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Spectral factorization of measurable rectangular matrix functions and the vector-valued Riemann problem

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Abstract. We define spectral factorization in L_p (or a generalized Wiener-Hopf factorization) of a measurable singular matrix function on a simple closed rectifiable contour Γ. Such factorization has the same uniqueness properties as in the nonsingular case. We discuss basic properties of the vector valued Riemann problem whose coefficient takes singular values almost everywhere on Γ. In particular, we introduce defect numbers for this problem which agree with the usual defect numbers in the case of a nonsingular coefficient. Based on the Riemann problem, we obtain a necessary and sufficient condition for existence of a spectral factorization in L_p.

1. Introduction.

Let Γ be a simple closed rectifiable contour which is the positively oriented boundary of a finitely connected region $\mathcal{D}_+$, and let $\mathcal{D}_- = \mathbb{C}_\infty \setminus (\mathcal{D}_+ \cup \Gamma)$. Let G and g be functions on Γ. The *Riemann problem* consists in finding functions ϕ_+ and ϕ_- which are analytic in $\mathcal{D}_+$ and $\mathcal{D}_-$, respectively, and whose nontangential boundary limits satisfy equation

$$\phi_+(t) + G(t)\,\phi_-(t) = g(t)\,. \tag{1.1}$$

This problem is also called a *Hilbert problem* [6], or a *barrier problem* [2], in the literature. The name Hilbert problem originates in [7], where the homogeneous version of the problem was considered under the assumptions that Γ is a smooth contour which is a boundary of a simply connected region, G is twice differentiable, and the scalar functions ϕ_+ and ϕ_- are continuous up to Γ.

A classical solution of the Riemann problem in the case where Γ is smooth and bounds a finitely connected region, G and g are Hölder continuous, and G does not vanish, is as follows. Assume $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_N$ where Γ_0 encloses $\Gamma_1 \cup \cdots \cup \Gamma_N$, and consider the homogeneous problem

$$\phi_+(t) = -G(t)\,\phi_-(t)\,. \tag{1.2}$$

Suppose the change of argument of $G(t)$ along the contour Γ_i is $2\pi\lambda_i$, $i = 0, 1, \ldots, N$. Assume $0 \in \mathcal{D}_+$, and pick a point α_i in the hole bordered by Γ_i ($i = 1, 2, \ldots, N$). Let

$$\pi(z) = (z-\alpha_1)^{\lambda_1}(z-\alpha_2)^{\lambda_2}\cdots(z-\alpha_N)^{\lambda_N}, \tag{1.3}$$

let $\kappa = \lambda_0 + \lambda_1 + \cdots + \lambda_N$, and let

$$G_0(t) = -t^{-\kappa}\pi(t)\,G(t)\,. \tag{1.4}$$

Then $\log G_0(t)$ is continuous on Γ and satisfies the Hölder condition. Consequently, if

$$\gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log G_0(t)}{t-z}\,dt \tag{1.5}$$

and $\gamma_\pm(z) = \gamma(z)$ for $z \in \mathcal{D}_\pm$, $\gamma_+(t) - \gamma_-(t) = \log G_0(t)$. Hence $e^{\gamma_+(t)} = e^{\gamma_-(t)}G_0(t)$, and

$$\varphi_+(z) = \frac{1}{\pi(z)}\,e^{\gamma(z)} \qquad \text{and} \qquad \varphi_-(z) = z^{-\kappa}e^{\gamma(z)} \tag{1.6}$$

are functions whose nontangential limits to Γ are Hölder continuous and satisfy equation (1.2). Functions φ_+ and φ_- can be used to obtain solution of the nonhomogeneous problem.

Equation (1.4) shows that the Riemann problem can be approached through factorization of its coefficient. Suppose we can find a factorization

$$G(t) = G_+(t)\, \Lambda(t)\, G_-(t)\,, \tag{1.7}$$

where $G_+(t)$ and $1/G_+(t)$ are boundary values of functions analytic in $\mathcal{D}_+$ and continuous up to Γ, $G_-(t)$ and $1/G_-(t)$ are boundary values of functions analytic in $\mathcal{D}_-$ and continuous up to Γ, and $\Lambda(t) = (t-t_+)^\kappa/(t-t_-)^\kappa$ for some points $t_+ \in \mathcal{D}_+$ and $t_- \in \mathcal{D}_-$ and an integer κ. Then (1.1) is equivalent to

$$\frac{\phi_+(t)}{G_+(t)} + \Lambda(t)\, G_-(t)\, \phi_-(t) = \frac{g(t)}{G_+(t)}\,. \tag{1.8}$$

The decomposition $g(t)/G_+(t) = g_+(t) + g_-(t)$, where g_+ (respectively g_-) is a boundary value of a function analytic in $\mathcal{D}_+$ (respectively $\mathcal{D}_-$) and continuous up to Γ, immediately yields all solutions of equation (1.1). We note that factorization (1.7) exists *e.g.* when G is Hölder continuous and does not vanish on Γ [2].

The factorization approach applies naturally to more general versions of the Riemann problem considered in the literature. The problem with $G(t)$ a square nonsingular matrix valued function has been treated in [6]. Factorability of an essentially bounded nonsingular matrix function G and the Riemann problem in L_p were considered in [12]. The case where G is a measurable nonsingular matrix function and ϕ_+ and ϕ_- are in $L_p(\Gamma)$ has been treated in [13] (see also [9]). Below, we extend some of the results presented in [9] to the case where G takes singular values. In particular, we relate the properties of the Riemann problem with a measurable singular matrix valued coefficient with existence of a factorization of the coefficient.

Let G be a continuous nonsingular matrix valued function on a simple closed rectifiable contour Γ. A (*left*) *standard factorization* of G relative to Γ is a factorization $G = G_+ \Lambda\, G_-$ where $G_+(z)$ and $G_+(z)^{-1}$ are analytic in $\mathcal{D}_+$ and continuous up to Γ, $G_-(z)$ and $G_-(z)^{-1}$ are analytic in $\mathcal{D}_-$ and continuous up to Γ, and

$$\Lambda(t) = \begin{pmatrix} \left(\dfrac{t-t_+}{t-t_-}\right)^{\kappa_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\dfrac{t-t_+}{t-t_-}\right)^{\kappa_n} \end{pmatrix} \tag{1.9}$$

for integers $\kappa_1 \geq \cdots \geq \kappa_n$. This factorization is also called a *Wiener-Hopf factorization* or a *spectral factorization relative to* Γ. The properties of a standard factorization relative to Γ are described in [2].

Let E_{p+} (respectively E_{p-}) be the space of functions f analytic in $\mathcal{D}_+$ (respectively $\mathcal{D}_-$) such that $\{\int_{\Gamma_k} |f|^p\}$ is bounded for some sequence of rectifiable contours Γ_k approaching Γ in $\mathcal{D}_+$ (respectively $\mathcal{D}_-$; see [5]). If the components of $G_+(z)$ and $G_+(z)^{-1}$ are in E_{p+} and E_{q+}, where $1/p + 1/q = 1$, the components of $G_-(z)$ and $G_-(z)^{-1}$ are in E_{q-} and E_{p-}, and Λ is given by (1.9), $G = G_+\Lambda G_-$ is called a (*left*) *factorization in* L_p [9]. We note that factorization with a different Λ has been considered in [14].

A function G may admit a left factorization in L_p although the space of all $g \in L_p(\Gamma)$ for which the problem (1.1) is solvable is not closed. Suppose the contour Γ is such that the operator of singular integration $(\mathcal{S}f)(t) = (1/\pi i) \int_\Gamma f(\tau)/(\tau - t)\, d\tau$ on the space $L_p(\Gamma)$ is bounded. Suppose G and its multiplicative inverse are essentially bounded, and $G = G_+\Lambda G_-$ is a factorization in L_p. Then the set of all $g \in L_p(\Gamma)$ for which problem (1.1) is solvable is a closed subspace of $L_p(\Gamma)$ if and only if the operator $G_+\mathcal{S}G_+^{-1}$ is bounded. If G and G^{-1} are bounded, a factorization $G = G_+\Lambda G_+^{-1}$ in L_p with the operator $G_+\mathcal{S}G_+^{-1}$ bounded is called in [2] a *generalized* (*left*) *standard factorization relative to* Γ.

The definition of a standard factorization relative to a contour has been extended to the singular case in [3] by requiring that G_+ have a left (respectively G_- a right) multiplicative inverse which is analytic in $\mathcal{D}_+$ (respectively in $\mathcal{D}_-$) and continuous up to the boundary, and that Λ be a square nonsingular diagonal matrix function as in (1.9). If G is a rational matrix function, a necessary and sufficient condition for existence of a canonical standard factorization ($\kappa_1 = \cdots = \kappa_k = 0$), together with realization formulas for the factors, has been obtained in [11]. Below, we apply this idea to factorization in L_p of measurable singular matrix valued functions. In addition to allowing functions to take singular matrix values, we make only general assumptions on contours. We assume that the contour Γ is simple, closed, and rectifiable. We do not require that Γ be regular [4] or Smirnov. Thus, the operator of singular integration on the space $L_p(\Gamma)$ is in general unbounded.

