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The phase of the Daubechies filters

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Abstract. We give the first term of the asymptotic development for the phase of the N-th (minimum-phased) Daubechies filter as N goes to $+\infty$. We obtain this result through the description of the complex zeros of the associated polynomial of degree $2N+1$.

0. Introduction.

The *Daubechies filters* $m_N(\xi)$ are defined in the following way [2]:

i) $m_N(\xi)$ is a trigonometric polynomial of degree $2N+1$

$$m_N(\xi) = \sum_{k=0}^{2N+1} a_{N,k}\, e^{-ik\xi} \tag{1}$$

with *real-valued* coefficients $a_{N,k}$.

ii) $\sqrt{2}\, m_N(\xi)$ and $\sqrt{2}\, e^{-i\xi}\, \overline{m}_N(\xi+\pi)$ are *conjugate quadrature filters*

$$|m_N(\xi)|^2 + |m_N(\xi+\pi)|^2 = 1\,. \tag{2}$$

iii) $m_N(\xi)$ satisfies at 0 and π

$$m_N(0) = 1\,, \tag{3}$$

$$\frac{\partial^p}{\partial \xi^p} m_N(\pi) = 0\,, \qquad \text{for } p \in \{0, 1, \dots, N\}\,. \tag{4}$$

The importance of those filters is due to the following facts: the associated *wavelet* ψ_N defined by

$$\hat{\psi}_N(\xi) = e^{-i\xi/2}\, \overline{m}_N\Big(\frac{\xi}{2}+\pi\Big) \prod_{j=2}^{+\infty} m_N\Big(\frac{\xi}{2^j}\Big)\,,$$

generates an orthonormal basis of $L^2(\mathbb{R})$ $\{2^{j/2}\psi_N(2^j x - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ and satisfies the cancellation properties

$$\int x^p \psi_N(x)\,dx = 0\,, \qquad \text{for } p \in \{0,1,\cdots,N\}\,,$$

and has a support of minimal length among all orthonormal wavelets satisfying (6).

Conditions (1) to (4) don't define m_N in an unique way. As a matter of fact, there is exactly $2^{[(N+1)/2]}$ solutions m_N (where $[x]$ is the integer part of x). Indeed, conditions (1) to (4) determine only the *modulus* of m_N

$$|m_N(\xi)|^2 = Q_N(\cos\xi)\,, \tag{7}$$

$$Q_N(X) = \Big(\frac{1+X}{2}\Big)^{N+1} \sum_{k=0}^{N} \binom{N+k}{k} \Big(\frac{1-X}{2}\Big)^k . \tag{8}$$

We are going to check easily the following result on the roots of Q_N.

Proposition 1. *The roots of Q_N are $X=-1$ with multiplicity $N+1$ and N roots $X_{N,1},\cdots,X_{N,N}$ with multiplicity 1 such that*

i) *for $1 \le k \le N$, $\operatorname{Re} X_{N,k} > 0$ and $X_{N,N+1-k} = \overline{X_{N,k}}$,*

ii) *for $1 \le k \le [N/2]$, $\operatorname{Im} X_{N,k} > 0$,*

iii) *if N is odd, $X_{N,(N+1)/2} > 1$.*

With help of Proposition 1, we may easily describe the solutions m_N of (1) to (4). Indeed, if $X_{N,k} = (z_{N,k} + 1/z_{N,k})/2$ with $|z_{N,k}| > 1$, then we have

$$m_N(\xi) = \prod_{k=1}^{[(N+1)/2]} S_{N,k}(\xi) \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1}, \tag{9}$$

where, for $1 \le k \le [N/2]$,

$$\text{(10)} \qquad S_{N,k}(\xi) = \frac{(e^{-i\xi} - z_{N,k})\,(e^{-i\xi} - \overline{z}_{N,k})}{|1 - z_{N,k}|^2} \quad \text{or} \quad S_{N,k}(\xi) = \frac{(1 - z_{N,k}\, e^{-i\xi})\,(1 - \overline{z}_{N,k}\, e^{-i\xi})}{|1 - z_{N,k}|^2}\,.$$

If N is odd,

$$\text{(11)} \qquad S_{N,(N+1)/2}(\xi) = \frac{e^{-i\xi} - z_{N,(N+1)/2}}{1 - z_{N,(N+1)/2}} \quad \text{or} \quad S_{N,(N+1)/2}(\xi) = \frac{1 - z_{N,(N+1)/2}\, e^{-i\xi}}{1 - z_{N,(N+1)/2}}\,.$$

The case where all the roots of $M_N(z)$ (the polynomial such that $m_N(\xi) = M_N(e^{-i\xi})$) are outside the unit disk is the *minimum-phased Daubechies filter*

$$\text{(12)} \qquad m_N(\xi) = \Big(\frac{1 + e^{-i\xi}}{2}\Big)^{N+1} \prod_{k=1}^{N} \frac{e^{-i\xi} - z_{N,k}}{1 - z_{N,k}}\,.$$

The aim of this paper is to describe the phase of the Daubechies filters as N goes to $+\infty$. Indeed, the modulus of m_N is described by (7) and (8) and one easily checks that

$$\text{(13)} \qquad \lim_{N \to +\infty} |m_N(\xi)| = \begin{cases} 1\,, & \text{if } |\xi| < \dfrac{\pi}{2}\,, \\[2mm] \dfrac{1}{\sqrt{2}}\,, & \text{if } |\xi| = \dfrac{\pi}{2}\,, \\[2mm] 0\,, & \text{if } \dfrac{\pi}{2} < |\xi| \le \pi\,. \end{cases}$$

The phase of m_N, on the other hand, is much more delicate to study: it depends of course on the choice of the factors $S_{N,k}$ in (9), but even for the case of minimum-phased filters we are not aware of any previous results on the behaviour of the phase.

We are going to give an approximate value of $z_{N,k}$ which allows the determination of the phase of m_N. More precisely, if $Z_1, \dots, Z_N$ are N complex numbers such that for $k \in \{1, \dots, N\}$, $|Z_k| \ne 1$ and if

$$\Pi(Z_1, \dots, Z_N)(\xi) = \prod_{k=1}^{N} \frac{e^{-i\xi} - Z_k}{1 - Z_k}\,,$$

we define *the phase* $\omega(Z_1, \ldots, Z_N)(\xi)$ as the C^∞ real-valued function such that $\omega(0) = 0$ and

$$\Pi(Z_1, \ldots, Z_N)(\xi) = \prod_{k=1}^{N} \Big| \frac{e^{-i\xi} - Z_k}{1 - Z_k} \Big| e^{-i\omega(Z_1,\ldots,Z_N)(\xi)} .$$

This function is easily computed as

$$\omega(Z_1, \ldots, Z_N)(\xi) = \operatorname{Im} \Big(\int_0^\xi \sum_{k=1}^{N} \frac{i\, e^{-is}}{e^{-is} - Z_k}\, ds \Big) . \tag{14}$$

Theorem 1. *Let $Q_N(X)$ be given by (8), $X_{N,1}, \ldots, X_{N,N}$ be its roots which are not equal to -1 ordered by:*

- *for $1 \le k \le [(N+1)/2]$, $\operatorname{Im} X_{N,k} \ge 0$ and $X_{N,N+1-k} = \overline{X_{N,k}}$,*
- *$|X_{N,1}| < |X_{N,2}| < \cdots < |X_{N,[(N+1)/2]}|$*

and let $z_{N,k}$ be defined by $X_{N,k} = (z_{N,k} + 1/z_{N,k})/2$ and $|z_{N,k}| > 1$.

For $1 \le k \le N$, we approximate $z_{N,k}$ by $Z_{N,k}$ where:

i) *for $1 \le k \le [(N^{1/5})/\operatorname{Log} N]$, $Z_{N,k} = i - \overline{\gamma_k}/\sqrt{N}$, where $\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots$ are the roots of $\operatorname{erfc}(z) = 1 - (2/\sqrt{\pi}) \int_0^z e^{-s^2}\, ds$, such that $\operatorname{Im} \gamma_k > 0$ and ordered by $|\gamma_1| < |\gamma_2| < \cdots < |\gamma_k| < \ldots$,*

ii) *for $[(N^{1/5})/\operatorname{Log} N] < k \le [(N+1)/2]$, $Z_{N,k} = \theta_{N,k} + \sqrt{\theta_{N,k}^2 - 1}$, where*

$$\operatorname{Im} \theta_{N,k} > 0\,, \tag{15.a}$$

$$1 - \theta_{N,k}^2 = \Big(1 + \frac{1}{N} \operatorname{Log}\,(2\sqrt{2N\pi \sin \varphi_{N,k}})\Big)\, e^{-2i\varphi_{N,k}}\,, \tag{15.b}$$

and

$$\varphi_{N,k} = \frac{8\,k - 1}{8N + 6}\, \pi\,, \tag{16}$$

iii) *for $[(N+1)/2] < k \le N$, $Z_{N,k} = \overline{Z}_{N,N+1-k}$.*

Then for any choice

$$m_N(\xi) = \Big(\frac{1 + e^{-i\xi}}{2}\Big)^{N+1} \Pi(z_{N,1}^{\varepsilon_1}, \ldots, z_{N,N}^{\varepsilon_N})(\xi)$$

of the Daubechies filter m_N *(where* $\varepsilon_k = \pm 1$ *and* $\varepsilon_{N+1-k} = \varepsilon_k$*), the approximation*

$$\tilde{m}_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \Pi(Z_{N,1}^{\varepsilon_1}, \dots, Z_{N,N}^{\varepsilon_N})(\xi)$$

satisfies

$$(17) \quad |\omega(z_{N,1}^{\varepsilon_1}, \dots, z_{N,N}^{\varepsilon_N})(\xi) - \omega(Z_{N,1}^{\varepsilon_1}, \dots, Z_{N,N}^{\varepsilon_N})(\xi)| \le C_0 \frac{(\mathrm{Log}\, N)^2}{N^{1/5}} ,$$

for all $\xi \in \mathbb{R}$*, where* C_0 *doesn't depend neither on* $N \ge 2$ *nor on* ξ *nor on the* ε_k*'s.*

Thus, due to Theorem 1, we may give the phase of m_N with an $o(1)$ precision! Of course, we need the knowledge of the roots of the complementary error function; these roots are described in [3] and our results give again the same estimates, as we shall see.

We may greatly simplify the approximating $Z_{N,k}$'s if we accept to get a greater error. For instance, we may characterize easily the minimum-phased filters with an $O(\sqrt{N})$ error:

Theorem 2. *Let*

$$m_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \Pi(z_{N,1}, \dots, z_{N,N})(\xi)$$

be the N*-th minimum-phased Daubechies filter. Then the phase*

$$\omega(z_{N,1}, \dots, z_{N,N})(\xi)$$

satisfies

$$(18) \quad |\omega(z_{N,1}, \dots, z_{N,N})(\xi) - N\omega(\xi)| \le C_0 \sqrt{N} , \qquad \text{for all } \xi \in \mathbb{R},$$

where C_0 *doesn't depend on* ξ *nor on* N *and where*

$$(19) \quad \omega(\xi) = \frac{1}{2\pi}\left(\mathrm{Li}_2(-\sin\xi) - \mathrm{Li}_2(\sin\xi)\right) = \frac{-1}{\pi} \sum_{k=0}^{+\infty} \frac{(\sin\xi)^{2k+1}}{(2k+1)^2} .$$

The Li_2 function is the polylogarithm of order 2

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_0^z \frac{1}{u} \,\mathrm{Log}\, \frac{1}{1-u}\, du\,. \tag{20}$$

The function $(\mathrm{Li}_2(z) - \mathrm{Li}_2(-z))/2$ is known under the name of Legendre's χ_2 function.

Theorem 2 will be proved by approximating m_N by

$$\tilde{m}_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \pi(\tilde{Z}_{N,1}, \dots, \tilde{Z}_{N,N})(\xi)$$

with

$$\tilde{Z}_{N,k} = \sqrt{e^{-i\theta_{N,k}}} + \sqrt{1+e^{-i\theta_{N,k}}}\,, \qquad \theta_{N,k} = -\pi + \frac{16k-2}{8N+6}\,\pi\,,$$

Then $\omega(\tilde{Z}_{N,1}, \dots, \tilde{Z}_{N,N})/N$ is identified with a Riemann sum for the integral

$$\frac{1}{2\pi}\,\mathrm{Im} \int_{-\pi}^{\pi} \mathrm{Log}\, \frac{1}{\sqrt{e^{-i\theta}} + \sqrt{1+e^{-i\theta}} - e^{-i\xi}}\, d\theta = \omega(\xi)\,.$$

This approximating $\tilde{Z}_{N,k}$ is a simplified version of the approximating $Z_{N,k}$ of Theorem 1, obtained by neglecting the term

$$\frac{1}{N}\,\mathrm{Log}\, 2\sqrt{2N\pi \sin\varphi_{N,k}}\,.$$

We will be also able to give a description of a family of almost linear-phased Daubechies filters:

Theorem 3. *Let*

$$m_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \pi(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi)$$

be the N-th Daubechies filter with $N = 4\,q$ and with the following choice of $\varepsilon_{N,k}$: for $1 \le p \le q$, $\varepsilon_{N,4p-3} = \varepsilon_{N,4p} = 1$ and $\varepsilon_{N,4p-2} = \varepsilon_{4p-1} = -1$ (so that $\varepsilon_{N,N+1-k} = \varepsilon_{N,k}$). Then the phase $\omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi)$ satisfies:

$$\Big|\omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi) - \frac{1}{2}\,N\xi\Big| \le C_0\,, \qquad \textit{for all } \xi \in \mathbb{R}, \tag{21}$$

where C_0 doesn't depend on ξ nor on N.

We are now going to prove Theorem 1 (and obtain theorems 2 and 3 as corollaries). Of course, it amounts to give a precise description of the roots $X_{N,k}$ of $Q_N(X)$. If we neglect the term $\operatorname{Log} 2\,\sqrt{2N\pi\sin\varphi_{N,k}}/N$ in $Z_{N,k}$, we obtain as a first approximation that the $z_{N,k}$ are close to the arc $\{|z-1| = \sqrt{2},\ \operatorname{Re} z \geq 0\}$ (which can be parameterized as $\{\sqrt{e^{-i\theta}} + \sqrt{1+e^{-i\theta}},\ -\pi \leq \theta \leq \pi\}$), or equivalently that the $X_{N,k}$ are close to the half-lemniscate $\{|1-X_{N,k}^2| = 1, \operatorname{Re} X_{N,k} \geq 0\}$. This will be obtained by representing $Q_N(X)$ as a Bernstein polynomial on $[-1,1]$ approximating the piecewise analytical function $\chi_{[0,1]}$

$$Q_N(X) = \sum_{k=N+1}^{2N+1} \binom{2N+1}{k} \Big(\frac{1+X}{2}\Big)^k \Big(\frac{1-X}{2}\Big)^{2N+1-k} \tag{22}$$

(a formula pointed by many authors [1], [6], [11]). In that form, $Q_N(X)$ corresponds to a Herrmann filter [4] and it is precisely the figure in Herrmann's paper representing the $z_{N,k}$'s for Q_{21} which lead us to conjecture the behaviour of the $z_{N,k}$'s.

A classical theorem of Kantorovitch [5], [7] on the behaviour of Bernstein polynomials of piecewise analytical functions ensures that $Q_N(X)$ converges to 0 uniformly on any compact subset of the interior of the half lemniscat $\{|1-x^2| < 1,\ \operatorname{Re} x < 0\}$ and to 1 uniformly on any compact subset of $\{|1-x^2| < 1,\ \operatorname{Re} x > 0\}$. We will use similar tools to study $Q_N(X)$ *outside* of the convergence subsets.

Near the critical point $X = 0$, the approximation by points on the lemniscat is no longer precise enough, and we will show that for the small roots $X_{N,k}$, $-\sqrt{N}X_{N,k}$ is to be approximated by a root of the complementary error function. Such an approximation occurs for instance in the study of the (spurious) zeros of the Taylor polynomials of the exponential function [12] and we will use quite similar tools to get our description. The main difference, however, is maybe that we are dealing with a divergent family of polynomials.

NOTATIONS. We will define as usually $\operatorname{Log} z$ and $\sqrt{z}$ as the reciprocal functions of

$$z = \operatorname{Log} w \in \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi\} \longmapsto w = e^z \in \{w \in \mathbb{C} : w \notin (-\infty, 0])\}\,,$$

$$z = \sqrt{w} \in \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \longmapsto w = z^2 \in \{w \in \mathbb{C} : w \notin (-\infty, 0]\}\,.$$

The paper will be organized in the following way:

1. Q_N as a Bernstein polynomial and other preliminary results.
2. Small roots of Q_N: first estimates.
3. Big roots of Q_N: first estimates.
4. Big roots of Q_N: further estimates.
5. Small roots of Q_N: further estimates.
6. The phase of a general Daubechies filter.
7. Minimum-phased Daubechies filters.
8. Almost linear-phased Daubechies filters.

1. Q_N as a Bernstein polynomial and other preliminary results.

We begin by proving a first localization result:

Result 1. *For $N \geq 2$ and $t \neq -1$, if $Q_N(t) = 0$ then $\mid 1 - t \mid < 1$.*

Proof. This will be the only time where we use the Daubechies formula (8) for $Q_N(X)$. This formula gives that if $Q_N(t) = 0$ and $t \neq -1$, then

$$\sum_{k=0}^{N} \frac{1}{2^k} \binom{N+k}{k} (1-t)^k = 0\,. \tag{23}$$

If we define α_k as $\alpha_k = \binom{N+k}{k}/2^k$, $0 \leq k \leq N$, then we have obviously $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_{N-1} = \alpha_N$, and we may apply a very classical lemma of Eneström, Kakeya and Hurwirtz (quoted by G. Pólya and Szegö [10, Exercise III-22]):

Lemma 1. *If $0 < a_0 < a_1 < \cdots < a_{N-1} = a_N$ and if $\sum_{k=0}^{N} a_k s^k = 0$ then $|s| < 1$.*

Proof of the lemma. If $s \geq 0$ then $\sum_{k=0}^{N} a_k s^k > 0$; if $s \notin [0, +\infty)$, then

$$\Big| a_0 + \sum_{k=1}^{N} (a_k - a_{k-1}) s^k \Big| < a_0 + \sum_{k=1}^{N} (a_k - a_{k-1})\, |s|^k\,,$$

thus if $|s| \geq 1$ (so that $|s|^k \leq |s|^{N+1}$) and $s \notin [0, +\infty)$, we get

$$\Big| (1-s) \sum_{k=0}^{N} a_k s^k \Big| > |s|^{N+1} \Big(a_N - \sum_{k=1}^{N} (a_k - a_{k-1}) - a_0 \Big) = 0\,.$$

Thus, we have shown that the roots t of Q_N such that $t \neq -1$ are located in the open disk of radius 1 and of center 1, and that the associated values $1 - t^2$ are located in the interior of a cardioid.

