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Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt's (A_p) condition

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Abstract. We describe the complete interpolating sequences for the Paley-Wiener spaces L^p_π $(1 < p < \infty)$ in terms of Muckenhoupt's (A_p) condition. For $p = 2$, this description coincides with those given by Pavlov [9], Nikol'skii [8], and Minkin [7] of the unconditional bases of complex exponentials in $L^2(-\pi, \pi)$. While the techniques of these authors are linked to the Hilbert space geometry of L^2_π, our method of proof is based on turning the problem into one about boundedness of the Hilbert transform in certain weighted L^p spaces of functions and sequences.

1. Introduction.

In this paper we study interpolation in the Paley-Wiener spaces L^p_π $(1 < p < \infty)$, which consist of all entire functions of exponential type at most π whose restrictions to the real line are in L^p. The Paley-Wiener spaces are Banach spaces when endowed with the natural $L^p(\mathbb{R})$-norms. We want to describe those sequences $\Lambda = \{\lambda_k\}$, $\lambda_k = \xi_k + i\,\eta_k$, in the complex plane $\mathbb{C}$ for which the interpolation problem

$$f(\lambda_k) = a_k \,, \tag{1}$$

has a unique solution $f \in L^p_\pi$ for every sequence $\{a_k\}$ satisfying

$$\sum_k |a_k|^p e^{-p\pi|\eta_k|}\,(1+|\eta_k|) < \infty\,. \tag{2}$$

Such sequences Λ are termed *complete interpolating sequences for* L^p_π. A classical example of a complete interpolating sequence for L^p_π ($1 < p < \infty$) is the sequence of integers $\mathbb{Z}$.

In the case $p = 2$ this problem is equivalent to that of describing all unconditional bases in $L^2(-\pi,\pi)$ of the form $\{\exp(i\,\lambda_k t)\}$. We refer to [4] for an account of this problem, including a detailed survey of its history. The unconditional basis problem was solved by Pavlov [9] under the additional restriction $\sup|\operatorname{Im}\lambda_k| < \infty$ and by Nikol'skii [8], assuming only $\inf \operatorname{Im}\lambda_k > -\infty$. Finally, Minkin [7] solved the problem without any a priori assumption on Λ.

The methods of [4], [7], [8], [9] are of a geometric nature and make crucial use of the Hilbert space structure of L^2_π. In this paper, we shall give a simpler proof, which works equally well for all p, $1 < p < \infty$. Incidentally, our method of proof shows that for $p = \infty$ or $0 < p \le 1$ there are no complete interpolating sequences. (See also [2], which "explains" this curious phenomenon). The core of our approach is a careful study of properties of the Hilbert transform in weighted spaces of functions and its discrete version in weighted spaces of sequences. More precisely, we turn our problem into one about boundedness of the discrete Hilbert transform in a weighted space, defined on a subsequence of Λ located in a horizontal strip, where the weight is expressed in terms of certain infinite products involving *all* the points of Λ.

As an application of our main theorem, we prove a counterpart of the well-known Kadets 1/4 theorem.

2. Preliminary observations and statement of the main result.

Suppose that Λ is a complete interpolating sequence for L^p_π. By a classical theorem of Plancherel and Pólya (see [6, Lecture 7, Theorem 4]),

$$\int_{-\infty}^{\infty} |f(x+i\,a)|^p\,dx \le e^{p\pi|a|}\|f\|^p_{L^p} \tag{3}$$

for every function $f \in L^p_\pi$ and each $a \in \mathbb{R}$, and so $\exp(i\,\pi\,z)\,f(z)$ belongs to the Hardy space H^p of $\mathbb{C}^+_a := \{z \in \mathbb{C} : \operatorname{Im} z > a\}$ for each

$a \in \mathbb{R}$. Hence the sequence $\Lambda \cap \mathbb{C}_a^+$ is H^p-interpolating in $\mathbb{C}_a^+$ (see [5, Chapter 9]). Similarly, $\Lambda \cap \mathbb{C}_a^-$ is H^p-interpolating in the half-plane $\mathbb{C}_a^- := \{z \in \mathbb{C} : \operatorname{Im} z < a\}$. So the sequences $\Lambda \cap \mathbb{C}_a^+$ and $\Lambda \cap \mathbb{C}_a^-$ satisfy the Carleson condition in the corresponding half-planes, *i.e.*,

$$\tag{4} \begin{gathered} \inf_{\operatorname{Im}\lambda_j > a} \prod_{\substack{\operatorname{Im}\lambda_k > a \\ k \neq j}} \Big| \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a} \Big| > 0\,, \\ \inf_{\operatorname{Im}\lambda_j < a} \prod_{\substack{\operatorname{Im}\lambda_k < a \\ k \neq j}} \Big| \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a} \Big| > 0\,. \end{gathered}$$

