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Rough maximal functions and rough singular integral operators applied to integrable radial functions

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Abstract. Let Ω be homogeneous of degree 0 in $\mathbb{R}^n$ and integrable on the unit sphere. A rough maximal operator is obtained by inserting a factor Ω in the definition of the ordinary maximal function. Rough singular integral operators are given by principal value kernels $\Omega(y)/|y|^n$, provided that the mean value of Ω vanishes. In an earlier paper, the authors showed that a two-dimensional rough maximal operator is of weak type (1,1) when restricted to radial functions. This result is now extended to arbitrary finite dimension, and to rough singular integrals.

1. Introduction.

Let $\Omega \geq 0$ be an integrable function on the unit sphere S^{n-1} in $\mathbb{R}^n$, and extend it to a function in $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree 0. The rough maximal operator corresponding to Ω is defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} \Omega(y)\, |f(x-y)|\, dy\,, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n)\,.$$

This operator is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, as seen by the method of rotations. It is, however, unknown whether it is of weak type (1,1).

Under (weak) additional assumptions on Ω, several authors have proved the weak type; see the authors' paper [S-S] for details. That paper contains a proof that M_Ω is of weak type (1,1) in the plane when restricted to radial functions f, for a general $\Omega \in L^1$. In fact, the same result is proved for $n = 2$ when M_Ω is replaced by the larger operator

$$M_\Omega^* f(x) = \int_{S^{n-1}} \Omega(\omega)\, M_\omega f(x)\, d\omega\,.$$

Here and below, $d\omega$ is the area measure on S^{n-1}. Further, M_ω is the one-dimensional maximal operator in the direction $\omega \in S^{n-1}$, defined by

$$M_\omega f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - t\,\omega)|\, dt\,.$$

As we pointed out in [S-S], M_Ω^* cannot be of weak type (1,1) on general functions even when Ω is the constant function 1. In this paper, we shall extend the above to $\mathbb{R}^n$, as follows.

Theorem 1. *The operator M_Ω^* is of weak type $(1,1)$ when restricted to radial functions in $\mathbb{R}^n$, for any nonnegative $\Omega \in L^1(S^{n-1})$ and any n. The same is true for M_Ω.*

Rough singular integral operators can be defined analogously. Now $\Omega \in L^1(S^{n-1})$ must have mean value 0. Let

$$T_\Omega f(x) = \text{p.v.} \int \frac{\Omega(y)}{|y|^n}\, f(x-y)\, dy = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon<|y|<R} \frac{\Omega(y)}{|y|^n}\, f(x-y)\, dy\,,$$

whenever the limit exists. The L^p boundedness of such operators (which is easy when Ω is odd) was proved by Calderón and Zygmund [C-Z] assuming $\Omega \in L \log L(S^{n-1})$. There is a nice proof due to J. Duoandikoetxea and J. L. Rubio de Francia [D-RF] when $\Omega \in L^q(S^{n-1})$, $q > 1$. With the same condition on Ω, S. Hofmann [H] proved the weak type (1,1) in the plane. The same was proved for $\Omega \in L \log L(S^{n-1})$ by M. Christ and J. L. Rubio de Francia [Ch-RF]. In an unpublished work, they also extended the result to dimension at most 7. More recently, A. Seeger [Se] has proved it in any dimension, again under the hypothesis $\Omega \in L \log L(S^{n-1})$. We remark that the L^p inequality, $1 < p < \infty$, cannot hold without additional assumptions on Ω, since the Fourier multiplier corresponding to T_Ω need not be bounded (*cf.* [St, Chapter II]). In our result, we have no additional assumption on Ω, but apply the operator only to radial functions.

Theorem 2. *Let $\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega\, d\omega = 0$. The operator $T_\Omega f$ is well defined for any radial function $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, in the sense that the principal value exists for almost every x. Moreover, when restricted to radial functions, T_Ω is of weak type $(1,1)$ and bounded on L^p, $1 < p < \infty$, and so is the maximal singular integral operator*

$$T^*_\Omega f(x) = \sup_{0<\varepsilon<R<\infty} \left| \int_{\varepsilon<|y|<R} \frac{\Omega(y)}{|y|^n}\, f(x-y)\, dy \right| .$$

To prove the two-dimensional estimate for M^*_Ω in [S-S, Theorem 3], we applied Theorems 1 and 2 of [S-S]. These two results say that $y^{-1}G(y\,\cdot) * f(x) \in L^{1,\infty}(y\, dx\, dy,\ y>0)$ for any $f \in L^1(\mathbb{R})$ and suitable $G \in L^1(\mathbb{R})$. Our method to prove Theorem 1 in the present paper is similar. Implicit in our proof is a version of Theorem 2 of [S-S], where $\mathbb{R}$ is replaced by S^{n-1}. We point out that a version with $\mathbb{R}$ replaced by $\mathbb{R}^n$ also follows from the arguments below. However, we leave it to the interested reader to state it explicitly.

Theorem 1 of this paper is proved in Section 2. It is then one of the tools used to prove Theorem 2 in Section 3.

Finally, with respect to the notation in this paper, an integral $\int_a^b$ with $a > b$ should be interpreted as 0. Further, C denotes many different positive finite constants.

