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# Estimates on the solution of an elliptic equation related to Brownian motion with drift (II)

**Joseph G. Conlon and Peder A. Olsen**

## 1. Introduction.

In this paper we continue the study of the Dirichlet problem for an elliptic equation on a domain in $\mathbb{R}^3$ which was begun in [5]. For $R > 0$ let $\Omega_R$ be the ball of radius $R$ centered at the origin with boundary $\partial\Omega_R$. The Dirichlet problem we are concerned with is the following

$$(1.1) \qquad (-\Delta - \mathbf{b}(x) \cdot \nabla)\, u(x) = f(x)\,, \qquad x \in \Omega_R\,,$$

with zero boundary conditions

$$(1.2) \qquad u(x) = 0\,, \qquad x \in \partial\Omega_R\,.$$

Since we shall be obtaining estimates on the solution of (1.1), (1.2) in terms of $R$ we shall think of the functions $\mathbf{b}(x)$, $f(x)$ as defined on all of $\mathbb{R}^3$. Thus we assume

$$\mathbf{b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad f : \mathbb{R}^3 \longrightarrow \mathbb{R}\,,$$

are Lebesgue measurable functions.

For $1 \le r \le q < \infty$ let $M_r^q$ be the Morrey space on $\mathbb{R}^3$ defined as follows: a function $g : \mathbb{R}^3 \longrightarrow \mathbb{C}$ is in $M_r^q$ if $|g|^r$ is locally integrable and there is a constant $C$ such that

$$\int_Q |g|^r \, dx \le C^r \, |Q|^{1-r/q} , \tag{1.3}$$

for all cubes $Q \subset \mathbb{R}^3$. Here $|Q|$ denotes the volume of $Q$. The norm of $g$, $\|g\|_{q,r}$ is defined as

$$\|g\|_{q,r} = \inf \{ C : \text{ (1.3) holds for } C \text{ and all cubes } Q \subset \mathbb{R}^3 \} .$$

In our previous paper we proved that the problem (1.1), (1.2) has a unique solution if $|\mathbf{b}| \in M_p^3$, $p > 1$, and $\|\mathbf{b}\|_{3,p}$ is sufficiently small. This is a perturbative result. We also had a nonperturbative theorem. This theorem stated that if $\mathbf{b}$ is locally in $M_p^3$ with the local Morrey norm being small then (1.1), (1.2) has a unique solution. The proof of the nonperturbative theorem required $p > 2$. In fact the estimates diverge as $p$ approaches 2. Our goal in this paper is to obtain nonperturbative theorems which are valid for $p > 1$.

To state our first nonperturbative theorem we need a quantity introduced by Fefferman [9]: suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes $Q$. A cube $Q$ is said to be minimal with respect to $\varepsilon$ if

$$\int_Q |\mathbf{b}|^p \, dx \ge \varepsilon^p \, |Q|^{1-p/3} ,$$

$$\int_{Q'} |\mathbf{b}|^p \, dx < \varepsilon^p \, |Q'|^{1-p/3} , \qquad Q' \subset Q ,$$

for all proper dyadic subcubes $Q'$ of $Q$. Then $N_\varepsilon(\mathbf{b})$ is the number of minimal cubes in the dyadic decomposition.

**Theorem 1.1.** *Suppose* $f \in M_r^q$, $1 < r \le q$, $r < p$, $p > 1$, $3/2 < q < 3$. *Then there exists* $\varepsilon > 0$ *depending only on* $p, q, r$ *such that if* $N_\varepsilon(\mathbf{b}) < \infty$ *the boundary value problem* (1.1), (1.2) *has a unique solution* $u(x)$ *in the following sense*:

a) $u$ *is uniformly Holder continuous on* $\Omega_R$ *and satisfies the boundary condition* (1.2).

b) *The distributional Laplacian* $\Delta u$ *of* $u$ *is in* $M_r^q$ *and the equation* (1.1) *holds for almost every* $x \in \Omega_R$.

REMARK 1.1. The restriction $q < 3$ is required by b) while $q > 3/2$ is required by a). Thus if $f$ is in $L^q$ for any $q > 3/2$ the solution has property a).

Next we turn to the problem of obtaining good $L^\infty$ estimates on the solution $u(x)$ given in Theorem 1.1. For $1 < p < 3$ and $n$ an integer define a function $a_{n,p} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$a_{n,p}(x) = \Big( 2^{n(3-p)} \int_{|x-y|<2^{-n}} |\mathbf{b}|^p(y)\, dy \Big)^{1/p} . \tag{1.4}$$

In [5] the following (Theorem 1.4) is proved:

**Theorem 1.2.** *Let $n_0$ be the integer which satisfies the inequality*

$$4R > 2^{-n_0} \ge 2R . \tag{1.5}$$

*Then there exists $\gamma$, $0 < \gamma < 1$, depending only on $p > 2$ such that $u$ satisfies the $L^\infty$ estimate*

$$\|u\|_\infty \le C_1\, R^{2-3/q}\, \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x\in\Omega_R} \exp\Big( C_2 \sum_{j=0}^{m} a_{n_0+j,p}(x) \Big) . \tag{1.6}$$