The paper is organized as follows. In Section 2 we indicate basic properties of factorization in L_p of singular matrix functions. In Section 3 we discuss the vector valued Riemann problem with singular matrix valued coefficient G. In Section 4 we relate the factorization of the coefficient G with the Riemann problem.

2. Spectral factorization in L_p.

Below, L_p with $p \geq 1$ will denote $L_p(\Gamma)$ (with respect to the usual Lebesgue measure). We will denote by L_{p+} and L_{p-} the closed subspaces of L_p formed by nontangential boundary limits of functions in E_{p+} and E_{p-}, where $E_{p\pm}$ are as defined above and $E_{\infty+}$ (respectively $E_{\infty-}$) is the space of functions analytic and bounded in $\mathcal{D}_+$ (respectively $\mathcal{D}_-$). We will identify L_{p+} and L_{p-} with E_{p+} and E_{p-}. $\dot{L}_{p-}$ will denote functions in E_{p-} which vanish at infinity. If $X \in \{L_p, L_{p+}, L_{p-}, \dot{L}_{p-}\}$, we will denote by $X^{m\times n}$ the space of $m \times n$ matrices over X. To simplify notation, we will write X^n instead of $X^{1\times n}$ or $X^{n\times 1}$.

Definition 2.1. *Let G be an $m \times n$ matrix valued function with measurable entries and let $p > 1$. By a (left) spectral factorization in L_p relative to Γ we will understand a factorization*

$$G = G_+ \Lambda\, G_- \,, \tag{2.1}$$

where

i) *$G_+ \in L_{p+}^{m\times k}$ and there exists $G_+^L \in L_{q+}^{k\times m}$ (with $q = p/(p-1)$) such that $G_+^L(t)\, G_+(t) = I$ almost everywhere on Γ,*

ii) *$G_- \in L_{q-}^{k\times n}$ and there exists $G_-^R \in L_{p-}^{n\times k}$ such that $G_-(t)\, G_-^R(t) = I$ almost everywhere on Γ,*

iii) *the middle factor*

$$\Lambda(t) = \begin{pmatrix} \left(\dfrac{t-t_+}{t-t_-}\right)^{\kappa_1} & & & \mathbf{0} \\ & \left(\dfrac{t-t_+}{t-t_-}\right)^{\kappa_2} & & \\ & & \ddots & \\ \mathbf{0} & & & \left(\dfrac{t-t_+}{t-t_-}\right)^{\kappa_k} \end{pmatrix}, \tag{2.2}$$

where t_+ is a point inside Γ, t_- is a point outside Γ, and $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_k$ are integers.

A right spectral factorization of G relative to Γ is a factorization $G = G_- \Lambda\, G_+$ with Λ as above and $G_- \in L_{p-}^{m\times k}$ and $G_+ \in L_{q+}^{k\times n}$

such that there exist functions $G_-^L \in L_{q-}^{k\times m}$ *and* $G_+^R \in L_{p+}^{k\times n}$ *for which* $G_-^L(t)\,G_-(t) = I$ *and* $G_+(t)\,G_+^R(t) = I$ *almost everywhere on* Γ*.*

Note that if a function G admits a spectral factorization in L_p relative to Γ, then the rank of G is constant almost everywhere on Γ. Also, since $\Lambda \in L_\infty^{k\times k}$, by Hölder's inequality $G \in L_1^{m\times n}$. To simplify notation, we will assume $0 \in \mathcal{D}_+$ and write

$$\Lambda(t) = \begin{pmatrix} t^{\kappa_1} & & & \mathbf{0} \\ & t^{\kappa_2} & & \\ & & \ddots & \\ \mathbf{0} & & & t^{\kappa_k} \end{pmatrix}. \tag{2.3}$$

We show first that the integers $\kappa_1, \kappa_2, \ldots, \kappa_k$ are unique.

Theorem 2.2. *Suppose* $1 < p_1 \le p_2 < \infty$ *and let* $G_{1+}\Lambda_1 G_{1-}$ *and* $G_{2+}\Lambda_2 G_{2-}$ *with*

$$\Lambda_1(t) = \begin{pmatrix} t^{\kappa_1^{(1)}} & & & \mathbf{0} \\ & t^{\kappa_2^{(1)}} & & \\ & & \ddots & \\ \mathbf{0} & & & t^{\kappa_k^{(1)}} \end{pmatrix}$$

and

$$\Lambda_2(t) = \begin{pmatrix} t^{\kappa_1^{(2)}} & & & \mathbf{0} \\ & t^{\kappa_2^{(2)}} & & \\ & & \ddots & \\ \mathbf{0} & & & t^{\kappa_k^{(2)}} \end{pmatrix}$$

be spectral factorizations in L_{p_1} *and* L_{p_2} *of a function* $G \in L_1^{m\times n}$ *relative to a contour* Γ*. Then* $\kappa_j^{(1)} \ge \kappa_j^{(2)}$ *for* $j = 1, 2, \ldots, k$*.*

Proof. Let $G_{2-}^R \in L_{p_2-}^{n\times k}$ and $G_{1+}^L \in L_{q_1+}^{k\times m}$ be right and left multiplicative inverses of G_{2-} and G_{1+}. Then

$$\Lambda_1 H_- = H_+ \Lambda_2\ , \tag{2.4}$$

where $H_+ = G_{1+}^L G_{2+} \in L_{p+}^{k\times k}$ and $H_- = G_{1-} G_{2-}^R \in L_{p-}^{k\times k}$ with $p = 1/(1/q_1 + 1/p_2) = 1/(1 - 1/p_1 + 1/p_2) \ge 1$. Also, G_{1+} and G_{2+} have the same column span almost everywhere on Γ, so H_+ takes nonsingular values almost everywhere on Γ. Similarly, H_- takes nonsingular values almost everywhere on Γ.

It follows from (2.4) that

$$t^{\kappa_i^{(1)}-\kappa_j^{(2)}} H_-(i,j) = H_+(i,j)\,. \tag{2.5}$$

Since $L_{p+} \cap L_{p-}$ consists of constants, $H_+(i,j) = 0$ if $\kappa_j^{(2)} > \kappa_i^{(1)}$ and $H_+(i,j)$ is a polynomial of degree at most $\kappa_i^{(1)} - \kappa_j^{(2)}$ otherwise. Suppose $\kappa_r^{(2)} > \kappa_r^{(1)}$. Then, for all $j \le r$ and $i \ge r$, $H_+(i,j) = 0$ contradicting nonsingularity of H_+ almost everywhere on Γ.

Corollary 2.3. *The integers* $\kappa_1, \kappa_2, \dots, \kappa_k$ *in* (2.2) *are unique.*

The integers $\kappa_1, \kappa_2, \dots, \kappa_k$ in (2.2) or (2.3) are called the *indices* of the factorization, and the sum of all indices is called the *total index* of the factorization. If all the indices of the factorization are equal to 0, the factorization is said to be *canonical*.

The proof of Theorem 2.2 actually gives the nonuniqueness of all the factors in a spectral factorization.

Theorem 2.4. *Suppose* $1 < p_2 \le p_1$,

$$G_{1+} \Lambda\, G_{1-} \tag{2.6}$$

is a spectral factorization in L_{p_1} *of a function* G *relative to a contour* Γ, *and* G *admits spectral factorization in* L_{p_2} *relative to* Γ *with the same total index. Then*

$$G_{2+} \Lambda\, G_{2-} \tag{2.7}$$

is a spectral factorization in L_{p_2} *of* G *relative to* Γ *if and only if*

$$G_{2+} = G_{1+} H_+ \qquad \text{and} \qquad G_{2-} = \Lambda^{-1} H_+^{-1} \Lambda\, G_{1-} \tag{2.8}$$

where H_+ *is a matrix polynomial such that* $\det H_+ \not\equiv 0$ *and*

i) $H_+(i,j) = 0$ *if* $\kappa_i < \kappa_j$,
ii) $H_+(i,j)$ *is a constant if* $\kappa_i = \kappa_j$,
iii) $\deg\, H_+(i,j) \le \kappa_i - \kappa_j$ *if* $\kappa_i > \kappa_j$.