From now until the end, we will use formula (22) instead of formula (8) to represent Q_N. The main interest in the representation of Q_N as a Bernstein polynomial is that Q_N is easily differentiated: (22) gives

$$\frac{d}{dt} Q_N(t) = \frac{(2N+1)!}{4^N (N!)^2} \frac{1}{2} (1-t^2)^N . \tag{24}$$

This expression can be easily related to the expression of $Q_N(\cos\xi)$ given by Y. Meyer ([8])

$$\begin{aligned} Q_N(\cos\xi) &= \int_{-1}^{\cos\xi} \frac{(2N+1)!}{4^N (N!)^2} \frac{1}{2} (1-t^2)^N \, dt \\ &= \int_{\xi}^{\pi} \frac{(2N+1)!}{4^N (N!)^2} \frac{1}{2} (\sin\theta)^{2N+1} \, d\theta . \end{aligned}$$

We will use intensively formula (24) in the following. If t is small, we approximate $Q_N(t)$ by $Q_N(0) = 1/2$ and obtain

$$Q_N(t) = \frac{1}{2}\Big(1 + \frac{(2N+1)!}{4^N (N!)^2} \int_0^t (1-s^2)^N \, ds\Big) , \tag{25}$$

while for a bigger t (with $\operatorname{Re} t > 0$) we approximate $Q_N(t)$ by $Q_N(1) = 1$ and obtain

$$Q_N(t) = 1 - \frac{1}{2} \frac{(2N+1)!}{4^N (N!)^2} \int_t^1 (1-s^2)^N \, ds . \tag{26}$$

Stirling's formula $N! = (N/e)^N \sqrt{2\pi N}(1 + 1/(12N) + O(1/N^2))$ allows one to simplify formulas (25) and (26)

$$\frac{(2N+1)!}{4^N (N!)^2} = 2\sqrt{\frac{N}{\pi}} \Big(1 + O\Big(\frac{1}{N^2}\Big)\Big) . \tag{27}$$

Thus $Q_N(t) = 0$ may be rewritten as

$$1 + \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{N} t} \Big(1 - \frac{s^2}{N}\Big)^N ds = 1 - 2\frac{\sqrt{N}}{\sqrt{\pi}} \frac{4^N (N!)^2}{(2N+1)!} = O\,(\frac{1}{N^2}) \tag{28}$$

or as

$$\sqrt{N}\int_t^1 (1-s^2)^N\,ds = 2\,\frac{4^N (N!)^2}{(2N+1)!} = \sqrt{\pi} + O\Big(\frac{1}{N^2}\Big)\,. \tag{29}$$

Formula (28) will be used for the small roots (sections 2 and 5) and formula (29) for the big roots (sections 3 and 4).

We mention a further application of (24) (which will not be used in the following): we may compute explicitly the generating series for $Q_N(t)$ when $\operatorname{Re} t < 0$:

Proposition 2. *Assume that* $\operatorname{Re} t < 0$ *and* $|(1-t^2)\,u| < 1$. *Then*

$$\sum_{N=0}^{+\infty} Q_N(t)\,u^N = \frac{1}{2}\,\frac{1-t^2}{\sqrt{1-u\,(1-t^2)}\,(-t+\sqrt{1-u\,(1-t^2)})}\,. \tag{30}$$

PROOF. We differentiate $\sum_{N=0}^{+\infty} Q_N(t)\,u^N$ with respect to t. Then (24) gives

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\sum_{N=0}^{+\infty} Q_N(t)\,u^N\Big) &= \sum_{N=0}^{+\infty} \frac{1}{2}\,\frac{(2N+1)!}{4^N N!}\,\frac{((1-t^2)\,u)^N}{N!}\\ &= \frac{1}{2}\,(1-u\,(1-t^2))^{-3/2}\,,\end{aligned}$$

hence

$$\sum_{N=0}^{+\infty} Q_N(t)\,u^N = \int_{-1}^{t} \frac{1}{2}\,\frac{ds}{(1-(1-s^2)\,u)^{3/2}}\,.$$

On the other hand, if we differentiate $t/(1-u\,(1-t^2))^{1/2}$, we get

$$\frac{\partial}{\partial t}\Big(\frac{t}{(1-u\,(1-t^2))^{1/2}}\Big) = \frac{1-u\,(1-t^2)-t^2 u}{(1-u\,(1-t^2))^{3/2}} = \frac{1-u}{(1-u\,(1-t^2))^{3/2}}\,.$$

Thus we have

$$\begin{aligned}\sum_{N=0}^{+\infty} Q_N(t)\,u^N &= \frac{1}{2\,(1-u)}\Big(\frac{t}{(1-u\,(1-t^2))^{1/2}}+1\Big)\\ &= \frac{1}{2\,(1-u)}\,\frac{1-u\,(1-t^2)-t^2}{(1-u\,(1-t^2))^{1/2}((1-u\,(1-t^2))^{1/2}-t)}\\ &= \frac{1}{2}\,\frac{1-t^2}{(1-u\,(1-t^2))^{1/2}((1-u\,(1-t^2))^{1/2}-t)}\,.\end{aligned}$$

As a corollary, we get:

Result 2. *If $t \in \mathbb{C}$ is such that $|1-t^2| > 1$, then*

$$\limsup_{N \to +\infty} |Q_N(t)| = +\infty \,.$$

PROOF. If $\operatorname{Re} t < 0$, this is obvious by formula (30); the right-hand term of equality (30) has $1/|1-t^2|$ as its radius of convergence in u, so that

$$\limsup_{N \to +\infty} |Q_N(t)|^{1/N} = |1-t^2| \,.$$

If $\operatorname{Re} t > 0$, then $Q_N(t) = 1 - Q_N(-t)$ so that again

$$\limsup_{N \to +\infty} |Q_N(t)|^{1/N} = |1-t^2| \,.$$

If $\operatorname{Re} t = 0$ and $t \neq 0$, then

$$|Q_N(t)| \sim \frac{1}{2}\, 2 \sqrt{\frac{N}{\pi}} \int_0^{|t|} (1+\rho^2)^N \, d\rho \longrightarrow +\infty \,, \qquad \text{as } N \longrightarrow +\infty \,.$$

A last (and direct) application of formula (24) is Proposition 1.

Result 3.

i) *If t is a root of $Q_N(t)$ and $t \neq -1$, then t has multiplicity 1.*

ii) *If N is even, $t = -1$ is the unique real root of Q_N.*

iii) *If N is odd, Q_N has only one other real root $x_{N,(N+1)/2} \neq -1$, and $x_{N,(N+1)/2} > 1$.*

PROOF. By (24), we know that the only roots of dQ_N/dt are 1 and -1, so i) is obvious. Moreover, if N is even, dQ_N/dt is non-negative on $\mathbb{R}$ and thus Q_N is increasing: -1 is the unique real root of Q_N. If N is odd, then Q_N decreases on $(-\infty, -1]$, vanishes at -1, increases between -1 and 1, and decreases again from the value 1 at $t = 1$ to the value $-\infty$ at $t = +\infty$: Q_N has another real root $x_{N,(N+1)/2} > 1$.

Results 1 and 3 imply obviously Proposition 1.

2. Small roots of Q_N: first estimates.

In this section, we are going to prove the following result:

Result 4. *Let $\varepsilon_0 \in (0,1/2)$ and $K = [\varepsilon_0 \operatorname{Log} N/(2\pi)]$. Then, if N is big enough, the number of roots t of $Q_N(t)$ such that $\operatorname{Im} t \geq 0$ and $|t| \leq \sqrt{2K\pi/N}$ is exactly K. Moreover, if we list those roots as $x_{N,1}, \ldots, x_{N,K}$ with $|x_{N,k}| < |x_{N,k+1}|$ and fix $\varepsilon_1 \in (\varepsilon_0, 1/2)$, we have*

$$\Big| x_{N,k} + \frac{1}{\sqrt{N}}\, \overline{\gamma}_k \Big| \leq C(\varepsilon_0, \varepsilon_1)\, \frac{1}{\sqrt{N}\, N^{1-2\varepsilon_1}} \,, \tag{31}$$

where $\gamma_1, \ldots, \gamma_K$ are the K first roots γ of $\operatorname{erfc}(\gamma) = 0$ *with* $\operatorname{Im}\gamma \geq 0$.

Proof. Assume that $|t| \leq \sqrt{\alpha_1 \operatorname{Log} N/N}$ for some fixed $\alpha_1 > 0$. Then, using formulas (25) and (27), we write

$$Q_N(t) = \Big(\frac{1}{2} + \eta_N\Big)\Big(1 + \eta'_N + \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{N}t} \Big(1 - \frac{s^2}{N}\Big)^N ds\Big),$$

where η_N, η'_N are two constants (depending only on N) which are $O(1/N^2)$. Now, if $|u| \leq \sqrt{\alpha_1 \operatorname{Log} N}$, we have

$$\frac{|u^4|}{N} \leq \alpha_1^2\, \frac{(\operatorname{Log} N)^2}{N} = o(1),$$

hence one may find $C_0 \geq 0$ so that for N big enough ($N \geq N_0$ where N_0 depends only on α_1)

$$\Big| \Big(1 - \frac{u^2}{N}\Big)^N - e^{-u^2} \Big| \leq C_0 \Big| e^{-u^2}\, \frac{u^4}{N} \Big| \leq C_0\, \alpha_1^2\, \frac{(\operatorname{Log} N)^2}{N^{1-\alpha_1}} \,.$$

Hence we get for fixed $\alpha_1 > 0$ and for $N \geq N_0(\alpha_1)$

$$\Big| \Big(\frac{1}{2} + \eta_N\Big)^{-1} Q_N(t) - \operatorname{erfc}(-\sqrt{N}\, t) \Big| \leq C_1\, \frac{(\operatorname{Log} N)^{5/2}}{N^{1-\alpha_1}} \,, \tag{32}$$

for $|t| \leq \sqrt{\alpha_1 \operatorname{Log} N/N}$, where C_1 depends only on α_1.

Now, assume that θ is such that $Q_N(\theta) = 0$ or $\operatorname{erfc}(-\sqrt{N}\,\theta) = 0$ and that $|\theta| \leq \sqrt{\alpha_1 \operatorname{Log} N/N}$; in every case we have

$$|\operatorname{erfc}(-\sqrt{N}\theta)| \leq C_1 \frac{(\operatorname{Log} N)^{5/2}}{N^{1-\alpha_1}} \,.$$

We are going to show that for δ_0 small enough, $\operatorname{erfc}(-\sqrt{N}\,\theta+z)$ is not too small on $|z|=\delta_0$. Indeed we have

$$\begin{aligned}|\operatorname{erfc}(-\sqrt{N}\,\theta+z)-\operatorname{erfc}(-\sqrt{N}\,\theta)| &= \frac{2}{\sqrt{\pi}}\Big|\int_0^z e^{-N\theta^2}e^{2\sqrt{N}\,\theta s}e^{-s^2}\,ds\Big| \\ &\geq \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\,|e^{-N\theta^2}|\,|z| \geq \frac{1}{\sqrt{\pi}}\,N^{-\alpha_1}|z|\,,\end{aligned}$$

provided that

$$|z| \leq \min\Big\{2\,\sqrt{\alpha_1 \operatorname{Log} N},\, \frac{1}{8\,C_2\sqrt{\alpha_1 \operatorname{Log} N}}\Big\}\,,$$

where $C_2=\max_{|w|\leq 1}|(e^w-1)/w|$.

Thus, if $|\theta| \leq \sqrt{\alpha_2 \operatorname{Log} N/N}$, where $\alpha_2<\alpha_1<1/2$, and if N is big enough so that

$$\sqrt{\alpha_2\,\frac{\operatorname{Log} N}{N}}+\frac{1}{8\,C_2\sqrt{\alpha_1 N \operatorname{Log} N}} < \sqrt{\alpha_1\,\frac{\operatorname{Log} N}{N}}$$

and

$$C_1\,\sqrt{\pi}\,\frac{(\operatorname{Log} N)^{5/2}}{N^{1-2\alpha_1}} < \frac{1}{8\,C_2\sqrt{\alpha_1 \operatorname{Log} N}} < 2\sqrt{\alpha_1 \operatorname{Log} N}\,,$$

we obtain that $Q_N(t)$ and $\operatorname{erfc}(-\sqrt{N}\,t)$ have the same number of zeros inside the open disk $D(\theta, C_1\sqrt{\pi}\,(\operatorname{Log} N)^{5/2}/N^{3/2-2\alpha_1})$ (by Rouché's theorem).

In order to conclude, we need some information on the zeros of $\operatorname{erfc}(z)$. A theorem by Fettis, Cuslin and Cramer ([3]) gives a development of γ_k

$$\tag{33}\begin{aligned}\gamma_k = e^{3i\pi/4}\Bigg(&\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi} \\ &-\frac{i}{2\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi}}\operatorname{Log}\Big(2\,\sqrt{\pi}\,\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi}\,\Big) \\ &+O\Big(\frac{(\operatorname{Log} k)^2}{k\,\sqrt{k}}\Big)\Bigg)\,.\end{aligned}$$

Thus if M_0 is a fixed number in $(-\pi/4, 3\pi/4)$, the number of roots γ of $\operatorname{erfc}(\gamma)=0$ such that $\operatorname{Im}\gamma \geq 0$ and $|\gamma| \leq \sqrt{2\,k\pi+M_0}$ is exactly k when k is large enough.

Now we may prove Result 4. Let $\varepsilon_0 < 1/2$ and $K = [\varepsilon_0 \mathrm{Log}\, N/(2\pi)]$. For each root t of $Q_N(s)$ such that $|\mathrm{Im}\, t| \geq 0$ and $|t| \leq \sqrt{2K\pi/N} \leq \sqrt{\varepsilon_0 \mathrm{Log}\, N/N}$ there is a root θ of $\mathrm{erfc}(-\sqrt{N}\, s)$ such that

$$|\theta - t| \leq C_1 \sqrt{\pi}\, \frac{(\mathrm{Log}\, N)^{5/2}}{N^{3/2-2\varepsilon_1}}\ ,$$

(where $\varepsilon_0 < \varepsilon_1 < 1/2$ and $N \geq N_1(\varepsilon_1)$). Then we have

$$\begin{aligned} |\sqrt{N}\, \theta| &\leq \sqrt{2K\pi} + C_1 \sqrt{\pi}\, \frac{(\mathrm{Log}\, N)^{5/2}}{N^{1-2\varepsilon_1}} \\ &\leq \sqrt{2K\pi} + \frac{\pi}{16\sqrt{2K\pi}} \\ &\leq \sqrt{\Big(2K + \frac{1}{8}\Big)\pi} \end{aligned}$$

provided that $N \geq N_2(\varepsilon_1)$. But we know that there are exactly $2K$ roots of $\mathrm{erfc}(-\sqrt{N}\, s)$ inside the disk $D(0, \sqrt{(2K+1/4)\pi}/\sqrt{N})$. Conversely, if θ is a root of $\mathrm{erfc}(-\sqrt{N}\, s)$ such that

$$|\theta| \leq \sqrt{\frac{2K\pi}{N}} - C_1 \sqrt{\pi}\, \frac{(\mathrm{Log}\, N)^{5/2}}{N^{3/2} - 2\,\varepsilon_1} \leq \sqrt{\varepsilon_0\, \frac{\mathrm{Log}\, N}{N}}\ ,$$

there is a root t of $Q_N(s)$ such that

$$|\theta - t| \leq C_1 \sqrt{\pi}\, \frac{(\mathrm{Log}\, N)^{5/2}}{N^{3/2-2\varepsilon_1}}\ ,$$

hence $|t| \leq \sqrt{2K\pi/N}$; moreover for $N \geq N_2(\varepsilon_1)$ we have

$$\sqrt{2K\pi} - C_1 \sqrt{\pi}\, \frac{(\mathrm{Log}\, N)^{5/2}}{N^{1-2\varepsilon_1}} > \sqrt{2K\pi} - \frac{\pi}{16\sqrt{2K\pi}} > \sqrt{\Big(2K - \frac{1}{8}\Big)\pi}\ ,$$

so that we have again $2K$ roots of $\mathrm{erfc}(-\sqrt{N}\, s)$ such that

$$|\theta| \leq \sqrt{\frac{2K\pi}{N}} - C_1 \sqrt{\pi}\, \frac{(\mathrm{Log}\, N)^{5/2}}{N^{1-2\varepsilon_1}}\ .$$

Finally, we conclude by noticing that (33) shows us that if $\mathrm{erfc}(-\sqrt{N}\theta_i) = 0$, $i = 1, 2$, $\theta_1 \neq \theta_2$ and $|\theta_i| \leq \sqrt{(2K+1/8)\pi/N}$ then $|\theta_1 - \theta_2| \geq$

$C_0/\sqrt{KN}$ and $|\mathrm{Im}\,\theta_i| \geq C_0\sqrt{K/N}$ for some positive C_0 which doesn't depend on K nor N; hence the balls

$$D\Big(\theta_i, C_1\sqrt{\pi}\,\frac{(\mathrm{Log}\,N)^{5/2}}{N^{3/2-2\varepsilon_1}}\Big)$$

are disjoint and don't meet the real axis (for N large enough). Thus (31) is proved, if we notice that

$$\frac{(\mathrm{Log}\,N)^{5/2}}{N^{1-2\varepsilon_1}} < \frac{1}{N^{1-2\varepsilon_1'}}$$

for $\varepsilon_1 < \varepsilon_1' < 1/2$ and N large enough.

3. Big roots of Q_N: first estimates.

In this section, we are going to devote our attention to formula (26). A straigthforward application of (26) is the following one:

Result 5. *For N large enough, if $t \neq -1$ and $Q_N(t) = 0$, then $|1-t^2| > 1$.*

PROOF. If $Q_N(t) = 0$, then we have $\sqrt{N}\int_t^1 (1-s^2)^N\,ds = \sqrt{\pi}\,(1+\eta_N)$ with $\eta_N = O(1/N^2)$. Now, since $\mathrm{Re}\,t > 0$ (due to Result 1), we may write

$$\begin{aligned}\int_t^1 (1-s^2)^N\,ds &= \int_0^{1-t^2} \omega^N\,\frac{d\omega}{2\sqrt{1-\omega}}\\ &= (1-t^2)^{N+1}\int_0^1 \lambda^N\,\frac{d\lambda}{2\sqrt{1-\lambda(1-t^2)}}\,.\end{aligned}$$

We write $\Omega = 1-t^2$. If $|\Omega| \leq 1$ then we will prove that

$$\inf_{\lambda\in[0,1]} |1-\lambda\Omega| \geq \frac{1}{2}\,|1-\Omega|\,.$$

This is obvious if $\mathrm{Re}\,\Omega \leq 0$: we have $|1-\lambda\Omega| \geq 1$ and $|1-\Omega| \leq 2$. If $\mathrm{Re}\,\Omega > 0$, $\Omega = \rho e^{i\varphi}$ $(0 < \rho \leq 1,\ \varphi \in (-\pi/2, \pi/2))$, we distinguish the case $\rho \leq \sin\varphi$ and $\rho > \sin\varphi$. If $\rho \leq \sin\varphi$, it is easily checked that $|1-\lambda\Omega| \geq |1-\Omega|$. If $\rho > \sin\varphi$, we have $|1-\lambda\Omega| \geq \sin\varphi$ and $|1-\Omega| \leq |1-e^{i\varphi}| = 2\,|\sin(\varphi/2)|$; hence

$$|1-\lambda\Omega| \geq \Big|\cos\frac{\varphi}{2}\Big|\,|1-\Omega| \geq \frac{\sqrt{2}}{2}\,|1-\Omega|\,.$$

Thus, we have for $\operatorname{Re} t > 0$ and $|1-t^2| \leq 1$

$$\Big|\int_t^1 (1-s^2)^N\,ds\Big| \leq \frac{|1-t^2|^{N+1}}{N+1}\,\frac{1}{|t|} \leq \frac{1}{\sqrt{N}}\Big(\frac{1}{\sqrt{N}\,|t|}\Big)\,.$$

If $|t\sqrt{N}| \geq 2/\sqrt{\pi}$, we get

$$\Big|\sqrt{N}\int_t^1 (1-s^2)^N\,ds\Big| \leq \frac{1}{2}\sqrt{\pi}\,,$$

and thus $Q_N(t) \neq 0$ (for N large enough so that $|\eta_N| < 1/2$). If $\sqrt{N}\,|t| \leq 2/\sqrt{\pi}$, then $t \sim -\overline{\gamma}/\sqrt{N}$ for a root γ of $\operatorname{erfc}(z)$ such that $|\gamma| \leq 2/\sqrt{\pi}$; but the roots of $\operatorname{erfc}(z)$ satisfy $\pi/2 < |\operatorname{Arg}\gamma| < 3\pi/4$ so that (for N large enough) $|\operatorname{Arg} t| > \pi/4$ and t cannot lie inside the lemniscate $|1-t^2| \leq 1$.