*The constant $C_1$ depends only on $p, q, r$ and $C_2$ only on $p > 2$.*

It is easy to see that the inequality (1.6) becomes stronger as $p$ decreases. We shall show in Section 3 that Theorem 1.2 does not hold for $1 < p < 2$. We will accomplish this by constructing a counterexample to (1.6) for $f \equiv 1$ and any $p < 2$. This is somewhat surprising since (1.6) does hold for $1 < p < 2$ if the drift is spherically symmetric. In that case one can obtain an explicit formula for the solution of (1.1), (1.2). The counterexample constructed in Section 3 has a drift which is far from being spherically symmetric. In fact it is concentrated on a set with dimension 1. By the recurrence property of Brownian motion the process hits this set with high probability. Once inside the set, the drift pulls the Brownian particle towards the center of the ball $\Omega_R$.

We wish to obtain a theorem which generalizes Theorem 1.2 to the case $1 < p < 2$. Let $s > 2$ be a parameter, and suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes $Q$ with $|Q| = 2^{-3m}$, $m$ an integer. For $m, n$ integers with $m \ge n$, and $x \in \mathbb{R}^3$ let

$$N_m(x) = \text{number of dyadic cubes } Q \text{ with } |Q| = 2^{-3m} ,$$

such that $Q$ is contained in the ball centered at $x$ with radius $2^{-n}$ and

$$\int_Q |\mathbf{b}|^p \, dx \geq \varepsilon^p \, |Q|^{1-p/3} \, ,$$

where $\varepsilon > 0$ is a given parameter. We define the function $a_{\varepsilon,n,s,p}(x)$ by

$$a_{\varepsilon,n,s,p}(x) = \Big( \frac{\sup_{m \geq n} N_m(x)}{2^{(m-n)(3-s)}} \Big)^{1/s} \, . \tag{1.7}$$

We may compare the functions $a_{n,p}$ and $a_{\varepsilon,n,s,p}$ defined by (1.4), (1.7) respectively. In fact by definition of $N_m(x)$ we have that

$$\varepsilon^p \, |Q_m|^{1-p/3} \, N_m(x) \leq \int_{|x-y|<2^{-n}} |\mathbf{b}|^p(y) \, dy = a_{n,p}(x)^p \, 2^{-n(3-p)} \, ,$$

whence

$$N_m(x) \leq \varepsilon^{-p} \, a_{n,p}(x)^p \, 2^{(m-n)(3-p)} \, ,$$

and so

$$\frac{N_m(x)}{2^{(m-n)(3-s)}} \leq \varepsilon^{-p} \, a_{n,p}(x)^p \, 2^{(m-n)(s-p)} \, .$$

We conclude that

$$a_{\varepsilon,n,p,p}(x) \leq \varepsilon^{-1} \, a_{n,p}(x) \, , \qquad x \in \mathbb{R}^3 \, . \tag{1.8}$$

**Theorem 1.3.** *Let $n_0$ be the integer satisfying* (1.5) *and suppose $2 < s \leq 3$, $1 < p \leq 3$. Then there exists $\varepsilon, \gamma$ with $\varepsilon > 0$, $0 < \gamma < 1$, depending only on $s, p$ such that the solution $u$ of* (1.1), (1.2) *satisfies an inequality*

$$\|u\|_\infty \leq C_1 R^{2-3/q} \, \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x \in \Omega_R} \exp \Big( C_2 \sum_{j=0}^{m} a_{\varepsilon, n_0+j,s,p}(x) \Big) \, .$$

*The constant $C_1$ depends only on $p, q, r, s$ and $C_2$ only on $s > 2$ and $p$, $1 < p \leq 3$.*

It follows from (1.8) that Theorem 1.3 implies Theorem 1.2. We shall show in Section 8 that Theorem 1.3 implies that for $1 < p \leq 3$, there exists $\varepsilon > 0$ and constants $C_1$, $C_2$ depending only on $p, q, r$ such that

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \exp\left(C_2 N_\varepsilon(\mathbf{b})\right). \tag{1.9}$$

Theorem 1.1 will be proved in Section 2. It will be sufficient to give a proof of Lemma 4.2 of [5] which is valid for $1 < p \leq 3$. The remainder of the argument of the proof of Theorem 1.1 is then exactly as for [5, Theorem 1.3]. In the new proof of Lemma 4.2 we will introduce the notion of a weighted Morrey space. This notion will play a key role in sections 4, 5, 6, 7, 8 where we prove Theorem 1.3.