PROOF. Suppose (2.7) is a spectral factorization in L_{p_2} of G. In the notation of the proof of Theorem 2.2,

$$H_+ = G_{1+}^L G_{2+} \tag{2.9}$$

where H_+ is a matrix polynomial whose determinant is not equal to zero identically and which satisfies properties i)-iii). In particular, $H_+ \in L_{\infty+}$. Multiplying both sides of (2.9) by G_{1+} we obtain

$$G_{1+}H_+ = G_{1+}G_{1+}^L G_{2+}\,. \tag{2.10}$$

Since $G_{1+}G_{1+}^L$ is a projection onto the column span of G_{1+}, and the column spans of G_{1+} and G_{2+} coincide almost everywhere on Γ,

$$G_{1+}(z)\, G_{1+}^L(z)\, G_{2+}(z) = G_{2+}(z) \tag{2.11}$$

for almost everywhere $z \in \Gamma$. Since a function analytic in $\mathcal{D}_+$ with nontangential boundary values equal to 0 on a set of positive measure is identically 0, equality (2.11) is valid inside Γ and

$$G_{1+}H_+ = G_{2+}\,. \tag{2.12}$$

Hence

$$G_{1+}\Lambda\, G_{1-} = G_{2+}\Lambda\, G_{2-} = G_{1+}H_+\Lambda\, G_{2-} = G_{1+}\Lambda\Lambda^{-1}H_+\Lambda\, G_{2-}$$

and the second equality in (2.8) holds as well.

Suppose now (2.6) is a spectral factorization of G in L_{p_1} relative to Γ and H_+ with $\det H_+ \not\equiv 0$ satisfies conditions i)-iii) of the theorem, and $G_{2\pm}$ satisfy (2.8). Then $\det H_+$ is a nonzero constant, and $H_+^{\pm 1} \in L_{\infty+}$. Also, $\Lambda^{-1}H_+\Lambda$ is a matrix polynomial in $1/z$ with a nonzero constant determinant, and $(\Lambda^{-1}H_+\Lambda)^{\pm 1} \in L_{\infty-}$. Hence $G_{2+} \in L_{p_1+}^{m\times k}$, $G_{2-} \in L_{q_1-}^{k\times n}$, and G_{2-}^R, a right multiplicative inverse of G_2, is an element of $L_{p_1-}^{k\times n}$. Suppose $\tilde{G}_{2+}\Lambda\tilde{G}_{2-}$ is a factorization of G in L_{p_1} relative to Γ and $\tilde{G}_{2+}^L \in L_{q_2+}^{k\times m}$ is a left multiplicative inverse of G_2. Then

$$\tilde{G}_{2+}^L G_{2+}\Lambda = \Lambda\, \tilde{G}_{2-}G_{2-}^R\ ,$$

or $G_+\Lambda = \Lambda\, G_-$ where $G_+ = \tilde{G}_{2+}^L G_{2+} \in L_{1+}^{k\times k}$ and $G_- = \tilde{G}_{2-}G_{2-}^R \in L_{1-}^{k\times k}$. By the same argument as above, G_+ is a unimodular matrix polynomial and G_{2+} has a left multiplicative inverse in $L_{q_2+}^{k\times m}$.

Since $G_{2+}\Lambda\, G_{2-} = \tilde{G}_{2+}\Lambda\, \tilde{G}_{2-}$ and the functions Λ, Λ^{-1}, and G_{+}^{-1} are bounded, $G_{2-} \in L_{q_2-}^{k\times n}$. Thus, (2.7) with G_{2+} and G_{2-} given by (2.8) is a spectral factorization of G in L_{p_2} relative to Γ.

In particular, Theorem 2.4 determines possible nonuniqueness of a spectral factorization in L_p of a function G relative to Γ. It also has the following corollary.

Corollary 2.5. *Suppose $1 < p_1 < p_2$ and a matrix function G admits spectral factorizations in L_{p_1} and L_{p_2} relative to Γ with the same total index. Then*

i) *G admits a spectral factorization in L_p relative to Γ for every $p \in [p_1, p_2]$,*

ii) *if $p_0 \in [p_1, p_2]$, a spectral factorization in L_{p_0} of G relative to Γ is a spectral factorization in L_p for all $p \in [p_1, p_2]$.*

A meromorphic matrix function W has a pole at a point $\lambda \in \mathbb{C}$ if it has a nonzero coefficient at a negative power of $z - \lambda$ in the Laurent expansion at λ. Equivalently, W has a pole at λ if at least one of its entries has a pole at λ. The function W has a zero at λ if each meromorphic multiplicative generalized inverse of W has a pole at λ. If the function W is analytic at λ, it has a zero at λ if the rank of $W(z)$ drops at $z = \lambda$. Every rational matrix function without poles or zeros on Γ admits a spectral factorization relative to Γ with all the factors rational (see [2] for the discussion of the regular case, that is, the case where the function is square and takes nonsingular values at all but a finite number of points; the argument in the nonregular case is similar). Later, we will need the following observation.

Proposition 2.6. *If $G \in L_1^{m\times n}$ admits a spectral factorization in L_p relative to Γ and F and H are rational $M \times m$ and $n \times N$ matrix functions analytic and with full column respectively row rank on Γ, then the function FGH also admits a spectral factorization in L_p relative to Γ.*

Proof. Let $\tilde{G}_{+}\tilde{\Lambda}\, \tilde{G}_{-}$ be a spectral factorization in L_p relative to Γ of the function G. Since F is a rational matrix function, there is a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_\eta\} \subset \mathcal{D}_{+}$ which contains all the poles and zeros of F in $\mathcal{D}_{+}$. Pick $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_\eta\}$. After multiplying $F\tilde{G}_{+}$ on the right by a unimodular matrix polynomial in $z-\lambda$, we can obtain a matrix function whose columns have linearly independent leading coefficients in the Laurent expansion at λ. Indeed, suppose $F\tilde{G}_+ = (\, f_1 \quad f_2 \quad \dots \quad f_k \,)$ and the leading coefficients in the Laurent expansions at λ of f_i's are linearly dependent. Then we can replace say f_i by

$$\tilde{f}_i(z) = f_i(z) - \sum_{\substack{j=1 \\ j\neq i}}^{n} c_j\,(z-\lambda)^{\gamma_j} f_j(z)\,, \tag{2.13}$$

with c_j's constants and γ_j's nonnegative integers such that $\tilde{f}_i$ has a pole at λ of a smaller order, or vanishes at λ to a higher order, than f_i. Since the columns of F are linearly independent over the field of scalar rational functions, for every function ϕ analytic and nonzero at λ the order of the zero at λ of the product $F\tilde{G}_+\phi$ is bounded by the largest partial multiplicity of the zero of F at λ. Hence the finite number of operations as in (2.13) can provide a matrix function whose columns have linearly independent leading coefficients in the Laurent expansions at λ. It follows that there exists a square rational matrix function R_1 whose determinant is not identically equal to zero and which has neither poles nor zeros on Γ such that $F\tilde{G}_+R_1 = \widehat{G}_+ \in L_{p+}^{m\times k}$ has full column rank at all points $z \in \mathcal{D}_+$ and $R_1^{-1}\tilde{G}_+^L F^{-1} \in L_{q+}^{k\times m}$.

Similarly, there exists a square rational matrix function R_2 whose determinant is not equal to zero identically and which does not have poles or zeros on Γ such that $R_2\tilde{G}_-H = \widehat{G}_- \in L_{q-}^{k\times n}$ has a right multiplicative inverse in $L_{p-}^{n\times k}$. If $\widehat{R}_1\Lambda\widehat{R}_2$ is a spectral factorization relative to Γ of the rational matrix function $R_1^{-1}\tilde{\Lambda}R_2^{-1}$, $(\widehat{G}_+\widehat{R}_1)\Lambda(\widehat{R}_2\widehat{G}_-)$ is a spectral factorization in L_p relative to Γ of the function G.

We illustrate the concepts of this section with an example.

EXAMPLE 2.7 Let Γ be the unit circle. Pick a branch of $z^{1/3}$ on $\mathbb{C}\setminus(-\infty,0)$, and let

$$G(t) = \begin{pmatrix} (t^{1/3})^2 \\ (t^{1/3})^5 \end{pmatrix}\,,$$

where the value of $t^{1/3}$ is determined almost everywhere by the selected branch. Let $\Phi(z)$ be a branch of $(z+1)^{2/3}$ which is analytic in $\mathbb{C}\setminus(-\infty,-1]$, and let $\Psi(z)$ be a branch of $(z/(z+1))^{2/3}$ which is analytic in $\mathbb{C}_\infty\setminus[-1,0]$, such that

$$G(t) = \begin{pmatrix} \Phi(t) \\ t\,\Phi(t) \end{pmatrix} (\,\Psi(t)\,) =: G_+(t)\,G_-(t)\,. \tag{2.14}$$

Let $p > 3$. Then $G_+ \in L_{p+}^{2\times 1}$ and $G_- \in L_{q-}$. Also, $G_-^{-1} \in L_{p-}$ and G_+ has a left multiplicative inverse $G_+^L(z) = \begin{pmatrix} \Phi(z)^{-1} & 0 \end{pmatrix} \in L_{q+}^{1\times 2}$. Thus, (2.14) is a canonical spectral factorization of G in L_p relative to the circle.

