\frac{1}{\varphi(q)^2}\int_{\xi(q)}|T(\eta)|^2\,d\eta+\int_{-1/(qQ)}^{1/(qQ)}|R(\eta,q,a)|^2\,d\eta\,,\tag{16}$$

where

$$\xi(q)=\begin{cases}\left(\dfrac{L^B}{N},\dfrac{1}{2}\right), & \text{if } q\le L^B\,,\\ \left(-\dfrac{1}{qQ},\dfrac{1}{qQ}\right), & \text{if } L^B<q\le Q\,.\end{cases}$$

Since $T(\eta)\ll\min\{N,1/\|\eta\|\}$, it is easy to see that

$$\frac{1}{\varphi(q)^2}\int_{\xi(q)}|T(\eta)|^2\,d\eta\ll NL^{-B}\,.\tag{17}$$

In order to estimate the second integral in the right hand side of (16) we proceed as in [7, Section 5]. We call a character χ good if $L(s,\chi)$ has no zeros in the rectangle

$$1-\frac{10\,(B/\varepsilon)\log L}{L}\le\sigma\le1\,,\qquad |t|\le N\,,\tag{18}$$

and bad otherwise. By the zero-free region of the Dirichlet L-functions, see Prachar [8, Chapter 8], and Siegel's theorem we have that $L(s,\chi)\neq 0$ in the region

$$\sigma>1-\frac{c(\varepsilon')}{\max\{q^{\varepsilon'},\log^{4/5}(|t|+1)\}}\,,\tag{19}$$

where $\varepsilon'>0$ is arbitrary. Hence the existence of a bad character implies that

$$q\gg_{\varepsilon'}L^{1/(2\varepsilon')}\,.\tag{20}$$

The density estimate

$$\sum_{\chi (\bmod q)} N(\sigma, T, \chi) \ll (qT)^{12(1-\sigma)/5} \log^{c_4} qT \,, \tag{21}$$

where $c_4 > 0$ is a suitable constant, see Huxley [2] and Ramachandra [9], implies that the number of bad characters for any modulus $q \leq Q$ is

$$\ll L^{25\, B/\varepsilon} . \tag{22}$$

Hence from (22) and the estimate $\tau(\chi) \ll q^{1/2}$ we have that

$$\begin{aligned} &\int_{-1/(qQ)}^{1/(qQ)} |R(\eta, q, a)|^2 \, d\eta \\ &\ll \frac{q}{\varphi(q)^2} L^{50\, B/\varepsilon} \max_{\chi \text{ bad}} \int_{-1/(qQ)}^{1/(qQ)} |W(\chi, \eta)|^2 \, d\eta \\ &\quad + \frac{q}{\varphi(q)} \sum_{\chi \text{ good}} \int_{-1/(qQ)}^{1/(qQ)} |W(\chi, \eta)|^2 \, d\eta + \frac{L^4}{qQ} . \end{aligned} \tag{23}$$

Choosing $\varepsilon' = \varepsilon/(200\, B)$, from (23) and the Parseval identity we get

$$\begin{aligned} &\int_{-1/(qQ)}^{1/(qQ)} |R(\eta, q, a)|^2 \, d\eta \\ &\ll \frac{q}{\varphi(q)} \sum_{\chi \text{ good}} \int_{-1/(qQ)}^{1/(qQ)} |W(\chi, \eta)|^2 \, d\eta + NL^{-B} . \end{aligned} \tag{24}$$

Now we argue as for [7, (21)-(26)], thus getting from (21) and (24) that

$$\begin{aligned} &\int_{-1/(qQ)}^{1/(qQ)} |R(\eta, q, a)|^2 \, d\eta \\ &\ll L^{c_5} \sup_{1/2 \leq \sigma \leq 1 - \frac{10 \frac{B}{\varepsilon} \log L}{L}} N^{2\sigma - 1} N^{12(5/6 - k\varepsilon/2)(1-\sigma)/5} + N L^{-B} \\ &\ll NL^{-B} , \end{aligned} \tag{25}$$

where $c_5 > 0$ is a suitable constant.

Theorem 1 follows now from (1), (4), (11), (14)-(17) and (25), choosing $B = B(A, F) > 0$ suitably large.