2. Proof of Theorem 1.

We write $x \in \mathbb{R}^n$ as $x = r\,\theta$ with $r \ge 0$ and $\theta \in S^{n-1}$, and denote as in [S-S] by $A(\omega,\theta) = A(\omega,x) \in [0,\pi)$ the angle between $\omega \in S^{n-1}$ and θ. Also let $s(\omega,\theta) = \max\{\sin A(\omega,\theta), A(\omega,\theta)/2\}$. With $0 \le g \in L^1(t^{n-1}\,dt)$ defined on $\mathbb{R}_+$, we follow [S-S] in defining

$$A_\omega g(x) = \frac{1}{r} \int_{rs(\omega,\theta)}^\infty g(t)\, \frac{t\, dt}{(t^2 - r^2 s(\omega,\theta)^2)^{1/2}} .$$

Consider the operator

$$Pg(x) = \int_{s(\omega,\theta)<\delta} \Omega(\omega)\, A_\omega g(x)\, d\omega ,$$

where $\delta > 0$ will be a small constant. The first part of the proof of [S-S, Theorem 3], which is carried out for each $n \geq 2$, now shows that we need only find an estimate

$$P : L^1([0,\infty),\, t^{n-1}\,dt) \longrightarrow L^{1,\infty}(S^{n-1} \times [0,\infty),\, r^{n-1}\,d\theta\,dr)\,.$$

Notice that

$$A_\omega g(x) \leq \frac{1}{r}\, G(r\, s(\omega,\theta))\,,$$

where

$$G(u) = \int_u^\infty g(t)\, \frac{t^{1/2}\,dt}{(t-u)^{1/2}}\,, \qquad u > 0\,.$$

Essentially as in [S-S, proof of Theorem 3], we majorize G by

$$G \leq C \sum_{\nu=0}^{\infty} 2^{-\nu/2} G_\nu + C\,h\,,$$

where

$$\begin{aligned} G_\nu(u) &= 2^\nu u^{1-n} \int_u^{2^{2^{-\nu}}u} g(t)\, t^{n-1}\,dt \\ &\leq C \sum_{k\in\mathbb{Z}} \int_{2^{-k2^{-\nu}}}^{2^{(2-k)2^{-\nu}}} g(t)\, t^{n-1}\,dt \\ &\qquad \cdot 2^\nu 2^{k2^{-\nu}(n-1)} \chi_{[2^{-k2^{-\nu}},\, 2^{(1-k)2^{-\nu}}]}(u) \end{aligned} \tag{2.1}$$

and

$$h(u) = \int_u^\infty g(t)\,dt\,.$$

This implies

$$\begin{aligned} Pg(x) &\leq C \sum_{\nu=0}^{\infty} 2^{-\nu/2} r^{-1} \int_{s(\omega,\theta)<\delta} \Omega(\omega)\, G_\nu(r\,s(\omega,\theta))\,d\omega \\ &\quad + C\,r^{-1} \int_{s(\omega,\theta)<\delta} \Omega(\omega)\, h(r\,s(\omega,\theta))\,d\omega \\ &= C \sum_{\nu=0}^{\infty} 2^{-\nu/2} P_\nu g(x) + C\,Qg(x)\,, \end{aligned} \tag{2.2}$$

the last equality defining P_ν and Q.

To extend the technique used to control P_ν and Q from [S-S], we need analogues of dyadic cubes in S^{n-1}. First, we divide S^{n-1} into a finite number of disjoint subsets E_s, $s = 1, \dots, s_0$, with piecewise smooth boundaries and of small diameters. In each E_s, we can then introduce coordinates simply by projecting E_s orthogonally onto a hyperplane of $\mathbb{R}^n$ tangent to E_s at some point of E_s. In this hyperplane, *i.e.* in $\mathbb{R}^{n-1}$, we introduce the ordinary hierarchy of dyadic cubes. Thus for each $j \in \mathbb{Z}$, we have a partition of $\mathbb{R}^{n-1}$ into cubes of side 2^{-j}. Some of these cubes have images in E_s under the inverse projection. These images will be denoted $(I_j^i)_i$ and called 2^{-j}-cubes. This is for $j \geq j_0$, some j_0. Suitably adapted near ∂E_s, all these sets will form a hierarchy of partitions of E_s and, hence, of S^{n-1}.

The conditional expectation at level j, $j \geq j_0$, of a function $f \in L^1(S^{n-1})$ is now defined by

$$E_j f(x) = |I_j^i|^{-1} \int_{I_j^i} f \,, \qquad x \in S^{n-1} \,,$$

where I_j^i is that 2^{-j}-cube in S^{n-1} which contains the given point x.

Now consider Q. The desired estimate

$$Q : L^1(t^{n-1}\, dt) \longrightarrow L^{1,\infty}(r^{n-1}\, d\theta\, dr) \,,$$

can be seen as a version of Theorems 1 and 4 of [S-S], where $\mathbb{R}$ and $\mathbb{R}^n$, respectively, are replaced by S^{n-1}. Instead of a convolution, we now have the integral defining Qg in (2.2). However, the proof technique carries over without problems. We can assume that the decreasing function h has the form $h = \sum a_k \chi_{[0, 2^{-k-C}]}$. Also, it is enough to consider dyadic values of r (*cf.* the inequality (2.3) below). One can now easily relate Q to the conditional expectation, essentially as in [S-S]. The estimates needed for conditional expectation carry over. This takes care of Q.