Suppose $p \in (1,3)$. From (2.14),

$$G(t) = \begin{pmatrix} \dfrac{1}{t+1}\,\Phi(t) \\ \dfrac{t}{t+1}\,\Phi(t) \end{pmatrix} (\,t\,) \left(\frac{t+1}{t}\,\Psi(t) \right) =: \widehat{G}_+(t)\,(\,t\,)\,\widehat{G}_-(t)\,. \tag{2.15}$$

Plainly, $\widehat{G}_+ \in L_{p+}^{2\times 1}$ and $\widehat{G}_- \in L_{q-}$. Also, $\widehat{G}_-^{-1} \in L_{p-}$ and $\widehat{G}_+$ has a left multiplicative inverse $\widehat{G}_+^L(z) = \begin{pmatrix} (z+1)/\Phi(z) & 0 \end{pmatrix} \in L_{q+}^{1\times 2}$. Thus, (2.15) is a spectral factorization of G in L_p relative to the circle.

Suppose G admits a spectral factorization in L_3 relative to the circle. By Theorem 2.2, the total index of the factorization is either 0 or 1. Then, by Corollary 2.5, either (2.14) or (2.15) is a spectral factorization of G in L_3 relative to the circle. Since $G_- \notin L_{3/2-}$ and $\widehat{G}_+ \notin L_{3+}$, this is a contradiction. Thus, G admits a spectral factorization in L_p relative to the circle if and only if $p \in (1,3) \cup (3,\infty)$.

3. Vector-valued Riemann problem with singular coefficient.

Suppose G is a measurable $m \times n$ matrix valued function on a contour Γ, and $p > 1$. The vector-valued Riemann problem consists in finding for a given function $g \in L_p^m$ a pair of functions (ϕ_+, ϕ_-) with $\phi_+ \in L_{p+}^m$ and $\phi_- \in \dot{L}_{p-}^n$ such that

$$\phi_+(t) + G(t)\,\phi_-(t) = g(t)\,. \tag{3.1}$$

For brevity, we will refer to this problem as the *Riemann problem* with *coefficient* G. The set of all functions $g \in L_p^m$ for which the problem is solvable is called the *image* of the problem. If the image of the Riemann problem is closed, the problem is said to be *normally solvable.* The set of all solutions of the homogeneous problem is called the *kernel* of the problem.

The *dual problem* consists in finding for a given $h \in L_q^n$ a pair of functions $\psi_- \in \dot{L}_{q-}^n$ and $\psi_+ \in L_{q+}^m$ such that

$$\psi_-(t) + G^T(t)\,\psi_+(t) = h(t)\,. \tag{3.2}$$

Here q is the conjugate exponent to p, that is, $1/p+1/q=1$. Similarly as in the case where G takes nonsingular values almost everywhere on Γ [9], there is a connection between the Riemann problem and its dual. Identify L_q^n with the dual space of L_p^n through the map

$$\langle f,\, g\rangle = \sum_{j=1}^{n} \int_\Gamma f_j(t) g_j(t)\, dt$$

for all $f(t)=\sum_{j=1}^n f_j(t)e_j \in L_p^n$ and all $g(t)=\sum_{j=1}^n g_j(t)e_j \in L_q^n$. If $\mathcal{L}\subset L_p^n$, the *annihilator* of $\mathcal{L}$ is the closed subspace of L_q^n

$$\{\, g\in L_q^n :\ \langle f,\, g\rangle = 0, \quad \text{for all } f\in \mathcal{L}\,\}\,.$$

Proposition 3.1. *The annihilator of the image of the Riemann problem with coefficient G contains the space of "+" components of elements in the kernel of its dual. If $G\in L_\infty^{m\times n}$, the two spaces coincide.*

PROOF. Suppose $\psi_- + G^T\psi_+ = 0$ for some $\psi_-\in \dot{L}_{q-}^n$ and $\psi_+\in L_{q+}^m$. Then $\psi_+^T G = -\psi_- \in \dot{L}_{q-}^n$, and hence

$$\langle \psi_+,\, (\phi_+ + G\phi_-)\rangle \;=\; \langle \psi_+,\, \phi_+\rangle \;-\; \langle \psi_-,\, \phi_-\rangle \;= 0\,,$$

for all $\phi_+\in L_{p+}^m$ and $\phi_-\in \dot{L}_{p-}$. Thus, ψ_+ annihilates the image of the problem.

Suppose $\langle \psi,\, \phi_+ + G\phi_-\rangle \;=0$ for all $\phi_+\in L_{p+}^m$ and all $\phi_-\in \dot{L}_{p-}^n$ such that $G\phi_-\in L_p^m$. Then $\langle\psi,\,\phi_+\rangle \;=\; 0$ for all $\phi_+\in L_{p+}^m$ and $\psi =: \psi_+\in L_{q+}^m$. If $G\in L_\infty^{m\times n}$, $G\phi_-\in L_p^m$ for all $\phi_-\in\dot{L}_{p-}^n$ and so $G^T\psi_+$ annihilates $\dot{L}_{p-}^n$. That is, $G^T\psi_+\in \dot{L}_{q-}^n$ and ψ_+ is the "+" component of an element in the kernel of the dual problem.

If the coefficient G of a Riemann problem takes almost everywhere nonsingular values, the defect numbers of the problem are the dimension α_R of the kernel and the co-dimension β_R of the closure of the image of the problem. If G takes singular values, both α_R and β_R are generically infinite. In view of Proposition 3.1, β_R can be defined as the co-dimension of $\{\psi_+\in L_{q+}^m :\ \psi_+ G = 0\}$ in the annihilator of the image of the problem. This definition discards the generic left kernel of G.

A similar observation holds for the dual problem.

Proposition 3.2. *The annihilator of the image of the dual problem contains the space of "$-$" components of elements in the kernel of the problem. If $G \in L_\infty^{m\times n}$, the two spaces coincide.*

Suppose G takes nonsingular values almost everywhere on Γ. Then

$$\begin{aligned}
&\{(\psi_+,\psi_-) \in L^n_{q+} \times \dot{L}^n_{q-} : \quad \psi_- + G^T\psi_+ = 0\} \\
&\qquad \cong \{\psi_+ \in L^n_{q+} : \; G^T\psi_+ \in \dot{L}^n_{q-}\} \\
&\qquad \cong \{\psi_- \in \dot{L}^n_{q-} : \; \psi_- + G^T\psi_+ = 0, \quad \text{for some } \psi_+ \in L^n_{q+}\}\,.
\end{aligned} \tag{3.3}$$

Indeed, if $G^T\psi_+ = 0$, then $\psi_+ = 0$. Hence the map $(\psi_+,\psi_-) \to \psi_-$ is a bijection from the first space in (3.3) to the third one. Plainly, the map $(\psi_+,\psi_-) \to \psi_-$ is a bijection from the first space in (3.3) to the second one. If G takes singular values on Γ, the same $\psi_- \in \dot{L}_{q-}$ may occur in several (in fact, infinitely many) elements in the kernel of the dual problem. Thus, the second congruence in (3.3) does not have to be valid. More precisely,

$$\begin{aligned}
&\{(\psi_+,\psi_-) \in L^m_{q+} \times \dot{L}^n_{q-} : \; \psi_- + G^T\psi_+ = 0\} \\
&\qquad \cong \{\psi_+ \in L^m_{q+} : \; G^T\psi_+ \in \dot{L}^n_{q-}\} \\
&\qquad \cong \{\psi_- \in \dot{L}^n_{q-} : \; \psi_- + G^T\psi_+ = 0 \;\text{ for some } \psi_+ \in L^m_{q+}\} \\
&\qquad\quad \dot{+}\; \{\psi_+ \in L^m_{q+} : \; G^T\psi_+ = 0\}\,.
\end{aligned}$$

The space on the right hand side of the preceding direct sum represents the generic kernel of G^T. The dimension of the space on the left hand side of this direct sum can be finite when the generic kernel of G^T is infinite dimensional. Similarly,

$$\begin{aligned}
&\{(\phi_+,\phi_-) \in L^m_{p+} \times \dot{L}^n_{p-} : \; \phi_+ + G\psi_- = 0\} \\
&\qquad \cong \{\phi_- \in \dot{L}^n_{p-} : \; G\phi_- \in L^m_{p+}\} \\
&\qquad \cong \{\phi_+ \in L^m_{p+} : \; \phi_+ + G\phi_- = 0 \;\text{ for some } \phi_- \in \dot{L}^n_{p-}\} \\
&\qquad\quad \dot{+}\; \{\phi_- \in \dot{L}^n_{p-} : \; G\phi_- = 0\}\,.
\end{aligned}$$

The direct summand on the right hand side of the last congruence can be finite dimensional although $\ker G$ is generically infinite dimensional.

Definition 3.3. *The defect numbers of a Riemann problem with coefficient G are the dimension α_R of the space of "$+$" components of elements in the kernel of the problem, and the co-dimension β_R of*

$$\{\psi_+ \in L^m_{q+} : \; G^T\psi_+ = 0\} \tag{3.4}$$

in the annihilator of its image. If α_R or β_R is finite, the difference $\alpha_R - \beta_R$ is called the index of the problem. The defect numbers of the dual problem are the dimension α_D of the space of "$-$" components of elements in the kernel of the dual problem, and the co-dimension β_D of

$$\{\phi_- \in \dot{L}^n_{p-} : \ G\phi_- = 0\} \tag{3.5}$$

in the annihilator of the image of the dual problem. If α_D or β_D is finite, the difference $\alpha_D - \beta_D$ is called the index of the dual problem.