3. Proof of Theorem 2.

We give only a brief sketch of the proof of Theorem 2, since the method is a simpler version of the one we have already used in Theorem 1.

Choose $P = L^B, Q = H^k L^{-B}$ and let B, $\mathfrak{M}(q,a)$, $S(\alpha)$ and $K(F,\alpha)$ be as defined at the beginning of the proof of Theorem 1. Write

$$\mathfrak{M} = \bigcup_{q \le P} \bigcup_{a=1}^{q}{}^{*} \mathfrak{M}(q,a) \qquad \text{and} \qquad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M}.$$

Then, by the Parseval identity, from the Lemma with $D = (B+k^2)/2-1$ we obtain that

$$\begin{aligned} &\sum_{N^{1/k} \le n \le N^{1/k}+H} R(F(n)) \\ &= \int_{\mathfrak{M}} S(\alpha)^2 K(F,-\alpha)\, d\alpha + O(NL \sup_{\alpha\in\mathfrak{m}} |K(F,\alpha)|) \\ &= \int_{\mathfrak{M}} S(\alpha)^2 K(F,-\alpha)\, d\alpha + O(HNL^{-(B-k^2-2K)/(2K)}), \end{aligned} \tag{26}$$

where $K = 2^{k-1}$.

By (15), the Cauchy-Schwarz inequality and the estimate $T(\eta) \ll \min\{N, 1/|\eta|\}$ for $|\eta| \le 1/2$ we have that

$$\begin{aligned} &\int_{\mathfrak{M}} S(\alpha)^2 K(F,-\alpha)\, d\alpha \\ &= \sum_{N^{1/k} \le n \le N^{1/k}+H} F(n) \sum_{q \le P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-F(n)) \\ &\qquad + O(H(\textstyle\sum(P) + (NL\sum(P))^{1/2})) + O(HNL^{-A}), \end{aligned} \tag{27}$$

where

$$\sum(P) = \sum_{q \le P} \sum_{a=1}^{q}{}^{*} \int_{-1/(qQ)}^{1/(qQ)} |R(\eta,q,a)|^2\, d\eta .$$

Arguing as for (3) and (4) we obtain that

$$\begin{aligned}\sum_{N^{1/k}\le n\le N^{1/k}+H} F(n)\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\,c_q(-F(n))\\ = \sum_{N^{1/k}\le n\le N^{1/k}+H} F(n)\,\mathfrak{S}(F(n)) + O(HNL^{-B+c_6}),\end{aligned} \tag{28}$$

where $c_6 = c_6(F) > 0$ is a suitable constant.

Since $H \le N^{1/k-\varepsilon}$ and $\mathfrak{S}(F(n)) \ll L$, we have that

$$\sum_{N^{1/k}\le n\le N^{1/k}+H} F(n)\,\mathfrak{S}(F(n)) = a_k N \sum(\mathfrak{S}) + O(HNL^{-A}), \tag{29}$$

where

$$\sum(\mathfrak{S}) = \sum_{N^{1/k}\le n\le N^{1/k}+H} \mathfrak{S}(F(n)),$$

hence from (26)-(29) we get

$$\begin{aligned}\sum_{N^{1/k}\le n\le N^{1/k}+H} R(F(n)) = a_k N \sum(\mathfrak{S})\\ + O(H(\textstyle\sum(P) + (NL\sum(P))^{1/2}))\\ + O(HNL^{-A})\end{aligned} \tag{30}$$

provided $B > 0$ is sufficiently large in terms of A and F.

By the orthogonality of the characters we have that

$$\sum(P) \ll \sum_{q\le P}\frac{q}{\varphi(q)}\sum_{\chi}\int_{-1/(qQ)}^{1/(qQ)} |W(\chi,\eta)|^2\,d\eta + \frac{L^4}{Q}\,. \tag{31}$$

Now we proceed as at the end of the proof of Theorem 1. Since in (29) we have $q \le L^B$, we can use the zero-free region (19) and the density estimate (21) in order to bound the sum over χ, and then we sum trivially over q. In this way we obtain that

$$\sum(P) \ll NL^{-2A-1}, \tag{32}$$

provided $Q \ge N^{1/6+\varepsilon}$, which is satisfied by our choice of H.