To control the operator P, we must also estimate the P_ν. It is enough to prove that each P_ν maps $L^1(t^{n-1}\, dt)$ boundedly into $L^{1,\infty}(r^{n-1}\, d\theta\, dr)$, with a constant that grows only polynomially in ν. This will allow summing in $L^{1,\infty}$. As in the proof of Theorem 2 in [S-S], we let r take only the values $r = 2^{2\nu j}$, $j \in \mathbb{Z}$, and prove that

$$\begin{aligned} \sum_j 2^{2\nu j n} |\{\theta \in S^{n-1} \,:\, P_\nu g(2^{2\nu j}\theta) > \lambda\}| \\ \leq C\,(1+\nu)^C \, \frac{1}{\lambda}\, \|g\|_{L^{1,\infty}(t^{n-1} dt)} \,. \end{aligned} \tag{2.3}$$

Here $|\cdot|$ is the area measure of S^{n-1}. This will complete the proof. To verify (2.3), it is enough, as in [S-S, proof of Theorem 2], to sum in (2.1) only over those k of the form $k = \ell\, 2^{\nu+1}\nu + \kappa$, $\ell \in \mathbb{Z}$, for each $\kappa = 0, \dots, 2^{\nu+1}\nu - 1$. For simplicity, we shall consider only $\kappa = 0$. The level set in (2.3) will thus be replaced by the set of those $\theta \in S^{n-1}$ for which

$$\begin{aligned}(2.4)\qquad 2^{-2\nu j} \int_{s(\omega,\theta)<\delta} \Omega(\omega) \Big(\sum_{\ell} \int_{2^{-2\nu\ell}}^{2^{2^{1-\nu}-2\nu\ell}} g(t)\, t^{n-1}\, dt \Big) \\ \cdot\, 2^{\nu} 2^{2(n-1)\nu\ell} \chi_{R_{\ell+j}(\theta)}(\omega)\, d\omega > \lambda\,,\end{aligned}$$

where $R_m(\theta)$ is the ring

$$R_m(\theta) = \{\omega \in S^{n-1} \,:\, 2^{-2\nu m} \le s(\omega,\theta) \le 2^{2^{-\nu}-2\nu m}\}\,.$$

Because of the condition $s(\omega,\theta) < \delta$ in the integral in (2.4), we need only consider $m \ge m_0$ here, for some $m_0 > 0$. This means that the sum in (2.4) is taken over $\ell \ge m_0 - j$. Notice that the radius and the width of $R_m(\theta)$ are approximately $2^{-2\nu m}$ and $2^{-\nu-2\nu m}$, respectively.

Next, we let the point θ move within a $2^{-\nu(1+2m)}$-cube $I^i_{\nu(1+2m)}$ and form

$$R^i_m = \bigcup_{\theta \in I^i_{\nu(1+2m)}} R_m(\theta)\,.$$

This set is contained in a ring of width at most $C\,2^{-\nu(1+2m)}$. Clearly, R^i_m is covered by those $2^{-\nu(1+2m)}$-cubes intersecting it. Their number is at most $C\,2^{(n-2)\nu}$. Among these $2^{-\nu(1+2m)}$-cubes, we discard those which are not in the same E_s as $I^i_{\nu(1+2m)}$. Then we enumerate the remaining ones as $I^{\lambda(i,q)}_{\nu(1+2m)}$, $q = 1, \dots, q_0 = \mathrm{O}(2^{(n-2)\nu})$, in a coherent way as i varies. By this we mean that the direction from the midpoint of $I^i_{\nu(1+2m)}$ (which is the approximate centre of the ring-like set R^i_m) to the midpoint of $I^{\lambda(i,q)}_{\nu(1+2m)}$ should not vary too much with i, for a fixed q. It is enough if two such directions never form an angle greater than $\pi/4$, say, measured in the coordinate system of each E_s.

In (2.4), we shall now replace $R_{\ell+j}(\theta)$ by $I^{\lambda(i,q)}_{\nu(1+2\ell+2j)}$ when $\theta \in I^i_{\nu(1+2\ell+2j)}$, for a fixed q. More precisely, this means that the level set in (2.3) is replaced by the union of those $I^i_{\nu(1+2\ell+2j)}$ for which

$$2^{-2\nu j}\int \Omega(\omega)\Big(\sum_{\ell\geq m_0-j}\int_{2^{-2\nu\ell}}^{2^{2^{1-\nu}-2\nu\ell}} g(t)\,t^{n-1}\,dt\Big) \cdot 2^{\nu}2^{2(n-1)\nu\ell}\chi_{I^{\lambda(i,q)}_{\nu(1+2\ell+2j)}}(\omega)\,d\omega>\lambda\,.$$

This version of (2.3), call it (2.3'), implies the theorem, since we can sum in q by means of the adding-up lemma in $L^{1,\infty}$ as in [S-S].