Note that if G takes nonsingular values almost everywhere on Γ, the spaces (3.4) and (3.5) are trivial and Definition 3.3 is equivalent to the usual definition of defect numbers. Also note that (3.4) and (3.5) are closed subspaces of L^m_q and $\dot{L}^n_p$. To see that (3.5) is closed, suppose $\phi \in L^n_p$ is such that $G\phi \neq 0$. Without loss of generality assume that G consists of a single row. Let $G^\dagger(t) = G(t)^*$ if $G(t) = 0$, and let

$$G^\dagger(t) = \frac{1}{G(t)G(t)^*}\, G(t)^*$$

otherwise. Then $G^\dagger$ is a measurable matrix function whose values are Moore-Penrose inverses of the values of G. We have

$$\phi = G^\dagger G\phi + (I - G^\dagger G)\phi =: \phi_1 + \phi_2$$

and $\|\phi_1\|_p > 0$. For any $\tilde{\phi} \in L^n_p$ such that $G\tilde{\phi} = 0$,

$$\|\phi - \tilde{\phi}\|_p = \|\phi_1 + (\phi_2 - \tilde{\phi})\|_p \geq \|\phi_1\|_p \ ,$$

and it follows that $\{\phi \in L^n_p : \ G\phi = 0\}$ is a closed subspace of L^n_p. Hence (3.5), the intersection of this space and $\dot{L}^n_{p-}$, is closed. The space (3.4) is closed by a similar argument.

The defect numbers of a Riemann problem and its dual are related as follows.

Proposition 3.4. *If α_R, β_R, α_D, and β_D are the defect numbers of a Riemann problem and its dual, then*

$$\alpha_R \leq \beta_D \qquad \text{and} \qquad \alpha_D \leq \beta_R \ . \tag{3.6}$$

Also, inequalities (3.6) are equalities if the indices of the problem and its dual are finite and opposite or if $G \in L^{m\times n}_\infty$.

PROOF. The space of " + " components of elements in the kernel of the Riemann problem is isomorphic to the quotient space of " − " components of elements in the kernel of the problem modulo $\{\phi_- \in \dot{L}^n_{p-} : G\phi_- = 0\}$. Hence, by Proposition 3.2, $\alpha_R \le \beta_D$ with equality if $G \in L^{m\times n}_\infty$. Similarly, by Proposition 3.1, $\alpha_D \le \beta_R$ with equality if $G \in L^{m\times n}_\infty$.

Suppose the indices of the problem and its dual are finite and opposite. Then

$$\alpha_R - \beta_D = \beta_R - \alpha_D \, .$$

Since by (3.6) $\alpha_R - \beta_D \le 0$ and $\beta_R - \alpha_D \ge 0$, it follows that $\alpha_R = \beta_D$ and $\alpha_D = \beta_R$.

We discuss now the homogeneous Riemann problem in the case where the coefficient G admits a spectral factorization in L_p relative to Γ.

Proposition 3.5. *Suppose $G_+\Lambda G_-$ is a spectral factorization in L_p relative to Γ of the coefficient G of a Riemann problem, let $G^L_+ \in L^{k\times m}_{q+}$ be a left multiplicative inverse of G_+, and let $G^R_- \in L^{n\times k}_{p-}$ be a right multiplicative inverse of G_-. Then*

i) *(ϕ_+, ϕ_-) is a solution of the homogeneous problem $\phi_+ + G\phi_- = 0$ if and only if*

$$(3.7) \qquad \phi_+ = G_+\rho_+ \quad \text{and} \quad \phi_- = r_- - G^R_-\Lambda^{-1}\rho_+ \, ,$$

where ρ_+ is a vector function with j^{th}-entry a polynomial of degree at most $\kappa_j - 1$ if $\kappa_j > 0$ and zero if $\kappa_j \le 0$, and $r_- \in \dot{L}^n_{p-}$ is such that $Gr_- = 0$,

ii) *(ψ_+, ψ_-) is a solution of the homogeneous dual problem $\psi_- + G^T\psi_+ = 0$ if and only if*

$$\psi_- = G^T_-\rho_- \quad \text{and} \quad \psi_+ = r_+ - (G^L_+)^T\Lambda^{-1}\rho_- \, ,$$

where ρ_- is a vector function with j^{th} entry zero if $\kappa_j \ge 0$ and a polynomial in z^{-1} of degree at most $-\kappa_j$ which vanishes at infinity if $\kappa_j < 0$, and $r_+ \in L^m_{q+}$ is such that $G^T r_+ = 0$.

PROOF. We verify assertion i). Suppose (ϕ_+, ϕ_-) is a solution of the homogeneous problem. Then

$$(3.8) \qquad G^L_+\phi_+ = -\Lambda G_-\phi_- =: \rho_+ \, .$$

Comparing both sides of the equality (3.8) we find out that ρ_+ is a vector polynomial satisfying the degree requirements. We have

$$G_+ G_+^L \phi_+ = G_+ \rho_+ \ .$$

Since $\phi_+ \in \text{im } G_+$ almost everywhere on Γ, $G_+ G_+^L \phi_+ = \phi_+$ and the first equality in (3.7) holds. By (3.8),

$$G_- \phi_- = -\Lambda^{-1} \rho_+ \ . \tag{3.9}$$

Since $-G_-^R \Lambda^{-1} \rho_+$ is a solution of equation $G_- x = -\Lambda^{-1} \rho_+$ in $\dot{L}_{p-}^n$, $r_- := \phi_- + G_-^R \Lambda^{-1} \rho_+ \in \dot{L}_{p-}^n$ is such that $G r_- = 0$. Thus, the second equality in (3.7) holds.

Conversely, suppose ϕ_+ and ϕ_- satisfy (3.7) with appropriate r_- and ρ_+. Then

$$\phi_+ + G\phi_- = G_+ \rho_+ + G r_- - G_+ \rho_+ = 0 \, ,$$

and (ϕ_+, ϕ_-) is a solution of the homogeneous problem.

It follows from Proposition 3.5 that if the coefficient G in a Riemann problem admits a spectral factorization in L_p relative to Γ, then α_R equals the sum of positive indices of the factorization, and α_D equals the absolute value of the sum of negative indices of the factorization. In fact, a stronger statement is true.

Theorem 3.6. *Suppose the coefficient G in a Riemann problem admits a spectral factorization in L_p relative to Γ with indices $\kappa_1, \kappa_2, \ldots, \kappa_k$. Then*

$$\alpha_R = \beta_D = \sum \{\kappa_i : \ \kappa_i > 0\}$$

and

$$\alpha_D = \beta_R = \sum \{-\kappa_i : \ \kappa_i < 0\} \, .$$

Proof. We show that β_D is the sum of the positive indices; the argument regarding β_R is similar. Since $\dot{L}_{q-}^n$ is contained in the image of the dual problem, the annihilator of the image of the dual problem is a subspace of $\dot{L}_{p-}^n$. Let $G = G_+ \Lambda G_-$ with Λ as in (2.3) be a spectral factorization in L_p relative to Γ, let j be such that $\kappa_j > 0 \geq \kappa_{j+1}$, and let $G_1, G_2, \ldots, G_j$ be the first j columns of $G_-^R \in L_{p-}^{n\times k}$. We show that the elements of the set

$$\{t^{-i} G_l(t) : \; 1 \le l \le j, \; 1 \le i \le \kappa_l\} \tag{3.10}$$

form a basis for a space which complements the space (3.5) in the annihilator of the image of the dual problem. Since $G_-^R(\infty)$ has linearly independent columns, the elements of the set (3.10) are linearly independent. Using the factorization $G = G_+ \Lambda G_-$, we can rewrite the space (3.5) as

$$\{\, \phi_- \in \dot{L}_{p-}^n : \; G_- \phi_- = 0 \,\}. \tag{3.11}$$

Since G_l's are the columns of a right multiplicative inverse of G_-, the span of the set (3.10) intersects trivially with the space (3.11). Now members of the set (3.10) annihilate $\dot{L}_{q-}^n$ and

$$t^{-i} G(t)\, G_l(t) \in L_{p+}^n, \qquad 1 \le l \le j, \quad 1 \le i \le \kappa_l \, .$$

Hence the members of the set (3.10) annihilate the image of the dual problem. Finally, consider an arbitrary $\phi_- \in \dot{L}_{p-}^n$ that annihilates the image of the dual problem. Choose f_- in the linear span of (3.10) such that $\Lambda G_-(\phi_- - f_-)(\infty) = (0)$ and let $\hat{\phi}_- = \phi_- - f_-$. Then $\hat{\phi}_- \in \dot{L}_{p-}^n$ and

$$\int_\Gamma \hat{\phi}_-(t)^T G_-(t)^T \Lambda(t)\, G_+(t)^T \psi_+(t)\, dt = 0 \tag{3.12}$$

for all $\psi_+ \in L_{q+}^m$ such that $G_-^T \Lambda G_+^T \psi_+ \in L_q^n$. In particular, (3.12) holds whenever $\psi_+ = (G_+^L)^T p$ with $G_+^L \in L_{q+}^{k\times m}$ a left multiplicative inverse of G_+ and p a vector polynomial. Hence

$$\int_\Gamma \big(\Lambda(t)\, G_-(t)\, \hat{\phi}_-(t)\big)^T p(t)\, dt = 0$$

for each vector polynomial p and $\Lambda G_- \hat{\phi} \in L_{1+}^k$. Since $\Lambda G_- \hat{\phi}_- \in \dot{L}_{1-}^k$, it follows that $\Lambda G_- \hat{\phi}_- = 0$ and $\phi_- = f_- + \hat{\phi}_-$ where f_- is in the span of (3.10) and $\hat{\phi}_-$ is a member of the space (3.11).