As a side remark, we mention that this condition may be expressed in different ways. For instance, by manipulating the Carleson condition in much the same way as in [1, p. 288-290] (we omit the details), we obtain the following equivalent condition

$$\tag{5} \sup_j \sum_{\substack{k \\ k \neq j}} \frac{(1 + |\eta_j|)\,(1 + |\eta_k|)}{|\lambda_j - \lambda_k|^2} < \infty\,.$$

Trivially, the inequalities in (4) imply that for each $a \in \mathbb{R}$

$$\begin{gathered} \inf_{\substack{\operatorname{Im}\lambda_j > a \\ \operatorname{Im}\lambda_k > a \\ k \neq j}} \Big| \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a} \Big| > 0\,, \\ \inf_{\substack{\operatorname{Im}\lambda_j < a \\ \operatorname{Im}\lambda_k < a \\ k \neq j}} \Big| \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a} \Big| > 0\,. \end{gathered}$$

Choosing respectively $a = -1$ and $a = 1$ in these two inequalities, we deduce that for some $\varepsilon > 0$ the disks

$$K(\lambda_k) := \{z : \ |z - \lambda_k| < 10\,\varepsilon\,(1 + |\eta_k|)\}$$

are pairwise disjoint. (We fix this value of ε until the end of the paper.)

Moreover, (4) implies that the measure

$$\mu_\Lambda^+ := \sum_{\eta_k > 0} \eta_k\,\delta_{\lambda_k}$$

(δ_λ is the unit point measure at λ) is a Carleson measure, *i.e.*,

$$\int_{\mathbb{C}^+} |f|^s \, d\mu_\Lambda^+ \leq C \, \|f\|_{H^s}^s$$

for each function f in the Hardy space $H^s(\mathbb{C}^+)$, $s \geq 1$ (see [1, p. 63]). Similarly, Λ generates a Carleson measure in the lower half-plane as well as in each of the half-planes $\mathbb{C}_a^\pm$.

If Λ is a complete interpolating sequence for L_π^p, then

$$\|f\|_{L^p(\mathbb{R})} \leq C\Big(\sum_k |f(\lambda_k)|^p \, e^{-p\pi|\eta_k|} \, (1+|\eta_k|)\Big)^{1/p}, \qquad f \in L_\pi^p \,. \tag{6}$$

Indeed, since the interpolation problem (1) has a solution $f \in L_\pi^p$ whenever (2) holds, the operator

$$T : f \longmapsto \{f(\lambda_k) \, e^{-\pi|\eta_k|} \, (1+|\eta_k|)^{1/p}\}$$

is bounded from L_π^p onto l^p. By the uniqueness of the solution of the interpolation problem, we have $\ker T = \{0\}$, and it suffices to apply the Banach theorem on inverse operators.

Given $x \in \mathbb{R}$, $r > 0$, let $Q(x,r)$ be the square with center at x, side length $2\,r$, and sides parallel to the coordinate axes. We say that a sequence $\Lambda \subset \mathbb{C}$ is *relatively dense* if there exists $r_0 > 0$ such that

$$\Lambda \cap Q(x, r_0) \neq \varnothing \qquad \text{for each } x \in \mathbb{R}.$$

If Λ is a complete interpolating sequence for L_π^p, (6) forces Λ to be relatively dense: if this is not the case and there exist sequences $\{x_j\} \subset \mathbb{R}$ and $r_j \to \infty$ such that $Q(x_j, r_j) \cap \Lambda = \varnothing$, then, setting

$$f_j(z) = \frac{\sin \frac{\pi}{2}\,(z - x_j)}{z - x_j} \,,$$

we find that

$$\sum_k |f_j(\lambda_k)|^p \, e^{-p\pi|\eta_k|} \, (1+|\eta_k|) \longrightarrow 0 \,, \qquad j \longrightarrow \infty \,,$$

while $\|f_j\|_{L^p}$ is independent of j.

Suppose that Λ is a complete interpolating sequence for L^p_π. Take $r > r_0$, where r_0 is as above, define

$$Q_j = Q(4\,r\,j, r)\,, \qquad j \in \mathbb{Z}\,,$$

and pick a sequence $\Gamma = \{\gamma_j\} \subset \Lambda$ such that $\gamma_j \in Q_j$. Let $\Sigma = \{\sigma_j\}$ be another sequence with $|\gamma_j - \sigma_j| = \varepsilon$. Suppose $w = \{w_j\}$ is a positive weight sequence. Associate with it the weighted space l^p_w consisting of all sequences $a = \{a_k\}$ satisfying