We may now enter the core of our computations. We are going to give a precise description of $\int_t^1 (1-s^2)^N\,ds$. Integration by parts gives us

$$\begin{aligned}\int_t^1 (1-s^2)^N\,ds &= \frac{(1-t^2)^{N+1}}{2\,t\,(N+1)} - \int_t^1 \frac{(1-s^2)^{N+1}}{2\,s^2(N+1)}\,ds\\ &= \frac{(1-t^2)^{N+1}}{2\,t\,(N+1)} - \frac{(1-t^2)^{N+2}}{4\,(N+1)}\int_0^1 \frac{\lambda^{N+1}\,d\lambda}{(1-\lambda\,(1-t^2))^{3/2}}\,.\end{aligned}$$

We then define $\eta(t)$ as

$$\eta(t) = \frac{|t^2|}{\inf_{\lambda\in[0,1]} |1-\lambda\,(1-t^2)|}\,. \tag{34}$$

We have

$$\int_t^1 (1-s^2)^N\,ds = \frac{(1-t^2)^{N+1}}{2\,t\,(N+1)}\Big(1 + \frac{(1-t^2)}{2\,(N+2)\,t^2}\,\mu_N(t)\Big)\,, \tag{35}$$

for $\operatorname{Re} t > 0$ with

$$|\mu_N(t)| \le \eta(t)^{3/2}\,. \tag{36}$$

Of course, (35) is a good formula if $\mu_N(t)$ cannot explode. As a matter of fact, we will show that in the neighbourhood of the roots of $Q_N(s)$ we have $|\eta(t)| \le C_0$ where C_0 doesn't depend on N nor t; but we are still far from being able to prove it! The only obvious estimations on η are the following ones: if $\operatorname{Re} t^2 \ge 1$, we have of course $|\eta(t)| = |t^2|$, while if $\operatorname{Re} t^2 < 1$ and $|1 - t^2| > 1$ we have

$$|\eta(t)| = \frac{|t^2|}{|\sin(\operatorname{Arg}(1-t^2))|}\,.$$

With help of formula (35) and a careful estimate of $\eta(t)$ in (36), we are going to prove:

Result 6. *Let $\varphi_{N,k} = (8k-1)\pi/(8N+6)$. Then for N large enough, the roots $x_{N,1}, \dots, x_{N,N}$ of Q_N such that $x_{N,k} \ne -1$, ordered by*

- *for $1 \le k \le [(N+1)/2]$, $\operatorname{Re} x_{N,k} \ge 0$ and $x_{N,N+1-k} = \overline{x_{N,k}}$*
- $|x_{N,1}| < |x_{N,2}| < \cdots < |x_{N,[(N+1)/2]}|$

satisfy

$$\begin{aligned}\Big| x_{N,k} &- \sqrt{2\sin\varphi_{N,k}}\; e^{i(\pi/4-\varphi_{N,k})/2} \\ &- \frac{e^{i(3\pi/4-3\varphi_{N,k}/2)}}{2N\sqrt{2\sin\varphi_{N,k}}}\operatorname{Log}\big(2\sqrt{2N\pi\sin\varphi_{N,k}}\big)\Big| \\ &\le C\,\frac{1}{\sqrt{N}}\,\max\Big\{\frac{(1+\operatorname{Log}k)^2}{k^{3/2}},\ \frac{(1+\operatorname{Log}N+1-k)^2}{(N+1-k)^{3/2}}\Big\}\,,\end{aligned} \tag{37}$$

where C doesn't depend on k nor N.

PROOF. Since $\varphi_{N,N+1-k} = \pi - \varphi_{N,k}$, it is enough to prove (37), for $1 \le k \le [(N+1)/2]$, *i.e.* for the roots which lie in the upper half-plane. The proof is decomposed in the following steps: one first proves that $\operatorname{Arg}(1 - x_{N,k}^2)$ cannot be too small, so that we have a first control on $\mu_N(x_{N,k})$; then one gives through (35) a first estimate on $x_{N,k}$ and on the related error; this gives us a more precise information on $\operatorname{Arg}(1 - x_{N,k}^2)$ and thus we may conclude with our final estimate.

Step 1. We want to estimate $\operatorname{Arg}(1-x_{N,k}^2)$. We fix $\theta_0 \in (\pi/4, \pi/2)$ so that the sector $\{z : \pi/2 \leq |\operatorname{Arg} z| \leq \pi - \theta_0\}$ contains no zero of $\operatorname{erfc}(z)$ (remember that $\lim_{k\to+\infty} \operatorname{Arg}\gamma_k = 3\pi/4$). We now distinguish the cases $\operatorname{Arg} x_{N,k} \in [0, \theta_0]$ and $\operatorname{Arg} x_{N,k} \in]\theta_0, \pi/2[$. If $\operatorname{Re} 1 - x_{N,k}^2 \leq 0$, we know that $\eta(x_{N,k}) \leq |x_{N,k}|^2 \leq 4$. If $\operatorname{Re} 1 - x_{N,k}^2 > 0$ and $\operatorname{Arg} x_{N,k} \in [0, \pi/4]$, then we see that $|x_{N,k}|^2 \leq |\tan \operatorname{Arg}(1-x_{N,k}^2)|$ (because $\omega = 1 - x_{N,k}^2$ satisfies $\operatorname{Re}\omega \in (0,1]$ and $|\omega| > 1$ so that $|\sin \operatorname{Arg}\omega| \leq |1-\omega| \leq |\tan \operatorname{Arg}\omega|$); moreover we have $|x_{N,k}|^2 \leq 4$; thus if $|\tan(\operatorname{Arg}(1-x_{N,k}^2)| \leq 4$, then we have

$$|\sin(\operatorname{Arg}(1-x_{N,k}^2))| = \frac{|\tan(\operatorname{Arg}(1-x_{N,k}^2)|}{\sqrt{1+\tan^2(\operatorname{Arg}(1-x_{N,k}^2))}} \geq \frac{|x_{N,k}|^2}{\sqrt{17}}$$

and $\eta(x_{N,k}) \leq \sqrt{17}$. On the other hand, if $|\tan(\operatorname{Arg}(1-x_{N,k}^2))| \geq 4$, then we have $|\operatorname{Arg}(1-x_{N,k}^2)| \in [\operatorname{Arg}\tan 4, \pi/2]$ and thus

$$|\sin(\operatorname{Arg}(1-x_{N,k}^2))| \geq \sin \operatorname{Arg}\tan 4 = \frac{4}{\sqrt{17}} \geq \frac{|x_{N,k}|^2}{\sqrt{17}}$$

and $\eta(x_{N,k}) \leq \sqrt{17}$ again.

If $\operatorname{Arg}(x_{N,k}) \in [\pi/4, \theta_0]$, we have

$$|\operatorname{Im}(1-x_{N,k}^2)| = |x_{N,k}^2|\,|\sin 2\operatorname{Arg} x_{N,k}|$$

so that

$$|\operatorname{Im}(1-x_{N,k}^2)| \geq |x_{N,k}|^2\,|\sin 2\,\theta_0|\,,$$

while

$$|\sin \operatorname{Arg}(1-x_{N,k}^2)| = \frac{|\operatorname{Im}(1-x_{N,k}^2)|}{|1-x_{N,k}^2|} \geq \frac{1}{3}\,|\operatorname{Im}(1-x_{N,k}^2)|\,,$$

so that

$$\eta(x_{N,k}) \leq \frac{3}{|\sin 2\,\theta_0|}\,.$$

The difficult case is when $\theta_0 \leq \operatorname{Arg} x_{N,k} \leq \pi/2$ (as a matter of fact, we will see in step 3 that this case never occurs when N is big enough!). For the moment, we will show that we have necessarily for such an $x_{N,k}$ (and provided N is large enough) the inequality

$$N\,|x_{N,k}|^4 \geq \frac{|\cos\theta_0|}{100\,C_0^2} = \varepsilon_1\,,$$

where C_0 is given by

$$C_0 = \max\Big\{ \sup_{|\sigma|\le 1/2} \Big|\frac{\sigma^2 + \mathrm{Log}\,(1-\sigma^2)}{\sigma^4}\Big|,\ \sup_{|\sigma|\le 1} \frac{|e^\sigma - 1|}{|\sigma|}\Big\}.$$

Indeed, let $A_0 > 0$ be large enough so that for $A \ge A_0$, $e^{3A^2 \cos(2\theta_0)/4}(1 + A^2/2) < 1/100$ (remember that $\cos 2\,\theta_0 < 0$), $4/(A^2|\cos 2\,\theta_0|) < 1/100$ and $A\, e^{A^2 \cos(2\theta_0)/4} < 1/100$. If $\sqrt{N}\,|x_{N,k}| \ge A_0$ and $N\,|x_{N,k}|^4 \le \varepsilon_1$, we write

$$Q_N(x_{N,k}) = \frac{1}{2} + \Big(1 + O\Big(\frac{1}{N^2}\Big)\Big)\sqrt{\frac{N}{\pi}} \int_0^{x_{N,k}} (1 - s^2)^N\, ds$$

and thus

$$|Q_N(x_{N,k})| \ge \frac{1}{10}\sqrt{N}\,\Big|\int_0^{x_{N,k}} (1 - s^2)^N\, ds\Big| - \frac{1}{2}\,.$$

We write

$$(1 - s^2)^N = e^{-Ns^2} e^{N(s^2 - \mathrm{Log}\,(1 - s^2))}\,,$$

since $|s| \le \sqrt{\varepsilon_1/N}/4$, we have $|s| \le 1/2$ for N large enough, thus

$$|N\,(s^2 - \mathrm{Log}\,(1 - s^2)| \le C_0|N\,s^4| \le \frac{1}{100}\,,$$

thus

$$|e^{N(s^2 - \mathrm{Log}(1 - s^2))} - 1| \le C_0^2\,|N\,s^4|\,.$$

Thus, writing $x_{N,k} = \rho_{N,k}\, e^{i\theta_{N,k}}$, we get

$$\begin{aligned} |Q_N(x_{N,k})| \ge{}& \frac{1}{10}\Big|\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\, ds\Big| \\ &- \frac{C_0^2}{10}\int_0^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\theta_{N,k}}\, \frac{s^4}{N}\, ds - \frac{1}{2} \\ \ge{}& \frac{1}{10}\Big|\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\, ds\Big| \\ &- \frac{C_0^2}{10}\,\frac{(\sqrt{N}\,\rho_{N,k})^3}{N\,|\cos 2\,\theta_{N,k}|} \end{aligned}$$

$$\cdot \int_0^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\,\theta_{N,k}} s\,|\cos 2\,\theta_{N,k}|\,ds - \frac{1}{2}$$

$$\geq \frac{1}{10}\Big|\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\,ds\Big|$$

$$- \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{2\sqrt{N}\,\rho_{N,k}}\Big(\frac{C_0^2\,(\sqrt{N}\,\rho_{N,k})^4}{10\,|\cos 2\,\theta_0|}\Big) - \frac{1}{2}\,.$$

We have now to estimate $\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\,ds$. We write

$$\begin{aligned}
&\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\,ds \\
&= e^{i\theta_{N,k}}\Big(\int_0^{\sqrt{N}\rho_{N,k}/2} e^{-s^2 e^{2i\theta_{N,k}}}\,ds + \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} e^{-s^2 e^{2i\theta_{N,k}}}\,ds\Big) \\
&= e^{i\theta_{N,k}}(I_1 + I_2)\,.
\end{aligned}$$

We have $|I_1| \leq e^{-N\rho_{N,k}^2 \cos(2\theta_{N,k})/4}\rho_{N,k}\sqrt{N}/2$, while

$$\begin{aligned}
I_2 &= \Big[\frac{e^{-s^2 e^{2i\theta_{N,k}}}}{-2\,s\,e^{2i\theta_{N,k}}}\Big]_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} - \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} \frac{e^{-s^2 e^{2i\theta_{N,k}}}}{2\,s^2 e^{2i\theta_{N,k}}}\,ds \\
&= \frac{e^{-N\rho_{N,k}^2\,e^{2i\theta_{N,k}}}}{-2\sqrt{N}\,\rho_{N,k}\,e^{2i\theta_{N,k}}} - \frac{e^{-N\rho_{N,k}^2 e^{2i\theta_{N,k}/4}}}{-\sqrt{N}\,\rho_{N,k}\,e^{2i\theta_{N,k}}} - I_3\,.
\end{aligned}$$

We have

$$\begin{aligned}
|I_3| &\leq \frac{1}{4\Big(\frac{1}{2}\sqrt{N}\,\rho_{N,k}\Big)^3|\cos 2\,\theta_{N,k}|} \\
&\quad \cdot \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\theta_{N,k}} 2\,s\,|\cos 2\,\theta_{N,k}|\,ds \\
&\leq \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{4\Big(\frac{1}{2}\sqrt{N}\,\rho_{N,k}\Big)^3|\cos 2\,\theta_0|}\,.
\end{aligned}$$

Thus we get

$$
\begin{aligned}
&|Q_N(x_{N,k})| \\
&\quad \geq \frac{1}{10}\,\frac{e^{-N\rho_{N,k}^2\cos 2\theta_{N,k}}}{2\sqrt{N}\,\rho_{N,k}} \\
&\qquad \cdot \Big(1 - 2\,e^{3N\rho_{N,k}^2\cos 2\theta_{N,k}/4} - \frac{4}{N\,\rho_{N,k}^2|\cos 2\theta_0|} \\
&\qquad\quad - N\,\rho_{N,k}^2\,e^{3N\rho_{N,k}^2\cos 2\theta_{N,k}/4} - \frac{C_0^2\,\varepsilon_1}{|\cos 2\,\theta_0|} \\
&\qquad\quad - 10\sqrt{N}\,\rho_{N,k}\,e^{N\rho_{N,k}^2\cos 2\theta_{N,k}}\Big) \\
&\quad \geq \frac{1}{10}\,\frac{e^{-N\rho_{N,k}^2\cos 2\theta_{N,k}}}{2\sqrt{N}\,\rho_{N,k}}\Big(1 - \frac{2}{100} - \frac{1}{100} - \frac{1}{100} - \frac{1}{100} - \frac{10}{100}\Big) > 0\,,
\end{aligned}
$$

which contradicts $Q_N(x_{N,k}) = 0$. Up to now, we have proved that if $\arg x_{N,k} > \theta_0$ then either $\sqrt{N}\,|x_{N,k}| \leq A_0$ or $N\,|x_{N,k}|^4 \geq \varepsilon_1$. But if $|x_{N,k}| \leq A_0/\sqrt{N}$ and N is large enough, Result 4 ensures that $-\sqrt{N}\,x_{N,k}$ is close to a zero of $\operatorname{erfc}(z)$. This is not possible for N large enough since the distance between $\{z : \pi/2 \leq |\operatorname{Arg} z| \leq \pi - \theta_0\}$ and $\{z : \operatorname{erfc}(z) = 0\}$ is positive.

Thus we must have $N\,|x_{N,k}|^4 \geq \varepsilon_1$. Write again $x_{N,k} = \rho_{N,k}\,e^{i\theta_{N,k}}$; since $|x_{N,k} - 1| \leq 1$ by Result 1, we have $\rho_{N,k} \leq 2\cos\theta_{N,k}$; thus $2\cos\theta_{N,k} \geq (\varepsilon_1/N)^{1/4}$ and

$$
|\operatorname{Im} x_{N,k}^2| = |x_{N,k}^2|\,|\sin 2\,\theta_{N,k}| \geq \sin\theta_0\Big(\frac{\varepsilon_1}{N}\Big)^{1/4}|x_{N,k}|^2\,.
$$

We thus have proved

$$
\eta(x_{N,k}) = \frac{|x_{N,k}|^2|1 - x_{N,k}^2|}{|\operatorname{Im} x_{N,k}^2|} \leq \frac{3\,N^{1/4}}{(\sin\theta_0)\,\varepsilon_1^{1/4}} = C_1^{N^{1/4}}\,.
$$

We thus have proved

- if $\operatorname{Arg} x_{N,k} < \theta_0$,

$$
|\mu_N(x_{N,k})| \leq \eta(x_{N,k})^{3/2} \leq C_2\,,
$$

- if $\operatorname{Arg} x_{N,k} > \theta_0$,

$$\begin{aligned} |\mu_N(x_{N,k})| &\le \eta(x_{N,k})^{3/2} \\ &\le (C_1 N^{1/4})^{3/2} \\ &= C_1^{3/2} \, \frac{(N\,|x_{N,k}^2)^{3/4}}{(N\,(|x_{N,k}|^4)^{3/8}} \\ &\le \frac{C_1^{3/2}}{\varepsilon_1^{3/8}} \, (N\,|x_{N,k}|^2)^{3/4} \,. \end{aligned}$$

In any case, we have

$$|\mu_N(x_{N,k})| \le C\,(N\,|x_{N,k}|^2)^{3/4} \,. \tag{38}$$

(Remember that $\lim_{N\to+\infty} \inf_k N\,|x_{N,k}|^2 = |\gamma_1|^2 > 0$).

Step 2. We are now able to give an estimate for $x_{N,k}$. Let us consider a root $y \neq -1$ of Q_N such that $\operatorname{Im} y \ge 0$. We have

$$\int_y^1 (1-s^2)^N \, ds = 2\, \frac{4^N (N!)^2}{(2N+1)!} \,,$$

hence from (35) and (36),

$$\frac{(1-y^2)^{N+1}}{2\,(N+1)\sqrt{\pi}\,y} \Big(1 + O\Big(\frac{\eta(y)^{3/2}}{N\,|y|^2}\Big)\Big) = \sqrt{\frac{\pi}{N}}\,\Big(1 + O\Big(\frac{1}{N^2}\Big)\Big) \,, \tag{39}$$

(where $\alpha = O\,(\varepsilon(N,y))$ means that $|\alpha|/\varepsilon(N,y) \le C$ for a positive constant C which doesn't depend neither on N nor on y). Taking the $(N+1)$-th root of the modulus of both terms of equality (39), we get

$$\begin{aligned} |1-y^2| = 1 &+ \frac{1}{N+1} \operatorname{Log}\Big(2\sqrt{N\pi}\,\frac{N+1}{N}\,|y|\Big) \\ &+ O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{1}{N^3}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big) \\ = 1 &+ \frac{1}{N} \operatorname{Log}\,(2\sqrt{N\pi}\,|y|) + O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big) \,. \end{aligned}$$

Now, we write $1 - y^2 = \rho\, e^{-i\varphi}$ ($\varphi \in [0,\pi]$, $\rho > 0$), so that $y = \sqrt{1-\rho\, e^{-i\varphi}}$. We have found

$$\begin{aligned}|1-\rho| &= O\Big(\frac{1}{N}\operatorname{Log}(\sqrt{N}\,|y|)\Big) + O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2\,|y|^2}\Big)\\ &= O\Big(\frac{1}{N}\operatorname{Log}(\sqrt{N}\,|y|)\Big),\end{aligned}$$

(since $1/CN \le \operatorname{Log}(\sqrt{N}\,|y|)/N \le C \operatorname{Log} N/N$, while $\eta(y)^{3/2}/(N^2|y|^2) \le C/(N(N|y|^2)) \le C'/N$). Thus $1 - \rho\, e^{-i\varphi} = 1 - e^{-i\varphi} + (1-\rho)\, e^{-i\varphi}$ with

$$\Big|\frac{(1-\rho)\, e^{-i\varphi}}{1-\rho\, e^{-i\varphi}}\Big| = O\Big(\frac{\operatorname{Log}(\sqrt{N}\,|y|)}{N|y|^2}\Big)$$

and we find

$$\begin{aligned}y &= \sqrt{(1-e^{-i\varphi})\Big(1 + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N|y|^2}\Big)\Big)}\\ &= \sqrt{2\sin\Big(\frac{\varphi}{2}\Big)}\; e^{i(\pi/4-\varphi/4)}\Big(1 + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N|y|^2}\Big)\Big).\end{aligned}$$

We insert this result in (39) and take the phase

$$-(N+1)\,\varphi - \frac{\pi}{4} + \frac{\varphi}{4} + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N|y|^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N|y|^2}\Big) = -2k\pi$$

or

$$\varphi = \frac{8k-1}{4N+3}\,\pi + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N^2|y|^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big). \tag{40}$$

If we assume $\sqrt{N}\,|y| \ge A_0$ where A_0 is big enough so that

$$O\Big(\frac{\operatorname{Log} A_0}{NA_0^2}\Big) + O\Big(\frac{1}{NA_0^{1/2}}\Big)$$

is less than $4\pi/(4N+3)$ (A_0 being chosen independently from N), we see that $0 \le \varphi \le \pi$ implies $0 \le k \le [(N+1)/2]$; moreover since

$$|y| = \sqrt{2\sin\Big(\frac{\varphi}{2}\Big)}\,\Big(1 + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N|y|^2}\Big)\Big)$$

we must have

$$2\sin\Big(\frac{\varphi}{2}\Big) \geq \frac{A_0^2}{N} + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N^2|y|^2}\Big)\,.$$

We take $A_0^2 = \sqrt{2K_0\pi}$, where K_0 is big enough; we then see that we must have $k > K_0$.

