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Existence and properties of the Green function for a class of degenerate parabolic equations

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1. Introduction.

It is known that degenerate parabolic equation exhibit somehow different phenomena when we compare them with their elliptic counterpart: see *e.g.* [CS1], [CS2], [CS3], [GW3], [GW4], [Fe2]. Thus, the problem of existence and properties of the Green function for degenerate parabolic boundary value problems is not completely solved, even after the contributions of [GN] and [GW4], in the sense that the existence problem is still open, even if the a priori estimates proved in [GN] will be crucial in our approach. Roughly speaking, we will consider the following Dirichlet problem

$$(\mathrm{P}_0) \qquad \begin{cases} \partial_t u - \sum_{i,j} \partial_i (a_{ij}\, \partial_j u) = g + \operatorname{div} \vec{f}, & \text{in } \Omega \times (0,T]\,, \\ u = 0\,, \quad \text{on the parabolic boundary of } \Omega \times (0,T]\,, \end{cases}$$

where $a_{ij} = a_{ji}$ are measurable functions such that

$$(1.1) \qquad \nu\, \omega(x)\, |\xi|^2 \le \sum_{i,j} a_{ij}(x,t)\, \xi_i \xi_j \le \frac{1}{\nu}\, \omega(x)\, |\xi|^2\,,$$

almost everywhere in $\Omega \times (0,T]$. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, the parabolic boundary of $\Omega \times (0,T]$ is the set $\Omega \times \{0\} \cup \partial\Omega \times (0,T)$, $t > 0$, $\nu \in (0,1)$, and ω is a weight function belonging to the Muckenhoupt class $A_{1+2/n}$ (see below for precise definitions). We will prove that the problem (P_0) admits a unique Green function $\gamma(x,t,\xi,\tau)$ belonging to the natural function space associated with the differential operator in (P_0). Moreover the Green function γ satisfies the classical properties concerning the adjoint equations and the representation formula for the solution of general Cauchy problems.

Our technique is inspired by Aronson's paper [A], where an analogous result is proved for non degenerate parabolic equations by approaching them with a sequence of parabolic equations with smooth coefficients. In our case, we approximate our degenerate equation by means of a sequence of non degenerate problems, for which Aronson's results are true and precise a priori estimates in terms of our weight function have already been proved in [CS1], [CS2], [CS3], [GN]. To this end, in Section 2, we prove a general approximation theorem for A_p weights ($p \geq 1$) by means of weights which are bounded away from 0 and infinity and whose "A_p-constants" depend only on the "A_p-constant" of ω (see Lemma 2.1). In Section 3, existence and properties of weak solutions of the Cauchy problem for the operator in (P_0) are proved. The crucial point consists of showing that a weak limit (in a suitable function space) of a sequence of solutions of approximate problems is in fact a solution of the original problem (see Theorem 3.14). Finally, in Section 4, our main existence and properties results for the Green functions are proved (Theorems 4.1 and 4.2)

1.1. Notations, definitions and basic inequalities.

We indicate by $(x,t) = (x_1, x_2, \dots, x_n, t)$ the points of $\mathbb{R}^{n+1}$, $n \geq 2$, $B = B(\xi, r)$ the usual Euclidean ball with center in ξ and radius r and by $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^n$. The symbols ∂_i, ∂_t and ∇ indicate the derivatives $\partial/\partial x_i$, $\partial/\partial t$ and the gradient $(\partial/\partial x_1, \dots, \partial/\partial x_n)$. If $f = (f_1, \dots, f_n)$ then $\operatorname{div} f = \sum_{i=1}^n \partial_i f_i$. If E is a measurable set in $\mathbb{R}^n$, we indicate by $|E|$ its Lebesgue measure and if $\omega : \mathbb{R}^n \to \mathbb{R}$ is a non negative, locally integrable function we put $\omega(E) = \int_E \omega(x)\,dx$.

For any $p \geq 1$, $L^p(E)$ is the usual Lebesgue space; $L^p_\omega(E)$ is the Lebesgue space with respect to the measure $\omega(x)\,dx$. If X is a normed space and $f \in X$, $\|f\,;X\|$ indicates the norm of f in X.

We recall that given $1 < p < \infty$, a locally integrable non negative function ω is called an A_p weight if there is a constant $c > 0$ such that for all cubes K in $\mathbb{R}^n$

$$(1.2) \qquad \Big(\frac{1}{|K|}\int_K \omega(x)\,dx\Big)\Big(\frac{1}{|K|}\int_K \omega(x)^{-1/(p-1)}\,dx\Big)^{p-1} \le c\,.$$

The infimum of the set of $c > 0$ such that (1.2) holds will be called the A_p constant of ω. We list now some basic properties of A_p weights (see [GC/RF]).

(1.3) Every A_p weight is doubling, *i.e.* $\omega(2K) \le c\,\omega(K)$ for any cube K in $\mathbb{R}^n$, with the constant c independent of the cube K. Here $2K$ denotes the cube with the same center as K and having twice the side length of K.

(1.4) If $\omega \in A_p$ with A_p constant c_p then there is $p_0 = p_0(n,p,c_p)$, $p_0 < p$ such that if $q \in (p_0,p)$ then $\omega \in A_q$ with A_q constant depending on q, c_p, n and p.

If $p = 1$, we say that a locally integrable non negative function w is a A_1 weight if there exists a constant $c > 0$, such that for all cubes K in $\mathbb{R}^n$,

$$\frac{1}{|K|}\int w(x)\,dx \le c \operatorname*{ess\,inf}_K w\,.$$

We now recall results proved in [CS2], namely Sobolev inequality and Sobolev interpolation inequality, that will be used throughout the paper.

Sobolev inequality. *Assume ω is an A_p weight in $\mathbb{R}^n$ for $p \le 2$, and let c_p be its A_p constant. Assume $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous function supported in $B = B(\xi, r)$. Then there exists $c > 0$ depending only on n, p and c_p such that*

$$(1.5) \qquad \Big(\frac{1}{\omega(B)}\int_B |u|^{2k}\,\omega\,dx\Big)^{1/2k} \le c\,r\Big(\frac{1}{\omega(B)}\int_B |\nabla u|^2\,\omega\,dx\Big)^{1/2},$$

for $1 \le k \le n/(n-2/p)$ (see [CS2, Theorem 2.7]).

Sobolev interpolation inequality 1. *Assume ω is a A_2 weight. Then for any cylinder $Q = B(\xi,r) \times I \subset \mathbb{R}^{n+1}$ and for any Lipschitz* *continuous function $f : Q \to \mathbb{R}$ which is compactly supported in $B(\xi, r)$ for any fixed t, we have*

$$\begin{aligned}(1.6)\qquad &\Big(\frac{1}{\omega(Q)}\iint_Q |f|^{2h}\,\omega\,dx\,dt\Big)^{1/h}\\ &\qquad\le c\Big(\sup_I \frac{1}{|B|}\int_B |f|^2\,dx + \frac{r^2}{\omega(Q)}\iint_Q |\nabla f|^2\omega\,dx\,dt\Big),\end{aligned}$$

where $h > 1$ and $c > 0$ depend only on n and c_2.

This inequality was proved in [CS2, Lemma 2.8].

Sobolev interpolation inequality 2. *Assume ω is a $A_{1+2/n}$ weight. Then for any cylinder $Q = B(\xi, r) \times I \subset \mathbb{R}^{n+1}$ and for any Lipschitz continuous function $f : Q \to \mathbb{R}$ which is compactly supported in $B(\xi, r)$ for any fixed t, we have*

$$\begin{aligned}(1.7)\qquad &\Big(\frac{1}{|Q|}\iint_Q |f|^{2h'}\,\omega\,dx\,dt\Big)^{1/h'}\\ &\qquad\le c\Big(\sup_I \frac{1}{|B|}\int_B |f|^2\,dx + \frac{r^2}{\omega(Q)}\iint_Q |\nabla f|^2\,\omega\,dx\,dt\Big),\end{aligned}$$

where $h' > 1$ and $c > 0$ depend only on n and $c_{1+2/n}$.

This inequality was proved in a more general context in [CS2, Lemma 2.9].

The next remark points out the choice of h' that will be important in the future (see Lemmas 3.7 and 3.11).

REMARK 1.8. If we look at the proof of [CS2, Lemma 2.9] we see that h' is chosen in the following way: let $q > n/2$ be such that $\omega \in A_{1+1/q}$ (see property (1.4)). Then inequality (1.5) holds for $k = n(q+1)/(n(q+1) - 2q)$. Put $\beta = 1/2k$, $\alpha = 1 - \beta(q+1)/q$; then inequality (1.7) holds for $h' = k\beta + \alpha = 1 + (2/n - 1/q)/2$.

2. Approximation of A_p weights.

The first step is to approximate the equation (P_0) by non-degenerate parabolic equations. To this end we have to prove the following lemma.

Lemma 2.1. *Let α, $\beta > 1$ be given and let w belong to some A_p class, $p \geq 1$, with A_p constant $c(w,p)$ and let $a_{ij} = a_{ji}$ be measurable functions satisfying*

$$\nu\, w(x)\, |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\, \xi_i \xi_j \leq \frac{1}{\nu}\, w(x)\, |\xi|^2 ,$$

for all $\xi \in \mathbb{R}^n$ and almost every $(x,t) \in \Omega \times (a,b)$. Then there exist $w_{\alpha\beta} \geq 0$ and measurable functions $a_{ij}^{\alpha\beta}(x,t)$ such that

i) *$c_1\, \beta^{-1} \leq w_{\alpha\beta} \leq c_2\, \alpha$ in Ω where c_1, c_2 depend only on w and Ω.*

ii) *$\tilde{w}_1 \leq w_{\alpha\beta} \leq \tilde{w}_2$, where $\tilde{w}_i$ is a fixed A_p weight and $c(\tilde{w}_i, p)$ depends only on $c(w,p)$, for $i = 1, 2$.*

iii) *$w_{\alpha\beta} \in A_p$ with $c(w_{\alpha\beta}, p)$ depending only on $c(w,p)$ uniformly on α and β.*

iv) *there exists a closed set $F_{\alpha\beta}$ such that $a_{ij}^{\alpha\beta} = a_{ij}$ in $F_{\alpha\beta}$, $w_{\alpha\beta} = w$ in $F_{\alpha\beta}$ and $w_{\alpha\beta} \sim \tilde{w}_1 \sim \tilde{w}_2$ in $F_{\alpha\beta}$ with equivalence constants depending on α and β (i.e. $c_{\alpha\beta} \leq w_{\alpha\beta}/\tilde{w}_i \leq C_{\alpha\beta}$ for some positive constants $c_{\alpha\beta}$ and $C_{\alpha\beta}$ and for $i = 1, 2$). Moreover, $F_{\alpha\beta} \subset F_{\alpha'\beta'}$ if $\alpha \leq \alpha'$, $\beta \leq \beta'$ and the complement of $\cup_{\alpha,\beta \geq 1} F_{\alpha\beta}$ has zero measure.*

v) *$w_{\alpha\beta} \longrightarrow w$ almost everywhere in $\mathbb{R}^n$ as α, β tend to infinity.*

vi)

$$\nu\, w_{\alpha\beta}(x)\, |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x,t)\, \xi_i \xi_j \leq \frac{1}{\nu}\, w_{\alpha,\beta}(x)\, |\xi|^2 ,$$

for any $\xi \in \mathbb{R}^n$ and almost every $(x,t) \in \Omega \times (a,b)$.