$$\|a\|^p_{w,p} := \sum_k |a_k|^p w_k < \infty\,.$$

We are interested in the boundedness of the *discrete Hilbert operator* $\mathcal{H}_{\Gamma,\Sigma}$ defined by the relation

$$\mathcal{H}_{\Gamma,\Sigma} : a = \{a_j\} \longmapsto \{(\mathcal{H}_{\Gamma,\Sigma} a)_j\}\,, \qquad (\mathcal{H}_{\Gamma,\Sigma} a)_j = \sum_k \frac{a_k}{\sigma_j - \gamma_k}\,,$$

on l^p_w. The following definitions are needed. We say that w satisfies the *discrete* (A_p) *condition* if

$$\sup_{\substack{k\in\mathbb{Z}\\ n>0}} \Big(\frac{1}{n} \sum_{j=k+1}^{k+n} w_j\Big)\Big(\frac{1}{n} \sum_{j=k+1}^{k+n} w_j^{-1/(p-1)}\Big)^{p-1} < \infty\,.$$

This condition is analogous to the classical continuous (A_p) condition for a positive weight $v(x) > 0$, $x \in \mathbb{R}$,

$$\sup_I \Big(\frac{1}{|I|}\int_I v\,dx\Big)\Big(\frac{1}{|I|}\int_I v^{-1/(p-1)}\,dx\Big)^{p-1} < \infty\,, \tag{7}$$

where I ranges over all intervals in $\mathbb{R}$ (see [3]). Recall that the latter condition is necessary and sufficient for boundedness of the classical Hilbert operator

$$\mathcal{H} : f \longmapsto (\mathcal{H}f)(t) = \frac{1}{i\,\pi}\int \frac{f(\tau)}{t-\tau}\,d\tau$$

on the weighted space of functions $L^p(\mathbb{R}; v)$ consisting of all functions f satisfying

$$\|f\|^p_{v,p} := \int |f(t)|^p\, v(t)\,dt < \infty\,.$$

We shall need the following lemma.

Lemma 1. *If $\mathcal{H}_{\Gamma,\Sigma}$ is bounded from l^p_w to l^p_w, then w satisfies the discrete (A_p) condition.*

PROOF. We adopt the proof for the continuous case (see [3]). Let $k \in \mathbb{Z}$ and $n > 0$ be given. For convenience, put $I_1 = \{k+1, k+2, \dots, k+n\}$, $I_2 = \{k+2n+1, k+2n+2, \dots, k+3n\}$. Suppose that a positive sequence a is supported on I_1. Then, for $j \in I_2$, we have

$$|(\mathcal{H}_{\Gamma,\Sigma}a)_j| \geq \sum_l a_l \frac{\operatorname{Re}(\sigma_j - \gamma_l)}{|\sigma_j - \gamma_l|^2} \geq \frac{C}{n} \sum_l a_l \ , \tag{8}$$

where C is independent of k and n. Putting $a_l = 1$, we get thus

$$\sum_{j \in I_2} w_j \leq C \sum_{l \in I_1} w_l \ ,$$

and by symmetry

$$\sum_{j \in I_1} w_j \asymp \sum_{l \in I_2} w_l \ . \tag{9}$$

Here and in what follows the sign $\asymp$ means that the ratio of the two sides lies between two positive constants. Now we put $a_l = w_l^\alpha$ for $l \in I_1$ and $a_l = 0$ otherwise, and get from (8) and the boundedness of $\mathcal{H}_{\Gamma,\Sigma}$

$$\Big(\sum_{j \in I_2} w_j\Big)\Big(\frac{1}{n} \sum_{l \in I_1} w_l^\alpha\Big)^p \leq C \sum_{m \in I_1} w_m^{1+\alpha p} \ .$$

Finally, we put $\alpha = -1/(p-1)$ and invoke (9), and the lemma is proved.

The converse of Lemma 1 is also true, but we will not need that fact. Note also that the boundedness of the operator $\mathcal{H}_{\Gamma,\Sigma}$ is independent of the choice of sequence Σ, provided the condition $|\gamma_j - \sigma_j| = \varepsilon$ holds.