If $\sqrt{N}\,|y| \leq \sqrt{2K_0\pi}$, we know that (provided N is big enough) $y \sim -\overline{\gamma}_k/\sqrt{N}$ for $k \in \{1,\dots,K_0\}$. We have moreover found candidates $y_{N,k}$ for the remaining roots $x_{N,k}$, $K_0 < k \leq [(N+1)/2]$, which are given by

$$1 - y_{N,k}^2 = \Big(1 + \frac{1}{N}\operatorname{Log} 2\,\sqrt{2N\pi\sin\varphi_{N,k}}\Big)e^{-2i\varphi_{N,k}}\,, \tag{41}$$

for $K_0 < k \leq [(N+1)/2]$ and $\varphi_{N,k} = (8k-1)\pi/(8N+6)$.

More precisely, we have shown that if $Q_N(y) = 0$, $\operatorname{Im} y \geq 0$, $y \neq -1$ and $\sqrt{N}\,|y| \geq \sqrt{2K_0\pi}$, then for some $k \in \{K_0+1,\dots,[(N+1)/2]\}$ we have

$$1-y^2 = 1-y_{N,k}^2 + O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big) + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N^2|y|^2}\Big)\,. \tag{42}$$

We are going now to prove that, provided that K_0 is fixed large enough (and provided thereafter that N is large enough), for each $y_{N,k}$ there is exactly one root y satisfying (42). Notice that $|y_{N,k}^2 - y_{N,k+1}^2| \geq C_0/N$ while

$$O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big) + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N^2|y|^2}\Big) \leq C\,\frac{1}{N}\Big(\frac{(\operatorname{Log} N)^2}{N} + \frac{1}{(\sqrt{N}\,|y|)^{1/2}}\Big)\,.$$

Indeed, let's write $s = \sqrt{y_{N,k}^2 - v}$ where $|v| = \eta_0/N$, η_0 small enough. We are going to estimate $Q_N(s)$. We know that

$$\int_s^1 (1-\sigma^2)^N\,d\sigma = \frac{(1-s^2)^{N+1}}{2\,s\,(N+1)}\,\Big(1 + O\Big(\frac{\eta(s)}{N|s|^2}\Big)\Big)\,,$$

where $\eta(s)$ is bounded independently of s provided that $|1-s| < 1$, $|1-s^2| > 1$ and $|\operatorname{Arg} s| < \theta_0$ (where $\theta_0 \in (\pi/4, \pi/2)$). Thus, we are going to estimate $|1-s|$, $|1-s^2|$ and $|\operatorname{Arg} s|$. We have obviously from (41)

$$y_{N,k}^2 = 1 - e^{-2i\varphi_{N,k}} + O\Big(\frac{\operatorname{Log} k}{N}\Big) = (1-e^{-2i\varphi_{N,k}})\Big(1+O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big)$$

and such an estimate holds as well for s^2. (We see also from (41) that

$$\begin{aligned}|1-s^2| &\geq 1 + \frac{1}{N}\operatorname{Log} 2\sqrt{2N\pi\sin\varphi_{N,k}} - \frac{\eta_0}{N}\\ &\geq 1 + \frac{1}{N}\operatorname{Log} 2\sqrt{4\pi K_0} - \frac{\eta_0}{N}\\ &> 1\end{aligned}$$

provided η_0 is small enough). Thus we find that

$$\operatorname{Arg} s^2 = \frac{\pi}{2} - \varphi_{N,k} + O\Big(\frac{\operatorname{Log} k}{k}\Big) < 2\,\theta_0 \ ,$$

if K_0 is large enough (so that $O(\operatorname{Log} K_0/K_0) < 2\,\theta_0 - \pi/2$) and thus

$$\operatorname{Arg} s = \frac{\pi}{4} - \frac{1}{2}\,\varphi_{N,k} + O\Big(\frac{\operatorname{Log} k}{k}\Big) \in (-\theta_0, \theta_0)\,.$$

Moreover,

$$|s| = \sqrt{2\sin\varphi_{N,k}}\Big(1+O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big)$$

and this latter estimate gives $|s| < 2\cos(\operatorname{Arg} s)$: if $\varphi_{N,k} > \varepsilon_0$ (where ε_0 is fixed small enough as we shall see below) and K_0 and N are large enough we have

$$\sqrt{2\sin\varphi_{N,k}}\,\Big(1+O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big) \leq \sqrt{2}\,\Big(1 + C\,\frac{\operatorname{Log} K_0}{K_0}\Big) \leq \sqrt{2}\,\Big(1+\frac{\varepsilon_0}{100}\Big)\,,$$

while

$$\begin{aligned}2\cos(\operatorname{Arg} s) &\geq 2\cos\Big(\frac{\pi}{4} - C\,\frac{\operatorname{Log} K_0}{K_0}\Big)\\ &\geq 2\cos\Big(\frac{\pi}{4} - \frac{\varepsilon_0}{3}\Big)\\ &\geq \sqrt{2}\,\Big(1 + \frac{2\,\varepsilon_0}{3\pi} - \frac{\varepsilon_0^2}{2}\Big)\,.\end{aligned}$$

On the other hand, if $\varphi_{N,k} < \varepsilon_0$ we find

$$\sqrt{2\sin\varphi_{N,k}}\Big(1+O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big) \leq \sqrt{2\,\varepsilon_0}\,\sqrt{1+C\,\frac{\operatorname{Log} K_0}{K_0}} \leq C'\sqrt{\varepsilon_0}\,,$$

while $2\cos(\operatorname{Arg} s) \geq 2\cos\theta_0$; thus if ε_0 is small enough to ensure $\varepsilon_0 < 4/(3\pi) - 1/50$ and $\varepsilon_0 < 4\cos^2\theta_0/C'^2$ we find $|s| < 2\cos(\operatorname{Arg} s)$. But this latter inequality is equivalent to $|1-s| < 1$. Thus we found

$$Q_N(s) = 1 - \Big(1+O\Big(\frac{1}{N^2}\Big)\Big)\sqrt{\frac{N}{\pi}}\,\frac{(1-s^2)^{N+1}}{2\,s\,(N+1)}\Big(1+O\Big(\frac{1}{|Ns^2|}\Big)\Big)\,.$$

We have moreover:

$$\begin{aligned}(1-s^2)^{N+1} &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{v}{1-y_{N,k}^2}\Big)^{N+1}\\ &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{Nv}{1-y_{N,k}^2}+O\,(N^2v^2)\Big)\end{aligned}$$

$$s = \sqrt{y_{N,k}^2 - v} = y_{N,k}\Big(1-\frac{v}{2\,y_{N,k}^2}+O\Big(\frac{v^2}{y_{N,k}^4}\Big)\Big)\,.$$

This gives, since $|s|$ has $\sqrt{k/N}$ as order of magnitude

$$\begin{aligned}Q_N(s) = 1-&\Big(1+O\Big(\frac{1}{k}\Big)\Big)\,\frac{(1-y_{N,k}^2)^{N+1}}{2\sqrt{N\pi}\,y_{N,k}}\\ &\cdot\Big(1+\frac{Nv}{1-y_{N,k}^2}+\frac{v}{2\,y_{N,k}^2}+O\,(N^2v^2)+O\Big(\frac{v^2}{y_{N,k}^4}\Big)\Big)\,.\end{aligned}$$

Moreover

$$|y_{N,k}| \geq 2\sqrt{\frac{8k-1}{8N+6}}\,\Big(1+O\Big(\frac{1}{k}\operatorname{Log} k\Big)\Big)$$

and

$$y_{N,k} = \sqrt{2\sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)}\;e^{i(\pi/4-(8k-1)\pi/(16N+12))}\Big(1+O\Big(\frac{1}{k}\operatorname{Log} k\Big)\Big)\,,$$

so that

$$\begin{aligned}
&\frac{(1-y_{N,k}^2)^{N+1}}{2\sqrt{N\pi}\,y_{N,k}}\\
&\quad= \frac{\Big(1+\dfrac{1}{N}\operatorname{Log}2\sqrt{2N\pi\sin\Big(\dfrac{8k-1}{8N+6}\,\pi\Big)}\,\Big)^N}{2\sqrt{2N\pi\sin\Big(\dfrac{8k-1}{8N+6}\,\pi\Big)}}\,\Big(1+O\Big(\frac{1}{k}\operatorname{Log}k\Big)\Big)\\
&\quad= \Big(1+O\Big(\frac{1}{N^2}(\operatorname{Log}k)^2\Big)\Big)^N\Big(1+O\Big(\frac{1}{k}\operatorname{Log}k\Big)\Big)
\end{aligned}$$

and finally

$$\begin{aligned}
Q_N(s) = 1-&\Big(1+O\Big(\frac{1}{k}(\operatorname{Log}k)^2\Big)\Big)\\
&\cdot\Big(1+\frac{Nv}{1-y_{N,k}^2}+\frac{v}{2\,y_{N,k}^2}+O\,(N^2v^2)+O\Big(\frac{v^2}{y_{N,k}^4}\Big)\Big)\,.
\end{aligned}$$

Now, we write

$$R_{N,k}(s) = N\,\frac{v}{1-y_{N,k}^2} = N\,\frac{y_{N,k}^2-s^2}{1-y_{N,k}^2}\,.$$

Since $|v|=\eta_0/N$, we have

$$|R_{N,k}(s)| = \eta_0\Big(1+O\Big(\frac{\operatorname{Log}k}{N}\Big)\Big)\,,$$

while

$$|Q_N(s)-R_{N,k}(s)| = O\Big(\frac{(\operatorname{Log}k)^2}{k}\Big)+O\Big(\frac{\eta_0}{k}\Big)+O\,(\eta_0^2)\,.$$

We choose η_0 small enough to ensure that the $O\,(\eta_0^2)$ term is smaller than $\eta_0/2$ (independently of N and k), and then choose K_0 large enough to ensure that $O\,((\operatorname{Log}k)^2/k)+O\,(\eta_0/k)$ is smaller than $\eta_0/4$ for $k>K_0$. For this choice of K_0, we get

$$|Q_N(s)-R_{N,k}(s)| < \frac{3}{4}\,\eta_0 < |R_{N,k}(s)|\,.$$

Thus, by Rouché's theorem, $Q_N(s)$ and $R_{N,k}(s)$ have the same number of roots inside the domain $\{|y_{N,k}^2 - s^2| \leq \eta_0/N, \operatorname{Re} s > 0\}$.

Step 3. We have thus found a number K_0 so that for N large enough we may list the roots $x_{N,1}, \dots, x_{N,[(N+1)/2]}$ of Q_N with $x_{N,k} \neq -1$, $\operatorname{Im} x_{N,k} \geq 0$, $|x_{N,k}| < |x_{N,k+1}|$ in the following way:

- for $k \leq K_0$, $|x_{N,k}| < \sqrt{2K_0\pi/N}$ and $x_{N,k} \sim -\overline{\gamma}_k/\sqrt{N}$,
- for $k \geq K_0$,

$$|x_{N,k}^2 - y_{N,k}^2| = O\Big(\frac{\eta(x_{N,k})^{3/2}}{N^2|x_{N,k}|^2}\Big) + O\Big(\frac{\operatorname{Log}(\sqrt{N}\,|x_{N,k}|)}{N^2|x_{N,k}|^2}\Big),$$

where $y_{N,k}$ is given by (41).

Moreover, we have seen in step 2 that in that case we must have $\operatorname{Arg} x_{N,k} < \theta_0$, hence $\eta(x_{N,k})$ is bounded independently of N and k. Moreover $x_{N,k}$ is of order of magnituge $\sqrt{k/N}$, hence

$$|x_{N,k}^2 - y_{N,k}^2| = O\Big(\frac{\operatorname{Log} k}{Nk}\Big).$$

Thus we find

$$\begin{aligned} 1 - x_{N,k}^2 = &\Big(1 + \frac{1}{N}\operatorname{Log} 2\sqrt{2N\pi\sin\Big(\frac{8k-1}{8N+6}\pi\Big)}\Big) \\ &\cdot e^{-2i\pi(8k-1)/(8N+6)} + O\Big(\frac{\operatorname{Log} k}{Nk}\Big) \end{aligned} \tag{43}$$

and thus

$$\begin{aligned} x_{N,k}^2 = &\Big(1 - e^{-2i\pi(8k-1)/(8N+6)}\Big) \\ &\cdot\Big(1 - \frac{e^{-2i\pi(8k-1)/(8N+6)}}{N(1-e^{-2i\pi(8k-1)/(8N+6)})}\operatorname{Log} 2\sqrt{2N\pi\sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)} \\ &+ O\Big(\frac{\operatorname{Log} k}{k^2}\Big)\Big) \end{aligned}$$

which gives

$$x_{N,k} = e^{i(\pi/4-(8k-1)\pi/(16N+12))}\sqrt{2\sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)}$$

(44)

$$\cdot\left(1+\frac{e^{i(\pi/2-(8k-1)\pi/(8N+6))}}{4N\sin\Big(\dfrac{8k-1}{8N+6}\,\pi\Big)}\operatorname{Log}2\sqrt{2N\pi\sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)}\right.$$
$$\left.+\,O\Big(\frac{\operatorname{Log}k}{k^2}\Big)\right),$$

which gives (37) for $k > K_0$. For $k \le K_0$, (37) says only that $x_{N,k}$ is $O\,(1/\sqrt{N})$, which we already known since $\sqrt{N}\,|x_{N,k}| \le \sqrt{2K_0\pi}$.

Thus we have proved Result 6.

A nice corollary of Result 6 is that we may recover formula (33) on the roots of $\operatorname{erfc}(z)$:

Corollary. *The k-th root γ_k of* $\operatorname{erfc}(z)$ *such that* $\operatorname{Im}\gamma_k > 0$ *is given by*

$$\gamma_k = e^{3i\pi/4}\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi}$$

(45)

$$\cdot\left(1-\frac{i}{2\Big(2k-\dfrac{1}{4}\Big)\pi}\operatorname{Log}2\sqrt{\pi}\,\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi}+O\Big(\frac{(\operatorname{Log}k)^2}{k^2}\Big)\right).$$

PROOF. It is enough to use formula (37) for $x_{N,k}$ with $N, k \to +\infty$ and $k < \operatorname{Log}N/8$: we have

$$x_{N,k} = -\frac{-\overline{\gamma}_k}{\sqrt{N}} + O\Big(\frac{1}{N}\Big) \qquad \text{and} \qquad \frac{k}{N} = O\Big(\frac{\operatorname{Log}N}{N}\Big),$$

thus we find γ_k. The only thing to check is the exact number of roots γ such that $|\gamma| \le \sqrt{2K_0\pi}$ (since we used formula (33) to give it). But this is an old and classical result of Nevanlinna [9], and thus we may recover formula (33) from formula (37).

4. Big roots of Q_N: further estimates.

Though Result 6 is enough for the proof of theorems 1 to 3 (provided we improve result n° 4 for the smaller roots), we may give even more precise estimations for the roots $x_{N,k}$. For instance, we may integrate by parts one step further formula (35) and thus get an $O\,((\operatorname{Log} k)^3 / Nk^2)$ error instead of $O\,(\operatorname{Log} k/Nk)$ for $1 - x_{N,k}^2$.

More generally, how far can we compute $\int_t^1 (1-s^2)^N\,ds$? We have

$$\int_t^1 (1-s^2)^N\,ds = (1-t^2)^{N+1} \int_0^1 \lambda^N \frac{d\lambda}{2\sqrt{1-\lambda(1-t^2)}}\,.$$

If we write

$$1-\lambda(1-t^2) = t^2\Big(1+\frac{1-t^2}{t^2}\,(1-\lambda)\Big)\,,$$

we see that if $\operatorname{Re} t^2 > 1/2$ (so that $|1-t^2| < t^2$), we may develop $(\sqrt{1-\lambda(1-t^2)})^{-1}$ as a Taylor series in $(1-\lambda)$ and find (for $\operatorname{Re} t^2 > 1/2$)

$$\frac{1}{\sqrt{1-\lambda(1-t^2)}} = \frac{1}{t}\sum_{k=0}^{+\infty}(-1)^k\,\frac{2k!}{4^k(k!)^2}\Big(\frac{(1-\lambda)\,(1-t^2)}{t^2}\Big)^k,$$

which gives

$$(46)\quad \begin{cases} \text{for } \operatorname{Re} t>0 \text{ and } \operatorname{Re} t^2 > \dfrac{1}{2}\,, \\[2mm] \displaystyle\int_t^1 (1-s^2)^N\,ds \\[2mm] \displaystyle\quad = \frac{(1-t^2)^{N+1}}{2\,t}\sum_{k=0}^{+\infty}(-1)^k\,\frac{(2k)!}{4^k(k!)^2}\,\frac{N!\,k!}{(N+k+1)!}\Big(\frac{1-t^2}{t^2}\Big)^k. \end{cases}$$

Unfortunately, we are mostly interested in small t's (remember that $x_{N,k} = O\,(\sqrt{k/N})$). (46) has to be replaced by an asymptotic formula (which is obtained by repeatedly integrating by parts)

$$(47)\quad \begin{cases} \text{for } \operatorname{Re} t>0 \text{ and } M\in\mathbb{N}, \\[2mm] \displaystyle\int_t^1 (1-s^2)^N\,ds \\[2mm] \displaystyle\quad = \frac{(1-t^2)^{N+1}}{2\,t}\sum_{k=0}^{M}(-1)^k\,\frac{(2k)!}{4^k(k!)^2}\,\frac{N!\,k!}{(N+k+1)!}\Big(\frac{1-t^2}{t^2}\Big)^k \\[2mm] \quad\quad +R_{M,N}(t)\,, \end{cases}$$

where the remainder

$$R_{M,N}(t) = (-1)^{M+1}(1-t^2)^{N+M+2}\,\frac{(2M+2)!}{4^{M+1}((M+1)!)^2} \cdot \frac{N!\,(M+1)!}{(N+M+2)!}\int_0^1 \frac{\lambda^{N+M+1}\,d\lambda}{(1-\lambda(1-t^2))^{1/2+M+1}}$$

may be estimated by

$$\text{(48)}\qquad |R_{M,N}(t)| \le \left|\frac{(1-t^2)^{N+1}}{2\,t}\right| \frac{(2M+2)!}{4^{M+1}((M+1)!)^2}\,\frac{(M+1)!\,N!}{(N+M+2)!} \cdot \left|\frac{1-t^2}{t^2}\right|^{M+1} \eta(t)^{1/2+M+1}\,.$$

$M=0$ gave Result 6. $M=1$ gives the following result:

Result 7. *Writing* $\varphi_{N,k} = (8k-1)\pi/(8N+6)$ *and*

$$\lambda_k = \operatorname{Log} 2\,\sqrt{2N\pi \sin\varphi_{N,k}}\ ,$$

we have more precisely for all $k \in \{1,\dots,N\}$

$$\text{(49)}\qquad 1-x_{N,k}^2 = e^{-2i\varphi_{N,k}} \cdot \Big(1+\frac{1}{N}\,\lambda_k + \frac{1}{N^2} + \frac{\lambda_k}{N^2} + \frac{\lambda_k^2}{2N^2} + \frac{i\,e^{-i\varphi_{N,k}}}{4N^2\sin\varphi_{N,k}}\,(\lambda_k - 1)\Big) + \varepsilon_{N,k}\ ,$$

where

$$|\varepsilon_{N,k}| \le C \max\Big\{\frac{1+(\operatorname{Log} k)^3}{Nk^2},\ \frac{1+\operatorname{Log}(N+1-k)^3}{N(N+1-k)^2}\Big\}$$

and C *doesn't depend neither on* N *nor on* K.