PROOF. Suppose first $w \in A_1$. Since we are interested to approximate in Ω, we may assume, without loss of generality, that $w \in L^1(\mathbb{R}^n)$. Then for each $\alpha > 1$, by Calderon-Zygmund decomposition, there exists a family of non-overlapping cubes $\{Q_j^\alpha\}$ consisting of those maximal dyadic cubes over which the average of w is greater than α. If we put

a) $\cup_{j=1}^\infty Q_j^\alpha = U_\alpha^+$, then

b) $\alpha < \dfrac{1}{|Q_j^\alpha|}\displaystyle\int_{Q_j^\alpha} w(x)\,dx \le 2^n\,\alpha$, for any j;

c) $w(x) \le \alpha$ for any $x \in F_\alpha^+ = (U_\alpha^+)'$;

d) $|U_\alpha^+| \le \dfrac{1}{\alpha}\displaystyle\int_{\mathbb{R}^n} w(x)\,dx$.

Moreover, if $M(w)$ denotes the usual Hardy-Littlewood maximal function, then

$$\{x \in \mathbb{R}^n : \; M(w)(x) > 4^n\,\alpha\} \subset \bigcup_j 3Q_j^\alpha$$

(see, for instance, [GC/RF, Chapter 2, Theorem 1.12]).

We explicitly note that, if $\alpha < \beta$, then $U_\beta^+ \subset U_\alpha^+$. In fact, let x belong to U_β^+; then there exists a (unique) dyadic cube $Q_{j_0}^\beta$ containing x such that b) holds. Let now $\mathcal{I}$ be the set of all indices $j \in \mathbb{N}$ such that $Q_j^\alpha \cap Q_{j_0}^\beta \neq \varnothing$. Note that $\mathcal{I} \neq \varnothing$, since otherwise we would have $Q_{j_0}^\beta \subset F_\alpha^+$ and hence, by c),

$$\beta < \frac{1}{|Q_{j_0}^\beta|}\int_{Q_{j_0}^\beta} w\,dx \le \alpha < \beta\,,$$

a contradiction. Since we are dealing with dyadic cubes, either $\mathcal{I} = \{j_1\}$ and $Q_{j_0}^\beta \subset Q_{j_1}^\alpha$ or $Q_j^\alpha \subset Q_{j_0}^\beta$ for any $j \in \mathcal{I}$. In the first case we are done. Let us show that the second case cannot occur. Indeed, for any j, Q_j^α is a maximal dyadic cube over which the average w is greater than α; on the other hand $Q_{j_0}^\beta$ is a dyadic cube and the average of w over it is greater than $\beta > \alpha$. Thus the assertion is completly proved.

Define

$$w_\alpha(x) = \sum_{k=1}^\infty \frac{1}{|Q_k^\alpha|}\int_{Q_k^\alpha} w(y)\,dy\;\chi_{Q_k^\alpha}(x) + w(x)\,\chi_{F_\alpha^+}(x)\,.$$

We will show that

I) $w_\alpha \in A_1$ and $c(w_\alpha, 1)$ depends only on $c(w, 1)$.

II) $w_\alpha \to w$ almost everywhere in $\mathbb{R}^n$ as $\alpha \to \infty$.

III) $\min\{1,w\} \in A_1$ and $c(\min\{1,w\},1)$ depends only on $c(w,1)$; moreover $\min\{w,1\} \le w_\alpha \le c\,w$, where c depends only on $c(w,1)$.

IV) If Q_0 is a fixed cube containing Ω, then

$$2^n\,\alpha \ge w_\alpha(x) \ge c = \frac{1}{|Q_0|}\int_{Q_0} \min\{w,1\}(y)\,dy\,, \qquad x \in \Omega\,.$$

In order to prove the above statements, first of all we note that, if $v \in L^1$ is a weight function such that for any dyadic cube Q we have $v(3Q) \le c_0\, v(Q)$, then we can restrict ourselves to test the A_1 condition only on dyadic cubes. In fact, denoting by $c^\star(v,1)$ the "A_1-constant for dyadic cubes", if $f \in L^1_v$ and $\{C^t_j\}$ is the Calderon-Zygmund decomposition for f, we have

$$\begin{aligned} v(\{x:\ M(f)(x) > 4^n t\}) &\le \sum_j v(3C^t_j) \\ &\le c_0 \sum_j v(C^t_j) \\ &\le c_0\, c^\star(v,1) \sum_j |C^t_j| \inf_{C^t_j} v \\ &\le c_0\, c^\star(v,1)\,\frac{1}{t} \sum_j \int_{C^t_j} |f(x)| \inf_{C^t_j} v\, dx \\ &\le c_0\, c^\star(v,1)\,\frac{1}{t} \int |f(x)|\, v(x)\,dx\,, \end{aligned}$$

and hence $v \in A_1$, by [GC/RF, Chapter 4, Theorem 2.1]. We note explicitly that $c(v,1)$ depends only on c_0 and $c^\star(v,1)$.

Thus, let us prove first that

$$w_\alpha(3Q) \le c_0\, w_\alpha(Q)\,,$$

for any dyadic cube Q and for any $\alpha > 1$, where c_0 depends only on n and $c(w,1)$.

First, let us suppose that $|Q \cap U^+_\alpha| < |Q|/2$. We will prove later, in III), that $w_\alpha \le \max\{1, c(w,1)\}\, w = c_1\, w$; we have:

$$w_\alpha(3Q) \le c_1\, w(3Q)$$

$$
\begin{aligned}
&= c_1\, 3^n\, |Q|\, \frac{1}{|3Q|} \int_{3Q} w(x)\, dx \\
&\le c_1\, c(w,1)\, 3^n\, |Q| \inf_{3Q} w \\
&\le c_1\, c(w,1)\, 3^n\, |Q| \inf_{Q\cap F_\alpha^+} w \\
&= c_1\, c(w,1)\, 3^n\, \frac{|Q|}{|Q\cap F_\alpha^+|}\, |Q\cap F_\alpha^+| \inf_{Q\cap F_\alpha^+} w \\
&\le 2\, c_1\, c(w,1)\, 3^n\, |Q\cap F_\alpha^+| \inf_{Q\cap F_\alpha^+} w \\
&\le 2\, c_1\, c(w,1)\, 3^n \int_{Q\cap F_\alpha^+} w_\alpha(x)\, dx
\end{aligned}
$$

since $w_\alpha \equiv w$ on F_α^+ and therefore

$$w_\alpha(3Q) \le 2\, c_1\, c(w,1)\, 3^n \int_Q w_\alpha(x)\, dx = 2\, c_1\, c(w,1)\, 3^n\, w_\alpha(Q)$$

and hence, in this case we are done.

Suppose now $|Q \cap U_\alpha^+| \ge |Q|/2$; then either $Q \subset Q_{j_0}^\alpha$ for some j_0 (which in turn is unique), or $Q_j^\alpha \subset Q$ for j belonging to a given set $\mathcal{J}$ of indices. By definition of w_α, it follows from b) and c) that $w_\alpha(x) \le 2^n\alpha$ almost everywhere; hence, if $Q \subset Q_{j_0}^\alpha$ we get

$$
\begin{aligned}
\int_{3Q} w_\alpha(x)\, dx &\le |3Q|\, 2^n\, \alpha \\
&= 3^n\, 2^n\, |Q|\, \alpha \\
&< 3^n\, 2^n\, |Q|\, \frac{1}{|Q_{j_0}^\alpha|} \int_{Q_{j_0}^\alpha} w(x)\, dx \\
&= 6^n \int_Q w_\alpha(x)\, dx\,,
\end{aligned}
$$

since $w_\alpha \equiv |Q_{j_0}^\alpha|^{-1} \int_{Q_{j_0}^\alpha} w(x)\, dx$ on Q. Otherwise

$$
\begin{aligned}
w_\alpha(3Q) &\le 3^n\, |Q|\, 2^n\, \alpha \\
&\le 2\, 6^n \int_{Q\cap U_\alpha^+} \alpha\, dx \\
&= 2\, 6^n \sum_{j\in\mathcal{J}} \int_{Q_j^\alpha} \alpha\, dx \\
&\le 2\, 6^n \sum_{j\in\mathcal{J}} \int_{Q_j^\alpha} w_\alpha(x)\, dx
\end{aligned}
$$

where in the last inequality we used b) and therefore we have

$$w_\alpha(3Q) \le 2\,6^n \int_{\cup Q_j^\alpha} w_\alpha(x)\,dx \le 2\,6^n \int_Q w_\alpha(x)\,dx$$

and the assertion is completely proved.

Now we are ready to prove I). Let Q be a fixed dyadic cube, then one of the three cases can happen

I$_1$) $Q \cap Q_j^\alpha = \varnothing$, for all j,

I$_2$) $Q \subset Q_j^\alpha$, for one and only one j,

I$_3$) $Q_j^\alpha \subset Q$, for some index $j \in J$.

In case I$_1$), $Q \subset F_\alpha^+$ and hence $w_\alpha \equiv w$ in Q and we are done. In case I$_2$),

$$w_\alpha(Q) = |Q|\,\frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha} w(x)\,dx = \inf_{x\in Q} w_\alpha(x)\,|Q|\,,$$

since $w_\alpha \equiv |Q_j^\alpha|^{-1}\int_{Q_j^\alpha} w(x)\,dx$ over Q_j^α. Finally in case I$_3$)

$$\begin{aligned} w_\alpha(Q) &= \sum_{j\in J}\frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha} w(x)\,dx\,|Q_j^\alpha| + w(Q\cap F_\alpha^+) \\ &= \sum_{j\in J}\int_{Q_j^\alpha\cap Q} w(x)\,dx + w(Q\cap F_\alpha^+) \\ &\le w(Q) \le c(w,1)\,|Q|\,\inf_{y\in Q} w(y)\,. \end{aligned}$$

On the other hand we note that if $y \in U_\alpha^+$, by definition

$$w_\alpha(y) = \frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha} \omega(x)\,dx > \alpha\,.$$

Thus, if $y \in Q \cap U_\alpha^+$ and Q_k^α is any cube contained in Q we have

$$\inf_Q w \le \inf_{Q_k^\alpha} w \le \frac{1}{|Q_k^\alpha|}\int_{Q_k^\alpha} w(z)\,dz \le 2^n\,\alpha < 2^n\,w_\alpha(y)\,.$$

In addition, if $y \in Q\cap F_\alpha^+$ then $w_\alpha(y) = w(y) \ge \inf_Q w$ so that $\inf_Q w \le 2^n \inf_Q w_\alpha$ and hence I) is completely proved.

To prove II), we note that $w_\alpha \equiv w$ in F_α^+ and that F_α^+ increases as α tends to infinity. Moreover, $|\cap (F_\alpha^+)'| = 0$.

Finally, to show III) we know that, for any cube Q, either $\inf_Q w \geq 1$ or $\inf_Q w < 1$. In the first case

$$\frac{1}{|Q|}\int_Q \min\{w,1\}(x)\,dx = \frac{1}{|Q|}\int_Q 1\,dx = 1 \leq \min\{w(y),1\},$$

for any $y \in Q$, whereas if $\inf_Q w \leq 1$ then

$$\frac{1}{|Q|}\int_Q \min\{w,1\}(x)\,dx \leq \frac{1}{|Q|}\int_Q w(x)\,dx \leq c(w,1)\inf_Q w\,.$$

Put $\lambda = \inf_Q w < 1$ and assume by contradiction that $\inf_Q \min\{w,1\} < \lambda$; then there exists $E \subset Q$, $|E| > 0$ such that $\min\{w,1\} < \lambda' < \lambda$ in E and hence, since $\lambda < 1$, $w < \lambda'$ in E, which is a contradiction. Thus we have proved the first part of III). To prove the second part we note that if $x \in F_\alpha$ then $w_\alpha(x) = w(x) \geq \min\{1,w\}(x)$; if $x \in Q_j^\alpha$, for some j, then

$$w_\alpha(x) = \frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha} w(y)\,dy > \alpha \geq \min\{1,w\}(x)\,.$$

Analogously, if $x \in Q_j^\alpha$ for some j, then

$$w_\alpha(x) \leq c(w,1)\inf_{Q_j^\alpha} w \leq c(w,1)\,w(x)\,.$$

Finally, assertion IV) follows straightforwardly from III) by using a) and c).