The main problem we need to solve to prove Theorem 1.3 is to estimate the exit probability from a spherical shell of Brownian motion with drift. Thus let us consider a particle started at $x \in \mathbb{R}^3$ with $|x| = R$ and let $P$ be the probability that the particle exits the shell $\{y : R/2 < |y| < 2R\}$ through the outer sphere. For Brownian motion one can explicitly compute $P = 2/3$. For the case of Brownian motion with drift $\mathbf{b}$ we need to obtain a lower bound on $P$ in terms of $\mathbf{b}$. In Section 4 we analyze this problem in the case when $\mathbf{b}$ is perturbative, that is when $\|\mathbf{b}\|_{3,p} \ll 1$. When $\mathbf{b}$ is not perturbative we estimate $P$ by first defining a length scale $\lambda \ll R$ in terms of $\mathbf{b}$. Then we construct paths from $x$, $|x| = R$, to the outer sphere $\{|y| = 2R\}$ which are linear on scales larger than $\lambda$ but diffusive on scales less than $\lambda$. Thus the paths of the drift process are confined to a cylinder of radius $\lambda$. The drift is propagated perturbatively on a length scale $\lambda$ and ballistically on larger scales.

In order to propagate the drift perturbatively on the length scale $\lambda$ we must limit the number of nonperturbative cubes on scales smaller than $\lambda$ to have dimension less than 1. The requirement that the constant $s$ in Theorem 1.3 exceeds 2 ensures that this holds on average. The analysis of this situation is in two parts. In sections 5, 6 we analyze the case when the number of nonperturbative cubes on a scale smaller than $\lambda$ actually has dimension less than 1. Then in Section 7 we use an induction argument to show that we may relax this requirement to having dimension less than 1 on average.

Once we have a lower bound on the probability $P$ of exiting from a spherical shell, Theorem 1.3 follows almost exactly as in the proof of [5, Theorem 1.4]. This is accomplished in Section 8.

The main task of this paper was to replace the use of the Cameron-Martin formula in [5]. The reason is that the Cameron-Martin formula involves integrals of $|\mathbf{b}|^2$ and hence cannot be used to estimate the solution of (1.1), (1.2) in terms of integrals of $|\mathbf{b}|^p$ with $p < 2$. In [5] we obtained a lower bound on the exit probability $P$ from a spherical shell by combining the Cameron-Martin formula with [4, Theorem 1.1.a)]. In Appendix A we give a new proof of [4, Theorem 1.1.a)] which brings out the relationship between the methods employed in this paper and in [5]. We show that Theorem 1.1.a) is a consequence of the fact that Brownian motion confined to a long cylinder of radius $\lambda$ behaves ballistically on length scales larger than $\lambda$. The proof of the ballistic behavior of Brownian motion depends on estimating accurately the Dirichlet Green's function for the heat equation on a disc of radius $\lambda$ at large time. Estimates of this type are already known [2], [8] for operators in divergence form with $L^\infty$ coefficients. It therefore seems reasonable that one might be able to generalize the results of Appendix A to the situation considered in [2], [8].

In the subsequent work we need to do more than simply estimate the exit probability from a spherical shell. We need to keep careful track of fluctuations of densities. The simplest example of this is as follows: Suppose we have a density $\rho_1$ on a sphere $|x| = R_1$ and that density is propagated by the drift process to a density $\rho_2$ on a sphere $|x| = R_2$, $R_1 < R_2$. In the case of Brownian motion the fluctuation of $\rho_2$ is smaller than $\rho_1$. Thus if $\|\cdot\|_q$ denotes the $L^q$ norm, normalized so that $\|\mathbf{1}\|_q = 1$ we have that if $\|\rho_1 - \operatorname{Av}\rho_1\|_q \le \delta \operatorname{Av}\rho_1$ then $\|\rho_2 - \operatorname{Av}\rho_2\|_q \le \delta \operatorname{Av}\rho_2$, where $\operatorname{Av}\rho_1$, $\operatorname{Av}\rho_2$ denotes the average value, and $\delta$ is arbitrary. We shall show in Section 4 that for a perturbative drift this still holds provided $(R_2 - R_1) \sim R_1$. If $(R_2 - R_1) \ll R_1$ it may not hold. We investigate this question further in [3].

There is now an extensive literature on elliptic equations with non-smooth coefficients. Within it there are roughly speaking two currents of thought. On the one hand there is the approach dominated by techniques from harmonic analysis as exemplified in [11], [12]. On the other hand there is the approach where functional integration and probability is at the fore as in [6], [7]. In the present paper the former approach dominates whereas in the previous paper [5] the latter approach was more prominent. See also [13] for results related to those of this paper.

## 2. Proof of Theorem 1.1.

Our goal in this section is to give a proof of [5, Lemma 4.2] which is valid for $p > 1$. Theorem 1.1 will follow from this and the proof of [5, Theorem 1.3].

