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L^p-bounds for spherical maximal operators on $\mathbb{Z}^n$

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Abstract. We prove analogue statements of the spherical maximal theorem of E. M. Stein, for the lattice points $\mathbb{Z}^n$. We decompose the discrete spherical "measures" as an integral of Gaussian kernels $s_{t,\varepsilon}(x) = e^{2\pi i |x|^2(t+i\varepsilon)}$. By using Minkowski's integral inequality it is enough to prove L^p-bounds for the corresponding convolution operators. The proof is then based on L^2-estimates by analysing the Fourier transforms $\hat{s}_{t,\varepsilon}(\xi)$, which can be handled by making use of the "circle" method for exponential sums. As a corollary one obtains some regularity of the distribution of lattice points on small spherical caps.

1. Introduction.

Let us denote by σ_λ the characteristic function of the sphere of radius $\lambda^{1/2}$ in $\mathbb{Z}^n$, *i.e.*

$$\sigma_\lambda = \chi_{\{x\in\mathbb{Z}^n : |x|^2=\lambda\}} \qquad \text{and} \qquad S_\lambda = \sum_{x\in\mathbb{Z}^n} \sigma_\lambda(x)\,.$$

Let Λ be a fixed positive number and define the spherical maximal operator as

$$M_\Lambda f(x) = \sup_{\Lambda\le\lambda<2\Lambda} \Big|\Big(\frac{\sigma_\lambda}{S_\lambda} * f\Big)(x)\Big|\,. \tag{1}$$

It is proved that

Theorem 1. *If* $n \geq 5$*,* $p > n/(n-2)$*,* $f \in L^p(\mathbb{Z}^n)$*, then*

$$\|M_\Lambda f\|_{L^p(\mathbb{Z}^n)} \leq c_{n,p}\, \|f\|_{L^p(\mathbb{Z}^n)}\ , \tag{2}$$

where the constant $c_{n,p}$ *is independent of* Λ.

We generalize estimate (2) to the case of the k-spheres, which are defined by

$$\sigma_\lambda = \chi_{\{x \in \mathbb{Z}^n : |x|^2 = \lambda\}}$$

and it is proved.

Theorem 2. *Let* $k \geq 2$*,* $K = 2^{k-1}$*, then for* $n > 4\,K\,k$*,* $p > n/(n - 4Kk)$ *we have*

$$\|M_{\Lambda,k} f\|_p \leq c_{n,k,p}\, \|f\|_p\ , \tag{3}$$

where the constant $c_{n,k,p}$ *is independent of* Λ.

It is well-known, that for $n \geq 5$, there exist constants $0 < c_n < C_n$ such that

$$c_n\, \lambda^{n/2-1} \leq S_\lambda \leq C_n\, \lambda^{n/2-1}\,. \tag{4}$$

We start with the decompositions

$$\sigma_\lambda(x) = \int_0^1 e^{2\pi i (|x|^2 - \lambda) t}\, dt$$

and

$$e^{-2\pi\varepsilon\lambda}\, \sigma_\lambda(x) = \int_0^1 e^{2\pi i |x|^2 (t + i\varepsilon)}\, e^{-2\pi i \lambda t}\, dt \tag{5}$$

and define the modified maximal operator as

$$\tilde{M}_{\lambda,\varepsilon} f(x) = \sup_{\Lambda \leq \lambda < 2\Lambda} |(e^{-2\pi\varepsilon\lambda} \lambda^{-n/2+1} \sigma_\lambda) * f(x)|\,. \tag{6}$$

From inequality (4) it follows, that if $\Lambda \leq \varepsilon^{-1}$ and $f \geq 0$, we have

$$M_\Lambda f(x) \leq c\, \tilde{M}_{\Lambda.\varepsilon} f(x)\,, \tag{8}$$

for every x, so it is enough to prove (2) for the modified maximal operator. Introducing the convolution operator

$$S_{t,\varepsilon} f = s_{t,\varepsilon} * f \,, \qquad \text{where} \qquad s_{t,\varepsilon}(x) = e^{2\pi i\,|x|^2 (t+i\varepsilon)} \,.$$

Minkowski's integral inequality together with formulae (4) and (5) imply

$$\|\tilde{M}_{\Lambda,\varepsilon} f\|_p \leq c_n\, \Lambda^{-n/2+1} \int_0^1 \|S_{t,\varepsilon} f\|_p \, dt \,. \tag{9}$$