PROOF. We assume $k \le [(N+1)/2]$. We write $1-x_{N,k}^2 = 1-y_{N,k}^2+v$ and the problem is to estimate v. We already know $v = O(\operatorname{Log} k/(Nk))$. Furthermore, we know that

$$\int_{x_{N,k}}^1 (1-s^2)^N\,ds = \frac{2\,4^N (N!)^2}{(2N+1)!} = \sqrt{\frac{\pi}{N}}\,\Big(1+O\Big(\frac{1}{N^2}\Big)\Big)$$

and

$$\int_{x_{N,k}}^{1} (1-s^2)^N \, ds = \frac{(1-x_{N,k}^2)^{N+1}}{2\,(N+1)\,x_{N,k}} \Big(1 - \frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2} + O\Big(\frac{1}{N^2 x_{N,k}^4}\Big)\Big).$$

Now, write

$$\begin{aligned} \frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2} &= \frac{1-y_{N,k}^2}{2\,(N+2)\,y_{N,k}^2} + O\Big(\frac{v}{k}\Big) + O\Big(\frac{Nv}{k^2}\Big) \\ &= \frac{1-y_{N,k}^2}{2\,(N+2)\,y_{N,k}^2} + O\Big(\frac{\operatorname{Log} k}{k^3}\Big) \end{aligned}$$

and

$$\begin{aligned} \frac{1-y_{N,k}^2}{2\,(N+2)\,y_{N,k}^2} &= \frac{e^{-2i\varphi_{N,k}}}{2\,(N+2)\,y_{N,k}^2} + O\Big(\frac{\operatorname{Log} k}{Nk}\Big) \\ &= \frac{e^{-2i\varphi_{N,k}}}{2N(1-e^{-2i\varphi_{N,k}})} + O\Big(\frac{\operatorname{Log} k}{k^2}\Big), \end{aligned}$$

so that

$$1 - \frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2} + O\Big(\frac{1}{N^2 x_{N,k}^4}\Big) = 1 + \frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}} + O\Big(\frac{\operatorname{Log} k}{k^2}\Big).$$

We now turn our attention to $(1-x_{N,k}^2)^{N+1}/(2\,(N+1)\,x_{N,k})$. We have

$$\begin{aligned} &2\,(N+1)\,x_{N,k}\sqrt{\frac{\pi}{N}} \\ &\quad = 2\Big(1+\frac{1}{N}\Big)\sqrt{N\pi}\,\sqrt{y_{N,k}^2 - v} \\ &\quad = 2\Big(1+\frac{1}{N}\Big)\sqrt{N\pi}\,\sqrt{1 - e^{-2i\varphi_{N,k}} - \frac{e^{-2i\varphi_{N,k}}}{N}\,\lambda_{N,k} + O\Big(\frac{\operatorname{Log} k}{Nk}\Big)} \\ &\quad = 2\Big(1+\frac{1}{N}\Big)\sqrt{N\pi}\,\sqrt{2\sin\varphi_{N,k}}\,e^{i(\pi/4-\varphi_{N,k}/2)} \\ &\qquad \cdot \Big(1 + \frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}}\,\lambda_{N,k} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big)\Big) \end{aligned}$$

and

$$\begin{aligned}(1-x_{N,k}^2)^{N+1} &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{v}{1-y_{N,k}^2}\Big)^{N+1}\\ &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{(N+1)\,v}{1-y_{N,k}^2}+O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big)\Big)\\ &= (1-y_{N,k}^2)^{N+1}\Big(1+Nv\,e^{2i\varphi_{N,k}}+O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big)\Big)\,.\end{aligned}$$

Finally we have

$$\begin{aligned}&\frac{(1-y_{N,k}^2)^{N+1}}{2\sqrt{2N\pi\sin\varphi_{N,k}}\,e^{i(\pi/4-\varphi_{N,k}/2)}}\\ &\qquad=\left(\frac{1+\dfrac{1}{N}\,\lambda_{N,k}}{1+\dfrac{1}{N+1}\,\lambda_{N,k}+\dfrac{1}{2\,(N+1)^2}\,\lambda_{N,k}^2+O\Big(\dfrac{(\operatorname{Log} k)^3}{N^3}\Big)}\right)^{N+1}\\ &\qquad=1-\frac{1}{N}\,\lambda_{N,k}-\frac{1}{2N}\,\lambda_{N,k}^2+O\Big(\frac{(\operatorname{Log} k)^3}{N^2}\Big)\,.\end{aligned}$$

We have thus obtained

$$\begin{aligned}&\Big(1+\frac{1}{N}\Big)\Big(1+O\Big(\frac{1}{N^2}\Big)\Big)\\ &\qquad=\frac{(1-x_{N,k}^2)^{N+1}}{2\sqrt{N\pi}\,x_{N,k}}\Big(1-\frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2}+O\Big(\frac{1}{N^2x_{N,k}^4}\Big)\Big)\\ &\qquad=1-\frac{\lambda_{N,k}}{N}-\frac{1}{2N}\,\lambda_{N,k}^2-\frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}}\,\lambda_{N,k}+Nv\,e^{2i\varphi_{N,k}}\\ &\qquad\quad+\frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}}+O\Big(\frac{(\operatorname{Log} k)^3}{k^2}\Big)\end{aligned}$$

which gives the value of v with an $O\left((\operatorname{Log} k)^3/(Nk^2)\right)$ error.

As a corollary, we find a further development of γ_k, which is exactly the formula given in [3]:

Corollary. *If* $\mu_k = (2k - 1/4)\pi$*, then*

$$\gamma_k = e^{-3i\pi/4}\sqrt{\mu_k}\Big(1 - \frac{i}{2\,\mu_k}\operatorname{Log} 2\sqrt{\pi\mu_k} - \frac{1}{4\,\mu_k^2}\operatorname{Log} 2\sqrt{\pi\mu_k} + \frac{1}{4\,\mu_k^2} + \frac{1}{8\,\mu_k^2}(\operatorname{Log} 2\sqrt{\pi\mu_k})^2 + O\Big(\frac{(\operatorname{Log} k)^3}{k^3}\Big)\Big). \tag{50}$$

PROOF. From (31) and (49), we get

$$1 - \frac{\overline{\gamma}_k^2}{N} = \Big(1 - i\,\frac{\mu_k}{N}\Big)\Big(1 + \frac{1}{N}\operatorname{Log} 2\sqrt{\pi}\,\sqrt{\mu_k} + \frac{i}{2N\mu_k}(\operatorname{Log} 2\sqrt{\pi}\,\sqrt{\mu_k} - 1)\Big) + O\Big(\frac{(\operatorname{Log} k)^3}{Nk^2}\Big),$$

hence

$$\gamma_k^2 = -i\,\mu_k - \operatorname{Log} 2\sqrt{\pi\mu_k} + \frac{i}{2\,\mu_k}\operatorname{Log} 2\sqrt{\pi\mu_k} - \frac{i}{2\,\mu_k} + O\Big(\frac{(\operatorname{Log} k)^3}{k^2}\Big)$$

and

$$\gamma_k = \sqrt{-i\,\mu_k}\Big(1 - \frac{i}{2\,\mu_k}\operatorname{Log} 2\sqrt{\pi\mu_k} - \frac{1}{4\,\mu_k^2}\operatorname{Log} 2\sqrt{\pi\mu_k} + \frac{1}{4\,\mu_k^2} + \frac{1}{8\,\mu_k^2}(\operatorname{Log} 2\sqrt{\pi\mu_k})^2 + O\Big(\frac{(\operatorname{Log} k)^3}{k^3}\Big)\Big)$$

and the corollary is proved.

5. Small roots of Q_N: further estimates.

We are now able to give a much better estimate for the small roots of Q_N. Indeed, we used the rough estimate $|e^{-Nx_{N,k}^2}| \le e^{N|x_{N,k}^2|}$ which is far from being good since $x_{N,k}$ accumulates on the line $x = y$ for k big (and $k^2 = O(N)$), so that $e^{-Nx_{N,k}^2}$ is much smaller than $e^{N|x_{N,k}|^2}$: indeed if $k^2 = O(N)$ we find that

$$x_{N,k}^2 = -\frac{1}{N}\operatorname{Log} 2\sqrt{\pi\Big(2k - \frac{1}{4}\Big)\pi} + \frac{i}{N}\Big(2k - \frac{1}{4}\Big)\pi + O\Big(\frac{\operatorname{Log} k}{Nk}\Big),$$

hence

$$|e^{-Nx_{N,k}^2}| = e^{\operatorname{Log} 2\sqrt{\pi(2k-1/4)\pi}}\, e^{O(\operatorname{Log} k/k)}$$
$$= 2\sqrt{\pi}\,\sqrt{\Big(2k-\frac{1}{4}\Big)\pi}\,\Big(1+O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big),$$

while

$$e^{N|x_{N,k}|^2} \geq e^{(2k-1/4)\pi}\Big(1+O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big).$$

Thus, we may improve Result 4 in an impressive manner: for a much bigger set of indexes k, $-\overline{\gamma}_k/\sqrt{N}$ provides a very precise approximation of $x_{N,k}$:

Result 8. *There exist $\eta_0 > 0$ and $C_0 > 0$ so that for N large enough and $k \leq \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ we have*

$$\Big|x_{N,k} + \frac{\overline{\gamma}_k}{\sqrt{N}}\Big| \leq C_0\,\frac{1}{N\sqrt{N}}\,\Big(\frac{k^{5/2}}{1+\operatorname{Log} k}\Big). \tag{51}$$

PROOF. We write

$$\tilde{Q}_N(t) = 4\sqrt{\frac{N}{\pi}}\,\frac{4^N(N!)^2}{(2N+1)!}\,Q_N(t) = 1+O\Big(\frac{1}{N^2}\Big)+2\sqrt{\frac{N}{\pi}}\int_0^t (1-s^2)^N\,ds$$

and approximate $(1-s^2)^N$ by e^{-Ns^2} (provided that Nt^4 remains bounded: $|Nt^4| \leq A_0$)

$$(1-s^2)^4 = e^{N\operatorname{Log}(1-s^2)} = e^{-Ns^2}(1+O(Ns^4)).$$

Thus

$$\tilde{Q}_N(t) = \operatorname{erfc}(-\sqrt{N}\,t) + O\Big(\frac{1}{N^2}\Big) + \sqrt{N}\int_0^t e^{-Ns^2}O(Ns^4)\,ds.$$

Let $\theta = \operatorname{Arg} t$ and assume $\theta \in (\pi/4, \pi/2)$. Then we have

$$\Big|\sqrt{N}\int_0^t e^{-Ns^2}O(Ns^4)\,ds\Big| \leq C\,N\sqrt{N}\,|t|^3\int_0^{|t|} e^{-N\lambda^2\cos 2\theta}\lambda\,d\lambda$$
$$\leq C\,\frac{|e^{-Nt^2}|\,\sqrt{N}\,|t|^3}{2\,|\cos 2\,\theta|}.$$

We have thus proved that for $|Nt^4| \leq A_0$ and $\operatorname{Arg} t \in (\pi/4, \pi/2)$ we have

$$|\tilde{Q}_N(t) - \operatorname{erfc}(-\sqrt{N}\,t)| \leq C\Big(\frac{1}{N^2} + \sqrt{N}\,|t|^3\,\frac{|e^{-Nt^2}|}{2\,|\cos 2\operatorname{Arg} t|}\Big).$$

Now, we write $t = x_{N,k} + \delta$, $|\delta| \leq \delta_0/N$. Remember that we have

$$|x_{N,k}| \approx \sqrt{\frac{\Big(2k - \frac{1}{4}\Big)\pi}{N}}$$

(hence we will look at $k \leq \sqrt{A_0 N/(2\pi)}$) and

$$\begin{aligned}\operatorname{Arg} x_{N,k} &= \frac{\pi}{4} - \frac{1}{2}\,\varphi_{N,k} + \operatorname{Arg}\Big(1 + \frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}}\operatorname{Log}\big(2\sqrt{2N\pi\sin\varphi_{N,k}}\big)\Big)\\ &\qquad + O\Big(\frac{(\operatorname{Log} k)}{k^2}\Big)\\ &= \frac{\pi}{4} + \frac{\operatorname{Log}\Big(2\sqrt{\pi}\,\sqrt{\Big(2k - \frac{1}{4}\Big)\pi}\,\Big)}{2\Big(2k - \frac{1}{4}\Big)\pi} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big),\end{aligned}$$

hence if $k \geq k_0$ where k_0 is large enough so that

$$O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big) = O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{1}{k}\Big)$$

is smaller than

$$\frac{1}{2}\,\frac{\operatorname{Log} 2\sqrt{\pi}\,\sqrt{\Big(2k - \frac{1}{4}\Big)\pi}}{2\Big(2k - \frac{1}{4}\Big)\pi},$$

we find that $\operatorname{Arg} x_{N,k} \in (\pi/4, \pi/2)$. (This is also true for $k \leq k_0$, if N is large enough, since $x_{N,k} \sim -\overline{\gamma}_k/\sqrt{N}$).

Moreover

$$\cos(2\operatorname{Arg} x_{N,k})$$
$$= -\sin\left(\frac{\operatorname{Log}\left(2\sqrt{\pi}\sqrt{\left(2k-\frac{1}{4}\right)\pi}\right)}{\left(2k-\frac{1}{4}\right)\pi} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big)\right)$$
$$= -\frac{\operatorname{Log}\left(2\sqrt{\pi}\sqrt{\left(2k-\frac{1}{4}\right)\pi}\right)}{\left(2k-\frac{1}{4}\right)\pi} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big),$$

hence $\cos(2\operatorname{Arg} x_{N,k})$ has order of magnitude $\operatorname{Log} k/k$. Thus we obtain for δ_0 small enough

- $t = x_{N,k}\Big(1 + O\Big(\frac{1}{\sqrt{Nk}}\Big)\Big)$,
- $e^{-Nt^2} = e^{-Nx_{N,k}^2}\left(1 + O\left(\sqrt{\frac{k}{N}}\right) + O\Big(\frac{1}{N}\Big)\right)$,
- $\operatorname{Arg} t = \operatorname{Arg} x_{N,k} + O\Big(\frac{1}{\sqrt{Nk}}\Big) = \operatorname{Arg} x_{N,k} + O\Big(\frac{1}{k\sqrt{k}}\Big)$,

thus we have

$$|\tilde{Q}_N(t) - \operatorname{erfc}(-\sqrt{N}\,t)| \le C\Big(\frac{1}{N^2} + \sqrt{N}\left(\frac{k}{N}\right)^{3/2}\frac{\sqrt{k}}{(\operatorname{Log} k)/k}\Big)$$
$$\le C'\,\frac{k^3}{N\operatorname{Log} k}\,.$$

On the other hand we have

$$|\operatorname{erfc}(-\sqrt{N}\,t) - \operatorname{erfc}(-\sqrt{N}\,x_{N,k})|$$
$$= \Big|2\sqrt{\frac{N}{\pi}}\int_{x_{N,k}}^{t} e^{-Ns^2}\,ds\Big|$$
$$= |e^{-Nx_{N,k}^2}|\,2\sqrt{\frac{N}{\pi}}\,\Big|\int_0^{\delta} e^{-2Nx_{N,k}s-Ns^2}\,ds\Big|.$$

We notice that

$$|2Nx_{N,k}\,s + Ns^2| \le 2\,|x_{N,k}|\,\delta_0 + \frac{\delta_0^2}{N} \le C\,\frac{\delta_0}{\sqrt{N}}\,,$$

so that if N is large enough,

$$|e^{-2Nx_{N,k}s-Ns^2}-1| \leq \frac{1}{2} \,,$$

which gives

$$|\mathrm{erfc}(-\sqrt{N}\,t)-\mathrm{erfc}(-\sqrt{N}\,x_{N,k})| \geq 2\sqrt{\frac{N}{\pi}}\;|e^{-Nx_{N,k}^2}|\;\frac{1}{2}\;|\delta| \geq C\sqrt{Nk}\;|\delta|\,.$$

Thus

$$(52)\qquad \begin{cases} |\mathrm{erfc}(-\sqrt{N}\,t)| \geq C_1\sqrt{N}\,k\,\delta - C_2\,\dfrac{k^3}{N\,\mathrm{Log}\,k}\,, \\ |\mathrm{erfc}(-\sqrt{N}\,t)-\tilde{Q}_N(t)| \leq C_2\,\dfrac{k^3}{\sqrt{N}\,\mathrm{Log}\,k}\,. \end{cases}$$

Now choose

$$\delta_{N,k} = \frac{3\,C_2}{C_1}\,\frac{k^{5/2}}{N^{3/2}\mathrm{Log}\,k}$$

(we have $\delta_{N,k} < \delta_0/N$ if $k^{5/2}/\mathrm{Log}\,k < \delta_0 C_1\sqrt{N}/(3C_2)$); we obtain that

$$\sup_{|t-x_{N,k}|=\delta_{N,k}} |\mathrm{erfc}(-\sqrt{N}\,t)-\tilde{Q}_N(t)| \leq \frac{1}{2}\inf_{|t-x_{N,k}|=\delta_{N,k}} |\,\mathrm{erfc}(-\sqrt{N}\,t)|\,,$$

hence by Rouché's theorem we find that $\tilde{Q}_N$ and $\mathrm{erfc}(-\sqrt{N}\,t)$ have the same number of roots in the disk $|t-x_{N,k}| < \delta_{N,k}$. Since

$$|x_{N,k}-x_{N,k+1}| \approx \sqrt{\frac{\pi}{2kN}}$$

and

$$\sqrt{kN}\,\delta_{N,k} = O\Big(\frac{k^3}{N\,\mathrm{Log}\,k}\Big) = O\Big(\frac{1}{N^{2/5}(\mathrm{Log}\,N)^{7/5}}\Big) = o\,(1)$$

(if $k \leq CN^{1/5}/(\mathrm{Log}\,N)^{2/5}$), we find: for $k \leq \eta_0 N^{1/5}/(\mathrm{Log}\,N)^{2/5}$ (η_0 small enough)

$$|x_{N,k}+\frac{\overline{\gamma}_k}{\sqrt{N}}| \leq C\,\frac{1}{N\sqrt{N}}\,\Big(\frac{k^{5/2}}{\mathrm{Log}\,k}\Big)\,.$$

Result 8 is proved.

Result 8 is enough for what we want to prove. But, of course, we may develop a bit further $(1-s^2)^N$ and get a better approximation for $x_{N,k}$:

Result 9. *For $k \leq \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ we have more precisely*

$$x_{N,k} = -\frac{\overline{\gamma}_k}{\sqrt{N}} + \frac{1}{N\sqrt{N}}\Big(\frac{1}{2}\,\overline{\gamma}_k^3 + \frac{3}{8}\,\overline{\gamma}_k + O\,(\sqrt{\operatorname{Log} k})\Big)\,.$$

PROOF. We write $\operatorname{Log}(1-s^2) = -s^2 - s^4/2 + O\,(s^6)$. Hence we have

$$(1-s^2)^N = e^{-Ns^2}\Big(1 - N\,\frac{s^4}{2} + O\,(Ns^6) + O\,(N^2 s^8)\Big)\,,$$

provided that $|s| \leq A_0/N^{1/4}$.