Suppose now $w \in A_p$, for $p > 1$. Then by Peter Jones' factorization theorem ([GC/RF, Theorem 5.2 and Corollary 5.3, Chapter 4]) there exist w_0, $w_1 \in A_1$ such that $w = w_0\, w_1^{1-p}$. In addition $c(w_i,1)$ depends only on $c(w,p)$, $i = 0,1$. Choose $\alpha, \beta > 1$ and define

$$w_{\alpha\beta} = (w_0)_\alpha\,((w_1)_{\beta^{1/(p-1)}})^{1-p}\,.$$

We need to show that $w_{\alpha\beta}$ satisfies properties i)-v). Obviously i) follows from IV); ii) from III) and [GC/RF, Theorem 5.2, Corollary 5.3] with $\tilde{w}_1 = \min\{w_0,1\}w_1^{1-p}$, $\tilde{w}_2 = w_0(\min\{w_1,1\})^{1-p}$; iii) from I) and [GC/RF, Theorem 5.2, Corollary 5.3]; v) from II); for iv), we define $F_{\alpha\beta} = F_\alpha^0 \cap F^1_{\beta^{1/(p-1)}}$, where $F_\alpha^0 = F_\alpha^+$ for the weight w_0 and

$F^1_{\beta^{1/(p-1)}} = (F_{\beta^{1/(p-1)}})^+$ for the weight w_1. By definition, $(w_0)_\alpha \equiv w_0$ and $(w_1)_{\beta^{1/(p-1)}} \equiv w_1$ on $F_{\alpha\beta}$. Hence to prove that $w_{\alpha\beta} \sim \tilde{w}_1$ (for instance) we can replace $w_{\alpha\beta}$ by w. Note now that (with the notation we used above) $\min\{w_0, 1\} \sim w_0$ in F_α. Obviously $\min\{w_0, 1\} \leq w_0$; moreover, if, for some $x \in F_\alpha$, $1 = \min\{w_0, 1\}$, then $w_0(x) \leq \alpha = \alpha \min\{w_0(x), 1\}$. An analogous argument shows that $w_1 \sim \min\{w_1, 1\}$ and hence

$$w_{\alpha\beta} = (w_0)_\alpha \big((w_1)_{\beta^{1/(p-1)}}\big)^{1-p} \sim \min\{w_0, 1\} \, \big((w_1)_{\beta^{1/(p-1)}}\big)^{1-p} = \tilde{w}_1 \, .$$

Finally to prove vi) we define

$$\begin{aligned} a^{\alpha\beta}_{ij}(x,t) = {} & \big((w_1)_{\beta^{1/(p-1)}}\big)^{1-p} \\ & \cdot \Big(\sum_{k=1}^{\infty} \frac{1}{|Q^\alpha_k|} \int_{Q^\alpha_k} a_{ij}(y,t)\, w_1^{p-1}(y)\, dy \; \chi_{Q^\alpha_k}(x) \\ & + a_{ij}(x,t)\, w_1^{p-1}(x)\, \chi_{F^+_\alpha}(x) \Big) \, , \end{aligned}$$

where $\{Q^\alpha_k\}$ is the Calderon-Zygmund decomposition for w_0. Now, by assumption,

$$a^{\alpha\beta}_{ij}(x,t)\, \xi_i \xi_j \leq \frac{1}{\nu} \big((w_1)_{\beta^{1/(p-1)}}\big)^{1-p} (w_0)_\alpha \, |\xi|^2 = \frac{1}{\nu}\, w_{\alpha\beta}(x)\, |\xi|^2 \, .$$

The lower estimate can be carried out in the same way.

REMARK. There is a different approach to the approximation theorem, see [G].

3. The weak solutions.

In this section we prove existence and properties of the weak solutions of the Cauchy-Dirichlet problems for the operator $\partial_t - L$. We will follow [A] and [FJK]. The crucial point is to prove that a weak limit (in a suitable function space) of a sequence of solutions of approximate non degenerate problems is in fact a solution of the original problem. This will be done in Theorem 3.14. We start this section by introducing some normed spaces in order to be able to define what we mean by a weak solution for Cauchy-Dirichlet problems for the operator $\partial_t - L$.

Put $Q = \Omega \times (0,T)$ and let ω belong to A_p for some $p > 1$. We denote by $H^{1,p}_{\omega}(Q)$ the closure of Lipschitz functions under the norm

$$\|u\|^p = \iint_Q |u(x,t)|^p \, \omega(x)\, dx\, dt + \iint_Q |\nabla u(x,t)|^p \, \omega(x)\, dx\, dt$$

and similarly $H^{1,p}_{0,\omega}(Q)$ denotes the closure of Lipschitz functions with compact support in Q under the same norm. We note that, since $\omega \in A_p$, ∇u in the limit sense belongs to L^1_{loc} and it coincides with the distribution gradient of u. Moreover we put

$$H^{-1,p}_{\omega}(Q) = \left\{ \operatorname{div} f : \ \frac{|\vec{f}|}{w} \in L^p_w \right\}.$$

The next theorem characterizes the dual space of $H^{1,p}_{\omega}(Q)$. We will assume that Ω is a smooth regular open subset of $\mathbb{R}^n$. This implies in particular that there exist $\alpha > 0$, $\rho_0 > 0$ such that for any $x_0 \in \partial\Omega$, $\rho < \rho_0$ we have

$$|B(x_0,\rho) \setminus \Omega| \geq \alpha |B(x_0,\rho)|.$$

We note that the smoothness assumption could be strongly relaxed; however we hold it to avoid a number of arguments at some points.

Theorem 3.1. *The space $H^{1,p}_{\omega}(Q)$ is a reflexive Banach space, for $p > 1$. Moreover*

i) $H^{-1,p}_{\omega}(Q) = (H^{1,p'}_{0,\omega}(Q))^*$.

ii) *Let ω_1 be another A_p weight; if $u \in H^{1,p}_{0,\omega_1}(Q)$ with respect to ω_1 and u, $|\nabla u| \in L^p_{\omega}$, then $u \in H^{1,p}_{0,\omega}(Q)$ with respect to ω (see also* [CP/SC])

PROOF. It is easy to see that $H^{1,p}_{\omega}(Q)$ is reflexive since it is isometrical to a closed subspace of the reflexive space $(L^p_{\omega}(Q))^{n+1}$.

Let us now prove ii). First, let $x \in \Omega$ be such that $d(x, \partial\Omega) < r$ for a given (small) $r > 0$, and let $y = y(x) \in \partial\Omega$ be such that $d(x,y) = d(x,\partial\Omega) = d$. If v is a Lipschitz function with compact support in Q, it can be continued by zero outside of $\bar{Q}$ and we have

$$B(y,d) \setminus \Omega \subset \{z \in B(y,2d) : \ v(z,t) = 0\},$$

for any $t \in [0,T]$, so that

$$|\{z \in B(y,2d) : \ v(z,t) = 0\}| \geq \alpha \, |B(y,d)| \geq c \, |B(y,2d)|.$$

Hence, keeping in mind that $x \in B(y, 2d)$, by standard arguments (see, *e.g.*, [KS] or arguing as in [FS, Lemma 4.3]) we obtain for $t \in [0, T]$

$$|v(x,t)| \leq c\, d\, M(|\nabla v(\cdot,t)|\, \chi_{B(y,\theta d)\cap\Omega})(x),$$

where Mf is the usual Hardy-Littlewood maximal function and $\theta > 0$ is an absolute constant. On the other hand, if $z \in B(y, \theta d)$ then $d(z, \partial\Omega) \leq d(z, y) \leq \theta\, d \leq \theta\, r$, so that

$$|v(x,t)| \leq c\, r\, M(|\nabla v(\cdot,t)|\, \chi_{\theta r})(x), \tag{3.2}$$

where $\chi_s(x)$ is the characteristic function of $\{z \in \Omega : \ d(z, \partial\Omega) < s\}$ for $s > 0$. On the other hand the function u is (by assumption) the limit in $H^{1,p}_{\omega_1}(Q)$ (with respect to ω_1) of a sequence $(v_k)_{k\in\mathbb{N}}$ of Lipschitz continuous functions supported in $\bar{Q}$. In particular, $v_k \to u$, as $k \to \infty$, for almost every $(x,t) \in Q$. Moreover (by [Mu])

$$\begin{aligned}
&\|M(|\nabla v_k - \nabla u|\, \chi_{\theta r}); L^p_{\omega_1}(\Omega; L^p([0,T]))\|^p \\
&\quad = \int_0^T \Big(\int_\Omega M(|\nabla v_k - \nabla u|\, \chi_{\theta r})^p(x)\, \omega_1(x)\, dx \Big)\, dt \\
&\quad \leq c \int_0^T \Big(\int_\Omega |\nabla v_k - \nabla u|^p\, \chi_{\theta r}\, \omega_1\, dx \Big)\, dt \\
&\quad \leq c\, \|v_k - u; H^{1,p}_{\omega_1}(Q)\|^p \to 0,
\end{aligned}$$

as $k \to \infty$. Hence

$$M(|\nabla v_k(\cdot, t) - \nabla u(\cdot, t)|\, \chi_{\theta r})(x) \to 0,$$

as $k \to \infty$ for almost every $(x,t) \in Q$. By applying (3.2) to $v = v_k$, we get

$$|v_k(x,t)| \leq c\, r\, \big(M(|\nabla u(\cdot,t)|\, \chi_{\theta r})(x) + M(|\nabla v_k(\cdot,t) - \nabla u(\cdot,t)|\, \chi_{\theta r})(x) \big)$$

and hence

$$|u(x,t)| \leq c\, r\, M(|\nabla u(\cdot,t)|\, \chi_{\theta r})(x), \tag{3.3}$$

for almost everywhere $(x,t) \in \Sigma_r = \{(x,t) \in Q : \ d(x, \partial\Omega) < r\}$.