In order to understand the integrand on the right side of inequality (9), we will apply the so-called "circle" method in the variable t (*cf.* [3]). First we decompose the interval $[0,1]$ into neighborhoods of rationals, whose denominator is smaller than a given number N as follows:

Let $N > 0$ be given and consider the set

$$H = \{p/q \,:\, 1 \leq q \leq N,\, 0 < p \leq q,\, (p,q) = 1\} \,,$$

and define the neighborhoods

$$V_{p,q} = \Big\{ t \in [0,1] \,:\, \Big| t - \frac{p}{q} \Big| = \min_{r \in H} |t - r| \Big\} \,.$$

From the obvious inequalities:

if $p/q \neq p_1/q_1$ then

$$\Big| \frac{p}{q} - \frac{p_1}{q_1} \Big| \geq \frac{1}{2Nq} + \frac{1}{2Nq_1} \,. \tag{I}$$

For every $t \in [0,1]$ there exists $p/q \in H$ such that,

$$\Big| t - \frac{p}{q} \Big| \leq \frac{1}{Nq} \,, \tag{II}$$

it follows

$$W^*_{p,q} \subseteq V_{p,q} \subseteq W_{p,q} \,,$$

where $W^*_{p,q} = \{t : |t - p/q| < 1/(2Nq)\}$ $W_{p,q} = \{t : |t - p/q| \leq 1/(Nq)\}$.

The crucial point is that one can estimate the Fourier transform $\hat{s}_{t,\varepsilon}(\xi)$ separately in each neighborhood $V_{p,q}$ by using Poisson summation and the properties of Gaussian sums, as it is shown below.

2. Fourier transform estimates.

Lemma 1. *Let $t = p/q + \tau$, $t \in V_{p,q}$, then*

$$\|S_t f\|_{L^2} \le c_n\, q^{-n/2} \min\{\varepsilon^{-n/2}, \tau^{-n/2}\}\, \|f\|_2\,. \tag{10}$$

PROOF. The Fourier transform of the function $s_t = s_{t,\varepsilon}$ is defined by

$$\hat{s}_t(\xi) = \sum e^{2\pi i(|x|^2(t+i\varepsilon)+x\xi)}\,, \qquad \xi \in \Pi^n$$

and inequality (10) is equivalent to

$$\sup_\xi |\hat{s}_t(\xi)| \le c_n\, q^{-n/2} \min\{\varepsilon^{-n/2}, \tau^{-n/2}\}\,. \tag{11}$$

Since $\hat{s}_t(\xi)$ is the product of n one dimensional functions, *i.e.* $\hat{s}_t(\xi) = \Pi_j \hat{s}_t(\xi_j)$ it is enough to prove formula (11) in case when $n = 1$. By Poisson summation and substituting $x = rq + s$, we have

$$\begin{aligned}
\hat{s}_t(\xi) &= \sum_x e^{2\pi i x^2 p/q}\, s_\tau(x) e^{2\pi i x\xi} \\
&= \sum_{s=0}^{q-1} e^{2\pi i s^2 p/q} \sum_r s_\tau(rq+s)\, e^{2\pi i(rq+s)\xi} \\
&= \sum_{s=0}^{q-1} e^{2\pi i s^2 p/q} \sum_l \frac{1}{q}\, e^{-2\pi i s/q}\, \tilde{s}_\tau\Big(\frac{l}{q} - \xi\Big)\,,
\end{aligned}$$

where $\tilde{s}_\tau(\xi) = \int_{\mathbb{R}} s_\tau(x)\, e^{-2\pi i x\xi}\, dx$ is simply the Fourier transform of s_τ as function on $\mathbb{R}$, which has the simple form

$$\tilde{s}_\tau(\xi) = \int_{\mathbb{R}} e^{2\pi i(x^2(\tau+i\varepsilon)-x\xi)}\, d\xi = (\varepsilon - i\tau)^{-1/2}\, e^{-\xi^2/(\varepsilon - i\tau)}\,.$$

So we have the formula

$$\begin{aligned}
\hat{s}_t(\xi) &= (\varepsilon - i\tau)^{-1/2} \\
&\quad \cdot \sum_l \Big(\frac{1}{q} \sum_{s=0}^{q-1} e^{2\pi i(p/qs^2 - l/qs)}\Big) e^{-\pi(\xi - l/q)^2/(2(\varepsilon - i\tau))}\,.
\end{aligned} \tag{12}$$