Thus we have for $|t| \leq A_0/N^{1/4}$ and $\operatorname{Arg} t \in (\pi/4, \pi/2)$

$$\begin{aligned}\Big|\tilde{Q}_N(t) - \operatorname{erfc}(-\sqrt{N}\,t) + 2\sqrt{\frac{N}{\pi}}\,N\int_0^t e^{-Ns^2} s^4\,ds\Big) \\ \leq C\Big(\frac{1}{N^2} + \sqrt{N}\,\Big|\frac{t^5 e^{-Nt^2}}{\cos(2\operatorname{Arg} t)}\Big| + N\sqrt{N}\,\Big|\frac{t^7 e^{-Nt^2}}{\cos(2\operatorname{Arg} t)}\Big|\Big)\,.\end{aligned}$$

Moreover we have

$$\begin{aligned}N\int_0^t e^{-Ns^2} s^4\,ds &= \Big[\frac{-e^{-Ns^2}s^3}{2}\Big]_0^t + \frac{3}{2}\int_0^t e^{-Ns^2} s^2\,ds \\ &= \frac{-e^{-Nt^2}t^3}{2} - \frac{3}{4N}\,e^{-Nt^2}t + \frac{3}{4N}\int_0^t e^{-Ns^2}\,ds\,.\end{aligned}$$

Now, we write $\eta = 1/\sqrt{2N|\cos(2\operatorname{Arg} t)|}$ (if $t \approx x_{N,k}$, we have $\eta \approx \sqrt{4k/(N\operatorname{Log} k)} < |t|$) and we write

$$\begin{aligned}\Big|\int_0^t e^{-Ns^2}\,ds\Big| &\leq \int_0^\eta |e^{-Nt^2}|\,ds + \int_\eta^{|t|} e^{-Ns^2\cos(2\operatorname{Arg} t)}\,\frac{s\,ds}{\eta} \\ &\leq \eta|e^{-Nt^2}| + \frac{|e^{-Nt^2}|}{2N|\cos(2\operatorname{Arg} t)|\,\eta} \\ &= \frac{2\,|e^{-Nt^2}|}{\sqrt{2N|\cos(2\operatorname{Arg} t)|}}\,.\end{aligned}$$

Finally we get

$$\begin{aligned}\operatorname{erfc}(-\sqrt{N}\,x_{N,k}) &= e^{-Nx_{N,k}^2}\sqrt{\frac{N}{\pi}}\,x_{N,k}^3 + e^{-Nx_{N,k}^2}\,\frac{3}{4\sqrt{N\pi}}\,x_{N,k}\\ &\quad + O\Big(\frac{1}{N^2}\Big) + O\Big(\frac{k^4}{N^2\log k}\Big) + O\Big(\frac{k^5}{N^2\log k}\Big)\\ &\quad + O\Big(\frac{\sqrt{\log k}}{N}\Big)\end{aligned}$$

and, assuming again $k < \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$,

$$\operatorname{erfc}(-\sqrt{N}\,x_{N,k}) = e^{-Nx_{N,k}^2}\sqrt{\frac{N}{\pi}}\,x_{N,k}^3\Big(1 + O\Big(\frac{1}{k}\Big)\Big)$$

On the other hand, we have $x_{N,k} = -\overline{\gamma}_k/\sqrt{N} + s$ with

$$s = O\Big(\frac{1}{N\sqrt{N}}\frac{k^{5/2}}{\operatorname{Log} k}\Big)$$

and we want a better estimate for s. We have

$$\sqrt{N}\,s\,\overline{\gamma}_k = O\Big(\frac{1}{N}\,\frac{k^3}{\operatorname{Log} k}\Big) = O\Big(\frac{1}{N^{2/5}}\Big)$$

and thus we may develop

$$\begin{aligned}\operatorname{erfc}(\overline{\gamma}_k - \sqrt{N}\,s) &= e^{-\overline{\gamma}_k^2}\,\frac{2}{\sqrt{\pi}}\int_0^{-\sqrt{N}\,s} e^{-2\overline{\gamma}_k u - u^2}\,du\\ &= -\frac{2}{\sqrt{\pi}}\,e^{-\overline{\gamma}_k^2}\,\sqrt{N}\,s\,(1 + O\,(\sqrt{N}\,s\,\overline{\gamma}_k) + O\,(Ns^2))\,.\end{aligned}$$

Hence we find

$$-\frac{2}{\sqrt{\pi}}\,e^{-\overline{\gamma}_k^2}\,\sqrt{N}\,s \sim \sqrt{\frac{N}{\pi}}\,x_{N,k}^3\,e^{-Nx_{N,k}^2}$$

and therefore

$$s \sim -\frac{1}{2}\,x_{N,k}^3 = O\Big(\frac{k^{3/2}}{N^{3/2}}\Big)\,,$$

so that

$$
\begin{aligned}
&- e^{-\overline{\gamma}_k^2}\,\frac{2}{\sqrt{\pi}}\,\sqrt{N}\,s\Big(1+O\Big(\frac{k^2}{N}\Big)+O\Big(\frac{k^3}{N^2}\Big)\Big)\\
&\quad=\sqrt{\frac{N}{\pi}}\,x_{N,k}^3\,e^{-Nx_{N,k}^2}+\frac{3}{4\sqrt{N\pi}}\,x_{N,k}\,e^{-Nx_{N,k}^2}+O\Big(\frac{\sqrt{\operatorname{Log} k}}{N}\Big),
\end{aligned}
$$

so that (since $e^{-Nx_{N,k}^2+\overline{\gamma}_k^2}=1+O(\sqrt{N}\,s\,\overline{\gamma}_k)=1+O\,(k^2/N)$)

$$
\begin{aligned}
s&=-\frac{1}{2}\,x_{N,k}^3-\frac{3}{8N}\,x_{N,k}+O\Big(\frac{\sqrt{\operatorname{Log} k}}{N\sqrt{N}}\Big)\\
&=\frac{1}{2}\,\frac{\overline{\gamma}_k^3}{N\sqrt{N}}+\frac{3\,\overline{\gamma}_k}{8N\sqrt{N}}+O\Big(\frac{\sqrt{\operatorname{Log} k}}{N\sqrt{N}}\Big)
\end{aligned}
$$

and Result 9 is proved.

6. The phase of a general Daubechies filter.

We have now almost achieved the proof of Theorem 1. Indeed, we have given estimates for $x_{N,k}$, hence for $z_{N,k}$, which is the solution of $x_{N,k}=(z_{N,k}+1/z_{N,k})/2$ with $\operatorname{Re} z_{N,k}>0$, hence which is given by $z_{N,k}=x_{N,k}+\sqrt{x_{N,k}^2-1}$. We thus have proved:

Proposition 3. *Let P_N be the N-th polynomial of I. Daubechies*

$$
P_N(z)=\Big(\frac{1+z}{2}\Big)^{2N+2}\sum_{k=0}^{N}(-1)^k\binom{N+k}{k}\Big(\frac{1-z}{2}\Big)^{2k} \tag{54}
$$

which is related to Q_N by

$$
e^{i(2N+1)\xi}P_N(e^{-i\xi})=Q_N(\cos\xi) \tag{55}
$$

or equivalently

$$
P_N(z)=z^{2N+1}Q_N\Big(\frac{1}{2}\Big(z+\frac{1}{z}\Big)\Big)\,. \tag{56}
$$

Then the roots of P_N are precisely given as the following ones:

- *$z=-1$ with multiplicity $2N+2$,*

• *$2N$ roots with multiplicity 1 which can be decomposed into*

$$\left\{z_{N,k}, \overline{z_{N,k}}, \frac{1}{z_{N,k}}, \frac{1}{\overline{z}_{N,k}}\right\}_{1\le k\le [N/2]},$$

(together with $\{z_{N,(N+1)/2}, 1/z_{N,(N+1)/2}\}$ if N is odd), where $\operatorname{Im} z_{N,k} \ge 0$, $\operatorname{Re} z_{N,k} \ge 0$, $|z_{N,k}| > 1$, $\operatorname{Im} z_{N,k} > 0$ *for* $k < [(N+1)/2]$ *and* $\operatorname{Im} z_{N,(N+1)/2} = 0$.

Moreover we have, for N large enough:

• *if $k \le \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ (where η_0 is fixed independently of N and is small enough)*

$$z_{N,k} = i - \frac{\overline{\gamma}_k}{\sqrt{N}} + O\Big(\frac{k}{N}\Big), \tag{57}$$

where γ_k is the k-th zero γ of $\operatorname{erfc}(z)$ *with* $\operatorname{Im}\gamma > 0$

• *for all k*

$$z_{N,k} = y_{N,k} + \sqrt{y_{N,k}^2 - 1} + O\Big(\frac{1+\operatorname{Log} k}{k\sqrt{Nk}}\Big), \tag{58}$$

where

$$y_{N,k} = \Bigg(1 - e^{-2i(8k-1)\pi/(8N+6)} - \frac{1}{N}\, e^{-2i(8k-1)\pi/(8N+6)} \operatorname{Log} 2 \sqrt{2N\pi \sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)}\Bigg)^{1/2}.$$

PROOF. Just write $z_{N,k} = x_{N,k} + \sqrt{x_{N,k}^2 - 1}$ and apply results 6 and 8.

Of course, we could give better estimates using results 7 and 9, but we won't need them. We have easy estimates for $1/z_{N,k}$ as well since $1/z_{N,k} = x_{N,k} - \sqrt{x_{N,k}^2 - 1}$.

We are now going to use proposition 3 in the estimation of the phase of a Daubechies filter. We want to approximate for $\xi \in [-\pi, \pi]$, $1/(e^{-i\xi} - \lambda_{N,k})$ where

$$\lambda_{N,k} \in \left\{z_{N,k}, \frac{1}{z_{N,k}}, \overline{z}_{N,k}, \frac{1}{\overline{z}_{N,k}}\right\}.$$

A direct consequence of Proposition 3 is the following proposition:

Proposition 4. *Let* $\xi \in [-\pi, \pi]$ *and let* $z_{N,k}$, $1 \leq k \leq [(N+1)/2]$ *be the roots of* P_N *described in Proposition* 3. *Let* $\lambda_{N,k} \in \{z_{N,k}, 1/z_{N,k}, \overline{z}_{N,k}, 1/\overline{z}_{N,k}\}$. *Then*

i) *for* $1 \leq k \leq \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ *we have, writing* $\widetilde{z_{N,k}} = i - \overline{\gamma}_k/\sqrt{N}$,

$$(59)\qquad \Big|\frac{1}{e^{-i\xi} - \lambda_{N,k}} - \frac{1}{e^{-i\xi} - \widetilde{\lambda_{N,k}}}\Big| \leq C\,\frac{k}{N}\,\frac{1}{\dfrac{k}{N} + |\cos\xi|^2}\,,$$

where C *doesn't depend neither on* N *nor on* k *nor on* ξ (*and where* $\widetilde{\lambda_{N,k}} = \widetilde{z_{N,k}}$ *if* $\lambda_{N,k} = z_{N,k}$, $1/\widetilde{z_{N,k}}$, *if* $\lambda_{N,k} = 1/z_{N,k}$ *and so on* ...).

ii) *for* $k \geq k_0$ (k_0 *large enough independently of* N) *we have, writing* $\widehat{z_{N,k}} = y_{N,k} + \sqrt{y_{N,k}^2 - 1}$ *as in formula* (58),

$$(60)\qquad \Big|\frac{1}{e^{-i\xi} - \lambda_{N,k}} - \frac{1}{e^{-i\xi} - \widehat{\lambda_{N,k}}}\Big| \leq C\,\frac{\operatorname{Log} k}{k\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N} + |\cos\xi|^2}\,.$$

PROOF. Of course, we may assume $\xi \in [0, \pi]$. If $\xi \in [\pi/2, \pi]$, the estimation is easy since $\operatorname{Re} e^{-i\xi} < 0$ and $\operatorname{Re}\lambda_{N,k} > 0$ (as well $\operatorname{Re}\widehat{\lambda_{N,k}}$ and $\operatorname{Re}\widetilde{\lambda_{N,k}}$). Thus,

$$|e^{-i\xi} - \lambda_{N,k}| \geq \operatorname{Re}(-e^{-i\xi} + \lambda_{N,k}) \geq C\sqrt{\frac{k}{N}} + |\cos\xi|$$

and the same for $|e^{-i\xi} - \widehat{\lambda_{N,k}}|$ and $|e^{-i\xi} - \widetilde{\lambda_{N,k}}|$. Of course, we must prove that $\min\{\operatorname{Re}\lambda_{N,k}, \operatorname{Re}\widehat{\lambda_{N,k}}, \operatorname{Re}\widetilde{\lambda_{N,k}}\} \geq C\sqrt{k/N}$. For $\operatorname{Re}\widetilde{\lambda_{N,k}}$, it is obvious, since

$$\operatorname{Re}\widetilde{\lambda_{N,k}} \geq \frac{-\operatorname{Re}\gamma_k}{\sqrt{N}\,\Big|i - \dfrac{\overline{\gamma}_k}{\sqrt{N}}\Big|^2} \approx \sqrt{\frac{k\pi}{N}}\,.$$

For $\operatorname{Re}\lambda_{N,k}$, if $k < \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$, we deduce that $\operatorname{Re}\lambda_{N,k} \geq C\sqrt{k/N}$ since

$$|\lambda_{N,k} - \widetilde{\lambda_{N,k}}| \leq |z_{N,k} - \widetilde{z_{N,k}}| \leq C\,\frac{k}{N} \leq \sqrt{\frac{k}{N}}\,C'\,N^{-2/5}\,.$$

We thus turn our attention to $\operatorname{Re}\widehat{\lambda_{N,k}} \geq \operatorname{Re}\widehat{z_{N,k}}/|\widehat{z_{N,k}}|^2$ and $\operatorname{Re}\lambda_{N,k} \geq \operatorname{Re} z_{N,k}/|z_{N,k}|^2$ for large k's. We define $\mu_{N,k} = \sqrt{1 - e^{-2i(8k-1)\pi/(8N+6)}}$ and $\xi_{N,k} = \mu_{N,k} + \sqrt{\mu_{N,k}^2 - 1}$. We have

$$\begin{aligned}\xi_{N,k} &= \sqrt{2\sin\Big(\frac{8k-1}{8N+6}\Big)}\, e^{i(\pi/4-(8k-1)\pi/(2(8N+6)))} \\ &\qquad + e^{i(\pi/2-(8k-1)\pi/(8N+6))} \\ &= 1+\sqrt{2}\, e^{i(\pi/4-(8k-1)\pi/(2(8N+6))+\arcsin\sqrt{2}\sin(\pi/4-(8k-1)\pi/2(8N+6)))}\end{aligned}$$

and thus we study $1+\sqrt{2}\, e^{i(\omega+\arcsin\sqrt{2}\sin\omega)}$ for $\omega \in [0, \pi/4]$. We have

$$\begin{aligned}&\operatorname{Re}\big(1+\sqrt{2}\, e^{i(\omega+\arcsin\sqrt{2}\sin\omega)}\big) \\ &\qquad = \sqrt{1-2\sin^2\omega}\,\big(\sqrt{1-2\sin^2\omega} + \sqrt{2\cos^2\omega}\,\big) \\ &\qquad = \sqrt{\cos 2\omega}\,\big(\sqrt{2\cos^2\omega} + \sqrt{1-2\sin^2\omega}\,\big) \geq \sqrt{\frac{4}{\pi}\Big(\frac{\pi}{4}-\omega\Big)}\,,\end{aligned}$$

which gives

$$\operatorname{Re}\xi_{N,k} \geq \sqrt{2\,\frac{8k-1}{8N+6}} \geq \sqrt{\frac{k}{N}}\,.$$

Now we have

$$|\widehat{z_{N,k}} - \xi_{N,k}| \leq C\sqrt{\frac{k}{N}}\,\frac{\operatorname{Log} k}{k}\,,$$

so that if k is large enough we have

$$\operatorname{Re}\widehat{z_{N,k}} \geq C'\sqrt{\frac{k}{N}}\,.$$

Moreover

$$|z_{N,k} - \widehat{z_{N,k}}| \leq C\sqrt{\frac{k}{N}}\,\frac{\operatorname{Log} k}{k^2}$$

and thus

$$\operatorname{Re} z_{N,k} \geq C''\sqrt{\frac{k}{N}}\,.$$

Finally, we control $|z_{N,k}|$ and $|\widehat{z_{N,k}}|$ by

$$|z_{N,k}| + |\widehat{z_{N,k}}| \leq 1+\sqrt{2} + O\Big(\sqrt{\frac{k}{N}}\,\frac{\operatorname{Log} k}{k}\Big) \leq C\,.$$

Thus we obtain

$$\operatorname{Re}\lambda_{N,k} \geq C\sqrt{\frac{k}{N}} \qquad \text{and} \qquad \operatorname{Re}\widehat{\lambda_{N,k}} \geq C\sqrt{\frac{k}{N}}\,.$$

We are going to prove that

$$|e^{-i\xi} - \lambda_{N,k}| \geq C\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)$$

and

$$|e^{-i\xi} - \widehat{\lambda_{N,k}}| \geq C\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)$$

holds for $\xi \in [0, \pi/2]$ as well. Notice that if $|\lambda_{N,k}| < 1$, we have

$$|\lambda_{N,k} - e^{-i\xi}| = \Big|\frac{1}{z_{N,k}}\Big|\,\Big|e^{-i\xi} - \frac{1}{\overline{\lambda_{N,k}}}\Big| \geq \frac{1}{C'}\Big|e^{-i\xi} - \frac{1}{\overline{\lambda_{N,k}}}\Big|$$

(and the same for $|e^{-i\xi} - \widehat{\lambda_{N,k}}|$) so that we may assume $|\lambda_{N,k}| > 1$. If $\lambda_{N,k} = z_{N,k}$, our equality is obvious: for $\xi_{N,k}$ we have either $\operatorname{Im}\xi_{N,k} \geq 1$ or $\operatorname{Re}\xi_{N,k} \geq 2$ and, since $\operatorname{Im} e^{-i\xi} < 0$, we find $|e^{-i\xi} - \xi_{N,k}| \geq 1$, hence (for k large), $|e^{-i\xi} - z_{N,k}| \geq 1/2$ and $|e^{-i\xi} - \widehat{z_{N,k}}| \geq 1/2$, while

$$\frac{1}{2} \geq \frac{1}{4}\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)\,.$$

Now if $\lambda_{N,k}$ is the conjugate of $z_{N,k}$ or $\widehat{z_{N,k}}$, we are going to show that

$$|e^{-i\xi} - \overline{\xi}_{N,k}| \geq C\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)\,,$$

which gives the control over $|e^{-i\xi} - \lambda_{N,k}|$ for large k's. Thus we are led to show that

$$\text{(61)}\qquad \begin{cases} \text{for } \xi \in \Big[0, \dfrac{\pi}{2}\Big] \text{ and } \omega \in \Big[0, \dfrac{\pi}{4}\Big]\,, \\ |e^{-i\xi} - 1 - \sqrt{2}\, e^{-i(\omega + \arcsin\sqrt{2}\,\sin\omega)}| \\ \qquad\qquad \geq C\Big(|\cos\xi| + \sqrt{\dfrac{\pi}{4} - \omega}\Big)\,. \end{cases}$$

We compute easily $\mu(\xi,\omega) = |e^{-i\xi} - 1 - \sqrt{2}\, e^{-i(\omega+\arcsin\sqrt{2}\sin\omega)}|^2$

$$\begin{aligned}
\mu(\xi,\omega) &= \Big(\cos\xi - \sqrt{1-2\sin^2\omega}\,\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big)\Big)^2 \\
&\quad + \Big(\sin\xi - \sqrt{2}\,\sin\omega\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big)\Big)^2 \\
&= 1 + \big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big)^2 \\
&\quad - 2\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\,\big) \\
&\qquad \cdot \big(\cos\xi\sqrt{1-2\sin^2\omega} + \sin\xi\sqrt{2}\,\sin\omega\big) \\
&= \big(\sqrt{2}\,\cos\omega - 1 + \sqrt{1-2\sin^2\omega}\big)^2 \\
&\quad + 2\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big) \\
&\qquad \cdot \big(1 - \cos(\xi - \arcsin(\sqrt{2}\,\sin\omega))\big) \\
&\geq 1 - 2\sin^2\omega + 2\left(1 - \cos(\xi - \arcsin(\sqrt{2}\,\sin\omega))\right).
\end{aligned}$$

We have

$$1 - 2\sin^2\omega = \cos 2\,\omega \geq \frac{2}{\pi}\Big(\frac{\pi}{2} - 2\,\omega\Big).$$

On the other hand, we have

$$\begin{aligned}
1 - \cos(\xi - \arcsin\sqrt{2}\,\sin\omega) &= 2\sin^2\Big(\frac{\xi}{2} - \frac{1}{2}\,\arcsin\sqrt{2}\,\sin\omega\Big) \\
&\geq \frac{2}{\pi^2}|\xi - \arcsin\sqrt{2}\,\sin\omega|^2.
\end{aligned}$$

Moreover we have

$$\frac{\pi}{2} - \arcsin\sqrt{2}\,\sin\omega = \arcsin\sqrt{\cos 2\,\omega} \leq \frac{\pi}{2}\,\sqrt{\cos 2\,\omega}\,,$$

hence we have (using $|a+b|^2 \geq a^2/3 - b^2/2$)

$$\begin{aligned}
\mu(\xi,\omega)^2 &\geq \cos 2\,\omega + \frac{4}{\pi^2}\Big|\xi - \frac{\pi}{2} + \frac{\pi}{2} - \arcsin\sqrt{2}\,\sin\omega\Big|^2 \\
&\geq \cos 2\,\omega + \frac{4}{3\pi^2}\Big|\xi - \frac{\pi}{2}\Big|^2 - \frac{2}{\pi^2}\Big|\frac{\pi}{2} - \arcsin\sqrt{2}\,\sin\omega\Big|^2 \\
&\geq \frac{1}{2}\,\cos 2\,\omega + \frac{4}{3\pi^2}\cos^2\xi \\
&\geq \frac{4}{3\pi^2}\Big(\cos^2\xi + \Big|\frac{\pi}{4} - \omega\Big|\Big)
\end{aligned}$$

and thus (61) is proved.