The mean value of Ω in $I^{\lambda(i,q)}_{\nu(1+2\ell+2j)}$ can be seen as an S^{n-1} version of the translated conditional expectation from the proof of Theorem 2 of [S-S]. In fact, the arguments used in that proof now carry over and prove (2.3'). We leave the details to the reader. This ends the proof of Theorem 1.

3. Proof of Theorem 2.

We start with the L^1 case. Let

$$T_\Omega^{\varepsilon,R}f(x)=\int_{\varepsilon<|y|<R}\frac{\Omega(y)}{|y|^n}\,f(x-y)\,dy\,.$$

Notice that all the conclusions follow from the weak type estimate for the maximal operator T^*_Ω. Also, in the definition of $T^*_\Omega f(x)$, we need only take $R\geq 10\,|x|=10\,\rho$. This is because in the case $R<10\,\rho$, one has

$$\int_{\varepsilon<|y|<R}\frac{\Omega(y)}{|y|}\,f(x-y)\,dy=T_\Omega^{\varepsilon,10\rho}f(x)-T_\Omega^{R,10\rho}f(x)\,.$$

Together with $T_\Omega^{\varepsilon,R}$, we consider

$$\tilde T_\Omega^{\varepsilon,R}f(x)=\int_{\substack{|\,|y-x|-|x|\,|>\varepsilon\\ |y|<R}}\frac{\Omega(y)}{|y|^n}\,f(x-y)\,dy\,. \tag{3.1}$$

We shall estimate the difference between these two operators.

The notation $x=\rho\,\theta$, $y=r\,\omega$, $A=A(\theta,\omega)$ will be as in Section 2. A radial function $f\in L^1$ will be written $f(x)=g(|x|)$, with $g\in L^1(\mathbb{R}_+;\rho^{n-1}\,d\rho)$. The distance $t=|x-y|$ satisfies

$$t^2=\rho^2+r^2-2\,\rho\,r\cos A\,. \tag{3.2}$$

Hence,

$$r = \rho \cos A \pm \sqrt{t^2 - \rho^2 \sin^2 A}\ . \tag{3.3}$$

Proposition 3. *The operator*

$$\tilde{T}^*_\Omega f(x) = \sup_{\substack{\varepsilon>0 \\ R>10|x|}} |\tilde{T}^{\varepsilon,R}_\Omega f(x)|\,,$$

is of weak type $(1,1)$ *when restricted to radial functions.*

Proposition 4. *The operator*

$$D^*_\Omega f(x) = \sup_{\substack{\varepsilon>0 \\ R>10|x|}} |T^{\varepsilon,R}_\Omega f(x) - \tilde{T}^{\varepsilon,R}_\Omega f(x)|\,,$$

is of weak type $(1,1)$ *when restricted to radial functions.*

It is clear that the L^1 part of Theorem 2 follows from these two results.

PROOF OF PROPOSITION 3. In the integral defining $\tilde{T}^{\varepsilon,R}_\Omega f(x)$, we pass to polar coordinates, getting

$$\tilde{T}^{\varepsilon,R}_\Omega f(x) = \int_{S^{n-1}} \Omega(\omega)\, d\omega \int_{\substack{|\,|x-r\omega|-\rho\,|>\varepsilon \\ 0<r<R}} \frac{g(|x - r\,\omega|)}{r}\, dr\,.$$

Next, we shall transform the inner integral here, using $t = |x - r\,\omega|$ as a new variable of integration. One has $dr = t\,dt/(r - \rho\cos A)$. The correspondence between r and t is not quite one-to-one, and the sign in (3.3) must be chosen correctly. As seen geometrically, one obtains a sum of four integrals. Indeed,

$$\tilde{T}^{\varepsilon,R}_\Omega f(x) = \int_{A>\pi/2} \Omega(\omega)\, d\omega \int_{\rho+\varepsilon}^{R_1(\rho)} \frac{g(t)}{\rho\cos A + \sqrt{t^2 - \rho^2\sin^2 A}} \cdot \frac{t\,dt}{\sqrt{t^2 - \rho^2 \sin^2 A}}$$

$$
\begin{aligned}
&+ \int_{A<\pi/2} \Omega(\omega)\, d\omega \int_{\rho+\varepsilon}^{R_2(\rho)} \frac{g(t)}{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad\qquad \cdot \frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&+ \int_{A<\pi/2} \Omega(\omega)\, d\omega \int_{\rho\sin A}^{\rho-\varepsilon} \frac{g(t)}{\rho\cos A - \sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad\qquad \cdot \frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&+ \int_{A<\pi/2} \Omega(\omega)\, d\omega \int_{\rho\sin A}^{\rho-\varepsilon} \frac{g(t)}{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad\qquad \cdot \frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&= I_1 + I_2 + I_3 + I_4\,.
\end{aligned}
$$

Here $R_j(\rho) \in [R-\rho, R+\rho]$ for $j=1,2$.