In order to estimate this expression, first we note that because of the properties of Gaussian sums one has

$$\Big|\frac{1}{q}\sum_{s=0}^{q-1} e^{2\pi i(s^2p/q\,-\,sl/q)}\Big| \le \sqrt{2}\,q^{-1/2}\,.$$

Now we choose $\varepsilon=\Lambda^{-1}$, $N=[\Lambda^{1/2}]$ (where $[x]$ denotes the integer part of x), and since $t=p/q+\tau\in V_{p,q}$ we have $\tau\le 1/(Nq)\le \varepsilon^{1/2}q^{-1}$. It follows

$$\frac{\varepsilon}{q^2(\varepsilon^2+\tau^2)} \ge \frac{1}{2\,q^2\varepsilon} \ge \frac{1}{2}\,, \qquad \text{if } \tau\le\varepsilon$$

and

$$\frac{\varepsilon}{q^2(\varepsilon^2+\tau^2)} \ge \frac{\varepsilon}{2\,q^2\tau^2} = \frac{1}{2}\,(\varepsilon^{1/2}q^{-1}\tau^{-1})^2 \ge \frac{1}{2}\,, \qquad \text{if } \varepsilon\le\tau\,.$$

Now it is easy to estimate the right hand side of formula (12)

$$\begin{aligned}
|\hat{s}_t(\xi)| &\le |\varepsilon-i\tau|^{-1/2}q^{-1/2}\sum_l e^{-\pi\varepsilon/(2(q\xi-l)^2(q^2(\varepsilon^2+\tau^2)))}\\
&\le c\,q^{1/2}(\varepsilon+\tau)^{-1/2}\Big(\sum_l e^{-\pi/4(q\xi-l)^l}\Big)\\
&\le c\,q^{-1/2}(\varepsilon+\tau)^{-1/2}\,,
\end{aligned}$$

where the constant c is independent of q and ξ.

This proves inequality (10) and Lemma 1 follows.

PROOF OF THEOREM 1. It is easy to see that

$$\|S_tf\|_1 \le \|S_t\|_1\,\|f\|_1 \le c_n\,\varepsilon^{-n/2}\,\|f\|_1\,. \tag{13}$$

Let $1<p\le 2$ and we choose the number α such that $1/p=\alpha/2+(1-\alpha)$. Interpolating between estimates (10) and (13), we have

$$\|S_tf\|_p \le c_n\,q^{-n\alpha/2}\varepsilon^{-n/2}\min\Big\{1,\Big(\frac{\tau}{\varepsilon}\Big)^{-n\alpha/2}\Big\}\|f\|_p\,.$$

This implies

$$\begin{aligned}
\int_{V_{p,q}}\|S_tf\|_p &\le c_n\,\Lambda^{-n/2+1}q^{-n\alpha/2}\\
&\qquad\cdot\Big(\int_0^{\varepsilon}\varepsilon^{-n/2}\,d\tau+\varepsilon^{-n/2}\int_{\varepsilon}^{\infty}\Big(\frac{\tau}{\varepsilon}\Big)^{-n\alpha/2}d\tau\Big)\\
&\le c_n\,q^{-n\alpha/2}(\varepsilon\,\Lambda)^{-n/2+1}\\
&\le c_n'\,q^{-n\alpha/2}\|f\|_p\,.
\end{aligned}$$

It follows when $n > 4$, $\alpha > 4/n$ or equivalently $p > n/(n-2)$

$$\begin{aligned}\|\tilde{M}_{\Lambda,\varepsilon} f\|_p &\leq c_n \Big(\sum_{p/q \in H} q^{-n\alpha/2} \Big) \|f\|_p \\ &\leq c_n \|f\|_p \Big(\sum_{q=1}^{\infty} q^{-n\alpha/2+1} \Big) \\ &\leq c_n \|f\|_p \,.\end{aligned}$$

This proves Theorem 1.

3. Estimates for k-spheres.

We now briefly describe how the L^2 estimate generalize to k-spheres. The extra complications arise are similar to those of the Waring problem. Indeed we refer to the analysis of Hardy-Littlewood in [3], where it was proved that for $n > 2^{k-1}k$

$$c_{n,k}\, \lambda^{n/k-1} \leq S_{\lambda,k} \leq C_{n,k}\, \lambda^{n/k-1}\,, \tag{14}$$

hence as for $k = 2$ one has

$$\|M_{\Lambda,k} f\|_p \leq c_{n,k}\, \Lambda^{-n/k+1} \int_0^1 \|S_t f\|_p \, dt\,,$$

where the kernel of the operator S_t is $s_t(x) = e^{2\pi i (\sum_j |x_j|^k)(t+i\varepsilon)}$.