If $\delta > 0$, let now $\sigma_\delta : [0, \infty) \to [0, 1]$ be a smooth function such that $\sigma_\delta(t) \equiv 0$ if $0 \leq t \leq \delta$, $\sigma_\delta(t) \equiv 1$ if $t \geq 2\delta$ and $|\sigma'_\delta(t)| \leq 2/\delta$ for any $t \geq 0$. Define

$$u = u\, \sigma_\delta(d(\cdot, \partial\Omega)) + u\, (1 - \sigma_\delta(d(\cdot, \partial\Omega))) = u_\delta + v_\delta\,.$$

Note that, by [GT, Lemma 14.16], $d(\cdot, \partial\Omega)$ is a smooth function in Ω if δ is small enough. Using our previous notations we have (by (3.3))

$$\begin{aligned}|\nabla v_\delta(x,t)| &\le |\nabla u(x,t)|\,\chi_{2\delta}(x) + \frac{2}{\delta}\,|u(x,t)|\,\chi_{2\delta}(x)\\ &\le |\nabla u(x,t)|\,\chi_{2\delta}(x) + c\,M(|\nabla u(\cdot\,,t)|\,\chi_{2\theta\delta})(x)\,.\end{aligned}$$

Hence

$$\begin{aligned}\|v_\delta; H^{1,p}_\omega(Q)\|^p &\le \iint_{\Sigma_{2\delta}} \left(|u(x,t)|^p + |\nabla u(x,t)|^p\right)\omega(x)\,dx\,dt\\ &\qquad + c\int_0^T \left(\int M(|\nabla u(\cdot\,,t)|\,\chi_{2\theta\delta})^p(x)\,\omega(x)\,dx\right)dt\\ &\le c\iint_{\Sigma_{2\theta\delta}} \left(|u(x,t)|^p + |\nabla u(x,t)|^p\right)\omega(x)\,dx\,dt\,,\end{aligned}$$

by [Mu]. Thus, by the absolute continuity of the integral, we can choose $\delta > 0$ such that

$$\|v_\delta; H^{1,p}_\omega(Q)\| < \varepsilon\,. \tag{3.4}$$

Let now δ be fixed so that (3.4) holds. If r, $\rho > 0$ and p_r, ψ_ρ are usual mollifiers in $\mathbb{R}$ and $\mathbb{R}^n$ respectively, we put

$$u_{\delta,r,\rho} = u_\delta * \psi_\rho * p_r\,,$$

where we have continued $u_\delta(x,\cdot)$ by zero outside of $[0,T]$. Obviously, $u_{\delta,r,\rho}$ belongs to $C_0^\infty(\Omega\times\mathbb{R})$ if ρ is small enough. Moreover

$$\begin{aligned}\|u_\delta - u_{\delta,r,\rho}; H^{1,p}_\omega(Q)\| &\le \|u_\delta - u_\delta * \psi_\rho; H^{1,p}_\omega(Q)\|\\ &\quad + \|u_\delta * \psi_\rho - u_\delta * \psi_\rho * p_r; H^{1,p}_\omega(Q)\|\\ &= \mathrm{I}_1 + \mathrm{I}_2\end{aligned}$$

Now

$$\mathrm{I}_1^p = \int_0^T \|u_\delta - u_\delta * \psi_\rho; H^{1,p}_\omega(\Omega)\|^p\,dt\,.$$

By assumption u_δ, $|\nabla u_\delta|$ belong to $L^p_\omega(\Omega)$ for almost every $t\in[0,T]$ and hence, arguing as in [CP/SC, Proposition 2.5],

$$\|u_\delta - u_\delta * \psi_\rho; H^{1,p}_\omega(\Omega)\| \to 0\,,$$

as $\rho \to 0$ for almost every $t \in [0,T]$. On the other hand

$$\begin{aligned}\|u_\delta * \psi_\rho; H^{1,p}_\omega(\Omega)\| &\leq \|u_\delta * \psi_\rho; L^p_\omega(\Omega)\| + \| \, |\nabla u_\delta| * \psi_\rho; L^p_\omega(\Omega)\| \\ &\leq \|Mu_\delta(\,\cdot\,,t); L^p_\omega(\Omega)\| + \|M(|\nabla u_\delta(\,\cdot\,,t)|); L^p_\omega(\Omega)\| \\ &\leq c\,\|u_\delta(\,\cdot\,,t); H^{1,p}_\omega(\Omega)\|\,,\end{aligned}$$

by [Mu]. Hence we can apply Lebesgue's dominate convergence theorem, since $t \to \|u_\delta(\,\cdot\,,t); H^{1,p}_\omega(\Omega)\|$ belongs to $L^p([0,T])$ and we can conclude that there exists $\rho > 0$ such that $I_1 < \varepsilon$.

Let now ρ be fixed as above; we will now show that there exists $r > 0$ such that $\mathrm{I}_2 < \varepsilon$. We have

$$\begin{aligned}\mathrm{I}_2 = &\int_\Omega \Big(\int_0^T |(u_\delta * \psi_\rho) - (u_\delta * \psi_\rho) * p_r|^p \, dt \Big)\, \omega(x)\, dx \\ &+ \int_\Omega \Big(\int_0^T |(\nabla u_\delta * \psi_\rho) - (\nabla u_\delta * \psi_\rho) * p_r|^p \, dt \Big)\, \omega(x)\, dx\,.\end{aligned}$$

We now denote by $\tilde{u}$ either $u_\delta * \psi_\rho$ or $\nabla u_\delta * \psi_\rho$. By assumption

$$|\tilde{u}(x,t)| \leq M\,(|u_\delta(\,\cdot\,,t)| + |\nabla u_\delta(\,\cdot\,,t)|)(x)$$

and hence, again by [Mu], $\tilde{u} \in L^p([0,T]; L^p_\omega(\Omega))$. On the other hand $w(x) \neq 0$ for almost every $x \in \Omega$ since it belongs to A_p and hence

$$\tilde{u}(x,\,\cdot\,) \in L^p([0,T])\,,$$

for almost $x \in \Omega$, so that, by standard results on convolution,

$$\int_0^T |\tilde{u}(x,t) - (\tilde{u}(x,\,\cdot\,) * p_r)(t)|^p \, dt \longrightarrow 0\,,$$

as $r \to 0^+$ for almost everywhere $x \in \Omega$. We can now argue as above by using the maximal function in $\mathbb{R}$ and hence apply the Lebesgue's dominate convergence theorem. Thus we get

$$I_2 < \varepsilon\,, \qquad \text{if } r \text{ is small enough.}$$

Combining these estimates with (3.4) we get

$$\|u - u_{\delta,r,\rho}\,; H^{1,p}_\omega(Q)\| < 3\,\varepsilon\,.$$

Thus we have proved that u can be approximated in $H^{1,p}_{\omega}(Q)$ by functions in $C_0^\infty([0,T]\times\Omega)$. In particular, u belongs to $H^{1,p}_{0,\omega}(Q)$.

Finally, let us prove i). Denote by T the application

$$T : H^{1,p'}_{0,\omega}(Q) \longrightarrow (L^{p'}(Q))^n$$

given by $Tu = w^{1/p'}\nabla u$. Note that $\|Tu\,;(L^{p'}(Q))^n\|$ is equivalent to $\|u\,;H^{1,p'}_{0,\omega}(Q)\|$ because of the Sobolev inequality (1.5) for compactly supported functions, so that the range of T is a closed subspace Y of $(L^{p'}(Q))^n$. Thus, if $F\in(H^{1,p'}_{\omega}(Q))^*$ then the application $\vec{v}\to F(T^{-1}(\vec{v}))$ is a linear continuous functional in Y which can be continued as an element of $((L^{p'}(Q))^n)^* = (L^p(Q))^n$ by Hahn-Banach theorem. Thus there exists $\vec{g}=(g_1,\dots,g_n)\in(L^p(Q))^n$ such that for any $u\in H^{1,p'}_{\omega}(Q)$ we can write

$$\begin{aligned} F(u) &= F(T^{-1}(\omega^{1/p'}\nabla u)) \\ &= \sum_j \iint_Q \partial_j u(x,t)\, g_j(x,t)\, \omega^{1/p'}(x)\, dx\, dt \\ &= \iint_Q \langle \nabla u(x,t), \vec{f}(x,t)\rangle\, dx\, dt \end{aligned}$$

where $\vec{f}(x,t) = \vec{g}(x,t)\omega^{1/p'}(x)$. Note that

$$\begin{aligned} \iint_Q \Big(\frac{|\vec{f}(x,t)|}{\omega}\Big)^p \omega\, dx\, dt &= \iint_Q |\vec{g}(x,t)|^p\, (\omega(x))^{p(1/p'-1)}\, \omega(x)\, dx\, dt \\ &= \iint_Q |\vec{g}(x,t)|^p\, dx\, dt < \infty, \end{aligned}$$

and the assertion is proved.

REMARK 3.5. If f_0 is such that $f_0/\omega\in L^p_\omega(Q)$, then f_0 can be identified with a linear functional on $H^{1,p'}_{0,\omega}(Q)$ (with respect to the duality between $\mathcal{D}$ and $\mathcal{D}'$) and hence we can write $f_0 = \operatorname{div}\vec{f}$ for some vector field $\vec{f}$ such that $|\vec{f}|/\omega\in L^p_\omega(Q)$.

If $Q=\Omega\times(0,T]$ is the parabolic cylinder, we will denote by Σ its lateral boundary, *i.e.* $\Sigma=\partial\Omega\times(0,T]$. We write

$$Lu = \sum_{i,j}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big).$$

Definition 3.6. *Let $S \in H^{-1,2}(Q)$ and $u_0 \in L^2(\Omega)$ be given. We say that $u \in H_0^{1,2}(Q)$ is a weak solution of the problem*

$$\begin{cases} \partial_t u - Lu = -S\,, \\ u(x,0) = u_0(x)\,, \\ u(x,t) = 0 \quad on\ \Sigma\,. \end{cases}$$

if

$$\iint_Q \left(a_{ij}(x,t)\,\frac{\partial u}{\partial x_j}\,\frac{\partial \varphi}{\partial x_i} - u\,\frac{\partial \varphi}{\partial t}\right) dx\,dt$$
$$= -\langle S,\varphi\rangle_{\mathcal{D}',\mathcal{D}} + \int_\Omega u_0(x)\,\varphi(x,0)\,dx\,,$$

for any $\varphi \in W = \{\varphi \in H_0^{1,2}(Q) :\ \partial\varphi/\partial t \in H^{-1,2}(Q)\}$ with $\varphi(T) = 0$.

For more details about space W see [CS3], where, it is proved, for instance, that $W \subset C([0,T], L^2(\Omega))$. Before we prove the main result of this section, Theorem 3.14, we need two preliminary results stated in Lemmas 3.7 and 3.11.

Lemma 3.7. *Let w be an A_2 weight, $S = \sum_i \partial_i fi \in H_w^{-1,2}(Q)$, $g/w \in L_w^p(Q)$ for $p > h/(h-1)$, where $h > 1$ is an index for which inequality (1.6) holds and let $u_0 \in L^2(\Omega)$. If $u \in H_w^{1,2}(Q)$ is a weak solution of the problem*

$$\begin{cases} \partial_t u - L_1 u = g - S & in\ Q\,, \\ u(x,0) = u_0(x) & in\ \Omega\,, \\ u(x,t) = 0 & in\ \Sigma\,, \end{cases}$$

where $L_1 = \sum \partial_i(b_{ij}\partial_j)$ and $b_{ji} = b_{ij}$ are measurable functions satisfying

$$\sum_{i,j=1}^{n} b_{ij}\,\xi_i\xi_j \sim w(x)\,|\xi|^2\,, \tag{3.8}$$

then

$$\sup_{t\in[0,T]} \int_\Omega u(x,t)^2\,dx + \iint_Q |\nabla u(x,t)|^2\,w\,dx\,dt$$
$$\le c\left(\|u_0\,; L^2(\Omega)\|^2 + \left\|\frac{g}{w}\,; L_w^{2p/(p+1)}(Q)\right\|^2 + \sum_{j=1}^{n}\left\|\frac{f_j}{w}\,; L_w^2(Q)\right\|^2\right), \tag{3.9}$$

where the constant c depends only on $c(w,2)$, the equivalence constant in (3.8), T and Ω.