The integrand is the same in I_1 and I_2, and one finds

$$
\begin{aligned}
I_1 + I_2 = \int_{S^{n-1}} \Omega(\omega)\, d\omega \int_{\rho+\varepsilon}^{R} g(t)\, \frac{\sqrt{t^2-\rho^2\sin^2 A} - \rho\cos A}{t^2-\rho^2} \\
\cdot \frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}} + E\,.
\end{aligned}
$$

Here the error E is due to the fact that $R_j(\rho)$ need not equal R, $j=1,2$. It follows that

$$
|E| \le \int_{R-\rho\le|y|\le R+\rho} \frac{|\Omega(y)|}{|y|^n}\, |f(x-y)|\, dy \le C\, M_\Omega f(x)\,,
$$

and Theorem 1 gives the weak type estimate for $\sup_{\varepsilon,R}|E|$. Thus we have

$$
\begin{aligned}
I_1 + I_2 &= \int_{\rho+\varepsilon}^{R} g(t)\, \frac{t}{\rho^2-t^2}\, dt \int_{S^{n-1}} \Omega(\omega)\, \frac{\rho\cos A}{\sqrt{t^2-\rho^2\sin^2 A}}\, d\omega + E \\
&= J_1 + E\,,
\end{aligned}
$$

where we used the equality $\int \Omega(\omega)\, d\omega = 0$. Moreover,

$$
\begin{aligned}
I_3 + I_4 &= \int_0^{\rho-\varepsilon} g(t)\, \frac{t}{\rho^2-t^2}\, dt \int_{\substack{A<\pi/2 \\ \sin A < t/\rho}} 2\,\Omega(\omega)\, \frac{\rho\cos A}{\sqrt{t^2-\rho^2\sin^2 A}}\, d\omega \\
&= J_2\,.
\end{aligned}
$$

That part of J_1 obtained by integrating only over $t > 2\,\rho$ is easy to control, since its absolute value is at most

$$C \int_{2\rho}^{R} |g(t)| \, \frac{1}{t} \, \frac{\rho}{t} \, dt \, \|\Omega\|_1 \, .$$

It is then enough to observe that

$$\int_0^\infty \rho^{n-1} \, d\rho \int_{2\rho}^\infty |g(t)| \, \frac{\rho}{t^2} \, dt \le C \int_0^\infty |g(t)| \, t^{n-1} \, dt \, .$$

This takes care of the supremum in R.

That part of J_2 which corresponds to $t < \rho/2$ can also be easily handled. Indeed, it equals what one gets by restricting the integral defining $\tilde{T}_\Omega^\varepsilon f(x)$ to the region $|y-x| < \min\{\rho/2, \rho-\varepsilon\}$. Since $|y| \sim |x|$ in this region, we can dominate by $M_\Omega f(x)$ and apply Theorem 1 to get the desired weak type estimate.

The remaining integrals are thus

$$J_1' = \int_{\rho+\varepsilon}^{2\rho} g(t) \, \frac{t}{\rho^2 - t^2} \, dt \int_{S^{n-1}} \Omega(\omega) \, \frac{\rho \cos A}{\sqrt{t^2 - \rho^2 \sin^2 A}} \, d\omega$$

and

$$J_2' = \int_{\rho/2}^{\rho-\varepsilon} g(t) \, \frac{t}{\rho^2 - t^2} \, dt \int_{\substack{A<\pi/2 \\ \sin A < t/\rho}} 2\,\Omega(\omega) \, \frac{\rho \cos A}{\sqrt{t^2 - \rho^2 \sin^2 A}} \, d\omega \, .$$

Notice that the value for $t = \rho$ of the inner integral in J_1' is

$$a(\theta) = \int_{S^{n-1}} \Omega(\omega) \operatorname{sgn} \cos A \, d\omega \, .$$

The corresponding quantity for J_2' is

$$\int_{A<\pi/2} 2\,\Omega(\omega) \, d\omega = a(\theta) \, ,$$

because of the vanishing mean value of Ω. Clearly a is a continuous function on S^{n-1}.

If we replace the inner integrals of J_1' and J_2' by $a(\theta)$, the resulting expressions will add up to

$$a(\theta)\int_{[\rho/2,2\rho]\setminus[\rho-\varepsilon,\rho+\varepsilon]} g(t)\,\frac{t}{\rho^2-t^2}\,dt\,.$$

This integral is a truncation of a smooth principal value singular integral on $\mathbb{R}_+$. By standard methods, it can be shown to define a weak type (1,1) operator for the measure $t^{n-1}\,dt$. So does the corresponding maximal singular integral, defined as the supremum in ε of the integral.

Since a is a bounded function, we also get a bounded operator from $L^1(t^{n-1}\,dt)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Thus, to prove Proposition 3, it only remains to estimate the difference operators arising when we subtract $a(\theta)$ from the inner integrals in J_1' and J_2'. For these operators, we shall actually derive strong type estimates.