For $t = p/q + \tau$ Poisson summation yields

$$\hat{s}_t(\xi) = \sum_{l=-\infty}^{\infty} \Big(\frac{1}{q} \sum_{s=0}^{q-1} e^{2\pi i (s^k p/q - sl/q)} \Big) \tilde{s}_\tau \Big(\frac{l}{q} - \xi \Big)\,,$$

where $\tilde{s}_\tau(\eta) = \int_{\mathbb{R}} s_\tau(x)\, e^{-2\pi i x\eta}\, dx$ is the Fourier transform on $\mathbb{R}$.

The decomposition into neighborhoods of rationals for $k > 2$ looks as follows

$$H_{k,0} = \Big\{ \frac{p}{q} \,:\, q \leq \Lambda^{1/k} \Big\}\,, \qquad H_{k,1} = \Big\{ \frac{p}{q} \,:\, \Lambda^{1/k} < q \leq \Lambda^{1-1/k} \Big\}\,.$$

$V_{p,q}$ is called a major arc if $p/q \in H_{k,0}$ and a minor arc if $p/q \in H_{k,1}$.

The reasoning of Theorem 1 generalizes to the major arcs as is shown in

Lemma 2. *Let $p/q \in H_{k,0}$, $t \in V_{p,q}$. Then we have for $p > n/(n-k+1)$*

$$\Lambda^{-n/k+1} \int_{V_{p,q}} \|S_t f\|_p \, dt \le c_{n,k,p}\, q^{-n\alpha/K} \|f\|_p \,, \tag{15}$$

where $\alpha = 2\,p/(p-1)$.

PROOF. We make use of the following estimates which are proved in [3] using slightly different notations.

$$|\tilde{s}_\tau(\eta)| \le c\,|\tau + i\,\varepsilon|^{-1/k} \tag{16}$$

holds uniformly in η. Let $\eta = l - q\,\xi$ then one has

$$\begin{aligned} \Big|\tilde{s}_\tau\Big(\xi - \frac{l}{q}\Big)\Big| &\le c_k\, |\tau + i\,\varepsilon|^{-1/(2(k-1))}\, q^{(k-2)/(2(k-1))} \\ &\qquad \cdot |\eta|^{-(k-2)/(2(k-1))}\, e^{-(c|\eta|^{k/(k-1)})} \,. \end{aligned} \tag{17}$$

From inequalities (16) and (17) it follows

$$\sum_{l=-\infty}^{\infty} \Big|\tilde{s}_\tau\Big(\xi - \frac{l}{q}\Big)\Big| \le c_k\, \big(|\tau + i\,\varepsilon|^{-1/k} + |\tau + i\,\varepsilon|^{-1/(2(k-1))}\, q^{(k-2)/(2(k-1))}\big)\,,$$

where inequality (16) is used when $|l - q\,\xi|$ is minimal. Also one has the standard estimate for the Weyl sum

$$\Big|q^{-1} \sum_{s=0}^{q-1} e^{2\pi i (s^k p/q - sl/q)}\Big| \le c\, q^{-1/K}\,,$$

which holds uniformly in l, when $K = 2^{k-1}$. Taking the n-th power of $\hat{s}_t$, we obtain (in n-dimension) on the major arcs

$$\begin{aligned} &\sup_\xi |\hat{s}_t(\xi)| \\ &\le c_{n,k}\, q^{-n/K} \big(|\tau + i\,\varepsilon|^{-n/k} + |\tau + i\,\varepsilon|^{-n/(2(k-1))} q^{n(k-2)/(2(k-1))}\big)\,. \end{aligned}$$

Let $1/p = \alpha/2 + 1 - \alpha$ and using the trivial estimate

$$\|s_t\|_1 \le \Big(\sum_{x \in Z} e^{-2\pi\varepsilon |x|^k}\Big)^n \le c_n\, \varepsilon^{-n/k}\,,$$

we obtain by interpolation

$$\|S_t\|_{p\to p} \le c_{n,k,p}\, q^{-n\alpha/K}\, \varepsilon^{-n/k} \cdot \Big(\Big|i+\frac{\tau}{\varepsilon}\Big|^{-n/k} + \Big|i+\frac{\tau}{\varepsilon}\Big|^{-n/(2(k-1))} (\varepsilon^{1/k}q)^{n(k-2)/(2(k-1))}\Big)^{\alpha}.$$