PROOF. For any $\tau \in (0,T)$, we put $\tilde{Q}_\tau = \Omega \times (0,\tau)$; arguing as in [CS3, proof of Theorem 2.4], we get

$$\iint_{\tilde{Q}_\tau} \sum_{i,j=1}^{n} b_{ij}\,\partial_i u\,\partial_j u\,dx\,dt + \frac{1}{2}\int_\Omega u(x,\tau)^2\,dx$$
$$= \frac{1}{2}\int_\Omega u(x,0)^2\,dx + \iint_{\tilde{Q}_\tau} g\,u\,dx\,dt + \iint_{\tilde{Q}_\tau} \sum_{i=1}^{n} \partial_i u\,f_i\,dx\,dt\,.$$

By standard arguments, keeping in mind (3.8), we can reduce ourselves to estimate the last two terms above by the right hand side of (3.9). For the last term it is quite easy. On the other hand, if we put $s = 2p/(p+1)$, for any $\varepsilon > 0$ we get

$$\begin{aligned}\iint_{\tilde{Q}_\tau} |g\,u|\,dx\,dt &= \iint_{\tilde{Q}_\tau} \frac{|g|}{\omega}\,\omega^{1/s}\,|u|\,\omega^{1/s'}\,dx\,dt \\ &\le \Big(\iint_{\tilde{Q}_\tau} \Big(\frac{|g|}{\omega}\Big)^s \omega\,dx\,dt\Big)^{1/s}\Big(\iint_{\tilde{Q}_\tau} |u|^{s'}\omega\,dx\,dt\Big)^{1/s'} \\ &\le \frac{1}{2\,\varepsilon^2}\Big(\iint_{\tilde{Q}_\tau} \Big(\frac{|g|}{\omega}\Big)^s \omega\,dx\,dt\Big)^{2/s} \\ &\quad + \frac{\varepsilon^2}{2}\Big(\iint_{\tilde{Q}_\tau} |u|^{s'}\,\omega\,dx\,dt\Big)^{2/s'}\,.\end{aligned}$$

Since $s' < 2h$, by (1.6) we have (let r = diameter of Ω)

$$\begin{aligned}\iint_{\tilde{Q}_\tau} |g\,u|\,dx\,dt \le \frac{1}{2\,\varepsilon^2}\Big(\iint_Q \Big(\frac{|g|}{\omega}\Big)^s \omega dx\,dt\Big)^{2/s} \\ + \frac{\varepsilon^2}{2}\,\omega(Q)^{2/s'}\Big(\sup_{t\in[0,T]} \frac{1}{|\Omega|}\int_\Omega |u|^2\,dx \\ + r^2\,\omega(Q)\iint_Q |\nabla u|^2\,\omega dx\,dt\Big)\,,\end{aligned}$$

and the assertion follows with a convenient choice of ε.

In order to establish the notation for the proof of Theorem 3.11 we will state, in a simpler context, a lemma proved in [CS2].

Lemma 3.10 (Chiarenza and Serapioni). *Assume ω is a $A_{1+2/n}$ weight. Then for any $(\xi,\tau) \in Q$ there is $R = R(\xi) > 0$ and a function $h : \Omega \times [0,T] \longrightarrow \mathbb{R}^+$, such that*

i) *$h(\xi, \cdot)$ is continuous, strictly increasing and $h(\xi,0) = 0$. (We will also denote $h(\xi,t)$ by $h_\xi(t)$),*

ii) *the set $Q_r(\xi,\tau) = \{(x,t) \in \mathbb{R}^{n+1} : x \in B(\xi,r),\ \tau - h(\xi,r) < t < \tau\}$ is contained in Q, for $r < R(\xi)$, (we will call $Q_r(\xi,\tau)$ a "standard cylinder"),*

iii) $\displaystyle\int_{Q_r(\xi,\tau)} \omega(x)\,dx\,dt \simeq r^{n+2}$,

iv) *there is a constant $\sigma_0 > 1$ such that $h(\xi;2r) < \sigma_0\, h(\xi,r)$, where σ_0 depends on $c(\omega, 1+2/n)$ and n, but it is independent of ξ, τ and r.*

v) *There is $\sigma \in (0,1)$ such that $h(\xi;\sigma r) \leq h(\xi,r)/4$. Here σ depends only on $c(\omega, 1+2/n)$ and n.*

It is also useful to define the following sets:

$$Q_r^+(\xi,\tau) = \left\{(x,t):\ |x-\xi| < \frac{r}{2},\ \tau - \frac{h(\xi,r)}{4} < t < \tau\right\},$$

$$Q_r^-(\xi,\tau) = \{(x,t):\ |x-\xi| < \frac{r}{2},\ \tau - \frac{7}{8}\,h(\xi,r) < t < \tau - \frac{5}{8}\,h(\xi,r)\Big\}.$$

Theorem 3.11 (Chiarenza and Serapioni). *Assume* (1.1) *holds with $\omega \in A_2$. For any $S \in H^{-1,2}(Q)$ there exists a unique $u = G(S) \in H_0^{1,2}(Q)$ which is a weak solution of the problem*

$$\begin{cases} \partial_t u - Lu = -S\,, \\ u(x,0) = u_0(x)\,. \end{cases}$$

Moreover, if $p > 2l/(l-1)$, where l is an index for which (1.5) *and* (1.6) *hold, $S = \operatorname{div} \vec{f} \in H^{-1,p}(Q)$, then*

$$\operatorname*{ess\,sup}_Q |u(x,t)| \leq C \left\| \frac{\vec{f}}{\omega}, L^p_\omega(Q) \right\| + \operatorname*{ess\,sup}_\Omega |u_0(x)|\,, \tag{3.12}$$

where C depends only on Ω, T, $c(\omega,2)$ and p. Finally, if $\omega \in A_{1+2/n}$ and l is chosen such that (1.7) *also holds then the solution u is Hölder* *continuous uniformly on the compact subsets of Q and, if $Q_r(\bar{x},\bar{t}) \subset Q_R(\bar{x},\bar{t}) \subset\subset Q$, then*

$$\operatorname*{osc}_{Q_r(\bar{x},\bar{t})} u \leq C_R \left(\frac{r}{R}\right)^\alpha \left(\operatorname*{osc}_{Q_R(\bar{x},\bar{t})} u + \left\| \frac{\vec{f}}{\omega}, L^p_\omega(Q) \right\| \right), \tag{3.13}$$

for some $\alpha \in (0,1)$ and $C_R > 0$ depending only on n, ν, $c(\omega, 1+2/n)$ and p.

PROOF. The existence and (3.12) are proved explicitly in [CS3, Theorems 2.3 and 2.4]. The last assertion is stated in [CS2, Theorem 3.7], but we prefer to give an explicit proof which stresses the dependence of the constants. Let $Q_r(\bar{x},\bar{t}) \subset\subset Q$ be a standard cylinder; by Lemma 3.10.v), there exists a constant $\sigma \in (0,1)$ such that $Q_{\sigma r}(\bar{x},\bar{t}) \subset Q_r^+(\bar{x},\bar{t})$. Denote now by $v \in H_0^{1,2}(Q_r(\bar{x},\bar{t}))$ the weak solution of $(\partial_t - L)v = S$ in $Q_r(\bar{x},\bar{t})$ and $v \equiv 0$ on the parabolic boundary of the standard cylinder and put $u = v + \tilde{v}$, so that $L\tilde{v} = 0$. Hence, by [CS2, Theorems 3.3 and 3.6],

$$\begin{aligned}
\operatorname*{osc}_{Q_r(\bar{x},\bar{t})} u &\leq \operatorname*{osc}_{Q_r(\bar{x},\bar{t})} \tilde{v} + 2 \sup_{Q_r(\bar{x},\bar{t})} |v| \\
&\leq (1-\varepsilon) \operatorname*{osc}_{Q_{r/\sigma}(\bar{x},\bar{t})} \tilde{v} + 2 \sup_{Q_r(\bar{x},\bar{t})} |v| \\
&\leq (1-\varepsilon) \operatorname*{osc}_{Q_{r/\sigma}(\bar{x},\bar{t})} u + 4 \sup_{Q_{r/\sigma}(\bar{x},\bar{t})} |v| \\
&\leq (1-\varepsilon) \operatorname*{osc}_{Q_{r/\sigma}(\bar{x},\bar{t})} u \\
&\quad + c \left[\left(\frac{1}{|Q_r(\bar{x},\bar{t})|} \int_{Q_r(\bar{x},\bar{t})} |v|^2 \, dx\, dt \right)^{1/2} \right. \\
&\qquad + \left(\frac{1}{\omega(Q_r(\bar{x},\bar{t}))} \int_{Q_r(\bar{x},\bar{t})} |v|^2 \, \omega \, dx\, dt \right)^{1/2} \\
&\qquad \left. + r \left(\frac{1}{\omega(Q_r(\bar{x},\bar{t}))} \int_{Q_r(\bar{x},\bar{t})} \left(\frac{|f|}{\omega} \right)^p \omega \, dx\, dt \right)^{1/p} \right],
\end{aligned}$$

where ε depends only on $c(\omega, 1+2/n, \nu, n)$. Let us consider now the terms between curly brakets. To estimate the last term, note that, if $r < R$ then

$$r\, \omega(Q_r(\bar{x},\bar{t}))^{-1/p} \simeq r^{1-(n+2)/p},$$

by Lemma 3.10.iii), with equivalence constants depending only on n, ν and $c(\omega, 1+2/n)$. But $p > 2l/(l-1) \geq 2h'/(h'-1)$ (where h' has been defined in Remark 1.8) and hence $1-(n+2)/p>0$, so that it can be estimated by $(r/R)^{\theta}$ (for some positive θ) times the average for $r=R$. Consider now the first term: it is estimated by

$$\begin{aligned}\Big(\frac{1}{|B_r(\bar{x})|}&\sup_{\{t:\,\bar{t}-h_r(\bar{x},\bar{t})<t<\bar{t}\}}\int_{B_r(\bar{x})}|v(x,t)|^2\,dx\Big)^{1/2}\\ &\le c\Big(\frac{1}{|B_r(\bar{x})|}\iint_{Q_r(\bar{x},\bar{t})}\Big(\frac{|f|}{\omega}\Big)^2\omega\,dx\,dt\Big)^{1/2}\\ &\le c\,r^{-n/2}\,\omega(Q_r(\bar{x},\bar{t}))^{(1-2/p)/2}\Big(\iint_{Q_r(\bar{x},\bar{t})}\Big(\frac{|f|}{\omega}\Big)^p\omega\,dx\,dt\Big)^{1/p}\\ &\le c\,r^{-n/2+(1-2/p)(n+2)/2}\Big(\iint_{Q_R(\bar{x},\bar{t})}\Big(\frac{|f|}{\omega}\Big)^p\omega\,dx\,dt\Big)^{1/p},\end{aligned}$$

where the first inequality follows from (3.9) and, if we look to the proof of Lemma 3.7, we see that the dependence of the constant c on the hight of Q appears only to bound the term in g that in the present case is zero. By remark 1.8, it is easy to see that we can choose $-n+(1-2/p)(n+2)>0$ and therefore we can repeat our previous arguments. Finally, by Sobolev inequality (1.5) and by the a priori estimate (3.9), the second term can be estimated by

$$\begin{aligned}c\,r\Big(\frac{1}{\omega(Q_r(\bar{x},\bar{t}))}&\int_{Q_r(\bar{x},\bar{t})}|\nabla v|^2\,\omega\,dx\,dt\Big)^{1/2}\\ &\le c\Big(\frac{r^2}{\omega(Q_r(\bar{x},\bar{t}))}\iint_{Q_r(\bar{x},\bar{t})}\Big(\frac{|f|}{\omega}\Big)^2\omega\,dx\,dt\Big)^{1/2}\\ &\le c\,r^{-n/2}\,\omega(Q_r(\bar{x},\bar{t}))^{(1-2/p)/2}\Big(\iint_{Q_r(\bar{x},\bar{t})}\Big(\frac{|f|}{\omega}\Big)^p.\end{aligned}$$

Note that all constants depend only on n, and $c(\omega,1+2/n)$. Thus we have proved that

$$\begin{aligned}\operatorname*{osc}_{Q_r(\bar{x},\bar{t})}u&\le(1-\varepsilon)\operatorname*{osc}_{Q_{r\sigma}(\bar{x},\bar{t})}u\\ &+c\Big(\frac{r}{R}\Big)^{\theta}\Big(\frac{1}{\omega(Q_R(\bar{x},\bar{t}))}\int_{Q_R(\bar{x},\bar{t})}\Big(\frac{|f|}{\omega}\Big)^p\omega\,dx\,dt\Big)^{1/p}\end{aligned}$$

and the assertion follows.