For the case of J_1', we write

$$\begin{aligned}\Big|\frac{\rho\cos A}{\sqrt{t^2-\rho^2\sin^2 A}}-\operatorname{sgn}\cos A\Big| &= \Big|\rho\cos A\Big(\frac{1}{\sqrt{t^2-\rho^2\sin^2 A}}-\frac{1}{\rho\,|\cos A|}\Big)\Big|\\ &\le \rho\,|\cos A|\,\frac{\sqrt{t^2-\rho^2\sin^2 A}-\rho\,|\cos A|}{\rho\,|\cos A|\sqrt{t^2-\rho^2\sin^2 A}}\\ &\le \frac{t^2-\rho^2}{t^2-\rho^2\sin^2 A}\,,\end{aligned}$$

where we multiplied and divided by the conjugate quantity of the numerator, to get the last inequality. Our difference operator is thus controlled by

$$V_1 g(\rho,\theta)=\int_\rho^{2\rho}|g(t)|\,t\,dt\int_{S^{n-1}}|\Omega(\omega)|\,\frac{d\omega}{t^2-\rho^2\sin^2 A}\,.$$

One finds

$$\begin{aligned}&\int_{S^{n-1}} V_1 g(\rho,\theta)\,d\theta\\ &\qquad\le \int_\rho^{2\rho}|g(t)|\,t\,dt\int_{S^{n-1}}|\Omega(\omega)|\,d\omega\int_{S^{n-1}}\frac{d\theta}{t^2-\rho^2\sin^2 A}\,.\end{aligned}$$

Writing $s = t/\rho \in (1,2)$, we see that the innermost integral here is

$$\begin{aligned}
C\rho^{-2}\int_0^{\pi/2} \frac{\sin^{n-2}\alpha\, d\alpha}{s^2-\sin^2\alpha} &= C\rho^{-2}\int_0^1 \frac{u^{n-2}\, du}{\sqrt{1-u^2}\,(s^2-u^2)} \\
&\le C\rho^{-2}\int_0^1 \frac{du}{\sqrt{1-u}\,(s-u)} \\
&= C\rho^{-2}\int_0^1 \frac{dv}{\sqrt{v}\,(s-1+v)} \\
&= C\rho^{-2}\Big(\int_0^{s-1} + \int_{s-1}^1\Big) \\
&\le \frac{C\rho^{-2}}{\sqrt{s-1}} \\
&= \frac{C\rho^{-3/2}}{\sqrt{t-\rho}}\ .
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_0^\infty \rho^{n-1}\, d\rho \int_{S^{n-1}} V_1 g(\rho,\theta)\, d\theta \\
&\qquad \le C\int_0^\infty |g(t)|\, t\, dt \int_{t/2}^t \rho^{n-1-3/2}\, \frac{d\rho}{\sqrt{t-\rho}}\, \|\Omega\|_1 \\
&\qquad = C\int_0^\infty |g(t)|\, t^{n-1}\, dt\, \|\Omega\|_1\ .
\end{aligned}$$

Since $V_1 g$ does not depend on ε, this is the desired strong type (1,1) estimate.

To deal with the difference operator coming from J_2', we observe that, almost as in the case of J_1',

$$\begin{aligned}
&\Big|\int_{\substack{A<\pi/2\\ \sin A<t/\rho}} 2\,\Omega(\omega)\, \frac{\rho\cos A}{\sqrt{t^2-\rho^2\sin^2 A}}\, d\omega - \int_{A<\pi/2} 2\,\Omega(\omega)\, d\omega\Big| \\
&\qquad \le 2\int_{\substack{A<\pi/2\\ \sin A<t/\rho}} |\Omega(\omega)|\, \frac{\rho^2-t^2}{\rho\cos A\sqrt{t^2-\rho^2\sin^2 A}}\, d\omega \\
&\qquad\quad + 2\int_{\substack{A<\pi/2\\ \sin A>t/\rho}} |\Omega(\omega)|\, d\omega \\
&\qquad = K_1 + K_2\ .
\end{aligned}$$

With $s = t/\rho \in (1/2, 1)$, we now get

$$\int_{S^{n-1}} d\theta \int_{\rho/2}^{\rho-\varepsilon} |g(t)| \frac{t}{\rho^2 - t^2}\, dt\, K_1$$
$$\leq 2 \int_{\rho/2}^{\rho} |g(t)|\, t\, dt\, \rho^{-2} \int_{S^{n-1}} |\Omega(\omega)|\, d\omega \int_{\substack{A<\pi/2 \\ \sin A < s}} \frac{d\theta}{\cos A \sqrt{s^2 - \sin^2 A}} \,.$$

Here the innermost integral is

$$\begin{aligned} C \int_0^{\arcsin s} \frac{\sin^{n-2}\alpha\, d\alpha}{\cos\alpha\, \sqrt{s^2 - \sin^2\alpha}} &= C \int_0^s \frac{u^{n-2}\, du}{(1-u^2)\sqrt{s^2 - u^2}} \\ &\leq C \int_0^s \frac{du}{(1-u)\sqrt{s-u}} \\ &= C \int_0^s \frac{du}{(1-s+u)\sqrt{u}} \\ &\leq \frac{C}{\sqrt{1-s}} \,. \end{aligned}$$

This implies

$$\begin{aligned} \int_0^\infty \rho^{n-1}\, d\rho \int_{S^{n-1}} d\theta \int_{\rho/2}^{\rho} |g(t)| \frac{t}{\rho^2 - t^2}\, dt\, K_1 & \\ \leq C \int_0^\infty |g(t)|\, t\, dt \int_t^{2t} \rho^{n-1-2+1/2} \frac{d\rho}{\sqrt{\rho - t}} \, \|\Omega\|_1 & \\ \leq C \int_0^\infty |g(t)|\, t^{n-1}\, dt\, \|\Omega\|_1 \,. & \end{aligned}$$