Corollary 3.7. *If the coefficient G of a Riemann problem admits a spectral factorization in L_p relative to Γ, then the index of the problem,* *and the opposite of the index of the dual problem, are both equal to the total index of the factorization.*

In particular, if G admits a spectral factorization in L_p, the indices of the Riemann problem and its dual are finite and opposite.

4. Condition for existence of a spectral factorization.

We will need below the following lemma. If G is a meromorphic matrix function defined on a connected domain $\mathcal{D}$, its rank is constant at all but a countable number of points in $\mathcal{D}$. This rank is usually called the *normal rank* of G.

Lemma 4.1. *Suppose* Γ *is a simple closed curve which forms a boundary of a connected domain* $\mathcal{D}_+$*, let* $p > 0$*, and suppose* $G \in L_p^{m\times n}$ *is formed by nontangential boundary values of a matrix function* G_+ *meromorphic in* $\mathcal{D}_+$ *with normal rank* k*. Then* $\operatorname{rank} G = k$ *almost everywhere on* Γ*.*

Proof. If $k < \min\{m, n\}$, let $H(t)$ be any $(k+1) \times (k+1)$ submatrix of $G(t)$ and form H_+ from the corresponding entries of G_+. Then $\det H_+ \equiv 0$ implies $\det H(t) = 0$ almost everywhere on Γ. Thus, $\operatorname{rank} G(t) \leq k$ for almost everywhere $t \in \Gamma$.

Choose a point $z_+ \in \mathcal{D}_+$ such that $\operatorname{rank} G_+(z_+) = k$, and pick matrices $A \in \mathbb{C}^{k\times m}$ and $B \in \mathbb{C}^{n\times k}$ such that $\operatorname{rank}(AG_+(z_+)B) = k$. Then $AG_+(z)B$ is a meromorphic $k\times k$ matrix function and $\det(AG_+(z)B) \not\equiv 0$. Hence $\det(AG(t)B) \neq 0$ and consequently $\operatorname{rank} G(t) \geq k$ almost everywhere on Γ. Thus, $\operatorname{rank} G = k$ almost everywhere on Γ.

One can formulate the following necessary and sufficient condition for existence of a canonical spectral factorization in L_p of a function G relative to Γ (*cf.* [14, Theorem 3.2] and [8]). Recall that if G admits a spectral factorization relative to Γ, then the rank of G is constant almost everywhere on Γ.

Theorem 4.2. *If* $G \in L_1^{m\times n}$ *with* $\operatorname{rank} G = k$ *almost everywhere on* Γ*, the following are equivalent:*

i) *there exist collections of linearly independent constant vectors* $\{a_1, a_2, \dots, a_k\}$ *and* $\{b_1, b_2, \dots, b_k\}$ *such that the image of the Riemann* *problem with coefficient G contains $\{t^{-1}a_1, t^{-1}a_2, \ldots, t^{-1}a_k\}$ and the image of the dual problem contains $\{b_1, b_2, \ldots, b_k\}$.*

ii) *the function G admits a canonical spectral factorization in L_p relative to Γ.*

Moreover, if the equivalent conditions i) *and* ii) *are satisfied, the image of either of the problems contains all rational vector functions in its closure.*

PROOF. Suppose first i) holds. Pick $\phi_{j+} \in E^m_{p+}$ and $\phi_{j-} \in \dot{E}^n_{p-}$ such that

$$\phi_{j+}(t) + G(t)\,\phi_{j-}(t) = t^{-1}a_j\ , \qquad j = 1, 2, \ldots, k\,, \tag{4.1}$$

and let $\Phi_- = (\phi_{1-}\ \phi_{2-}\ \ldots\ \phi_{k-})$. Then $F(t) := t\,G(t)\Phi_-(t) \in L^{m\times k}_{p+}$ and $F(0) = (a_1\ a_2\ \ldots\ a_k)$. Similarly, pick $\psi_{j+} \in E^m_{q+}$ and $\psi_- \in \dot{E}^n_{q-}$ such that

$$\psi_{j-}(t) + G^T(t)\,\psi_{j+}(t) = b_j\ , \qquad j = 1, 2, \ldots, k\,, \tag{4.2}$$

and let $\Psi_+ = (\psi_{1+}\ \psi_{2+}\ \ldots\ \psi_{k+})$. Then $H = G^T\Psi_+ \in E^{n\times k}_{q-}$ and

$$H(\infty) = (b_1\ b_2\ \ldots\ b_k)\,.$$

Let $S(t) = t\Psi^T_+(t)G(t)\Phi_-(t)$. Since

$$S(t) = \Psi^T_+(t)F(t) = H^T(t)(t\Phi_-(t)), \tag{4.3}$$

$S(t) \in L^{k\times k}_{1+} \cap L^{k\times k}_{1-}$. Thus, $S(t) = S$ is a constant. Also, $\det S \neq 0$. Indeed, by Lemma 4.1, $F(t)$ has linearly independent columns for almost everywhere $t \in \Gamma$. Since $\operatorname{rank} G = k$ almost everywhere on Γ, the column spans of F and G are equal almost everywhere on Γ. Thus, to prove that S is nonsingular it suffices to show $\operatorname{rank}(\Psi^T_+G) = k$ almost everywhere on Γ. But this follows from Lemma 4.1 and the fact that

$$(G^T\Psi_+)(\infty) = H(\infty) = (b_1\ b_2\ \ldots\ b_k)\,.$$

Let

$$G_+(t) = F(t)\,, \qquad G^L_+(t) = S^{-1}\Psi^T_+(t)\,,$$
$$G_-(t) = S^{-1}H^T(t)\,, \qquad G^R_-(t) = t\,\Phi_-(t)\,.$$

Then $G_+ \in L^{m\times k}_{p+}$, $G^L_+ \in L^{k\times m}_{q+}$, $G_- \in L^{k\times n}_{q-}$, and $G^R_- \in L^{n\times k}_{p-}$. By (4.3),

$$G^L_+(t)\,G_+(t) = I \qquad \text{and} \qquad G_-(t)\,G^R_-(t) = I\,.$$

By (4.3) and the definition of F,

$$G_+^L(t)\,G(t)\,G_-^R(t) = I$$

almost everywhere on Γ. Hence

$$G_-^R G_+^L G\, G_-^R G_+^L = G_-^R G_+^L\,,$$

or $G^\times G\, G^\times = G^\times$ where $G^\times = G_-^R G_+^L$. Since $\operatorname{rank} G^\times = \operatorname{rank} G$ almost everywhere on Γ, $G\, G^\times G = G$ (see [1, Theorem 1.5.2]; *cf.* [10, Lemma 3.8]). Thus,

$$G(t) = G(t)\, t\,\Phi_-(t)\, S^{-1}\Psi_+^T(t)\, G(t) = G_+(t)\, G_-(t)$$

almost everywhere on Γ and it follows that G admits a canonical spectral factorization in L_p relative to Γ.

Conversely, suppose ii) holds and let $G = G_+G_-$ be a canonical factorization. Let $G_-^R \in L_{p-}^{n\times k}$ be a right multiplicative inverse of G_-. Then $t^{-1}G_-^R(t) \in \dot{L}_{p-}^{n\times k}$, and

$$G(t)\,(t^{-1}G_-^R(t)) = t^{-1}G_+(t) = t^{-1}G_+(0) + t^{-1}(G_+(t) - G_+(0))\,.$$

Hence the columns of $t^{-1}G_+(0)$ are in the image of the problem. Similarly, if $G_+^L \in L_{q+}^{m\times k}$ is a left multiplicative inverse of G_+, $G^T(G_+^L)^T = G_-^T$ and so the columns of $G_-^T(\infty)$ are in the image of the problem. Thus, ii) implies i) and the conditions are equivalent.