First we need a generalization of [5, Theorem 1.2]. Let $\Omega_R$ be a ball in $\mathbb{R}^3$ with radius $R$ and boundary $\partial\Omega_R$. For an arbitrary cube $Q \subset \mathbb{R}^3$ define $d(Q)$ by

$$d(Q) = \sup\{d(x, \partial\Omega_R) \, : \, x \in Q\}\,.$$

We define the Morrey space $M_r^q(\Omega_R)$ where $1 \le r \le q < \infty$ as follows: a measurable function $g : \Omega_R \to \mathbb{C}$ is in $M_r^q(\Omega_R)$ if $(R - |x|)^r \, |g(x)|^r$ is integrable on $\Omega_R$ and there is a constant $C > 0$ such that

$$R^{-r} \int_{Q \cap \Omega_R} (R - |x|)^r \, |g(x)|^r \, dx \le C^r \, |Q|^{1 - r/q}\,, \tag{2.1}$$

for all cubes $Q \subset \mathbb{R}^3$. The norm of $g$, $\|g\|_{q,r,R}$ is defined as

$$\|g\|_{q,r,R} = \inf\{C : \ (2.1) \text{ holds for all cubes } Q\}\,.$$

Let $\chi_R$ be the characteristic function of the set $\Omega_R$. Evidently $g$ is in $M_r^q(\Omega_R)$ if and only if the function $(1 - |x|/R)\, \chi_R(x)\, g(x)$ is in the Morrey space $M_r^q$ of [5].

Let $T$ be an integral operator on functions with domain $\Omega_R$ which has kernel $k_T : \Omega_R \times \Omega_R \longrightarrow \mathbb{C}$. Thus for measurable $g : \Omega_R \longrightarrow \mathbb{C}$ one defines $Tg$ by

$$Tg(x) = \int_{\Omega_R} k_T(x,y)\, g(y)\, dy\,, \qquad x \in \Omega_R\,.$$

**Proposition 2.1.** *Suppose the kernel $k_T$ of the integral operator $T$ satisfies the inequality*

$$|k_T(x,y)| \le \frac{|\mathbf{b}(x)|}{|x-y|^2} \min\Big\{1, \frac{R - |y|}{|x-y|}\Big\}\,, \qquad x, y \in \Omega_R\,,$$

*where $|\mathbf{b}| \in M_p^3$, $1 < p \le 3$. Then for any $r, q$ which satisfy the inequalities*

$$1 < r < p\,, \ r \le q < 3\,,$$

*the operator $T$ is a bounded operator on the space $M_r^q(\Omega_R)$. The norm of $T$ satisfies the inequality*

$$\|T\| \le C\, \|\mathbf{b}\|_{3,p}\,,$$

*where the constant $C$ depends only on $r, p, q$.*

REMARK. Observe that [5, Theorem 1.2] follows from Proposition 2.1 by letting $R \longrightarrow \infty$.

The proof of Proposition 2.1 follows the same lines as the proof of [5, Theorem 1.2]. Define an integer $n_0$ by

$$2^{-n_0-1} < 8\,R \le 2^{-n_0}\,.$$

Let $Q_0(\xi)$ be a cube with side of length $2^{-n_0}$ and centered at $\xi$. It is clear that if $\xi \in \Omega_R$ then $\Omega_R \subset Q_0(\xi)$. Let $K$ be one of the cubes $Q_0(\xi)$ with $\xi \in \Omega_R$. We define an operator $T_K$ on functions $u : \Omega_R \longrightarrow \mathbb{C}$ which have the property that $(R - |x|)\,u(x)$ is integrable. To do this we decompose $K$ into a dyadic decomposition of cubes $Q_n$ with sides of length $2^{-n}$, $n \ge n_0$. For any dyadic cube $Q \subset K$ with volume $|Q|$ let $u_Q$ be defined by

$$u_Q = R^{-1}\,|Q|^{-1} \int_{\Omega_R \cap Q} (R - |x|)\,|u(x)|\,dx\,.$$

If $Q$ is a distance of order $R$ from $\partial\Omega_R$ then $u_Q$ is comparable to the average of $|u|$ on $Q$. Otherwise $u_Q$ can be much smaller than the average. For $n \ge n_0$ define the operator $S_n$ by

$$S_n u(x) = 2^{-n}\Big(\frac{R}{d(Q_n)}\Big) u_{Q_n}\,, \qquad x \in Q_n\,.$$

The operator $T_K$ is then given by

$$T_K u(x) = \sum_{n=n_0}^{\infty} |\mathbf{b}(x)|\,S_n u(x)\,, \qquad x \in \Omega_R\,.$$

It follows now by Jensen's inequality that there is a universal constant $C$ such that for any $r \ge 1$ and cube $Q$ there is the inequality

$$\int_{Q\cap\Omega_R} (R - |x|)^r\,|Tu(x)|^r\,dx \le \frac{C^r}{|\Omega_R|} \int_{\Omega_R} d\xi \int_{Q\cap\Omega_R} (R - |x|)^r\,|T_{Q_0(\xi)}u(x)|^r\,dx\,. \tag{2.2}$$

Hence it is sufficient to prove Proposition 2.1 with the operator $T$ replaced by $T_K$ where $K = Q_0(\xi)$ and $\xi \in \Omega_R$ is arbitrary.