Theorem 3.14. *Assume $\omega\in A_2$, (1.1) holds, $S=\sum_i\partial_i f_i\in H^{-1,2}_{\omega}(Q)$, $g/\omega\in L^p_{\omega}$ for $p>2l/(l-1)$ and $u_0\in L^2(\Omega)$, where $l>1$ is an index for* *which inequalities* (1.5) *and* (1.6) *hold. Then the solution* $u \in H^{1,2}_{\omega}(Q)$ *of the problem*

$$(\mathrm{P}) \qquad \begin{cases} (\partial_t - L)u = g + S & \text{in } Q\,, \\ u(x,0) = u_0 & \text{in } \Omega\,, \\ u(x,t) = 0 & \text{on } \Sigma\,, \end{cases}$$

is the weak limit in $H^{1,2}_{\tilde{\omega}_1}(Q)$ *of a sequence of solutions* $u_m \in H^{1,2}_{\omega_m}(Q)$ *of the problems*

$$(\mathrm{P}_m) \qquad \begin{cases} (\partial_t - L_m)u_m = g_m + S_m & \text{in } Q\,, \\ u_m(x,0) = u_0 & \text{in } \Omega\,, \\ u_m(x,t) = 0 & \text{on } \Sigma\,, \end{cases}$$

where $\omega_m = \omega_{mm}$ *and* $\tilde{\omega}_1$ *are as in Lemma* 2.1, $L_m = \sum_i \partial_i(a_{ij}^{mm}\partial_j)$, $g_m = g(\omega/\omega_m)^{(1-p)/p}$, $S_m = \sum_i \partial_i f_{mi}$, $f_{mi} = f_i(\omega/\omega_m)^{-1/2}$. *Moreover, if* $\omega \in A_{1+2/n}$ *and* l *is chosen such that* (1.7) *also holds then* u *is the uniform limit of* $(u_m)_{m\in\mathbb{N}}$ *in any compact subset of* Q.

PROOF. Obviously, $g_m/\omega_m \in L^p_{\omega_m}(Q)$ and $S_m \in H^{-1,p}_{\omega_m}(Q)$. Since u_m is a solution of (P_m), by (3.9),

$$\begin{aligned} &\sup_{t\in[0,T]} \int_\Omega u_m(x,t)^2\,dx + \iint_Q |\nabla u_m(x,t)|^2\,\omega_m\,dx\,dt \\ &\le c\left(\|u_0; L^2(\Omega)\|^2 + \left\|\frac{g_m}{\omega_m}; L^{2p/(p+1)}_{\omega_m}(Q)\right\|^2 + \sum_{j=1}^n \left\|\frac{f_{mj}}{\omega_m}; L^2_{\omega_m}(Q)\right\|^2\right), \end{aligned}$$

where the constant c depends only on $c(\omega,2)$, (since $c(\omega_m,2)$ depends only on $c(\omega,2)$), ν, T and Ω.

Note that

$$\left\|\frac{f_{mj}}{\omega_m}; L^2_{\omega_m}(Q)\right\| = \left\|\frac{f_j}{\omega}; L^2_{\omega}(Q)\right\|,$$

$$\begin{aligned} \left\|\frac{g_m}{\omega_m}; L^{2p/(p+1)}_{\omega_m}(Q)\right\| &\le (\omega_m(\Omega)\,T)^{(p-1)/2p} \left\|\frac{g_m}{\omega_m}; L^p_{\omega_m}(Q)\right\| \\ &\le (\tilde{\omega}_2(\Omega)\,T)^{(p-1)/2p} \left\|\frac{g}{\omega}; L^p_{\omega}(Q)\right\|, \end{aligned}$$

where $\tilde{\omega}_2$ was defined in Lemma 2.1.ii). Thus

$$\begin{aligned}(3.15)\qquad &\sup_{t\in[0,T]}\int_\Omega u_m(x,t)^2\,dx + \|\,|\nabla u_m|;L^2_{\omega_m}(Q)\|^2 \\ &\le c\left(\|u_0;L^2(\Omega)\|^2 + (\tilde{\omega}_2(\Omega)\,T)^{(p-1)/p}\left\|\frac{g}{\omega};L^p_\omega(Q_T)\right\|^2 \right. \\ &\qquad \left. +\sum_{j=1}^n\left\|\frac{f_j}{\omega};L^2_\omega(Q)\right\|^2\right) = C_1\,.\end{aligned}$$

In particular, since $\omega_m \ge \tilde{\omega}_1$ (see Lemma 2.1.ii)), we have

$$\sup_{t\in[0,T]}\|u_m(\,\cdot\,,t);L^2(\Omega)\|^2 + \|\,|\nabla u_m|;L^2_{\tilde{\omega}_1}(Q)\|^2 \le C_1\,.$$

By (1.5), in view of the weak compactness of bounded sets in $H^{1,2}_{0,\tilde{\omega}_1}(Q)$ there exists a subsequence of $(u_m)_{m\in\mathbb{N}}$, again denoted by $(u_m)_{m\in\mathbb{N}}$, which converges weakly to an element u in $H^{1,2}_{0,\tilde{\omega}_1}(Q)$.

First we note that $u \in H^{1,2}_{0,\omega}(Q)$. In fact, fix $F_k = F_{k,k}$ (see Lemma 2.1). We know that $\partial u_m/\partial x_i \to \partial u/\partial x_i$ weakly in $L^2_{\tilde{\omega}_1}(Q)$ and that if $\varphi \in L^2_\omega(Q)$ then $\varphi\,\chi_{F_k} \in L^2_{\tilde{\omega}_1}$ (since on F_k, $\omega \simeq \tilde{\omega}_1$). Thus,

$$\iint \frac{\partial u_m}{\partial x_i}\,\chi_{F_k}\,\varphi\,\tilde{\omega}_1\,dx\,dt \longrightarrow \iint \frac{\partial u}{\partial x_i}\,\chi_{F_k}\,\varphi\,\tilde{\omega}_1\,dx\,dt\,,$$

for any $\varphi \in L^2_\omega(Q)$ and hence $(\chi_{F_k}\,\partial u_m/\partial x_i)_{m\in\mathbb{N}}$ converges weakly to a function in $L^2_\omega(Q)$, again since $\omega \simeq \tilde{\omega}_1$ on F_k. Therefore

$$\|\nabla u;L^2_\omega(F_k)\| \le \limsup_{m\to\infty}\|\nabla u_m;L^2_\omega(F_k)\|$$

and since for $m \ge k$, $\omega = \omega_m$ over F_k, it follows that

$$\|\nabla u;L^2_\omega(F_k)\| \le \limsup_{m\to\infty}\|\nabla u_m;L^2_{\omega_m}(F_k)\| \le C_1\,,$$

for any $k \in \mathbb{N}$. By monotone convergence theorem

$$\|\nabla u;L^2_\omega(Q)\| \le C_1\,.$$

On the other hand the same argument shows that $u \in L^2_\omega(Q)$ and hence u belongs to $H^{1,2}_{0,\omega}(Q)$ by Theorem 3.1.

Next we have to show that u is a weak solution to the problem (P). Suppose $\varphi \in W$, $\varphi(T) = 0$. By a density argument (see [CS3, proof of Theorem 2.3]) we can suppose $\varphi \in C^\infty(\bar{Q})$ and $\varphi(\cdot, t)$ compactly supported in Ω for any t. We have

$$\iint_Q \big(\langle A\,\nabla u, \nabla\varphi\rangle - u\,\varphi_t\big)\,dx\,dt = \lim_{k\to\infty} \iint_{F_k} \big(\langle A\,\nabla u, \nabla\varphi\rangle - u\,\varphi_t\big)\,dx\,dt\,.$$

It is easy to see that the linear functional

$$u \longrightarrow \iint_Q \big(\langle A\,\nabla u, \nabla\varphi\rangle - u\,\varphi_t\big)\,\chi_{F_k}(x)\,dx\,dt$$

is continuous in $H^{1,2}_{0,\tilde{\omega}_1}(Q)$ (since $\tilde{\omega}_1 \simeq \omega$ in F_k by Lemma 2.1.iv). Since $u_m \to u$ weakly in $H^{1,2}_{0,\tilde{\omega}_1}(Q)$, $A^m = A$ in F_k for any $m \geq k$ and u_m is a solution of (P_m), it follows that

$$\begin{aligned}
&\int_0^T \int_{F_k} \big(\langle A\,\nabla u, \nabla\varphi\rangle - u\,\varphi_t\big)\,dx\,dt \\
&\quad= \lim_{m\to\infty} \int_0^T \int_{F_k} \big(\langle A^m\,\nabla u_m, \nabla\varphi\rangle - u_m\,\varphi_t\big)\,dx\,dt \\
&\quad= \lim_{m\to\infty} \Big(\iint_Q \big(\langle A^m\,\nabla u_m, \nabla\varphi\rangle - u_m\,\varphi_t\big)\,dx\,dt \\
&\qquad\qquad - \int_0^T \int_{F_k'} \big(\langle A^m\,\nabla u_m, \nabla\varphi\rangle - u_m\,\varphi_t\big)\,dx\,dt \Big) \\
&\quad= \lim_{m\to\infty} \Big(\int_\Omega u_0(x)\,\varphi(x,0)\,dx \\
&\qquad\qquad + \sum_{j=1}^n \iint_Q f_{mj}(x,t)\,\frac{\partial\varphi}{\partial x_j}(x,t)\,dx\,dt \\
&\qquad\qquad - \iint_Q g_m(x,t)\,\varphi(x,t)\,dx\,dt \\
&\qquad\qquad - \int_0^T \int_{F_k'} \big(\langle A^m\,\nabla u_m, \nabla\varphi\rangle - u_m\,\varphi_t\big)\,dx\,dt \Big)\,.
\end{aligned}$$

Note now that both integrals over Q converge as $m \to \infty$. Indeed $f_{mj}(x,t) \to f(x,t)$ almost everywhere by Lemma 2.1.v) and

$$\iint_Q f_{mj}(x,t)\,\frac{\partial\varphi}{\partial x_j}(x,t)\,dx\,dt = \iint_Q \frac{f_j(x,t)}{\omega^{1/2}(x)}\,\frac{\partial\varphi}{\partial x_j}(x,t)\,\omega_m^{1/2}(x)\,dx\,dt,$$

where $f_j(x,t)\,\omega^{-1/2}(x) \in L^2(Q_T)$ (by hypothesis) and

$$\left|\frac{\partial\varphi}{\partial x_j}(x,t)\right| \omega_m^{1/2}(x) \le c_\varphi\, \tilde\omega_2^{1/2} \in L^2(\Omega)$$

(by Lemma 2.1.ii)), since $\tilde\omega_2 \in A_2$. Thus, the conclusion follows by Lebesgue dominated convergence theorem. Analogously, to prove the corresponding assertion for the integral containing g_m, we note that $g(x,t)\,\omega^{1/p-1}(x) \in L^p(Q)$ and $\omega_m^{1-1/p} \le \tilde\omega_2^{1-1/p} \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$, since $\tilde\omega_2 \in A_2$. By difference, also the integrals over F_k' converge as $m \to \infty$ to $l(k) \in \mathbb{R}$, so that

$$\begin{aligned}\int_0^T \int_{F_k} \big(\langle A\,\nabla u, \nabla\varphi\rangle - u\,\varphi_t\big)\,dx\,dt = \int_\Omega u_0(x)\,\varphi(x,0)\,dx \\ + \sum_{j=1}^n \iint_Q f_j(x,t)\,\frac{\partial\varphi}{\partial x_j}(x,t)\,dx\,dt \\ - \iint_Q g(x,t)\,\varphi(x,t)\,dx\,dt + l(k)\,.\end{aligned}$$