In order to treat $\sum(\mathfrak{S})$ we define

$$f(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } 2 \mid n, \\ \mu(n)^2 \prod_{p|n} \left(\frac{1}{p-2}\right), & \text{if } 2 \nmid n, \end{cases}$$

hence f is multiplicative and for $2|F(n)$

$$\mathfrak{S}(F(n)) = \mathfrak{S} \sum_{d|F(n)} f(d).$$

Choosing $c_7 = c_7(F) > 0$ a suitable constant, we thus obtain that

$$\begin{aligned} \sum(\mathfrak{S}) &= \mathfrak{S} \sum_{\substack{N^{1/k} \le n \le N^{1/k}+H \\ 2|F(n)}} \Big(\sum_{\substack{d|F(n) \\ (d,2)=1}} f(d) \Big) \\ &= \mathfrak{S} \sum_{\substack{d \le c_7 N \\ (d,2)=1}} f(d) \Big(\sum_{\substack{N^{1/k} \le n \le N^{1/k}+H \\ F(n) \equiv 0 \pmod{2d}}} 1 \Big). \end{aligned} \tag{33}$$

It is easy to see that

$$\sum_{\substack{N^{1/k} \le n \le N^{1/k}+H \\ F(n) \equiv 0 \pmod{2d}}} 1 = H \frac{\varrho_F(2d)}{2d} + O(\varrho_F(2d)), \tag{34}$$

see, *e.g.*, Halberstam-Richert [1]. Moreover, it is well known that ϱ_F is multiplicative and satisfies

$$\varrho_F(m) \ll m^{\varepsilon} \tag{35}$$

for every $\varepsilon > 0$. This follows from Nagell [5, Theorem 54] if F is primitive and its discriminant is different from 0, and the general case is an easy consequence of this special case.

Since

$$f(d) \ll \frac{\log^2 d}{d}, \tag{36}$$

from (33)-(36) we get

$$\begin{aligned}\sum(\mathfrak{S}) &= \frac{\varrho_F(2)}{2} H \mathfrak{S} \sum_{\substack{d=1\\(d,2)=1}}^{\infty} \frac{f(d)\,\varrho_F(d)}{d} \\ &\quad + O\Big(\sum_{d\le c_7 N} f(d)\,\varrho_F(d)\Big) + O\Big(H \sum_{d>c_7 N} \frac{f(d)\,\varrho_F(d)}{d}\Big) \\ &= \frac{\varrho_F(2)}{2} H \mathfrak{S} \prod_{p>2} \Big(1 + \frac{\varrho_F(p)}{p(p-2)}\Big) + O(N^{\varepsilon}).\end{aligned} \tag{37}$$

Theorem 2 follows now from (30), (32) and (37).

References.

[1] Halberstam, H., Richert, H.-E., *Sieve Methods.* Academic Press, 1974.

[2] Huxley, M. N., Large values of Dirichlet polynomials, III. *Acta Arithm.* **26** (1975), 435-444.

[3] Kaczorowski, J., Perelli, A., Pintz, J., A note on the exceptional set for Goldbach's problem in short intervals. *Mh. für Math.* **116** (1993), 275-282; corrigendum **119** (1995), 215-216.

[4] Mikawa, H., On the exceptional set in Goldbach's problem. *Tsukuba J. Math.* **16** (1992), 513-543.

[5] Nagell, T., *Introduction to Number Theory.* Chelsea Publ., 1964.

[6] Nair, M., Multiplicative functions of polynomial values in short intervals. *Acta Arithm.* **62** (1992), 257-269.

[7] Perelli, A., Pintz, J., On the exceptional set for Goldbach's problem in short intervals. *J. London Math. Soc.* **47** (1993), 41-49.

[8] Prachar, K., *Primzahlverteilung.* Springer Verlag, 1957.

[9] Ramachandra, K., On the number of Goldbach numbers in short intervals. *J. Indian Math. Soc.* **37** (1973), 157-170.

[10] Vaughan, R. C., *The Hardy-Littlewood Method.* Cambridge Univ. Press, 1981.

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