Similarly,

$$\int_{S^{n-1}} d\theta \int_{\rho/2}^{\rho-\varepsilon} |g(t)| \frac{t}{\rho^2 - t^2}\, dt\, K_2$$
$$\leq 2 \int_{\rho/2}^{\rho} |g(t)| \frac{t}{\rho^2 - t^2}\, dt \int_{S^{n-1}} |\Omega(\omega)|\, d\omega \int_{\substack{A<\pi/2 \\ \sin A > s}} d\theta \,.$$

Here the innermost integral is found to be $\mathrm{O}(\sqrt{1-s})$. Integrating the above against $\rho^{n-1}\, d\rho$, we get at most

$$C \int |g(t)|\, t^{n-1}\, dt\, \|\Omega\|_1 \,,$$

as before. This strong type estimate ends the proof of Proposition 3.

PROOF OF PROPOSITION 4. Observe that $T_\Omega^{\varepsilon,R} f(x) - \tilde{T}_\Omega^{\varepsilon,R} f(x)$ is independent of R. One has

$$D_\Omega^* f(x) \le \sup_{\varepsilon>0} \int\limits_{\substack{|y|>\varepsilon \\ |\,|x-y|-|x|\,|<\varepsilon}} \frac{|\Omega(y)|}{|y|^n}\, |f(x-y)|\, dy\,. \tag{3.4}$$

We assume that $f, g \ge 0$. Notice that $r = \varepsilon$ is equivalent to $t = t_\varepsilon$, where

$$t_\varepsilon^2 = \rho^2 + \varepsilon^2 - 2\,\rho\,\varepsilon \cos A\,. \tag{3.5}$$

One can assume that $\varepsilon < \rho/2$, since otherwise the integral in (3.4) is taken over a region where $\varepsilon < |y| < C\varepsilon$. Then the rough maximal operator of Theorem 1 applies.

As in the preceding proof, we write the integral in (3.4) in polar coordinates and replace the integration in r by integration in t. Again, we divide the resulting integral into four parts, though not quite in the same way as before. For the supremum of each part, we shall derive a strong or weak type (1,1) estimate.

Part 1: $A > \pi/2$. Then $\cos A < 0$, and $t > \rho$. This part of the integral in (3.4) is dominated in absolute value by

$$\begin{aligned}
&\int_{A>\pi/2} |\Omega(\omega)|\, d\omega \int_{t_\varepsilon}^{\rho+\varepsilon} \frac{g(t)}{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}}\, \frac{t\, dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad = \int_{A>\pi/2} |\Omega(\omega)|\, d\omega \int_{t_\varepsilon}^{\rho+\varepsilon} \frac{\rho\,|\cos A| + \sqrt{t^2-\rho^2\sin^2 A}}{t^2-\rho^2} \\
&\qquad\qquad\qquad \cdot \frac{g(t)\, t\, dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad \le 2 \int_{A>\pi/2} |\Omega(\omega)|\, d\omega \int_{t_\varepsilon}^{\rho+\varepsilon} \frac{g(t)\, t\, dt}{t^2-\rho^2}\,,
\end{aligned} \tag{3.6}$$

since here $\rho\,|\cos A| \le \sqrt{t^2-\rho^2\sin^2 A}$.

The last inner integral is no larger than

$$\int_{t_\varepsilon}^{\rho+\varepsilon} g(t)\, \frac{dt}{t-\rho} \le \int_\rho^{\rho+\varepsilon} g(t) \min\left\{\frac{1}{t-\rho}, \frac{1}{t_\varepsilon-\rho}\right\} dt\,.$$

Since the minimum here is decreasing in t for $\rho < t < \rho + \varepsilon$, it is well known that the right hand integral is dominated by the maximal function of g at ρ times

$$\int_\rho^{\rho+\varepsilon} \min\Big\{\frac{1}{t-\rho}, \frac{1}{t_\varepsilon - \rho}\Big\}\, dt = 1 + \log \frac{\varepsilon}{t_\varepsilon - \rho}\,.$$

Instead of the ordinary maximal function $Mg(\rho)$, we can here use

$$M_\ell g(\rho) = M(g\chi_{[\rho/2, 2\rho]})(\rho)\,,$$

since $\varepsilon < \rho/2$. Because of (3.5), we have

$$\log \frac{\varepsilon}{t_\varepsilon - \rho} = \log \frac{\varepsilon\,(t_\varepsilon + \rho)}{\varepsilon^2 + 2\,\rho\,\varepsilon\,|\cos A|} \le \log \frac{1}{|\cos A|}\,.$$

Altogether, the expressions in (3.6) are majorized by

$$2\, M_\ell g(\rho) \int_{S^{n-1}} |\Omega(\omega)| \Big(1 + \log \frac{1}{|\cos A|}\Big)\, d\omega\,.$$

Here the first factor is in $L^{1,\infty}(\rho^{n-1}\, d\rho)$ and the second in $L^1(S^{n-1})$ as a function of θ, as shown via Fubini's theorem. A product of this type belongs to $L^{1,\infty}(\rho^{n-1}\, d\rho\, d\theta)$. Since the product is independent of ε, this ends Part 1.