The argument from the last paragraph can be used in a more general situation. Suppose G_+G_- is a canonical spectral factorization in L_p relative to Γ. Let $G_+^L \in L_{q+}^{k\times m}$ and $G_-^R \in L_{p-}^{n\times k}$ be one-sided multiplicative inverses of G_+ and G_-, and let $r \in \dot{L}_{\infty-}^k$ be a rational vector function. Then $G_-^R r \in \dot{L}_{p-}^{n\times k}$, and

$$G(G_-^R r) = G_+ r$$

differs from a rational vector function by an element in L_{p+}^m. Hence $\mathcal{Q}(G_+r)$, where $\mathcal{Q}$ is a canonical projection of $L_{p+}^m \dot{+} \dot{L}_{p-}^m$ onto $\dot{L}_{p-}^m$, is a rational vector function in the image of the problem. We claim that any rational vector function in the intersection of $\dot{L}_{p-}^m$ and the closure of the image of the problem arises in this way. Indeed, let $f_- \in \dot{L}_{\infty-}^m$ be a rational vector function such that

$$f_- \notin \{\mathcal{Q}(G_+r) :\ r \in \dot{L}_{\infty-}^k \text{ is a rational vector function}\}\,. \tag{4.4}$$

We may assume f_- has a single pole, located at $\lambda \in \mathcal{D}_+$. Suppose the leading coefficient in the Laurent expansion of f_- at λ is contained in the image of $G_+(\lambda)$. Then after subtracting from f_- an element in the set on the right hand side of (4.4), we obtain a strictly proper rational vector function analytic in $\mathbb{C}\setminus\{\lambda\}$ with the pole at λ of smaller order. By induction, there exists a strictly proper rational vector function with the only pole at λ whose leading coefficient in the Laurent expansion at λ is not contained in the image of $G_+(\lambda)$. Call this function again f_-.

Consider a problem

$$\phi_+ + G\phi_- = g\,, \tag{4.5}$$

where $\phi_- \in \dot{L}^n_{p-}$ is such that $G\phi_- \in L^m_p$ and $\phi_+ \in L^m_{\infty+}$. The image of the problem (4.5) is contained in the image of the Riemann problem. Since rational functions without poles on Γ are dense in L_p, and the projection $\mathcal{P}$ is bounded on $L_{1+}\dot{+}\dot{L}_{1-}$, $L_{\infty+}$ is dense in L_{p+}. Hence the closures of the images of both problems coincide. Now

$$(I - G_+(t)\,G^L_+(t))\,G(t) = \big(G_+(t) - G_+(t)\,G^L_+(t)\,G_+(t)\big)\,G_-(t) = 0$$

almost everywhere on Γ and, since $I - G_+(\lambda)G^L_+(\lambda)$ is an $m\times m$ matrix of rank $m-k$ whose null space coincides with the image of $G_+(\lambda)$,

$$(I - G_+(z)\,G^L_+(z))f_-(z)$$

has a pole at $z = \lambda$. Consequently, there exists a function $\psi_+ \in L^{1\times m}_{1+}$ such that $\psi^T_+ f_-$ has a simple pole at λ and $\int_\Gamma \psi^T_+ g$ equals zero for all functions g in the image of the problem (4.5). Let X be a subspace of L^m_p spanned by f_- and the image of the problem (4.5). Then

$$x \longrightarrow \int_\Gamma \psi_+(t)^T x(t)\,dt$$

is a continuous linear functional on the space X whose kernel contains the image of the problem (4.5) and which has nonzero value at f_-. By the Hahn-Banach Theorem, there exists a continuous linear functional Ψ on L^m_p which annihilates the image of the problem (4.5) and such that $\Psi(f_-) \neq 0$. Hence f_- is not in the closure of the image of the problem (4.5).

In order to obtain a condition for existence of a spectral factorization of a function G in a non-canonical case, we will need the following lemma.

Lemma 4.3. *Suppose the defect numbers α_R and β_D of the Riemann problem with coefficient G and its dual are finite and positive. Then there exists a square rational matrix function H with a nonzero determinant and without poles or zeros on Γ such that the Riemann problem with coefficient GH and its dual have the corresponding defect numbers smaller by 1. Moreover, the Riemann problem with coefficient G (respectively its dual) contains all rational vectors functions in its closure if and only if the image of the Riemann problem with coefficient GH (respectively its dual) contains all rational vector functions in its closure.*

Proof. Pick $(\varphi_+, \varphi_-) \in L^m_{p+} \dot{+} \dot{L}^n_{p-}$ such that $\varphi_+ \neq 0$ and

$$\varphi_+ + G\varphi_- = 0\,.$$

Then $\varphi_- \notin \{\phi \in \dot{L}^n_{p-} :\ G\phi = 0\}$ and there exists a point $z_0 \in \mathcal{D}_-$ such that $\varphi_-(z_0)$ is not a member of

$$\operatorname{span}\{\phi_-(z_0) :\ \phi_- \in \dot{L}^n_{p-} \text{ and } G\phi_- = 0\}\,. \tag{4.6}$$

After adding to φ a linear combination of functions in $\{\phi_- \in \dot{L}^n_{p-} :$ $G\phi_- = 0\}$, and multiplying G on the right by a nonsingular constant matrix, we may assume $\varphi_-(z_0) = e_1$ and

$$\operatorname{span}\{\phi_-(z_0) :\ \phi_- \in \dot{L}^n_{p-} \text{ and } G\phi_- = 0\} \subset \operatorname{span}\{e_2, e_3, \ldots, e_n\}\,.$$

As usual, we assume $0 \in \mathcal{D}_+$. Let

$$H(z) = \begin{pmatrix} \dfrac{z - z_0}{z} & & & \mathbf{0} \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{pmatrix}.$$

We show that the space of "+" components of the members of the kernel of the problem

$$\phi_+ + GH\phi_- = g \tag{4.7}$$

has dimension one less than the corresponding number for the problem with coefficient G. First, note that φ_+ is not a "+" component of a member of the kernel of problem (4.7). Indeed, suppose there exists $\phi_- \in \dot{L}^n_{p-}$ such that $\varphi_+ + GH\phi_- = 0$, and let $f_- = H\phi_- - \varphi_-$. Then $f_- \in \dot{L}^n_{p-}$, $Gf_- = 0$, and

$$f_-(z_0) \notin \operatorname{span}\{e_2, \ldots, e_n\},$$

a contradiction. Secondly, suppose (ϕ_+, ϕ_-) is in the kernel of the Riemann problem with coefficient G. If $\phi_-(z_0) = (0, *, \ldots, *)$, the element $(\phi_+, H^{-1}\phi_-)$ is in the kernel of the problem (4.7). If $\phi_-(z_0) = (\lambda, *, \ldots, *)$ with $\lambda \neq 0$,

$$\left(\varphi_+ - \frac{1}{\lambda}\phi_+, H^{-1}\left(\varphi_- - \frac{1}{\lambda}\phi_-\right)\right)$$

is contained in the kernel of the problem (4.7). Thus, each "+" component of a member of the kernel of the Riemann problem with coefficient G is a linear combination of φ_+ and a "+" component of a member of the kernel of the problem (4.7). Finally, if (ϕ_+, ϕ_-) belongs to the kernel of the problem (4.7), $(\phi_+, H\phi_-)$ satisfies the homogeneous Riemann problem.

Consider now the problem dual to (4.7),

$$\psi_- + (GH)^T\psi_+ = h. \tag{4.8}$$

After multiplying both sides of (4.8) by H^{-1}, we obtain a new problem

$$H^{-1}\psi_- + G^T\psi_+ = h, \qquad \psi_- \in \dot{L}^n_{q-},\ \psi_+ \in L^m_{q+},\ \text{and } h \in L^n_q. \tag{4.9}$$

Let $\mathcal{W}$ be the image of the problem dual to the Riemann problem with coefficient G. Then the image of the problem (4.9) equals $\mathcal{W} + \operatorname{span}(z - z_0)^{-1}e_1$. Since

$$\int_\Gamma \varphi_-(z)^T(z - z_0)^{-1}e_1\, dz = -2\pi i,$$

by Proposition 3.2 $(z - z_0)^{-1}e_1 \notin \operatorname{cl}\mathcal{W}$. We have $\operatorname{cl}(\mathcal{W} + \operatorname{span}\{(z - z_0)e_1\} = \operatorname{cl}\mathcal{W} + \operatorname{span}\{(z - z_0)e_1\}$. Since multiplication by H is an isomorphism $L^n_p \to L^n_p$, it follows that the closure of the image of problem (4.8) equals

$$H(\operatorname{cl}\mathcal{W}) + \operatorname{span}\{H(z)(z - z_0)^{-1}e_1\} = H(\operatorname{cl}\mathcal{W}) + \operatorname{span}\left\{\frac{1}{z}\, e_1\right\}. \tag{4.10}$$

Now the space $\{\phi_- \in \dot{L}^n_{p-} : \ G\phi_- = 0\}$ has a finite co-dimension β_D in the annihilator of $\mathcal{W}$. Hence the co-dimension of the space

$$\{H^{-1}\phi_- : \ \phi_- \in \dot{L}^n_{p-} \text{ and } G\phi_- = 0\} \tag{4.11}$$

in the annihilator of $H(\operatorname{cl}\mathcal{W})$ equals β_D. Consequently, the co-dimension of the space (4.11) in the annihilator of (4.10) equals $\beta_D - 1$. Since

$$\{\phi_- \in \dot{L}^n_{p-} : \ GH\phi_- = 0\} = \{H^{-1}\phi_- : \ \phi_- \in \dot{L}^n_{p-} \text{ and } G\phi_- = 0\}\,,$$

the co-dimension of the closure of $\{\phi_- \in \dot{L}^n_{p-} : \ GH\phi_- = 0\}$ in the annihilator of the space (4.10) equals $\beta_D - 1$.