Proposition 4 is then obvious since

$$\Big|\frac{1}{e^{-i\xi}-\lambda_{N,k}}-\frac{1}{e^{-i\xi}-\tilde\lambda_{N,k}}\Big| = \frac{|\lambda_{N,k}-\tilde\lambda_{N,k}|}{|e^{-i\xi}-\lambda_{N,k}|\,|e^{-i\xi}-\tilde\lambda_{N,k}|}$$

and since we control each term due to (61) or to Proposition 3.

We may now obtain Theorem 1 as a corollary of Proposition 4:

Corollary. *With the same notation as in Proposition* 4, *if* $k_0 \le k_N \le \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ *then*

$$\int_0^{2\pi}\Big|\sum_{k=1}^{[(N+1)/2]}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\lambda_{N,k}} - \sum_{k=1}^{k_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\tilde\lambda_{N,k}} - \sum_{k_N+1}^{[(N+1)/2]}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\hat\lambda_{N,k}}\Big|\,d\xi \le C\Big(\frac{k_N^{3/2}}{\sqrt N}+\frac{\operatorname{Log} k_N}{k_N}\Big)\,. \tag{62}$$

PROOF. Using Proposition 4, and writing $I_N(\xi)$ for

$$I_N(\xi)=\sum_{k=1}^{N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\lambda_{N,k}}-\sum_{k=1}^{k_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\tilde\lambda_{N,k}}-\sum_{k_N+1}^{[(N+1)/2]}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\tilde\lambda_{N,k}}\ ,$$

we get

$$I_N(\xi)\le\sum_{k=1}^{k_N}C\,\frac{k}{N}\,\frac{1}{\dfrac{k}{N}+|\cos\xi|^2}+\sum_{k_N+1}^{[(N+1)/2]}C\,\frac{\operatorname{Log} k}{k\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N}+|\cos\xi|^2}\ .$$

Thus we have to estimate

$$\begin{aligned}\int_0^{2\pi}\frac{d\xi}{k+N|\cos\xi|^2} &\le 4\int_0^{\arccos\sqrt{k/N}}\frac{d\xi}{N\cos^2\xi}+4\int_{\arccos\sqrt{k/N}}^{\pi/2}\frac{d\xi}{k}\\ &=\frac{4}{N}\tan\Big(\arccos\sqrt{\frac{k}{N}}\,\Big)+\frac{4}{k}\Big(\frac{\pi}{2}-\arccos\sqrt{\frac{k}{N}}\,\Big)\\ &\le\frac{4}{\sqrt{Nk}}+\frac{2\pi}{\sqrt{Nk}}\ ,\end{aligned}$$

so that

$$\int_0^{2\pi} I_N(\xi)\, d\xi \leq C'\Big(\sum_{k=1}^{k_N} \sqrt{\frac{k}{N}} + \sum_{k_N+1}^{[(N+1)/2]} \frac{\operatorname{Log} k}{k^2}\Big) \leq C''\Big(\frac{k_N^{3/2}}{\sqrt{N}} + \frac{\operatorname{Log} k_N}{k_N}\Big).$$

Now Theorem 1 is proved with $k_N = [N^{1/5}/\operatorname{Log} N]$. At least, we have proved it for $\xi \in [0, 2\pi]$. But $\omega(z_{N,1}^{\varepsilon_1}, \ldots, z_{N,N}^{\varepsilon_N}) - \omega(Z_{N,1}^{\varepsilon_1}, \ldots, Z_{N,N}^{\varepsilon_N})$ is 2π-periodical, since $\omega(Z_1, \ldots, Z_N)(\xi + 2\pi) - \omega(Z_1, \ldots, Z_N)(\xi) = 2i\pi M$ where M is the number of Z_k's which lie inside the open disk $|Z| < 1$.

7. Minimum-phased Daubechies filters.

This section is devoted to the proof of Theorem 2.

Result 10. *We have the following inequality*

$$(63)\quad \Big|\frac{d}{d\xi}\omega(z_{N,1}, \ldots, z_{N,N})(\xi) - \frac{N}{2\pi}\operatorname{Im}\int_{-\pi}^{\pi} \frac{i\, e^{-i\xi}}{e^{-i\xi} - \xi(\omega)}\, d\omega\Big| \leq C\sqrt{N}\ ,$$

where $\xi(\omega) = \sqrt{e^{-i\omega}} + \sqrt{1 + e^{-i\omega}}$.

PROOF. We approximate $z_{N,k}$ by $Z_{N,k} = Z((8k-1)\pi/(8N+6))$, $(1 \leq k \leq N)$ where

$$Z(\omega) = \sqrt{2\sin\omega}\; e^{i(\pi/4 - \omega/2)} + e^{i(\pi/2 - \omega)}\ .$$

We have shown that for $k_0 \leq k \leq [(N+1)/2]$, ($k_0$ large enough) we have

$$\Big|\frac{1}{e^{-i\xi} - z_{N,k}} - \frac{1}{e^{-i\xi} - Z_{N,k}}\Big| \leq C\,\frac{\operatorname{Log} k}{\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N} + \cos^2\xi}$$

and

$$\Big|\frac{1}{e^{-i\xi} - \overline{z}_{N,k}} - \frac{1}{e^{-i\xi} - \overline{Z}_{N,k}}\Big| \leq C\,\frac{\operatorname{Log} k}{\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N} + \cos^2\xi}$$

(notice that $z_{N,N+1-k} = \overline{z}_{N,k}$ and $Z_{N,N+1-k} = \overline{Z}_{N,k}$). If $k < k_0$, we have to prove similarly

$$\Big|\frac{1}{e^{-i\xi} - z_{N,k}} - \frac{1}{e^{-i\xi} - Z_{N,k}}\Big| \leq C\,\frac{1}{\sqrt{N}}\,\frac{1}{\dfrac{1}{N} + \cos^2\xi}$$

and
$$\Big|\frac{1}{e^{-i\xi}-\overline{z}_{N,k}}-\frac{1}{e^{-i\xi}-\overline{Z}_{N,k}}\Big|\le C\,\frac{1}{\sqrt{N}}\,\frac{1}{\dfrac{1}{N}+\cos^2\xi}\,.$$

We have of course
$$|z_{N,k}-Z_{N,k}|\le|z_{N,k}|+|Z_{N,k}|\le\frac{C}{\sqrt{N}}\,,$$

so that we only have to check that
$$|e^{-i\xi}-Z_{N,k}|\ge\frac{1}{C}\Big(\frac{1}{\sqrt{N}}+|\cos\xi|\Big)$$

(which is an easy consequence of (61)) and that
$$|e^{-i\xi}-z_{N,k}|\ge\frac{1}{C}\Big(\frac{1}{\sqrt{N}}+|\cos\xi|\Big)\,.$$

If $|\xi+\pi/2|\ge 3\,|\gamma_{k_0}|/\sqrt{N}$ and $\xi\in[-2\pi,0]$, we find
$$e^{-i\xi}-z_{N,k}=2\,e^{-i(\xi/2+\pi/4)}\sin\Big(\frac{\xi}{2}+\frac{\pi}{4}\Big)-\frac{\overline{\gamma}_k}{\sqrt{N}}+O\Big(\frac{1}{N}\Big)\,,$$

hence
$$\begin{aligned}|e^{-i\xi}-z_{N,k}|&\ge\Big|\sin\Big(\frac{\xi}{2}+\frac{\pi}{4}\Big)\Big|-\frac{|\gamma_{k_0}|}{\sqrt{N}}+O\Big(\frac{1}{N}\Big)\\&\ge\frac{1}{2}\Big|\sin\Big(\frac{\xi}{2}+\frac{\pi}{4}\Big)\Big|\\&\ge\max\Big\{\frac{1}{4}\,|\cos\xi|,\frac{6}{\pi}\,\frac{|\gamma_{k_0}|}{\sqrt{N}}\Big\}\,.\end{aligned}$$

On the other hand, if $|\xi+\pi/2|\le 3\,|\gamma_{k_0}|/\sqrt{N}$, we have
$$e^{-i\xi}-z_{N,k}=-\Big(\frac{\xi}{2}+\frac{\pi}{4}\Big)-\frac{\overline{\gamma}_k}{\sqrt{N}}+O\Big(\frac{1}{N}\Big)\,,$$

hence
$$|e^{-i\xi}-z_{N,k}|\ge\frac{1}{2}\,\frac{\inf\operatorname{Im}\gamma_k}{\sqrt{N}}=\frac{c_0}{\sqrt{N}}\ge C_0\max\Big\{\frac{1}{\sqrt{N}},\frac{1}{6\,|\gamma_{k_0}|}\,|\cos\xi|\Big\}\,.$$

Thus we have obtained

$$\Big|\frac{d}{d\xi}\,\omega(z_{N,1},\dots,z_{N,N})(\xi)-\sum_{k=1}^{N}\operatorname{Im}\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z_{N,k}}\Big|$$

$$\leq C\sum_{k=1}^{N}\frac{(1+\operatorname{Log}k)}{\sqrt{Nk}}\;\frac{1}{\dfrac{k}{N}+\cos^2\xi}$$

$$\leq C\sqrt{N}\sum_{1}^{\infty}\frac{1+\operatorname{Log}k}{k\sqrt{k}}\,.$$

Now we look at

$$S_N(\xi)=\operatorname{Im}\sum_{k=1}^{N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z_{N,k}}$$

as at a Riemann sum: we have

$$\frac{\pi}{N}\,S_N(\xi)\underset{N\to\infty}{\longrightarrow}\operatorname{Im}\int_0^{\pi}\frac{i\,e^{-i\xi}\,d\omega}{e^{-i\xi}-Z(\omega)}\,.$$

If $\xi\neq\pm\pi/2$, we have a proper Riemann integral; if $\xi=\pm\pi/2$, the integrand is unbounded at 0 ($\xi=-\pi/2$) or π ($\xi=\pi/2$); but for $\xi=-\pi/2$ we have $e^{-i\xi}-Z(\omega)=e^{i\pi/4}\sqrt{2\,\omega}+O\,(\omega)$ near $\omega=0$ and thus

$$\int_0^{\pi}\frac{1}{|i-Z(\omega)|}\,d\omega<+\infty\,.$$

It is easy to evaluate the distance between $\pi S_N/N$ and the integral. We have

$$\Big|\int_0^{7\pi/(8N+6)}\frac{d\omega}{e^{-i\xi}-Z(\omega)}\Big|\leq C\int_0^{7\pi/(8N+6)}\frac{d\omega}{\sqrt{\omega}}\leq C'\,\frac{1}{\sqrt{N}}\,,$$

$$\Big|\int_{(8N-1)\pi/(8N+6)}^{\pi}\frac{d\omega}{e^{-i\xi}-Z(\omega)}\Big|\leq C\int_{(8N-1)\pi/(8N+6)}^{\pi}\frac{d\omega}{\sqrt{\pi-\omega}}$$

$$\leq C'\,\frac{1}{\sqrt{N}}\,,$$

$$\frac{1}{N}\left|\frac{1}{e^{-i\xi}-Z\Big(\dfrac{8N-1}{8N+6}\,\pi\Big)}\right|\leq C'\,\frac{\sqrt{N}}{N}\,,$$

and finally for $1 \leq k < N$

$$\left| \int_{(8k-1)\pi/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{1}{e^{-i\xi} - Z(\omega)}\, d\omega - \frac{8\pi}{8N+6} \frac{1}{e^{-i\xi} - Z\Big(\frac{8k-1}{8N+6}\Big)\pi} \right|$$

$$\leq C \int_{(8k-1)/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{\Big|Z(\omega) - Z\Big(\frac{8k-1}{8N+6}\Big)\pi\Big|}{|e^{-i\xi} - Z(\omega)|\, \Big|e^{-i\xi} - Z\Big(\frac{8k-1}{8N+6}\pi\Big)\Big|}\, d\omega$$

$$\leq C' \int_{(8k-1)\pi/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{\frac{1}{\sqrt{Nk}}}{\sqrt{\frac{k}{N}}\,\sqrt{\frac{k}{N}}}\, d\omega$$

$$\leq C'' \frac{1}{k^{3/2}\sqrt{N}}$$

and thus

$$\Big| \frac{\pi}{N} S_N(\xi) - \mathrm{Im} \int_0^\pi i\, e^{-i\xi} \frac{d\omega}{e^{-i\xi} - Z(\omega)} \Big| \leq C \frac{1}{\sqrt{N}} .$$

Thus, Result 10 is proved since writing $-e^{-2i\omega} = e^{-i\sigma}$ gives

$$\int_0^\pi i\, e^{-i\xi} \frac{d\omega}{e^{-i\xi} - \sqrt{2\sin\omega}\; e^{i(\pi/4-\omega/2)} - e^{i(\pi/2-\omega)}}$$
$$= \frac{1}{2} \int_{-\pi}^{\pi} i\, e^{-i\xi} \frac{d\sigma}{e^{-i\xi} - \sqrt{e^{-i\sigma}} - \sqrt{1+e^{-i\sigma}}} .$$

We will easily prove Theorem 2 if we know the value of $I(\xi) = \int_{-\pi}^{\pi} i\, e^{-i\xi}\, d\sigma/(e^{-i\xi} - \xi(\sigma))$:

Result 11. *Let* $\xi(\sigma) = \sqrt{e^{-i\xi}} + \sqrt{1+e^{-i\sigma}}$ *and* $\xi \in [-\pi, \pi]$. *Then*

$$\int_{-\pi}^{\pi} i\, e^{-i\xi} \frac{d\sigma}{e^{-i\xi} - \xi(\sigma)}$$

(64)
$$= \begin{cases} -\pi \tan\Big(\frac{\xi}{2}\Big) + i \frac{\cos\xi}{\sin\xi} \mathrm{Log}\Big(\frac{1-\sin\xi}{1+\sin\xi}\Big) , & \text{if } |\xi| \leq \frac{\pi}{2} , \\ -\pi \,\mathrm{cotan}\Big(\frac{\xi}{2}\Big) + i \frac{\cos\xi}{\sin\xi} \mathrm{Log}\Big(\frac{1-\sin\xi}{1+\sin\xi}\Big), & \text{if } |\xi| \geq \frac{\pi}{2} . \end{cases}$$

We find that $I(\xi)$ is continuous, which is obvious since by (61)

$$|e^{-i\xi} - \xi(\sigma)| \geq C\sqrt{\pi^2 - \sigma^2}\,,$$

so that we may apply Lebesgue's dominated convergence theorem.

PROOF. Since $\xi(\sigma) = \overline{\xi}(-\sigma)$, we find that

$$I(-\xi) = -\overline{\int_{-\pi}^{\pi} \frac{i\,e^{-i\xi}}{e^{-i\xi} - \overline{\xi}(\sigma)}\,d\sigma} = -\overline{I(\xi)}\,,$$

so that it is enough to compute $I(\xi)$ for $\xi \in [0, \pi]$.

Writing $e^{-i\sigma} = u$, we may write

$$I(\xi) = \int_{-1+i0}^{-1-i0} \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u}\,,$$

where u runs clockwise on the circle $|u| = 1$. The function

$$f(z) = \frac{e^{-i\xi}}{z\,(\sqrt{z} + \sqrt{1+z} - e^{-i\xi})}$$

is analytical on $\mathbb{C}\backslash(-\infty, 0]$ and may be extended continuously to $(-\infty, 0] + i\,0$ and $(-\infty, 0] - i\,0$ but at three points: $z = 0$ (both a pole and a branching point), $z = -1$ (a branching point) and if $\xi \in [0, \pi/2]$ at $-\sin^2\xi - i\,0 = z_\xi$. Thus we may write:

- for $\xi \in [\pi/2, \pi]$

$$\begin{aligned} I(\xi) = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon} &\frac{e^{-i\xi}}{\sqrt{u+i\,0} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u} \\ &+ \int_{-\varepsilon}^{-1} \frac{e^{-i\xi}}{\sqrt{u-i\,0} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u} \\ &+ \int_{-\varepsilon+i0}^{-\varepsilon-i0} \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u} \\ &= 2\,i\int_0^1 \frac{dt}{\cos\xi - \sqrt{1-t^2}} - 2\,i\,\pi\,\frac{e^{-i\xi}}{1 - e^{-i\xi}} \\ &= 2\,i\int_0^{\pi/2} \frac{\cos\alpha}{\cos\xi - \cos\alpha}\,d\alpha - \pi\,\mathrm{cotan}\Big(\frac{\xi}{2}\Big) + \pi\,i\,. \end{aligned}$$

• if $\xi \in (0,\pi/2)$ we have, writing $t_\varepsilon^+ = \sqrt{\sin^2\xi+\varepsilon}$ and $t_\varepsilon^- = \sqrt{\sin^2\xi-\varepsilon}$

$$I(\xi) = \lim_{\varepsilon\to 0} A_\varepsilon + B_\varepsilon + C_\varepsilon \ ,$$

where

$$\begin{aligned} A_\varepsilon &= \int_{-1}^{-(t_\varepsilon^+)^2} + \int_{-(t_\varepsilon^-)^2}^{-\varepsilon} \frac{e^{-i\xi}}{\sqrt{u+i\,0}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\ &\quad + \int_{-\varepsilon}^{-(t_\varepsilon^-)^2} + \int_{-(t_\varepsilon^+)^2}^{-1} \frac{e^{-i\xi}}{\sqrt{u-i\,0}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\ &= 2\,i\int_{\sqrt{\varepsilon}}^{t_\varepsilon^-} + \int_{t_\varepsilon^+}^{1} \frac{dt}{\cos\xi-\sqrt{1-t^2}} \end{aligned}$$

$$\begin{aligned} B_\varepsilon &= \int_{-\varepsilon+i\,0}^{-\varepsilon-i\,0} \frac{e^{-i\xi}}{\sqrt{u}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\ &= -2\,i\,\pi\,\frac{e^{-i\xi}}{1-e^{-i\xi}} + O\left(\sqrt{\varepsilon}\right) \\ &= -\pi\,\mathrm{cotan}\Big(\frac{\xi}{2}\Big) + i\,\pi + O\left(\sqrt{\varepsilon}\right), \end{aligned}$$

$$\begin{aligned} C_\varepsilon &= \int_{-(t_\varepsilon^+)^2}^{-(t_\varepsilon^-)^2} \frac{e^{-i\xi}}{\sqrt{u+i\,0}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\ &\quad + \int_{z_\xi+\varepsilon}^{z_\xi-\varepsilon} \frac{e^{-i\xi}\,du}{(\sqrt{u}+\sqrt{1+u}-e^{-i\xi})\,u} \\ &= -i\,\pi\,2\,i\,\mathrm{cotan}\,\xi + O\left(\varepsilon\right) \\ &= 2\pi\,\mathrm{cotan}\,\xi + O\left(\varepsilon\right), \end{aligned}$$

since the residue of

$$f(\xi) = \frac{e^{-i\xi}}{\sqrt{u}+\sqrt{1+u}-e^{-i\xi}}\,\frac{1}{u}$$

at $z_\xi = -\sin^2\xi - i\,0$ is equal to

$$\frac{e^{-i\xi}}{\frac{1}{2}\,\frac{1}{\sqrt{z_\xi}}+\frac{1}{2}\,\frac{1}{\sqrt{1+z_\xi}}}\,\frac{1}{z_\xi} = \frac{2\sqrt{z_\xi}\,\sqrt{1+z_\xi}}{z_\xi} = 2\,i\,\mathrm{cotan}\,\xi\,.$$

Hence we have

$$
\begin{aligned}
I(\xi) &= \pi\Big(2\,\mathrm{cotan}\,\xi - \mathrm{cotan}\Big(\frac{\xi}{2}\Big)\Big) \\
&\qquad + i\,\pi + 2\,i \lim_{\varepsilon\to 0} \int_0^{t_\varepsilon^-} + \int_{t_\varepsilon^+}^1 \frac{dt}{\cos\xi - \sqrt{1-t^2}} \\
&= -\pi \tan\Big(\frac{\xi}{2}\Big) + i\,\pi + 2\,i \lim_{\varepsilon\to 0} \int_0^{\alpha_\varepsilon^-} + \int_{\alpha_\varepsilon^+}^{\pi/2} \frac{\cos\alpha\, d\alpha}{\cos\xi - \cos\alpha}\ ,
\end{aligned}
$$

where $\alpha_\varepsilon^- = \arcsin t_\varepsilon^-$ and $\alpha_\varepsilon^+ = \arcsin t_\varepsilon^+$.