Let Λ be a complete interpolating sequence for L^p_π. It may be that $0 \in \Lambda$, in which case we assume that $\lambda_0 = 0$. If the function $f_0 \in L^p_\pi$ solves the interpolation problem $f_0(\lambda_k) = \delta_{0,k}$, $k \in \mathbb{Z}$, then $f_0(\mu) \neq 0$ for $\mu \in \mathbb{C} \setminus \Lambda$, since otherwise the function $(z - \lambda_0)(z - \mu)^{-1} f_0(z)$ belongs to L^p_π and vanishes on Λ, contradicting the uniqueness of the solution of the interpolation problem (1). Since $f_0 \in L^p_\pi$, f_0 belongs to the Cartwright class $\mathcal{C}$ (see [6, Lecture 15]) and, in particular, the limit

$$S(z) = (z - \lambda_0) \lim_{R\to\infty} \prod_{|\lambda_k|<R, k\neq 0} \Big(1 - \frac{z}{\lambda_k}\Big) \tag{10}$$

exists and defines the *generating function* of the sequence Λ. Besides, the solution $f_k \in L^p_\pi$ of the interpolation problem $f_k(\lambda_n) = \delta_{k,n}$ has the form

$$f_k(z) = \frac{S(z)}{S'(\lambda_k)\,(z - \lambda_k)}\,. \tag{11}$$

We may now formulate our main theorem.

Theorem 1. $\Lambda = \{\lambda_k\}$, *where* $\lambda_k = \xi_k + i\eta_k$, *is a complete interpolating sequence for* L^p_π *if and only if the following three conditions hold.*

i) *The sequences* $\Lambda \cap \mathbb{C}^+$ *and* $\Lambda \cap \mathbb{C}^-$ *satisfy the Carleson condition in* $\mathbb{C}^+$ *and* $\mathbb{C}^-$ *respectively, i.e.* (4) *holds with* $a = 0$, *and also* $\inf_{k\neq j} |\lambda_k - \lambda_j| > 0$.

ii) *The limit* $S(z)$ *in* (10) *exists and represents an entire function of exponential type* π.

iii) *There exists a relatively dense subsequence* $\Gamma = \{\gamma_j\} \subset \Lambda$ *such that the sequence* $\{|S'(\gamma_j)|^p\}$ *satisfies the discrete* (A_p) *condition.*

Defining $F(x) = |S(x)|/\operatorname{dist}(x, \Lambda)$ $(x \in \mathbb{R})$, *we may replace statement* iii) *by the following*:

iii') $F^p(x)$ $(x \in \mathbb{R})$ *satisfies the (continuous)* (A_p) *condition.*

Note that that condition i) is equivalent to the statement that, for each $a \in \mathbb{R}$, the sequences $\Lambda \cap \mathbb{C}^\pm_a$ satisfy the Carleson condition (4). Another, more compact way of expressing i), is given by (5).

3. Proof of Theorem 1: necessity.

We have already proved the necessity of i) and ii), and also the existence of a relatively dense sequence $\Gamma = \{\gamma_j\} \subset \Lambda$. We prove now that iii) is necessary as well. Let ε be as above. Then, for every j, we can find a point σ_j with $|\sigma_j - \gamma_j| = \varepsilon$ and

$$|S(\sigma_j)| = \varepsilon\, |S'(\gamma_j)|\,.$$

This follows from the fact that $S(z)\,(z-\gamma_j)^{-1} \neq 0$ for $|z-\gamma_j| \leq \varepsilon$, hence

$$\min_{|z-\gamma_j|=\varepsilon} |S(z)\,(z-\gamma_j)^{-1}| \leq |S'(\gamma_j)| \leq \max_{|z-\gamma_j|=\varepsilon} |S(z)\,(z-\gamma_j)^{-1}|\,.$$

Set $\Sigma = \{\sigma_j\}$. The Plancherel-Pólya inequality (see [6, Lecture 20]) yields

$$\sum_j |f(\sigma_j)|^p \leq C\, \|f\|_{L^p}^p\,, \qquad f \in L_\pi^p\,. \tag{12}$$

Now let $a = \{a_j\}$ be a finite sequence. By (11), the unique solution of the interpolation problem $f(\gamma_j) = a_j$, $f(\lambda_k) = 0$, $\lambda_k \notin \Gamma$ has the form

$$f(z) = \sum_j \frac{a_j}{S'(\gamma_j)}\, \frac{S(z)}{(z-\gamma_j)}\,.$$

By (6) and (12), we have

$$\sum_j |f(\sigma_j)|^p \leq C \sum_j |a_j|^p\,.$$

Now, by our particular choice of the sequence Σ, we obtain iii) by observing that Lemma 1 applies with $w_j = |S'(\gamma_j)|^p$.

To prove that iii) implies iii'), we need the following lemma.

Lemma 2. *Suppose* $x \in \mathbb{R}$ *and* $\operatorname{Re}\gamma_j \leq x \leq \operatorname{Re}\gamma_{j+1}$. *Then there exists an* $\alpha = \alpha(x) \in [0,1]$ *such that*

$$|S'(\gamma_j)|^\alpha |S'(\gamma_{j+1})|^{1-\alpha} \asymp \frac{|S(x)|}{\operatorname{dist}(x,\Lambda)}\,,$$

uniformly with respect to $x \in \mathbb{R}$.