The following lemma generalizes [5, Lemma 2.4]. It is proved in an exactly similar fashion.

**Lemma 2.1.** *Let $Q' \subset K$ be an arbitrary dyadic subcube of $K$ with side of length $2^{-n_{Q'}}$. Suppose $r, p$ satisfy the inequality $1 \le r < p$. Then there are constants $\varepsilon$, $C > 0$ depending only on $r$ and $p$ such that*

$$|Q|^{1/3+\varepsilon}\, u_Q \le |Q'|^{1/3+\varepsilon}\, u_{Q'}\ ,$$

*for all dyadic subcubes $Q$ of $Q'$ implies the inequality*

$$\int_{Q'} (R-|x|)^r \Big( \sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\Big)^r\, dx \le C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'| R^r\, u_{Q'}^r\ .$$

If we replace the function $u(x)$ by the function $(R-|x|)\,u(x)$ in the argument of [5] and use the previous lemma we conclude:

**Corollary 2.1.** *For any dyadic subcube $Q' \subset K$ one has*

$$\int_{Q'} (R-|x|)^r \Big( \sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\Big)^r\, dx \le C^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q'} (R-|x|)^r\, |u(x)|^r\, dx\,,$$

*for some constant $C$ depending only on $r$ and $p$.*

Proposition 2.1 for $T_K$ follows now from the last corollary in the same way as the corresponding theorem in [5] from [5, Lemma 2.4].

Next let $g \in L^q(\partial\Omega_R)$, $1 \le q < \infty$. We define a function $Bg(x)$ for $x \in \Omega_R$ by

$$Bg(x) = |\mathbf{b}(x)| \int_{|y|=R} \frac{|g(y)|}{|x-y|^3}\, dy\,, \qquad |x| < R\,. \tag{2.3}$$

**Lemma 2.2.** *Suppose $\mathbf{b} \in M_p^3$ with $1 < p < 3/2$, and $r, q$ are numbers which satisfy the inequalities*

$$1 < r < p\,, \qquad \frac{1}{r} > \frac{1}{p} + \frac{2}{q}\ . \tag{2.4}$$

*Then $B$ is a bounded operator from $L^q(\partial\Omega_R)$ to $M_r^{q_1}(\Omega_R)$ where*

$$\frac{1}{q_1} = \frac{1}{3} + \frac{2}{3\,q}\,. \tag{2.5}$$

*Furthermore the norm of $B$ satisfies an inequality*

$$\|B\| \le CR^{-1}\,\|\mathbf{b}\|_{3,p} \text{ and } C \text{ is a universal constant.} \tag{2.6}$$

PROOF. From (2.1) we need to estimate the integral

$$R^{-r}\int_{Q\cap\Omega_R} (R-|x|)^r\,|Bg(x)|^r\,dx \tag{2.7}$$

on an arbitrary cube $Q$. From Holder's inequality this integral is bounded by

$$R^{-r}\Big(\int_{Q\cap\Omega_R} |\mathbf{b}(x)|^p\,dx\Big)^{r/p} \cdot\Big(\int_{Q\cap\Omega_R} (R-|x|)^{rp'}\Big(\int_{|y|=R}\frac{|g(y)|}{|x-y|^3}\,dy\Big)^{rp'}\,dx\Big)^{1/p'},$$

where $r/p + 1/p' = 1$.

Again from Holder we have

$$\int_{|y|=R}\frac{|g(y)|}{|x-y|^3}\,dy \le \|g\|_q\Big(\int_{|y|=R}\frac{dy}{|x-y|^{3q'}}\Big)^{1/q'},$$

where $1/q + 1/q' = 1$. Using the fact that

$$\int_{|y|=R}\frac{dy}{|x-y|^{3q'}} \le \frac{C^{q'}}{(R-|x|)^{3q'-2}}\,,$$

for some universal constant $C$, we conclude that (2.7) is bounded by

$$R^{-r}\Big(\int_{Q\cap\Omega_R} |\mathbf{b}(x)|^p\,dx\Big)^{r/p} \cdot C^r\,\|g\|_q^r\Big(\int_{Q\cap\Omega_R} (R-|x|)^{-2rp'/q}\,dx\Big)^{1/p'}. \tag{2.8}$$

The inequality (2.4) implies that $2\,r\,p'/q < 1$.

Hence if we use the fact that $\mathbf{b} \in M^3_p$ then (2.8) implies that (2.7) is bounded by

$$C^r R^{-r}\, \|\mathbf{b}\|^r_{3,p}\, \|g\|^r_q\, |Q|^{r/p - r/3}\, d(Q)^{-2r/q}\, |Q|^{1/p'} \le C^r R^{-r}\, \|\mathbf{b}\|^r_{3,p}\, \|g\|^r_q\, |Q|^{1-r/q_1}\,,$$

on using the fact that $d(Q) \ge |Q|^{1/3}$.