Using the facts that on a major arc $\varepsilon^{1/k}q = \Lambda^{-1/k}q \le 1$ and the simple estimate

$$\int_{\mathbb{R}} \Big|i+\frac{\tau}{\varepsilon}\Big|^{-\beta} d\tau \le c_\beta\, \varepsilon\,, \qquad \text{for } \beta > 1\,.$$

Estimate (15) follows when $n\alpha/(2(k-1)) > 1$. This proves Lemma 2.

On the minor arcs one can give direct estimates for $\hat{s}_t(\xi)$ exploiting that the denominator q is large

Lemma 3. *Let $p/q \in H_{k,1}$, $t \in V_{p,q}$ then we have*

$$\sup_{\xi} \hat{s}_{t,\varepsilon}(\xi) \le c_{n,k}\, \Lambda^{n/k - n/(4kK)}\,. \tag{18}$$

PROOF. It is enough to prove (18) in one dimension. Let $L = \Lambda^{1/k+\delta}$, for some $\delta > 0$, then one has

$$\hat{s}_t(\xi) = \sum_{x=0}^{L} e^{2\pi i(x^k(t+i\varepsilon)+x\xi)} + O\Big(\sum_{x>L} e^{-\varepsilon x^k}\Big) \tag{19}$$

and

$$\sum_{x>L} e^{-\varepsilon x^k} \le e^{-\varepsilon\Lambda^{1+\delta}}\varepsilon^{-1/k} \le e^{-\Lambda^{\delta}}\Lambda^{1/k} = O(1)\,.$$

To estimate the maim term of formula (19) we use partial summation. Let us define the sums $s_{l,\xi} = \sum_{x=0}^{l} e^{2\pi i(x^k p/q + x\xi)}$, we have

$$\begin{aligned} &\sum_{l=0}^{L}(s_{l,\xi} - s_{l-1,\xi})\, e^{2\pi i l^k(\tau+i\varepsilon)} \\ &\qquad = \sum_{l=0}^{L} s_{l,\xi}\,\big(e^{2\pi i l^k(\tau+i\varepsilon)} - e^{2\pi i(l-1)^k(\tau+i\varepsilon)}\big)\,. \end{aligned} \tag{20}$$

Since on the minor arcs $\tau \le \Lambda^{1/k-1}q^{-1} \le \Lambda^{-1}$, it follows

$$|e^{2\pi i(\tau+i\varepsilon)l^k} - e^{2\pi i(\tau+i\varepsilon)l^{k-1}}| \le c_k\, \Lambda^{-1/k+k\delta}\,,$$

so the sum in the formula (19) is less than equal

$$|\hat{s}_t(\xi)| \le c_{k,\delta}\big(\max_{l\le L}|s_{l,\xi}|\big)\Lambda^{(k+1)\delta}\,.$$

Using the standard estimate for Weyl sums (*cf.* [2, Chapter 6]), one has

$$\Big|\sum_{x=0}^{l} e^{2\pi(x^k p/q+x\xi)}\Big| \le c_{k,\delta}\,\Lambda^{(1/k)(1-1/(2K))+2k\delta} \tag{21}$$

holds uniformly in ξ and p, when $\Lambda^{1/k}\le q\le \Lambda^{1-1/k}$, $l\le \Lambda^{1/k+\delta}$.

The above estimates imply for $\delta\le\delta(k)$ the estimate

$$|\hat{s}_{t,\varepsilon}(\xi)| \le c_{k,\delta}\,\Lambda^{1/k-1/(2kK)+4k\delta} \le c_k\,\Lambda^{1/k-1/(4kK)} \le c_{n,k} \tag{22}$$

holds uniformly in ξ , and Lemma 3 follows.