In fact, assuming this lemma to hold, we see that (7) with $v = F^p$ follows from iii) and the inequality $t^\alpha s^{1-\alpha} \leq t + s$, $t, s > 0$, $\alpha \in [0,1]$.

Proof of Lemma 2. We assume that $x \in [\operatorname{Re}\gamma_j, \operatorname{Re}\gamma_{j+1}]$ and, for simplicity, $x \notin \Lambda$. Set $\Lambda(x) = \{\lambda \in \Lambda : |\lambda - x| < 30\, r\}$. (Here r is the number used for constructing Γ.) For $\alpha \in [0,1]$ we have

$$\begin{aligned}
\rho &:= \frac{|S'(\gamma_j)|^\alpha |S'(\gamma_{j+1})|^{1-\alpha}}{|S(x)|\operatorname{dist}(x,\Lambda)^{-1}} \\
&= \Bigg(\frac{\Big|\dfrac{1}{\gamma_j} \displaystyle\prod_{\lambda_k\in\Lambda(x)\setminus\{\gamma_j\}} \Big(1-\frac{\gamma_j}{\lambda_k}\Big)\Big|^\alpha}{\Big|\displaystyle\prod_{\lambda\in\Lambda(x)}\Big(1-\frac{x}{\lambda}\Big)\Big|} \\
&\qquad \cdot \Big|\frac{1}{\gamma_{j+1}} \prod_{\lambda_k\in\Lambda(x)\setminus\{\gamma_{j+1}\}} \Big(1-\frac{\gamma_{j+1}}{\lambda_k}\Big)\Big|^{1-\alpha} \operatorname{dist}(x,\Lambda)\Bigg) \\
&\quad \cdot \Big(\prod_{\lambda_k\in\Lambda\setminus\Lambda(x)} \frac{|\gamma_j-\lambda_k|^\alpha|\gamma_{j+1}-\lambda_k|^{1-\alpha}}{|x-\lambda_k|}\Big) \\
&= \Pi_1(x)\,\Pi_2(x)\,.
\end{aligned}$$

Writing

$$\begin{aligned}
\Pi_1(x) &= \frac{|\gamma_j-\gamma_{j+1}|}{\max\{|x-\gamma_j|, |x-\gamma_{j+1}|\}} \\
&\qquad \cdot \prod_{\lambda_k\in\Lambda(x)\setminus\{\gamma_j,\gamma_{j+1}\}} \frac{|\lambda_k-\gamma_j|^\alpha|\lambda_k-\gamma_{j+1}|^{1-\alpha}}{|x-\lambda_k|}\,,
\end{aligned}$$

we see that $\Pi_1(x) \asymp 1$ uniformly with respect to $\alpha \in [0,1]$.

To estimate $\Pi_2(x)$, we begin by writing

$$\gamma_j = x - x_j + i\, y_j\,, \qquad \gamma_{j+1} = x + x_{j+1} + i\, y_{j+1}\,.$$

The values x_j and x_{j+1} depend on x and also satisfy the inequalities $0 \le x_j,\ x_{j+1} \le 8\,r$. Recall also that $|y_j| < r$ for all j. We may then write

$$\begin{aligned}
\rho^2 \asymp \prod_{\lambda_k\notin\Lambda(x)} & \frac{((x-x_j-\xi_k)^2+(y_j-\eta_k)^2)^\alpha}{(x-\xi_k)^2+\eta_k^2} \\
&\cdot ((x+x_{j+1}-\xi_k)^2+(y_{j+1}-\eta_k)^2)^{1-\alpha}
\end{aligned}$$

$$= \prod_{\lambda_k \notin \Lambda(x)} \Big(1 - \frac{2\, x_j\, (x-\xi_k) + 2\, y_j\, \eta_k + O\,(1)}{(x-\xi_k)^2 + \eta_k^2}\Big)^{\alpha}$$

$$\cdot \Big(1 + \frac{2\, x_{j+1}(x-\xi_k) - 2\, y_{j+1}\, \eta_k + O\,(1)}{(x-\xi_k)^2 + \eta_k^2}\Big)^{1-\alpha} .$$

Choosing $\alpha = \alpha(x)$ so that $\alpha\, x_j - (1-\alpha)\, x_{j+1} = 0$, *i.e.*,

$$\alpha = \frac{x_{j+1}}{(x_j + x_{j+1})} \,,$$

we find that

$$K_1 \exp\Big(c_1 \sum_{\lambda_k \notin \Lambda(x)} \frac{|\eta_k|}{(x-\xi_k)^2 + \eta_k^2}\Big) \le \rho^2$$

$$\le K_2 \exp\Big(c_2 \sum_{\lambda_k \notin \Lambda(x)} \frac{|\eta_k|}{(x-\xi_k)^2 + \eta_k^2}\Big)$$

for some c_1, c_2, K_1, K_2 independent of x. By Carleson's condition (5), the sum is uniformly bounded, and we are done.