Hence $Bg \in M^{q_1}_r(\Omega_R)$ and its norm satisfies the inequality (2.6).

Suppose $G_D(x,y), x, y \in \Omega_R$ is the Dirichlet kernel, whence

$$G_D(x,y) = \frac{1}{4\pi\,|x-y|} - \frac{1}{4\pi}\Big(\frac{R}{|y|}\Big)\frac{1}{|x-\overline{y}|}\,,$$

where $\bar{y}$ is the conjugate of y in the sphere $\partial\Omega_R$. Let $g \in M^q_1(\Omega_R)$, $1 \le q < \infty$ and define $Hg$ by

$$Hg(x) = \int_{\Omega_R} G_D(x,y)\, g(y)\, dy\,, \qquad x \in \Omega_{R/2}\,.$$

**Lemma 2.3.** *Suppose $m > 1$ satisfies the inequality*

$$\frac{2}{3} + \frac{1}{m\,q} > \frac{1}{q}\,.$$

*Then $H$ is a bounded operator from $M^q_1(\Omega_R)$ to $L^m(\Omega_{R/2})$ and the norm of $H$, $\|H\|$, satisfies an inequality*

$$\|H\| \le C_{q,m}\, R^{2+3/m-3/q}\,, \tag{2.9}$$

*where $C_{q,m}$ is a constant depending only on $q$ and $m$.*

PROOF. We write $Hg = H_1 g + H_2 g$, where

$$H_1 g(x) = \int_{\Omega_{3R/4}} G_D(x,y)\, g(y)\, dy\,.$$

Since we are restricting $x$ to the region $|x| < R/2$, there is a universal constant $C$ such that

$$\|H_2 g\|_\infty \le \frac{C}{R^2}\int_{\Omega_R} (R - |y|)\, |g(y)|\, dy \le \frac{C}{R}\, |\Omega_R|^{1-1/q}\, \|g\|_{q,1,R}\,.$$

It follows that $H_2 g$ is in $L^m(\Omega_{R/2})$ for any $m \ge 1$ and

$$\|H_2\| \le C\, R^{2+3/m-3/q}\,, \qquad \text{for some universal constant } C\,. \tag{2.10}$$

Next we bound $H_1 g$ by using the method of proof for the John-Nirenberg inequality [10]. For any $\alpha$, $0<\alpha<1$, we have the inequality

$$\begin{aligned} |H_1 g(x)| &\le \frac{1}{4\pi}\int_{\Omega_{3R/4}} \frac{|g(y)|}{|x-y|}\,dy \\ &\le \frac{1}{4\pi}\Big(\int \frac{|g(y)|}{|x-y|^{\alpha m'}}\,dy\Big)^{1/m'}\Big(\int \frac{|g(y)|}{|x-y|^{(1-\alpha)m}}\,dy\Big)^{1/m}, \end{aligned} \tag{2.11}$$

where $1/m + 1/m' = 1$. Now

$$\begin{aligned} \int_{\Omega_{3R/4}} \frac{|g(y)|}{|x-y|^{\alpha m'}}\,dy &= \frac{1}{\alpha m'}\int_0^\infty \frac{d\rho}{\rho^{\alpha m'+1}} \int_{\Omega_{3R/4}\cap\{y:|x-y|<\rho\}} |g(y)|\,dy \\ &\le \frac{1}{\alpha\, m'}\int_0^{2R} \frac{d\rho}{\rho^{\alpha m'+1}}\, \|g\|_{q,1,R}\,\rho^{3-3/q} \\ &\quad + \frac{1}{\alpha\, m'}\int_{2R}^\infty \frac{d\rho}{\rho^{\alpha m'+1}}\, \|g\|_{q,1,R}\,(2R)^{3-3/q} \\ &\le C\,\|g\|_{q,1,R}\,R^{3-3/q-\alpha m'}\,, \end{aligned} \tag{2.12}$$

for some constant C provided

$$3 - \frac{3}{q} - \alpha\, m' > 0\,. \tag{2.13}$$

On the other hand

$$\begin{aligned} \int_{\Omega_{R/2}} dx \int_{\Omega_{3R/2}} \frac{|g(y)|\,dy}{|x-y|^{(1-\alpha)m}} &\le C\,R^{3-(1-\alpha)m}\int_{\Omega_{3R/2}} |g(y)|\,dy \\ &\le C\,R^{3-(1-\alpha)m}\,\|g\|_{q,1,R}\,R^{3-3/q}\,, \end{aligned} \tag{2.14}$$

for some constant $C$ depending on $\alpha\, m$, provided

$$(1-\alpha)\,m < 3. \tag{2.15}$$

It is possible to choose an $\alpha$, $0 < \alpha < 1$, satisfying both (2.13) and (2.15) provided $m$ and $q$ satisfy the inequality (2.9). Choosing such an $\alpha$ yields the inequality

$$\int_{\Omega_{R/2}} |H_1 g(x)|^m \, dx \leq C_{q,m}^m \, R^{2m+3-3m/q} \, \|g\|_{q,1,R}^m \,, \tag{2.16}$$

upon using (2.11)-(2.14). Here the constant $C_{q,m}$ depends on $q$ and $m$. Taking the $m$-th root of (2.16) and combining with (2.10) yields the result.