Part 2: $A < \pi/2$ and $r < (\rho\cos A)/2$. Since

$$r = \rho\cos A - \sqrt{t^2 - \rho^2 \sin^2 A}\,,$$

this implies

$$\sqrt{t^2 - \rho^2\sin^2 A} > \frac{1}{2}\,\rho\cos A\,. \tag{3.7}$$

We can assume that

$$\frac{1}{2}\,\rho\cos A > \varepsilon\,, \tag{3.8}$$

because otherwise we get nothing. The part of the integral in (3.4) we get is

$$\begin{aligned}
&\int_{A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho-\varepsilon}^{t_\varepsilon} \frac{g(t)}{\rho\cos A - \sqrt{t^2-\rho^2\sin^2 A}}\, \frac{t\, dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&= \int_{A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho-\varepsilon}^{t_\varepsilon} \frac{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}}{\rho^2-t^2}\, \frac{g(t)\, t\, dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&\le C \int_{A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho-\varepsilon}^{t_\varepsilon} \frac{g(t)\, t\, dt}{\rho^2-t^2}\,,
\end{aligned}$$

the last step because of (3.7).

We proceed as in Part 1. The logarithm to be estimated is now

$$\log \frac{\varepsilon}{\rho - t_\varepsilon} = \log \frac{\varepsilon\,(\rho + t_\varepsilon)}{2\,\rho\,\varepsilon\cos A - \varepsilon^2}\,.$$

By means of (3.8), we get rid of the ε^2 term in the denominator, and the logarithm is seen to be dominated by $\log(1/\cos A)$. The rest is like Part 1.

Part 3: $A < \pi/2$ and $(\rho\cos A)/2 < r < 2\,\rho\cos A$. This part of the integral in (3.4) is dominated by the rough maximal function $M^*_\Omega f(x)$. We apply Theorem 1.

Part 4: $A < \pi/2$ and $r > 2\,\rho\cos A$. Notice that this inequality for r is equivalent to $t > \rho$. We can assume that $A > \pi/4$, because otherwise $\rho/C \le r \le C\rho$ for some C, and M^*_Ω will apply.

The integral we now get is

$$\begin{aligned}
&\int_{\pi/4<A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho}^{\rho+\varepsilon} \frac{g(t)}{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}}\, \frac{t\, dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad\qquad \le \int_{\pi/4<A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho}^{2\rho} \frac{g(t)\, t\, dt}{t^2-\rho^2\sin^2 A}\,.
\end{aligned}$$

Notice that the last expression does not contain ε. Its integral with respect to $dx = C\rho^{n-1}\,d\rho\,d\theta$ is

$$
\begin{aligned}
C\int_0^\infty \rho^{n-1}\,d\rho \int_{S^{n-1}} d\theta \int_{\pi/4<A<\pi/2} |\Omega(\omega)|\,d\omega \int_\rho^{2\rho} \frac{g(t)\,t\,dt}{t^2-\rho^2\sin^2 A} \\
\le C\int_0^\infty g(t)\,t\,dt \int_{S^{n-1}} |\Omega(\omega)|\,d\omega \int_{t/2}^{t} \rho^{n-1}\,d\rho \\
\cdot \int_{\pi/4<A<\pi/2} \frac{d\theta}{t^2-\rho^2+\rho^2\cos^2 A}\,.
\end{aligned}
$$

The innermost integral here is

$$
C\int_{\pi/4}^{\pi/2} \frac{\sin^{n-2}\alpha\,d\alpha}{t^2-\rho^2+\rho^2\cos^2\alpha} \le C\int_0^{1/\sqrt{2}} \frac{du}{t^2-\rho^2+\rho^2 u^2} \le \frac{C}{\rho\sqrt{t^2-\rho^2}}\,.
$$

It follows that the fourfold integral is no larger than

$$
C\int g(t)\,t^{n-1}\,dt\,\|\Omega\|_1\,.
$$

This ends Part 4 and the proof of Proposition 4.

For the L^p part of Theorem 2, it is clearly enough to prove versions of Propositions 3 and 4 with strong type (p,p) instead of weak type $(1,1)$. This requires only small modifications of the proofs just given. For instance, in the proof of Proposition 3 one obtains several strong type $(1,1)$ inequalities by integrating various expressions with respect to $\rho^{n-1}\,d\rho\,d\theta$. For the L^p inequality, one can instead estimate these expressions by quantities like

$$
C\,M_\ell g(\rho)\int_{S^{n-1}} |\Omega(\omega)|\Big(1+\log\frac{1}{|\cos A|}\Big)\,d\omega\,,
$$

which is in $L^p(\rho^{n-1}\,d\rho\,d\theta)$ if $g\in L^p(\rho^{n-1}\,d\rho\,d\theta)$. We leave the details of the rest of the L^p case to the reader.

This ends the proof of Theorem 2.

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