4. Proof of Theorem 1: sufficiency.

We will now prove that i), ii), iii') imply that Λ is a complete interpolating sequence.

To begin with, note that

$$\int (F(x))^p \, \frac{dx}{1+|x|^p} < \infty \tag{13}$$

and

$$\int (F(x))^p \, dx = \infty\,. \tag{14}$$

The first relation follows from the fact that $\int (F(x))^p |\mathcal{H} f(x)|^p \, dx < \infty$ for each bounded finite function f; it suffices to take $f = \chi_{[0,1]}$. The second is a direct consequence of [3, Lemma 2]. (Alternatively, we may apply the operator $\mathcal{H}$ to an appropriate δ-sequence $\{\delta_n(x)\}$.)

First, we check that Λ is a uniqueness set. To this end, we need to estimate $|S(z)|$ from below.

Lemma 3. *Let as above* $\varepsilon > 0$ *be such that the disks*

$$K(\lambda_k) := \{z : |z - \lambda_k| < 10\,\varepsilon\,(1 + |\eta_k|)\}$$

are pairwise disjoint. Then

$$|S(z)| \geq C(1+|z|)^{-1/p}\, e^{\pi |\operatorname{Im} z|}\,, \qquad \textit{for } \operatorname{dist}(z, \Lambda) > \varepsilon\,(1+|\operatorname{Im} z|)\,. \tag{15}$$

Proof of Lemma 3. Put $\Lambda' = \Lambda \cap \{z : |\operatorname{Im} z| < \varepsilon\}$ and consider the auxiliary function

$$S_1(z) = S(z) \prod_{\lambda \in \Lambda'} \frac{z - \lambda + 2\,i\,\varepsilon}{z - \lambda}\,.$$

It is plain that

$$|S_1(z)| \asymp |S(z)|\,, \qquad |\operatorname{Im} z| > 3\,\varepsilon\,, \tag{16}$$

and, besides, iii') implies that $|S_1(x)|^p$ satisfies the (A_p) condition because $|S_1(x)| \asymp F(x)$.

The function $e^{i\pi z} S_1(z)/(z+i)$ belongs to H^p of the upper half-plane, as follows from (13), ii), and the Plancherel-Pólya theorem (3). Hence we have the following inner-outer factorization of S_1

$$S_1(z) = e^{-i\pi z} G(z)\, B_1(z)\,, \qquad \operatorname{Im} z > 0\,. \tag{17}$$

Here the Blaschke product B_1 corresponds to the Carleson sequence $(\Lambda \cap \mathbb{C}^+) \setminus \Lambda'$ and, in particular,

$$|B_1(z)| > c > 0\,, \qquad \text{for } \operatorname{dist}(z, \Lambda) > \varepsilon\,|\operatorname{Im} z|\,. \tag{18}$$

Moreover, G is an outer function and $|G(x)|^p$ satisfies the (A_p) condition. Therefore, $|G(x)|^{-q}$ is an (A_q) weight (here $1/p + 1/q = 1$), $G(x)^{-1}(1+|x|)^{-1} \in L^q(\mathbb{R})$, and thus

$$\frac{1}{(z+i)\,G(z)} = \frac{1}{2\pi i} \int \frac{1}{(t+i)\,G(t)}\, \frac{dt}{t-z}\,, \qquad \operatorname{Im} z > 0\,.$$

It follows that

$$\frac{1}{|G(z)|} \leq C\,(1+|z|)^{1/p}\,. \tag{19}$$

Combining relations (16)-(19), we obtain (15) for $\operatorname{Im} z > 3\,\varepsilon$. The estimate for $\operatorname{Im} z < -3\,\varepsilon$ is similar, and to fill the gap $-3\,\varepsilon < \operatorname{Im} z < 3\,\varepsilon$, we may repeat the construction, taking another horizontal line instead of $\mathbb{R}$.