Thus, for proving Result 11, we just have to estimate for $\xi \in (0,\pi)$, $\xi \neq \pi/2$

$$
A(\xi) = \lim_{\varepsilon\to 0} \int_0^{\alpha_\varepsilon^-} + \int_{\alpha_\varepsilon^+}^{\pi/2} \frac{\cos\alpha\, d\alpha}{\cos\xi - \cos\alpha}
$$

with $\alpha_\varepsilon^- = \arcsin\sqrt{\sin^2\xi - \varepsilon}$ and $\alpha_\varepsilon^+ = \arcsin\sqrt{\sin^2\xi + \varepsilon}$. We do the usual change of variable $\beta = \tan(\alpha/2)$. Then

$$
A(\xi) = \lim_{\varepsilon\to 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \frac{2\,(1-\beta^2)}{(1+\beta^2)\,((1+\beta^2)\cos\xi - (1-\beta^2))}\, d\beta\,.
$$

We write

$$
\begin{aligned}
(1+\beta^2)\cos\xi - (1-\beta^2) &= \beta^2(1+\cos\xi) - (1-\cos\xi) \\
&= 2\,\beta^2\cos^2\Big(\frac{\xi}{2}\Big) - 2\sin^2\Big(\frac{\xi}{2}\Big)\ ,
\end{aligned}
$$

hence

$$
\begin{aligned}
A(\xi) &= \frac{1}{\cos^2\Big(\dfrac{\xi}{2}\Big)} \lim_{\varepsilon\to 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \frac{1-\beta^2}{(1+\beta^2)\Big(\beta^2 - \tan^2\Big(\dfrac{\xi}{2}\Big)\Big)}\, d\beta \\
&= \frac{1}{\cos^2\Big(\dfrac{\xi}{2}\Big)} \lim_{\varepsilon\to 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \Bigg(\frac{-2}{1+\tan^2\Big(\dfrac{\xi}{2}\Big)}\, \frac{1}{1+\beta^2} \\
&\qquad\qquad + \frac{1-\tan^2\Big(\dfrac{\xi}{2}\Big)}{1+\tan^2\Big(\dfrac{\xi}{2}\Big)}\, \frac{1}{\beta^2 - \tan^2\Big(\dfrac{\xi}{2}\Big)}\, d\beta\Bigg)
\end{aligned}
$$

$$= \lim_{\varepsilon\to 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \Bigg(\frac{-2}{1+\beta^2} + \frac{\cos\xi}{\sin\xi}\bigg(\frac{1}{\beta - \tan\left(\frac{\xi}{2}\right)} - \frac{1}{\beta + \tan\left(\frac{\xi}{2}\right)} \bigg)\Bigg)\, d\beta$$

$$= \lim_{\varepsilon\to 0} -\frac{\pi}{2} + \frac{\cos\xi}{\sin\xi} \operatorname{Log}\left|\frac{1-\tan\left(\frac{\xi}{2}\right)}{1+\tan\left(\frac{\xi}{2}\right)}\right| - \frac{\cos\xi}{\sin\xi} \operatorname{Log}\left|\frac{\beta_\varepsilon^+ - \tan\left(\frac{\xi}{2}\right)}{\beta_\varepsilon^+ + \tan\left(\frac{\xi}{2}\right)}\right| + \frac{\cos\xi}{\sin\xi} \operatorname{Log}\left|\frac{\beta_\varepsilon^- - \tan\left(\frac{\xi}{2}\right)}{\beta_\varepsilon^- + \tan\left(\frac{\xi}{2}\right)}\right|$$

$$= -\frac{\pi}{2} + \frac{\cos\xi}{2\sin\xi} \operatorname{Log}\left(\frac{1-\tan\left(\frac{\xi}{2}\right)}{1+\tan\left(\frac{\xi}{2}\right)}\right)^2 + \frac{\cos\xi}{\sin\xi} \lim_{\varepsilon\to 0} \operatorname{Log}\left|\frac{\beta_\varepsilon^- - \tan\left(\frac{\xi}{2}\right)}{\beta_\varepsilon^+ - \tan\left(\frac{\xi}{2}\right)}\right|.$$

Now we have

$$\left(\frac{1-\tan\left(\frac{\xi}{2}\right)}{1+\tan\left(\frac{\xi}{2}\right)}\right)^2 = \frac{\cos^2\left(\frac{\xi}{2}\right) - 2\sin\left(\frac{\xi}{2}\right)\cos\left(\frac{\xi}{2}\right) + \sin^2\left(\frac{\xi}{2}\right)}{\cos^2\left(\frac{\xi}{2}\right) + 2\sin\left(\frac{\xi}{2}\right)\cos\left(\frac{\xi}{2}\right) + \sin^2\left(\frac{\xi}{2}\right)} = \frac{1-\sin\xi}{1+\sin\xi}\,,$$

while we have for $\xi \in (0, \pi/2)$

$$\beta_\varepsilon^- - \tan\left(\frac{\xi}{2}\right) \sim \frac{1}{2}\Big(1+\tan^2\left(\frac{\xi}{2}\right)\Big)(\alpha_\varepsilon^- - \xi) \sim \frac{1}{2}\Big(1+\tan^2\left(\frac{\xi}{2}\right)\Big)\frac{\sqrt{\sin^2\xi - \varepsilon} - \sin\xi}{\cos\xi} \sim \frac{-\varepsilon\Big(1+\tan^2\left(\frac{\xi}{2}\right)\Big)}{4\sin\xi\cos\xi}$$

and

$$\beta_\varepsilon^+ - \tan\Big(\frac{\xi}{2}\Big) \sim \frac{+\varepsilon\Big(1+\tan^2\frac{\xi}{2}\Big)\Big)}{4\sin\xi\cos\xi} \sim -\Big(\beta_\varepsilon^- - \tan\Big(\frac{\xi}{2}\Big)\Big)\,.$$

Thus

$$A(\xi) = -\frac{\pi}{2} + \frac{\cos\xi}{2\sin\xi}\operatorname{Log}\frac{1-\sin\xi}{1+\sin\xi}$$

and Result 11 is proved.

Now, (63) gives

$$\Big|\frac{d}{d\xi}\,\omega(z_{N,1},\dots,z_{N,N})(\xi) - \frac{N}{2\pi}\,\frac{\cos\xi}{\sin\xi}\operatorname{Log}\frac{1-\sin\xi}{1+\sin\xi}\Big| \le C\sqrt{N}\,.$$

Integrating this for $\xi \in [-\pi,\pi]$ we get

$$\Big|\omega(z_{N,1},\dots,z_{N,N})(\xi) - \frac{N}{2\pi}\left(\mathrm{Li}_2(-\sin\xi) - \mathrm{Li}_2(\sin\xi)\right)\Big| \le C\sqrt{N}\,.$$

Since both functions are 2π-periodical, this inequality can be extended to all $\xi \in \mathbb{R}$ and Theorem 2 is proved.

8. Almost linear-phased Daubechies filters.

In this section, we prove Theorem 3. The proof is very easy. Indeed, we want to estimate for $N = 4q$, $\omega(z_{N,1}^{\varepsilon_{N,1}},\dots,z_{N,N}^{\varepsilon_{N,N}})(\xi)$ with $\varepsilon_{N,k} = 1$ if $k = 0 \bmod 4$ or $k = 1 \bmod 4$, and $\varepsilon_{N,k} = -1$ otherwise.

We have (writing ω_N for $\omega(z_{N,1}^{\varepsilon_{N,1}},\dots,z_{N,N}^{\varepsilon_{N,N}})$, K_N for $\{k \in \mathbb{N} : 1 \le k \le N,\ \varepsilon_{N,k} = 1\}$ and $\tilde K_N$ for $\{k \in \mathbb{N} : 1 \le k \le N,\ \varepsilon_{N,k} = -1\}$)

$$\frac{d\omega_N}{d\xi} = \operatorname{Im}\sum_{k\in K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}} + \sum_{k\in\tilde K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\frac{1}{\overline{z}_{N,k}}}$$

(we have used that for $k \in \tilde K_N$, $N+1-k \in \tilde K_N$ and $z_{N,k} = \overline{z}_{N,N+1-k}$). Hence we have

$$\frac{d\omega_N}{d\xi} = \operatorname{Im}\Big(\sum_{k\in K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}} - \sum_{k\in\tilde K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}}\Big)$$
$$+\operatorname{Im}\Big(\sum_{k\in\tilde K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}} + \frac{i\,e^{-i\xi}}{e^{-i\xi}-\dfrac{1}{\overline{z}_{N,k}}}\Big)\,.$$

But we have

$$\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z}+\frac{i\,e^{-i\xi}}{e^{-i\xi}-\dfrac{1}{\overline{Z}}}=\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z}+\frac{i\,\overline{Z}}{\overline{Z}-e^{+i\xi}}$$

$$=\frac{i\,e^{-i\xi}(e^{i\xi}-\overline{Z})+i\,\overline{Z}\,(-e^{-i\xi}+Z)}{|e^{-i\xi}-Z|^2}$$

$$=\frac{i\,(1-2\,\overline{Z}\,e^{-i\xi}+|Z|^2)}{|Z-e^{-i\xi}|^2}$$

$$=i+\frac{i\,(Z\,e^{i\xi}-\overline{Z}\,e^{-i\xi})}{|Z-e^{-i\xi}|^2}\;,$$

hence

$$\operatorname{Im}\left(\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z}+\frac{i\,e^{-i\xi}}{e^{-i\xi}-\dfrac{1}{\overline{Z}}}\right)=1\,.$$

Thus, we have obtained

$$\frac{d\omega_N}{d\xi}=\frac{N}{2}+\operatorname{Im}\sum_{k=1}^{q} i\,e^{-i\xi}\Big(\frac{1}{e^{-i\xi}-z_{N,4k-3}}-\frac{1}{e^{-i\xi}-z_{N,4k-2}}$$

$$-\frac{1}{e^{-i\xi}-z_{N,4k-1}}+\frac{1}{e^{-i\xi}-z_{N,4k}}\Big)\,.$$

Now we write, for $r\in\{1,2,3\}$

$$\frac{1}{e^{-i\xi}-z_{N,4k-r}}=\frac{1}{e^{-i\xi}-z_{N,4k}}+\frac{z_{N,4k-r}-z_{N,4k}}{(e^{-i\xi}-z_{N,4k})\,(e^{-i\xi}-z_{N,4k-r})}$$

$$=\frac{1}{e^{-i\xi}-z_{N,4k}}+\frac{z_{N,4k-r}-z_{N,4k}}{(e^{-i\xi}-z_{N,4k})^2}$$

$$+\frac{(z_{N,4k-r}-z_{N,4k})^2}{(e^{-i\xi}-z_{N,4k})^2(e^{-i\xi}-z_{N,4k-r})}\;.$$

We have, writing $\tilde{k}=\min\{k,q+1-k\}$

$$\left|\frac{(z_{N,4k-r}-z_{N,4k})^2}{(e^{-i\xi}-z_{N,4k})^2(e^{-i\xi}-z_{N,4k-r})}\right|\le C\,\frac{\dfrac{1}{N\tilde{k}}}{\left(\dfrac{\tilde{k}}{N}+\cos^2\xi\right)^{3/2}}\le\frac{C\,\dfrac{1}{\tilde{k}}\sqrt{\dfrac{1}{N\tilde{k}}}}{\dfrac{\tilde{k}}{N}+\cos^2\xi}$$

and

$$\int_{-\pi}^{\pi} \frac{d\xi}{\dfrac{\tilde{k}}{N} + \cos^2 \xi} \leq 4 \int_0^{\arccos \sqrt{\tilde{k}/N}} \frac{d\xi}{\cos^2 \xi} + \frac{4N}{\tilde{k}} \int_{\arccos \sqrt{\tilde{k}/N}}^{\pi/2} d\xi$$

$$= 4\sqrt{\frac{N}{\tilde{k}}} \sin\left(\arccos \sqrt{\frac{\tilde{k}}{N}}\right) + \frac{4N}{\tilde{k}} \arcsin \sqrt{\frac{\tilde{k}}{N}}$$

$$\leq 4\sqrt{\frac{N}{\tilde{k}}} + 2\pi\sqrt{\frac{N}{\tilde{k}}} ,$$

so that

$$\int_{-\pi}^{\pi} \Big| \frac{d\omega_N}{d\xi} - \frac{N}{2} - \mathrm{Im} \sum_{k=1}^{q} \frac{z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}}{(e^{-i\xi} - z_{N,4k})^2} \Big|$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{k^2} = C' < +\infty$$

and

$$\int_{-\pi}^{\pi} \Big| \frac{d\omega_N}{d\xi} - \frac{N}{2} \Big| \, d\xi$$

$$\leq C' + C \sum_{k=1}^{q} \sqrt{\frac{N}{\tilde{k}}} \, |z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}| \, .$$

When $\tilde{k} \leq k_0$, we write

$$|z_{N,4k-r} - z_{N,4k+1-r}| = O\Big(\frac{1}{\sqrt{N\tilde{k}}}\Big)$$

and obtain

$$\sum_{\tilde{k} \leq k_0} \sqrt{\frac{N}{\tilde{k}}} \, |z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}| \leq C \, \mathrm{Log} \, k_0 \ .$$

When $\tilde{k} \geq k_0$, we may write as in formula (58)

$$z_{N,4k-r} = y_{N,4k-r} + \sqrt{y_{N,4k-r}^2 - 1} + O\Big(\frac{\mathrm{Log}\, \tilde{k}}{\tilde{k}\sqrt{N\tilde{k}}}\Big)$$

$$= \sqrt{\omega_{N,4k-r}} + \sqrt{\omega_{N,4k-r} + 1} + O\Big(\frac{\mathrm{Log}\, \tilde{k}}{\tilde{k}\sqrt{N\tilde{k}}}\Big) ,$$

where

$$\omega_{N,\ell} = -e^{-2i\pi(8\ell-1)/(8N+6)}$$
$$- \frac{1}{N}\, e^{-2i\pi(8\ell-1)/(8N+6)} \operatorname{Log}\left(2\sqrt{2N\pi \sin\left(\frac{8\ell-1}{8N+6}\,\pi\right)}\,\right).$$

We write

$$\sqrt{\alpha+\beta} = \sqrt{\alpha} + \frac{\beta}{\sqrt{\alpha}+\sqrt{\alpha+\beta}} = \sqrt{\alpha} + \frac{\beta}{2\sqrt{\alpha}} - \frac{\beta^2}{2\sqrt{\alpha}\,(\sqrt{\alpha}+\sqrt{\alpha+\beta})^2}\ .$$

Now, we have: $\omega_{N,\ell}$ is order of magnitude 1, $\omega_{N,\ell}+1$ is of order of magnitude $\min\{\sqrt{\ell/N}, \sqrt{(N+1-\ell)/N}\}$ and $\omega_{N,\ell+1}-\omega_{N,\ell}$ is of order of magnitude $1/N$. Thus, we may write

$$\sqrt{\omega_{N,4k-r}} = \sqrt{\omega_{N,4k}} + O\Big(\frac{1}{N}\Big)$$

$$\sqrt{1+\omega_{N,4k-r}} = \sqrt{1+\omega_{N,4k}} + \frac{\omega_{N,4k-r}-\omega_{N,4k}}{2\sqrt{1+\omega_{N,4k}}} + O\Big(\frac{1}{\tilde{k}\sqrt{N\tilde{k}}}\Big)$$
$$= \sqrt{1+\omega_{N,4k}} + \frac{e^{-2i\pi(32k-1)/(8N+6)}\big(1-e^{2i8r\pi/(8N+6)}\big)}{2\sqrt{1+\omega_{N,4k}}}$$
$$+ O\Big(\frac{\operatorname{Log}\tilde{k}}{N^2}\Big) + O\Big(\frac{1}{\tilde{k}\sqrt{N\tilde{k}}}\Big)$$

and finally

$$\frac{\sqrt{N}}{\tilde{k}}\,|z_{N,4k-3}-z_{N,4k-2}-z_{N,4k-1}+z_{N,4k}|$$
$$= \sqrt{\frac{N}{\tilde{k}}}\,\Big|\frac{e^{2i24\pi/(8N+6)}-e^{2i16\pi/(8N+6)}-e^{2i8\pi/(8N+6)}+1}{2\sqrt{1+\omega_{N,4k}}}\Big|$$
$$+ O\Big(\frac{\operatorname{Log}\tilde{k}}{\tilde{k}^2}\Big) + O\Big(\frac{1}{\sqrt{N\tilde{k}}}\Big) + O\Big(\frac{\operatorname{Log}\tilde{k}}{N\sqrt{N\tilde{k}}}\Big)$$
$$= O\Big(\frac{1}{N\tilde{k}}\Big) + O\Big(\frac{\operatorname{Log}\tilde{k}}{\tilde{k}^2}\Big) + O\Big(\frac{1}{\sqrt{N\tilde{k}}}\Big) + O\Big(\frac{\operatorname{Log}\tilde{k}}{N\sqrt{N\tilde{k}}}\Big).$$

We thus have proved Theorem 3, since

$$\sum_1^N \frac{1}{N\tilde{k}} \leq C\,\frac{\operatorname{Log} N}{N} = o\,(1)\,,$$

$$\sum_1^N \frac{1}{\sqrt{N\tilde{k}}} \leq C\,\frac{\sqrt{N}}{\sqrt{N}} = C < +\infty\,,$$

$$\sum_1^\infty \frac{\operatorname{Log}\tilde{k}}{\tilde{k}^2} < +\infty\,,$$

$$\sum_1^N \frac{\operatorname{Log}\tilde{k}}{N\sqrt{N\tilde{k}}} \leq C\,\frac{1}{N\sqrt{N}}\,\sqrt{N}\,\operatorname{Log} N = o\,(1)\,.$$

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