PROOF OF [5, LEMMA 4.2] FOR $p > 1$. We shall freely use the notation of [5]. Let us suppose that $p$ and $q$ satisfy the inequalities

$$\frac{1}{p} + \frac{2}{q} < 1 \,, \qquad 1 < p < \frac{3}{2} \,. \tag{2.17}$$

It will be sufficient for us to show that for any $\delta > 0$ there exists $\varepsilon > 0$ depending only on $p, q$ such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies that the operator $Q_n^*$ is a bounded operator from $L_\mu^q(A_{n-1})$ to $L_\mu^q(A_n)$ and satisfies the inequality

$$\|Q_n^* f\|_{q,\mu} \leq \delta \, \|f\|_{q,\mu} \,, \tag{2.18}$$

where $\| \cdot \|_{q,\mu}$ is the norm in the space $L_\mu^q$. To do this observe that $Q_n^* f(x)$ is given by the formula

$$Q_n^* f(x) = \frac{2^{n-1}}{(\sqrt{2}-1)} \int_{2^{-n+1}}^{2^{-n+3/2}} (-\Delta_{D,\lambda})^{-1} (I - T_\lambda)^{-1} \, \mathbf{b} \cdot \nabla P_\lambda f(x) \, d\lambda \,,$$

where $2^{-n} < |x| < 2^{-n+1/2}$. This follows from (3.38) of [5].

Now let us assume for the moment that $\lambda$ is fixed and $f$ is in $L^q(\partial\Omega_\lambda)$ with norm $\|f\|_{q,\partial\Omega_\lambda}$. It is easy to see from the explicit formula for the Poisson kernel that

$$|\mathbf{b}(x) \cdot \nabla P_\lambda f(x)| \leq CBf(x) \,, \qquad x \in \Omega_\lambda \,,$$

where $C$ is a universal constant and $B$ is the operator defined by (2.3). In view of (2.17) we can choose $r > 1$ such that (2.4) holds. Hence by Lemma 2.2, $\mathbf{b} \cdot \nabla P_\lambda f$ is in the space $M_r^{q_1}(\Omega_R)$ where $q_1$ is determined from (2.5). It is easy to verify that the operator $T_\lambda$ has kernel which satisfies the conditions for Proposition 2.1. Hence the function $(I - T_\lambda)^{-1}\,\mathbf{b}\cdot\nabla P_\lambda f$ is also in the space $M_r^{q_1}(\Omega_R)$ provided $\|\mathbf{b}\|_{3,p} < \varepsilon$ and $\varepsilon$ is sufficiently small. Now Lemma 2.3 tells us that the function

$$g_\lambda(x) = (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1}\,\mathbf{b}\cdot\nabla P_\lambda f(x)$$

is in the space $L^m(\Omega_{2^{-n+1/2}})$ provided $m$ satisfies the inequality

$$\frac{2}{3} + \frac{1}{m\,q_1} > \frac{1}{q_1}\,, \tag{2.19}$$

with $q_1$ given by (2.5). Furthermore, the norm of $g_\lambda$ satisfies an inequality

$$\|g_\lambda\|_m \le C_{p,q,m}\,\varepsilon\,2^{-n(1+3/m-3/q_1)}\,\|f\|_{q,\partial\Omega_\lambda}\,, \tag{2.20}$$

where the constant $C_{p,q,m}$ depends only on $p, q, m$. It is clear that the inequality (2.19) implies that $1/m > (2-q)/(2+q)$. Taking $m = q$, we have from (2.20) the inequality

$$\|g_\lambda\|_q \le C_{p,q}\,\varepsilon\,2^{-n/q}\,\|f\|_{q,\partial\Omega_\lambda}\,.$$

The triangle inequality now yields

$$\begin{aligned}
&\left\|\frac{2^{n-1}}{(\sqrt{2}-1)}\int_{2^{-n+1}}^{2^{-n+3/2}} g_\lambda\,d\lambda\right\|_q \\
&\quad\le \frac{2^{n-1}}{(\sqrt{2}-1)}\int_{2^{-n+1}}^{2^{-n+3/2}} \|g_\lambda\|_q\,d\lambda \\
&\quad\le \frac{2^{n-1}}{(\sqrt{2}-1)}\int_{2^{-n+1}}^{2^{-n+3/2}} C_{p,q}\,\varepsilon\,2^{-n/q}\,\|f\|_{q,\partial\Omega_\lambda}\,d\lambda\,.
\end{aligned} \tag{2.21}$$

From Jensen's inequality we see that

$$\frac{2^{n-1}}{(\sqrt{2}-1)}\int_{2^{-n+1}}^{2^{-n+3/2}} \|f\|_{q,\partial\Omega_\lambda}\,d\lambda \le C^{1/q}\,2^{-2n/q}\,\|f\|_{q,\mu}\,, \tag{2.22}$$

where $C$ is a universal constant. Putting (2.21), (2.22) together gives us the inequality (2.18) with $\delta$ proportional to $\varepsilon$.