PROOF OF THEOREM 2. Interpolation between the trivial L^1, and the L^2 estimate (18), shows that for $1/p = 1-\alpha/2$ on a minor arc we have

$$\Lambda^{-n/k+1}\|S_t\|_{p\to p} \le c_{n,k,p}\,\Lambda^{n/k-n\alpha/(4Kk)}\,.$$

Hence for $n>4Kk$, $p>n/(n-4Kk)$ one has

$$\begin{aligned}\|M_{\Lambda,k}f\|_p &\le c_{n,k,p}\Big(\sum_{p/q\in H_{0,k}} q^{-n\alpha/K} + \Lambda^{-n/k+1+n/k-n\alpha/(4kK)}\Big)\|f\|_p\\ &\le c'_{n,k,p}\,\|f\|_p\ ,\end{aligned}$$

since $n\alpha/K>2$ and $n\alpha/(4Kk)>1$. This proves Theorem 2.

We would like to point out how these estimates are connected with the distribution of integer points on spherical caps. More precisely we define the maximal function

$$s^*_{\lambda,l} = \sup_{\lambda\le\mu<2\lambda} |S^{n-1}_\mu \cap (x+D^n_l)|\,,$$

where $x+D^n_l = \{u\in\mathbb{Z}^n : |x-u|\le l^{1/2}\}$. On the other hand the average number of integer points on a spherical cap of radius $l^{1/2}$ lying on the sphere of radius $\lambda^{1/2}$ is clearly

$$\tilde{S}_{\lambda,l} = l^{(n-1)/2}\lambda^{-1/2}\,.$$

Theorem 1. implies the following

Corollary 1. *Let $\varepsilon > 0$. If $l > \lambda^{1-\varepsilon/2}$ then*

$$\lambda^{-n/2}|\{x \in D_\lambda^n \,:\, s^*_{\lambda,l}(x) > \lambda^\varepsilon\, \tilde{s}_{\lambda,l}\}| < c_{n,\varepsilon}\, \lambda^{-\varepsilon n/(2(n-2))}\,. \tag{23}$$

Proof. Let $\lambda > 0$, $f(x) = |x|^{-n+2}$. First we estimate $M_\lambda f(x)$ from below as follows

$$\begin{aligned}
&M_\lambda f(x)\\
&\quad\geq c_n\, \lambda^{-n/2+1} \sup_{\lambda\leq\mu<2\lambda} \sum_{l=1}^{\infty} l^{-n/2+1}|\{y \in \mathbb{Z}^n \,:\, |x-y|^2 = l,\ |y|^2 = \mu\}|\\
&\quad\geq c_n\, \lambda^{-n/2+1}\\
&\qquad\cdot \sup_{\lambda\leq\mu<2\lambda} \sum_{L \text{ dyadic}} L^{-n/2+1}|\{y \in \mathbb{Z}^n \,:\, L \leq |x-y|^2 < 2L,\ |y|^2 = \mu\}|\\
&\quad\geq c_n\, \lambda^{-n/2+1}\\
&\qquad\cdot \sup_{\lambda\leq\mu<2\lambda} \sum_{L \text{ dyadic}} L^{-n/2+1}|\{y \in Z^n \,:\, |x-y|^2 \leq L,\ |y|^2 = \mu\}|\,,
\end{aligned}$$

where the last inequality was obtained by partial summation. This immediately implies

$$M_\lambda f(x) \geq c_n\, \lambda^{-n/2+1}\, \frac{s^*_{\lambda,L}(x)}{\tilde{s}_{\lambda,L}}\, \frac{L^{1/2}}{\lambda^{1/2}}\,, \tag{24}$$

for every dyadic value $L = 2^l$, but it remains true for every integer l since the function $s^*_{\lambda,l}(x)$ is monotone increasing in l.

Choosing $p = n/(n-2) + \eta$, it follows for $l > \lambda^{1-\varepsilon/2}$

$$\begin{aligned}
\|M_\lambda f\|_p^p &\geq \lambda^{-\eta(n/2-1)}\lambda^{-n/2} \sum_{x\in D_\lambda^n} \Big(\frac{s^*_{\lambda,l}(x)}{\tilde{s}_{\lambda,l}}\, \frac{l^{1/2}}{\lambda^{1/2}}\Big)^{n/(n-2)+\eta}\\
&\geq \lambda^{3\varepsilon n/4(n-2)-\eta n/2}\lambda^{-n/2}|\{x \in D_\lambda^n \,:\, s^*_{\lambda,l} > \lambda^\varepsilon \tilde{s}_{\lambda,l}\}|\,.
\end{aligned}$$

Choosing η small enough estimate (23) follows immediately, since $f \in L^p(\mathbb{Z}^n)$ and the maximal operator M_λ is bounded in $L^p(\mathbb{Z}^n)$ by Theorem 1.

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