It remains to verify the assertion about the images. First, note that the images of the Riemann problems with coefficients G and GH coincide. Indeed, since $H\dot{L}^n_{p-} \subset \dot{L}^n_{p-}$, the image of the problem with coefficient GH is contained in the image of the problem with coefficient G. Since

$$\phi_+ + G\phi_- = \phi_+ - \lambda\varphi_+ + GH(H^{-1}(\phi_- - \lambda\varphi_-))$$

for any scalar λ, and for each $\phi_- \in \dot{L}^n_p$ there exists λ such that $H^{-1}(\phi_- - \lambda\varphi_-) \in \dot{L}_{p-}$, the image of the problem with coefficient G is contained in the image of the problem with coefficient GH.

Suppose the image of the problem dual to the Riemann problem with coefficient G contains all rational vector functions in its closure, and let a rational vector function f be a member of the set (4.10). Then $H^{-1}(f(z) - z^{-1}e_1) \in \operatorname{cl}\mathcal{W}$, so $H^{-1}(f(z) - z^{-1}e_1) \in \mathcal{W}$ and

$$f \in H(\mathcal{W}) + \operatorname{span}\{H(z)(z-z_0)^{-1}e_1\}\,.$$

Thus, f is a member of the image of problem (4.8). Conversely, suppose the image of the problem (4.8) contains all rational vector functions in its closure, and let $f \in \operatorname{cl}\mathcal{W}$ be a rational vector function. Then $Hf \in H(\operatorname{cl}\mathcal{W}) \subset H(\operatorname{cl}\mathcal{W} + \operatorname{span}\{H(z)(z-z_0)^{-1}e_1\}$, so $Hf \in H\mathcal{W} + \operatorname{span}\{H(z)(z-z_0)^{-1}e_1\}$. Thus, $f \in \mathcal{W} + \operatorname{span}\{(z-z_0)^{-1}e_1\}$. Since $(z-z_0)^{-1}e_1 \notin \operatorname{cl}\mathcal{W}$, $f \in \mathcal{W}$.

In a similar way one can show the following dual version of Lemma 4.3. We omit the details of the proof.

Lemma 4.4. *Suppose the defect numbers α_D and β_R of the Riemann problem with coefficient G and its dual are finite and positive. Then there exists a square rational matrix function F with a nonzero determinant and without poles or zeros on Γ such that the Riemann problem with coefficient FG and its dual have the corresponding defect numbers smaller by 1. Moreover, the image of the Riemann problem with coefficient G (respectively its dual) contains all rational vector functions in its closure if and only if the image of the problem with coefficient FG (respectively its dual) contains all rational vector functions in its closure.*

We can give now a necessary and sufficient condition for existence of a spectral factorization in L_p of a summable singular matrix valued function (*cf.* [13, Theorem 3.1]).

Theorem 4.5. *If $G \in L_1^{m\times n}$ and $\operatorname{rank} G = k$ almost everywhere on Γ, the following are equivalent:*

i) *the indices of the Riemann problem with coefficient G and its dual are finite and opposite, and the image of each of the problems contains all rational vector functions in its closure,*

ii) *G admits a spectral factorization in L_p relative to Γ.*

Proof. Suppose first i) holds. By Proposition 3.4, $\alpha_R = \beta_D$ and $\alpha_D = \beta_R$. Applying Lemmas 4.3 and 4.4 a finite number of times, we can find regular rational matrix functions F and H without poles or zeros on Γ such that

1) the annihilator of the image of the Riemann problem with coefficient $\widehat{G} = FGH$ coincides with $\{\psi_+ \in L_{q+}^m : \ \widehat{G}^T\psi_+ = 0\}$,

2) the annihilator of the image of the dual problem equals $\{\phi_- \in \dot{L}_{p-}^n : \ \widehat{G}\phi_- = 0\}$,

3) the image of either of the problems contains all rational vector functions in its closure.

Let

$$\Omega_+ = \operatorname{span}\{\psi_+(0) : \ \psi_+ \in L_{q+}^m \text{ and } \widehat{G}^T\psi_+ = 0\}.$$

Since $\operatorname{rank}\widehat{G} = k$ almost everywhere on Γ, by Lemma 4.1 dim $\Omega_+ \le m-k$. Hence there exist linearly independent vectors $\{a_1, a_2, \dots, a_k\}$ such that $\omega a_i = 0$ whenever $\omega \in \Omega_+$ and $i = 1, 2, \dots, k$. Suppose

$\psi_+ \in L^m_{q+}$ and $\widehat{G}^T\psi_+ = 0$. Then $\psi_+(t)^T t^{-1} a_i \in L_{q+}$, and so

$$\int_\Gamma \psi_+(t)^T t^{-1} a_i \, dt = 0 \,, \qquad \text{for} \quad i = 1, 2, \ldots, k \,.$$

It follows that the set

$$\left\{ \frac{1}{t}\, a_1, \frac{1}{t}\, a_2, \ldots, \frac{1}{t}\, a_k \right\}$$

is in the closure of the image, and hence in the image, of the Riemann problem with coefficient $\widehat{G}$.

Similarly, let

$$\Omega_- = \operatorname{span}\{\phi_-(\infty) : \ \phi_- \in L^n_{p-} \text{ and } \widehat{G}\phi_- = 0\}$$

and pick linearly independent vectors $\{b_1, b_2, \ldots, b_k\}$ such that $\omega_- b_j = 0$, for $j = 1, 2, \ldots, k$ and all $\omega_- \in \Omega_-$. Suppose $\phi_- \in \dot{L}^n_{p-}$ is such that $\widehat{G}\phi_- = 0$. Then $z\phi_-(z) b_j \in \dot{L}_{p-}$ and hence

$$\int_\Gamma \phi_-(t) b_j \, dt = \int_\Gamma (t\phi_-(t) b_j)\, t^{-1} dt = 0 \,,$$

for $j = 1, 2, \ldots, k$. Thus, the set $\{b_1, b_2, \ldots, b_k\}$ is contained in the closure of the image, and consequently in the image, of the problem dual to the Riemann problem with coefficient $\widehat{G}$. Consequently, by Theorem 4.2, the function $\widehat{G} = FGH$ admits a canonical spectral factorization in L_p relative to Γ. Hence, by Proposition 2.6 the function G admits a spectral factorization in L_p relative to Γ.

Conversely, suppose ii) holds. By Theorem 3.6, the indices of the problem and its dual are finite and opposite. Applying Lemmas 4.3 and 4.4 a finite number of times, we can find square rational matrix functions F and H whose determinants are not equal to zero identically and which have neither poles nor zeros on Γ such that the Riemann problem with coefficient FGH and the dual problem have defect numbers

$$\alpha_R = \beta_R = \alpha_D = \beta_D = 0 \,.$$

By Proposition 2.6 and Theorem 3.6, the function FGH admits a canonical spectral factorization in L_p relative to Γ. By Theorem 4.2, the image of the Riemann problem with coefficient FGH and the image of the dual problem each contain all rational vector functions in their closures. By Lemmas 4.3 and 4.4, the image of the Riemann problem with coefficient G (respectively image of the dual problem) contains all rational vector functions in its closure.

We note that the part of condition i) in Theorem 4.5 involving rational vector functions cannot be in general omitted. Indeed, suppose Γ is the unit circle, $p = 3$, and let

$$G(t) = \begin{pmatrix} t^{2/3} \\ t^{5/3} \end{pmatrix}$$

be as in Example 2.7. Since G admits a spectral factorization in L_p for p in a deleted neighborhood of 3, by Theorem 3.6 the numbers α_R and α_D are finite when the problem is considered in L_{p_1} or L_{p_2} with $p_1 < 3 < p_2$. Since $L_3 \subset L_{p_1}$ and $L_{3/2} \subset L_{p_2/(p_2-1)}$, α_R and α_D are finite when $p = 3$. Since $G \in L_\infty$, by Proposition 3.4 $\alpha_R = \beta_D$ and $\alpha_D = \beta_R$. Thus, the indices of the problem and its dual are finite and opposite although G does not admit a spectral factorization in L_3 relative to the circle.

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