But, by Cauchy-Schwarz inequality, Lemma 2.1.vi) and (3.15), we have

$$\begin{aligned}&\left|\int_0^T \int_{F_k'} \big(\langle A^m\,\nabla u_m, \nabla\varphi\rangle - u_m\,\varphi_t\big)\,dx\,dt\right| \\ &\quad\le c_\varphi \sqrt{T}\,\bigg(\frac{1}{\nu}\Big(\iint_Q |\nabla u_m|^2\,\omega_m\,dx\,dt\Big)^{1/2}\,\omega_m(\Omega\cap F_k')^{1/2} \\ &\qquad + \Big(\iint_Q u_m^2\,\omega_m\,dx\,dt\Big)^{1/2}\Big(\int_{F_k'\cap\Omega}\frac{1}{\omega_m}\,dx\Big)^{1/2}\bigg) \\ &\quad\le c(\varphi,T,\nu,\text{data})\,\bigg(\big(\omega_m(\Omega\cap F_k')\big)^{1/2} + \Big(\int_{F_k'\cap\Omega}\frac{1}{\omega_m}\,dx\Big)^{1/2}\bigg).\end{aligned}$$

Note now that $\omega_m \le \tilde\omega_2$ and $1/\omega_m \le 1/\tilde\omega_1$ and that both $\tilde\omega_2$ and $1/\tilde\omega_1$ belong to A_∞. Hence (considering, for instance, the first case) there exists $\delta > 0$ such that, if Q_0 is a cube containing $\bar\Omega$, then ([GC/RF, Theorem 2.9, Chapter IV])

$$\omega_m(\Omega\cap F_k') \le \tilde\omega_2(\Omega\cap F_k') \le \tilde\omega_2(Q_0)\,\Big(\frac{|F_k'|}{|Q_0|}\Big)^\delta,$$

which is independent of m and tends to zero as $k \to \infty$, by Lemma 2.1.iv). Thus, taking the limit as $k \to \infty$, we conclude that u is a weak solution (which in turn is unique: and hence it is the limit of the original sequence, since our argument can be carried out in the same way starting from any subsequence of the original one).

Now, by (3.13) (uniform Hölder continuity of the solutions), keeping in mind that the norms of g_m and S_m are uniformly bounded with respect to m (as we showed at the beginning of the present proof), we obtain that the sequence $(u_m)_{m\in\mathbb{N}}$ is locally equicontinuous. In addition, it is locally uniformly bounded, since we can combine the local boundedness of the solutions ([CS2, Theorem 3.3]) with the a priori estimate arguing as in the proof of Hölder continuity (*i.e.* noting that the L^2_ω norm of u can be estimated by the analogous norm of its gradient, by Sobolev inequality (1.5)). Hence, we can apply Arzelà-Ascoli theorem and conclude that u_m converges to u uniformly on compact subsets (note that all converging subsequences converge to u).

4. The Green function.

In this section we will prove that there exists a weak Green function $\gamma(x,t;\xi,\tau)$ for $\partial_t u - Lu = 0$, where $Lu = \sum_{i,j}\partial_i(a_{ij}\partial_j u)$ satisfies condition (1.1), for any bounded cylinder $Q = \Omega\times(0,T)$. In particular, γ belongs to $H^{1,p'}_{0,\omega}(Q)$ and it gives a representation formula for the solutions of the Cauchy-Dirichlet problem. We will also derive some additional properties of γ which are analogous to the corresponding ones in the non-degenerate problems (see [A, Theorem 9]). The derivation is based on the fact that γ can be approximated, in convenient spaces, by Green functions in the sense of Aronson (see [A]). In the sequel, we will use the following notations: if τ is an arbitrary point in $[0,T)$ we set $Q_\tau = \Omega\times(\tau,T]$, while if t is an arbitrary point in $(0,T]$ we set $\tilde{Q}_t = \Omega\times[0,t)$.

Next, we prove the existence of a Green function for $\partial_t - L$ in Q, by following the abstract argument given by Aronson.

Theorem 4.1. *Suppose $\omega\in A_{1+2/n}$ and $p > 2l/(l-1)$, where l is an index for which* (1.5), (1.6) *and* (1.7) *hold. Then there exists a function $\gamma = \gamma(x,t;\xi,\tau)$ such that:*

i) $\gamma(x,t;\,\cdot\,,\,\cdot\,)\in H^{1,p'}_{0,\omega}(Q)$ *for* $(x,t)\in Q$ *and*

$$\|\gamma(x,t;\,\cdot\,,\,\cdot\,);H^{1,p'}_{0,\omega}(Q)\| \le C\,,$$

where C depends only on Ω, T, ν, $c(\omega,2)$ and p;

ii) *If u is a solution of the problem*

$$\begin{cases} (\partial_t - L)u = f_0 - \operatorname{div}\vec{f} \ \ \text{in } Q\,, \\ u \equiv 0 \ \ \text{on the parabolic boundary of } Q\,, \end{cases}$$

for some f_0, $\vec{f}$ such that f_0/ω, $|\vec{f}|/\omega \in L^p_\omega(Q)$, then

$$u(x,t) = \sum_{j=1}^{n} \iint_Q \partial_{\xi_j}\gamma(x,t;\xi,\tau)\, f_j(\xi,\tau)\, d\xi\, d\tau + \iint_Q \gamma(x,t;\xi,\tau)\, f_0(\xi,\tau)\, d\xi d\tau\,.$$

We will say that γ is the weak Green function for $\partial_t - L$.

PROOF. Let $\vec{f} = (f_1,\dots,f_n)$ be such that $|\vec{f}|/\omega \in L^p_\omega$ for $p > 2l/(l-1)$. By Theorem 3.11, if $(x,t) \in Q$ and u is a solution of the problem stated there with $u_0 \equiv 0$ and $S = \operatorname{div}\vec{f}$, then the linear functional

$$\vec{f} \longmapsto u(x,t)$$

is well defined and is continuous on $H^{-1,p}_\omega(Q)$ by (3.13). By Theorem 3.1, there exists a function $\gamma(x,t;\,\cdot\,,\cdot\,)$ in $H^{1,p'}_0(Q)$ such that

$$u(x,t) = \iint_Q \sum_{j=1}^{n} \partial_{\xi_j}\gamma(x,t;\xi,\tau)\, f_j(\xi,\tau)\, d\xi\, d\tau\,,$$

and

$$\|\gamma(x,t;\,\cdot\,,\cdot\,); H^{1,p'}_0(Q)\| \le C\,,$$

where C is the constant of (3.12) and hence it depends only on Ω, T, ν, $c(\omega,2)$ and p. Then the assertion follows from the Remark 3.5.

Theorem 4.2. *Suppose $\omega \in A_{1+2/n}$, condition (1.1) holds, $u_0 \in L^2(\Omega)$ and G such that $G/\omega \in L^p_\omega(Q)$, $p > 2l/(l-1)$, where l is an index for which (1.5) and (1.6) hold. Then the weak Green function $\gamma(x,t;\xi,\tau)$ of $\partial_t - Lu$ in Q has the following properties:*

i) *$\gamma(x,t;\xi,\tau) = \tilde{\gamma}(\xi,\tau;x,t)$ in $Q\times Q$ for $t > \tau$, where $\tilde{\gamma}$ is the weak Green function for the adjoint problem in Q,*

ii) *For fixed* $(\xi,\tau) \in \tilde{Q}_T$ *let* Z *denote an arbitrary open domain such that* $\bar{Z} \subset \Omega\backslash\{\xi\}$. *Then the function* $\gamma(\,\cdot\,,\,\cdot\,;\xi,\tau)$ *is a weak solution of* $\partial_t - Lu = 0$ *in* $Z\times(\tau,T)$ *with initial value zero on* $t=\tau$ *and vanishing on the lateral boundary. For fixed* $(x,t) \in Q_0$ *let* Z *denote an arbitrary open domain such that* $\bar{Z} \subset \Omega\backslash\{x\}$. *Then the function* $\gamma(x,t;\,\cdot\,,\,\cdot\,)$ *is a weak solution of the adjoint problem in* $Z\times(0,t)$ *with initial value zero on* $\tau = t$ *and vanishing on the lateral boundary.*

iii) *The weak solution of the boundary value problem*

$$\text{(P)} \qquad \begin{cases} (\partial_t - L)u = G(x,t) \quad \textit{in } Q\,, \\ u(x,0) = u_0(x) \quad \textit{for } x \textit{ in } \Omega\,, \\ u(x,t) = 0 \quad \textit{for } (x,t) \in \Sigma\,, \end{cases}$$

is given by

$$u(x,t) = \int_\Omega \gamma(x,t;\xi,0)\,u_0(\xi)\,d\xi + \iint_Q \gamma(x,t;\xi,\tau)\,G(\xi,\tau)\,d\xi\,d\tau\,.$$

iv) *The weak solution of the adjoint boundary value problem*

$$(\tilde{\text{P}}) \qquad \begin{cases} (-\partial_t - L)v = G(x,t) \quad \textit{in } Q\,, \\ v(\xi,T) = u_0(\xi) \quad \textit{for } \xi \textit{ in } \Omega\,, \\ v(\xi,t) = 0 \quad \textit{for } (x,t) \in \Sigma\,, \end{cases}$$

is given by

$$v(\xi,\tau) = \int_\Omega \gamma(x,T;\xi,\tau)\,u_0(x)\,dx + \iint_Q \gamma(x,t;\xi,\tau)\,G(x,t)\,dx\,dt\,.$$

PROOF. We denote by γ_m the Green function of $\partial_t - L^m$ (with the notations of Theorem 3.14). First, we assert that

$$\gamma_m(x,t;\,\cdot\,,\,\cdot\,)\,\omega_m^{1/p'} \longrightarrow \gamma(x,t;\,\cdot\,,\,\cdot\,)\,\omega^{1/p'} \tag{4.3}$$

and

$$\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau)\,\omega_m^{1/p'} \longrightarrow \tilde{\gamma}(\xi,\tau;\,\cdot\,,\,\cdot\,)\,\omega^{1/p'} \tag{4.4}$$

weakly in $L^{p'}(Q)$, for any p' whose Hölder conjugate satisfies $p > 2l/(l-1)$ (*i.e*, $p' < 2l/(l+1)$). First, we prove (4.3). Fix $f_0 \in L^p(Q)$ and put $g_0 = \omega^{1/p'} f_0$, so that $g_0/\omega \in L^p_\omega(Q)$. Then, as in Theorem 3.14, consider the solution u_m of the Dirichlet problem

$$\begin{cases} (\partial_t - L^m)u_m = (g_0)_m \ \text{ in } Q\,, \\ u_m \equiv 0 \ \text{ on the parabolic boundary of } Q\,. \end{cases}$$

By Theorem 3.14, we know that $\{u_m\}_{m\in\mathbb{N}}$ converges uniformly on compact subsets of Q to the solution u of the problem

$$\begin{cases} (\partial_t - L)u = g_0 \ \text{ in } Q\,, \\ u \equiv 0 \ \text{ on the parabolic boundary of } Q\,. \end{cases}$$

So, by definition of the Green function, Theorem 4.1, and [A, Theorem 9],

$$\iint_Q \gamma_m(x,t;\xi,\tau)\,(g_0)_m(\xi,\tau)\,d\xi\,d\tau \longrightarrow \iint_Q \gamma(x,t;\xi,\tau)\,g_0(\xi,\tau)\,d\xi\,d\tau\,.$$

Recalling that $(g_0)_m = g_0(\omega_m/\omega)^{1/p'} = f_0\,\omega_m^{1/p'}$ we are done. In addition, (4.4) follows by applying the same argument to the adjoint equation.