By (14) and the fact that $|S_1(x)|^p$ is an (A_p) weight, we have

$$\int |S_1(x)|^p\,dx = \infty\,.$$

Applying the Plancherel-Pólya theorem (3) to the function $f(z) = S_1(z+i\,a)$, we therefore obtain

$$\int |S_1(x+i\,a)|^p\,dx = \infty\,, \qquad a \in \mathbb{R}\,.$$

Hence, by (16),

$$\int |S(x+i)|^p\,dx = \infty\,.$$

By a second application of the Plancherel-Pólya theorem, we find that

$$\int |S(x)|^p\,dx = \infty\,. \tag{20}$$

We are now in position to prove the uniqueness. Indeed, if $f \in L^p_\pi$ and $f(\lambda) = 0$, $\lambda \in \Lambda$, then $\phi(z) = f(z)/S(z)$ is an entire function of exponential type 0. By (15) and the pointwise bound

$$|f(z)| \leq C_p\,\|f\|_{L^p}(1+|\operatorname{Im} z|)^{-1/p}\,e^{\pi|\operatorname{Im} z|}\,,$$

it follows that $|\phi(z)|$ is uniformly bounded for z satisfying $\operatorname{dist}(z,\Lambda) > \varepsilon(|\operatorname{Im} z|+1)$. By the classical Phragmén-Lindelöf theorem, we get $\phi(z) \equiv C$, which is incompatible with (20), unless $C = 0$.

It remains only to check that we can actually solve the interpolation problem (1) for each sequence $a = \{a_k\}$ satisfying (2). It suffices to consider a finite sequence a and bound the norm of the solution by a constant times the left-hand side of (2). After doing so, we can apply a limit procedure. If a is a finite sequence, then, by (13), the unique solution of the interpolation problem (1) has the form

$$f(z) = \sum_k \frac{a_k}{S'(\lambda_k)} \frac{S(z)}{(z-\lambda_k)} \,. \tag{21}$$

We split the sum (21) into two parts, corresponding to points lying in $\mathbb{C}^+ \cup \mathbb{R}$ and in $\mathbb{C}^-$, respectively. We may estimate the norm of each sum separately, so let us assume that all the λ_k corresponding to $a_k \neq 0$ are in $\mathbb{C}^+ \cup \mathbb{R}$. Clearly, we may estimate the L^p integral along $\operatorname{Im}(z) = -1/2$. Let us, however, for conventional reasons, estimate it along $\mathbb{R}$ and assume all the points λ_k satisfy $\eta_k \geq 1/2$. Now let

$$B(z) = \prod_k \frac{z + \frac{i}{2} - \Big(\lambda_k + \frac{i}{2}\Big)}{z + \frac{i}{2} - \Big(\overline{\lambda_k} - \frac{i}{2}\Big)} \,.$$

Writing $S(z) = B(z)\, e^{-i\pi z} G(z)$, where G is an outer function in $\mathbb{C}^+$, we observe that iii') is equivalent to $|G(x)|^p$ satisfying the (A_p) condition. Since

$$S'(\lambda_k) = G(\lambda_k)\, \frac{e^{-i\pi\lambda_k}}{i\,(1+\eta_k)} \prod_{j\neq k} \frac{\lambda_k - \lambda_j}{\lambda_k - \overline{\lambda_j} + i} \,,$$

the Carleson condition (4) implies

$$|S'(\lambda_k)| \asymp |G(\lambda_k)|\, \frac{e^{\pi\eta_k}}{\eta_k} \,.$$

Thus it is enough to consider the L^p boundedness of

$$\tilde{f}(x) = \sum_k \frac{a_k\, \eta_k\, e^{-\pi\eta_k}}{G(\lambda_k)} \frac{G(x)}{x-\lambda_k} \,.$$

By duality,

$$\begin{aligned}
\|\tilde{f}\|_p &\asymp \sup_{\substack{\|h\|_q=1\\ h\in H^q}} \Big| \sum_k \frac{a_k\, \eta_k\, e^{-\pi\eta_k}}{G(\lambda_k)} \int_{\mathbb{R}} \frac{G(x)\, h(x)}{x-\lambda_k}\, dx \Big| \\
&\leq \sup_{\substack{\|h\|_q=1\\ h\in H^q}} \Big| \sum_k \frac{a_k\, \eta_k\, e^{-\pi\eta_k}}{G(\lambda_k)}\, (\mathcal{H}Gh)(\lambda_k) \Big| \\
&\leq \sup_{\substack{\|h\|_q=1\\ h\in H^q}} \Big(\sum_k |a_k|^p \eta_k\, e^{-p\pi\eta_k} \Big)^{1/p} \Big(\sum_k \Big| \frac{(\mathcal{H}Gh)(\lambda_k)}{G(\lambda_k)} \Big|^q \eta_k \Big)^{1/q} .
\end{aligned}$$

Since $|G(x)|^{-q}$ satisfies the (A_q) condition, G is an outer function in $\mathbb{C}^+$, and $h \in H^q, \|h\|_q \le 1$, we have $(\mathcal{H}Gh)(z)/G(z) \in H^q$, and $\|(\mathcal{H}Gh)(z)/G(z)\|_q \le C$. Since $\sum_k \eta_k \, \delta_{\lambda_k}$ is a Carleson measure, the last sum is uniformly bounded, and we get the desired conclusion.