Now, let $(\xi,\tau) \in \tilde{Q}_T$ be fixed and let $\delta > 0$ be such that $\tau < T - \delta < T$; from [A, Theorem 9.v)], we know that $\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau)$ is a solution of the Dirichlet problem

$$\begin{cases} (\partial_t - L^m)v = 0 \ \text{ in } Q_{\tau+\delta}\,, \\ v(x,\tau+\delta) = \gamma_m(x,\tau+\delta;\xi,\tau) \ \text{ in } \Omega\,, \\ v \equiv 0 \ \text{ on } \partial\Omega\times]\tau+\delta,T]\,. \end{cases}$$

By Lemma 3.7 we have

$$\sup_{\tau+\delta\le t\le T} \|\gamma_m(\,\cdot\,,t;\xi,\tau);L^2(\Omega)\|^2 + \|\nabla_x\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau);L^2_{\omega_m}(Q_{\tau+\delta})\|^2 \le c\,\|\gamma_m(\,\cdot\,,\tau+\delta;\xi,\tau);L^2(\Omega)\|^2\,,$$

where c does not depend on m. Now, by [GN] we have (note that Green function γ_m satisfies the assumptions of [GN] and, in addition, a fundamental solution Γ_m exists by [A], since L^m is an usual elliptic operator)

$$\gamma_m(x,\tau+\delta;\xi,\tau) \le \Gamma_m(x,\tau+\delta;\xi,\tau) \le c\left(\frac{1}{(h_x)_m^{-1}(\delta)} + \frac{1}{(h_\xi)_m^{-1}(\delta)}\right),$$

where the constant c is independent of m and the function $(h_x)_m$ corresponds to h_x at the step m. By [CS1, Proposition 1.1] there exist positive numbers l_1 and l_2 (both depending only on $c(\omega, 1+2/n)$) and there are two constants c_1 and c_2 (depending only on $c(\omega, 1+2/n)$, l_1, l_2 and Ω) such that

$$c_1\, r^{l_1} \le (h_{x_0})_m(r) \le c_2\, r^{l_2}\,.$$

In particular,

$$c_2'\, t^{1/l_2} \le ((h_{x_0})_m)^{-1}(t) \le c_1'\, t^{1/l_1}\,, \tag{4.5}$$

where c_1' and c_2' depend only on $c(\omega, 1+2/n)$ and therefore

$$\gamma_m(x, \tau+\delta; \xi, \tau) \le C\,\delta^{-1/l_2}$$

and

$$\|\gamma_m(\,\cdot\,, \tau+\delta; \xi, \tau); L^2(\Omega)\|^2 \le C\,\delta^{-2/l_2}\,.$$

Thus we have

$$\sup_{\tau+\delta\le t\le T} \|\gamma_m(\,\cdot\,, t; \xi, \tau); L^2(\Omega)\|^2 + \|\nabla_x\gamma_m(\,\cdot\,,\,\cdot\,; \xi, \tau); L^2_{\omega_m}(Q_{\tau+\delta})\|^2 \le C\,\delta^{-2/l_2}$$

and since $\omega_m \ge \tilde\omega_1$,

$$\sup_{\tau+\delta\le t\le T} \|\gamma_m(\,\cdot\,, t; \xi, \tau); L^2(\Omega)\|^2 + \|\nabla_x\gamma_m(\,\cdot\,,\,\cdot\,; \xi, \tau); L^2_{\tilde\omega_1}(Q_{\tau+\delta})\|^2 \le C\,\delta^{-2/l_2}\,.$$

Hence (keeping in mind the Sobolev inequality for compact support functions in space variables), there exists a subsequence of $\gamma_m(\,\cdot\,,\,\cdot\,; \xi, \tau)$ which converges weakly to a limit function γ^* in $H^{1,2}_{0,\tilde\omega_1}(Q_{\tau+\delta})$. Next we show that γ^* is $\tilde\gamma(\xi, \tau, \cdot\,, \cdot\,)$. By (4.4),

$$\gamma_m(\,\cdot\,,\,\cdot\,; \xi, \tau)\,\omega_m^{1/p'} \longrightarrow \tilde\gamma(\xi, \tau; \,\cdot\,,\,\cdot\,)\,\omega^{1/p'}$$

weakly in $L^{p'}(Q_{\tau+\delta})$, $p' < 2l/(l+1)$. On the other hand, $\omega_m \ge \tilde\omega_1$ (by Lemma 2.1.iv)) and hence, for any $g \in L^p(Q_{\tau+\delta})$, the functions $(\tilde\omega_1/\omega_m)^{1/p'} g$ are uniformly bounded by a constant times g for any m and converge to $(\tilde\omega_1/\omega)^{1/p'} g$ as m tends to infinity by Lemma 2.1.v).

Then, by Lebesgue Theorem, $(\tilde{\omega}_1/\omega_m)^{1/p'}g$ converges to $(\tilde{\omega}_1/\omega)^{1/p'}g$ in $L^p(Q_{\tau+\delta})$ and hence

$$\iint_{Q_{\tau+\delta}} \gamma_m(x,t,\xi,\tau)\,\tilde{\omega}_1^{1/p'}(x)\,g(x,t)\,dx\,dt = \iint_{Q_{\tau+\delta}} \gamma_m(x,t,\xi,\tau)\,\omega_m^{1/p'}(x)\left(\frac{\tilde{\omega}_1}{\omega_m}\right)^{1/p'}(x)\,g(x,t)\,dx\,dt$$

tends to

$$\iint_{Q_{\tau+\delta}} \tilde{\gamma}(x,t,\xi,\tau)\,\omega^{1/p'}(x)\left(\frac{\tilde{\omega}_1}{\omega}\right)^{1/p'}(x)\,g(x,t)\,dx\,dt$$

as $m \to \infty$, *i.e.*

$$\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau)\,\tilde{\omega}_1^{1/p'} \longrightarrow \tilde{\gamma}(\xi,\tau;\,\cdot\,,\cdot\,)\,\tilde{\omega_1}^{1/p'}$$

weakly in $L^{p'}(Q_{\tau+\delta})$, $p' < 2l/(l+1)$. Or equivalently

$$\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau) \longrightarrow \tilde{\gamma}(\xi,\tau;\,\cdot\,,\,\cdot\,)$$

weakly in $L^{p'}_{\tilde{\omega}_1}(Q_{\tau+\delta})$, $p' < 2l/(l+1)$. But

$$\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau) \longrightarrow \gamma^*(\xi,\tau;\,\cdot\,,\,,\cdot\,)$$

weakly in $L^2_{\tilde{\omega}_1}(Q_{\tau+\delta})$ and therefore $\tilde{\gamma} = \gamma^*$ almost everywhere. The same argument can be carried out starting from any subsequence $\{\gamma_{m_k}\}_{k\in\mathbb{N}}$ and hence $\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau)$ converges weakly in $H^{1,2}_{0,\tilde{\omega}_1}$ to $\tilde{\gamma}(\xi,\tau;\,\cdot\,,\,\cdot\,)$. As in the proof of Theorem 3.14, $\tilde{\gamma}(\xi,\tau;\,\cdot\,,\,\cdot\,)$ is a weak solution of $Lu = 0$ for $(x,t) \in Q_\tau$. If we hold $(x,t) \in Q_0$ fixed and apply the same argument to γ_m considered as function of (ξ,τ), we find that $\gamma(x,t;\cdot\,,\,\cdot\,)$ is a weak solution of $\tilde{L}v = 0$ for $(\xi,\tau) \in \tilde{Q}_t$.

On the other hand, for fixed $(\xi,\tau) \in \tilde{Q}_T$ it follows from [GN, (1.3)], (4.5) and (3.13) that the sequence $\{\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau)\}_{m\in\mathbb{N}}$ is uniformly bounded and equicontinuous for (x,t) in any compact subset of Q_τ. Thus,

$$\gamma_m(\,\cdot\,,\,\cdot\,;\xi,\tau) \longrightarrow \tilde{\gamma}(\xi,\tau;\,\cdot\,,\,\cdot\,)$$

uniformly in any compact subset of Q_τ. Similarly, for each $(x,t) \in Q_0$,

$$\gamma_m(x,t;\,\cdot\,,\,\cdot\,) \longrightarrow \gamma(x,t;\,\cdot\,,\,\cdot\,)$$

uniformly in any compact subset of $\tilde{Q}_t$. Thus, in particular, i) holds. From i) and what we proved before ii) also holds.

According to Theorem 3.14, if u is a solution of the problem (P) then for $(x,t) \in Q$ we have $u_m(x,t) \to u(x,t)$ where u_m is the solution of the approximate problem (P_m). Here we are using the notation introduced in Theorem 3.14. If γ_m is the Green function associated with problem (P_m) then we have

$$u_m(x,t) = \int_\Omega \gamma_m(x,t;\xi,0)\, u_0(\xi)\, d\xi + \iint_Q \gamma_m(x,t;\xi,\tau)\, G_m(\xi,\tau)\, d\xi\, d\tau\,.$$

Now let Γ_m denotes the fundamental solution associated with problem (P_m). Then

$$\gamma_m(x,t;\xi,\tau) \leq \Gamma_m(x,t;\xi,\tau)\,,$$

and by [GN, (1.3)] it follows that

$$\gamma_m(x,t,\xi,0) \leq C\left(\frac{1}{[(h_x)_m^{-1}(t)]^n} + \frac{1}{[(h_\xi)_m^{-1}(t)]^n}\right)$$

where C depends only on ν, n, $c(\omega, 1{+}1/2n)$. By (4.5) $\{\gamma_m(x,t;\cdot,0)\}_{m\in\mathrm{N}}$ is uniformly bounded in $L^2(\Omega)$ and therefore

$$\int_\Omega \gamma_m(x,t;\xi,0)\, u_0(\xi)\, d\xi \longrightarrow \int_\Omega \gamma(x,t;\xi,0)\, u_0(\xi)\, d\xi\,,$$

as m tends to infinity. Moreover, $\{\gamma_m(x,t;\,\cdot\,,\,\cdot\,)\,\omega_m^{1/p'}\}_{m\in\mathrm{N}}$ is uniformly bounded in $L^{p'}(Q)$ for $p' < 2l/(l+1)$ (by (4.3)). Since

$$\iint_Q \gamma_m(x,t;\xi,\tau)\, G_m(\xi,\tau)\, d\xi\, d\tau$$
$$= \iint_Q \gamma_m(x,t;\xi,\tau)\, \omega_m^{1/p'}\, \frac{G}{\omega}\, \omega^{1/p}\, d\xi\, d\tau\,,$$

and $(G/\omega)\,\omega^{1/p} \in L^p$ we have that

$$\iint_Q \gamma_m(x,t;\xi,\tau)\, G_m(\xi,\tau)\, d\xi\, d\tau \longrightarrow \iint_Q \gamma(x,t;\xi,\tau)\, G(\xi,\tau)\, d\xi\, d\tau\,,$$

as m tends to infinity. This proves iii), and the proof of iv) is similar.

REMARK 4.6, If $\omega \equiv 1$ then the lower bound of the set of p such that Theorem 3.14 and Theorem 4.2 hold does not coincide with the analogous bound established in [A, Theorem 9 and Theorem 1], since we have followed the results proved in [CS1], [CS2] and [CS3]. On the other hand, in the degenerate case, the optimal value of l and hence of p is rather implicit, since it depends on the lower bound of the set of q such that $\omega \in A_{1+q}$ (see for instance [W] and [F/SC]).

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