The sum corresponding to points in $\mathbb{C}^-$ is treated similarly.

5. A stability result.

We will now show how Theorem 1 can be used to obtain a result similar to the Kadets 1/4 theorem. The same technique implies more sophisticated stability results for L^p_π, similar to the theorems of Avdonin and Katsnelson for L^2_π; see [4] for the latter results.

For $1 < p < \infty$ we denote by q the conjugate exponent, $1/p+1/q = 1$, and put

$$p' = \max\{p, q\}\,.$$

We may now prove:

Theorem 2. *Suppose $\{\delta_k\}_{k\in\mathbb{Z}}$ is a sequence of real numbers, and put $\lambda_k = k+\delta_k$, $k \in \mathbb{Z}$. If $|\delta_k| \le d < 1/(2\,p')$ for every k, then $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for L^p_π. If merely $|\delta_k| < 1/(2\,p')$ for every k, then $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is not necessarily a complete interpolating sequence for L^p_π.*

Note that for $p = 2$ this is precisely the Kadets theorem (see [4]).

Proof of Theorem 2. We prove first that the inequality

$$|\delta_k| < \frac{1}{2\,p'}$$

is not sufficient. If $\delta_0 = 1$ and otherwise $\delta_k = \operatorname{sgn}(k)\,\delta$, $-1 < \delta < 1$, standard estimates of infinite products yield

$$F(x) \asymp (1+|x|)^{-2\delta}\,.$$

For $1 < p < 2$ we choose $\delta = -1/(2\,q)$. Then

$$\frac{1}{|x|}\int_0^x F^p\,dt \Big(\frac{1}{|x|}\int_0^x F^{-q}\,dt\Big)^{p-1} \ge C\,(\log(1+|x|))^{p-1}\,,$$

and the (A_p) condition fails. We obtain the same conclusion if $|\delta_k| < 1/(2\,q)$ and δ_k tends sufficiently fast to $\operatorname{sgn}(k)/(2\,q)$ as k tends to $\pm\infty$. If $2 < p < \infty$, we put $\delta = -1/(2\,p)$, and argue similarly.

With Λ as required in the theorem, define

$$\lambda_{\alpha,k} = k + \alpha\,\delta_k \qquad \text{and} \qquad \Lambda_\alpha = \{\lambda_{\alpha,k}\}\,,$$

where α is a real number. Suppose that $\delta < 1/2$ and $|\alpha|\,\delta < 1/2$, so that the distance between any two distinct points of Λ, and likewise the distance between any two distinct numbers of Λ_α, exceeds a certain positive number. Then estimates of infinite products show that

$$F_\alpha(x) \asymp (F(x))^\alpha\,, \tag{22}$$

where $F_\alpha(x) = |S_\alpha(x)|/\operatorname{dist}(x, \Lambda_\alpha)$ and S_α is the generating function of Λ_α.

Suppose first that $1 < p < 2$. If $d < 1/(2\,q)$, then $F^2_{q/2}$ satisfies the (A_2) condition, according to the classical 1/4 theorem. By (22), it means that F^q satisfies the (A_2) condition, which implies, by Hölder's inequality, that F^p satisfies the (A_p) condition.

If $2 < p < \infty$, put $\alpha = p/2$ and argue similarly.

References.

[1] Garnett, J. B., *Bounded Analytic Functions.* Academic Press, 1981.

[2] Eoff, C., The discrete nature of Paley-Wiener spaces. *Proc. Amer. Math. Soc.* **123** (1995), 505-512.

[3] Hunt, R., Muckenhoupt, B., Wheeden, R., Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176** (1973), 227-251.

[4] Khrushchev, S. V., Nikol'skii, N. K., Pavlov, B. S., *Unconditional bases of exponentials and reproducing kernels.* In "Complex Analysis and Spectral Theory", Lecture Notes in Math. **864**, Springer-Verlag (1981), 214-335.

[5] Koosis, P., *Introduction to H_p Spaces.* Cambridge University Press, 1980.

[6] Levin, B., *Lectures on Entire Functions.* American Mathematical Society, 1996.

[7] Minkin, A. M., Reflection of exponents, and unconditional bases of exponentials. *St. Petersburg Math. J.* **3** (1992), 1043-1068.

[8] Nikol'skii, N. K., Bases of exponentials and the values of reproducing kernels. *Dokl. Akad. Nauk SSSR* **252** (1980), 1316-1320. English transl. in *Sov. Math. Dokl.* **21** (1980), 937-941.

[9] Pavlov, B. S., Basicity of an exponential system and Muckenhoupt's condition. *Dokl. Akad. Nauk SSSR* **247** (1979), 37-40. English transl. in *Sov. Math. Dokl.* **20** (1979), 655-659.

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