## 3. A counterexample.

Let $r_0, R_0$ be two numbers which satisfy $0 < r_0 < R_0 < \infty$, and let $v$ be the solution of the two dimensional boundary value problem,

$$\begin{cases} \Delta v(x) = 0\,, & r_0 < |x| < R_0\,, \\ v(x) = 1\,, & |x| = r_0\,, \\ v(x) = 0\,, & |x| = R_0\,. \end{cases} \tag{3.1}$$

The function $v$ is explicitly given by the formula,

$$v(x) = \frac{\log\left(\dfrac{R_0}{|x|}\right)}{\log\left(\dfrac{R_0}{r_0}\right)}\,. \tag{3.2}$$

For $a \in \mathbb{R}^2$ and $r > 0$, let $D(a,r)$ be the disc centered at $a$ with radius $r$ and let $\overline{D}(a,r)$ denote the closure of $D(a,r)$. We can extend $v$ to $\mathbb{R}^2 \backslash \overline{D}(0,r_0)$ by setting $v$ to be zero for $|x| \geq R_0$. In that case $v$ is a subharmonic function, and so in the distributional sense one has

$$\Delta v(x) \geq 0\,, \qquad |x| > r_0\,. \tag{3.3}$$

**Lemma 3.1.** *Let $U \subset \mathbb{R}^2$ be a domain and suppose $a_j \in U$, $j = 1,\ldots,k$. Let $r_0 > 0$ be such that all the sets $\overline{D}(a_j, r_0)$ are disjoint and contained in $U$, $j = 1,\ldots,k$. Let $W$ be the domain*

$$W = U \backslash \bigcup_{j=1}^{k} \overline{D}(a_j, r_0)\,.$$

*Let $u(x)$ be the solution of the equation*

$$\begin{cases} \Delta u(x) = 0\,, & x \in W\,, \\ u(x) = 0\,, & x \in \partial U\,, \\ u(x) = 1\,, & x \in \partial D(a_j, r_0)\,,\ j = 1,\ldots,k\,. \end{cases}$$

*For $R_0 > r_0$, suppose $S_0$ is a subset of $\{1,\ldots,k\}$ with the property that*

$$\overline{D}(a_j, R_0) \subset U\,, \qquad j \in S_0\,.$$

*For $x \in W$ define a function $\overline{u}(x)$ by*

$$\overline{u}(x) = \frac{\sum_{j\in S_0} v(x-a_j)}{\sup\left\{ \sum_{j\in S_0} v(a_i - a_j + \delta) : |\delta| = r_0 \,,\ 1 \le i \le k \right\}} \,,$$

*where $v$ is given by* (3.2). *Then there is the inequality*

$$u(x) \ge \overline{u}(x)\,, \qquad x \in W\,. \tag{3.4}$$

PROOF. From (3.3) it follows that $\overline{u}$ is subharmonic on $W$. By definition of $S_0$ one has $\overline{u}(x) = 0$, $x \in \partial U$. Furthermore one has $\overline{u}(x) \le 1$ if $x \in \partial D(a_j, r_0)$, $j = 1, \ldots, k$. The maximum principle now yields the inequality (3.4).

Next let $\mathbb{Z}^2_\lambda$ be the lattice

$$\mathbb{Z}^2_\lambda = \{\lambda(n,m) :\ n, m \text{ integers}\}\,.$$

For $2\,r_0 < \lambda < R$ let $W_{\lambda,R}$ be the set

$$W_{\lambda,R} = D(0,R) \backslash \cup \{\overline{D}(a,r_0) :\ a \in \mathbb{Z}^2_\lambda,\ \overline{D}(a,r_0) \subset D(0,R)\}\,. \tag{3.5}$$

Consider the function $u(x)$ which is the solution of the boundary value problem

$$\begin{cases} \Delta u(x) = 0\,, & x \in W_{\lambda,R}\,, \\ u(x) = 0\,, & |x| = R\,, \\ u(x) = 1\,, & x \in \partial D(a,r_0)\,,\ a \in \mathbb{Z}^2_\lambda\,,\ \overline{D}(a,r_0) \subset D(a,R)\,. \end{cases} \tag{3.6}$$

Evidently $u(x)$ is the probability that Brownian motion started at $x \in D(0,R)$ hits one of the discs radius $r_0$, centered at $a \in \mathbb{Z}^2_\lambda$, before exiting the region $D(0,R)$. Let us consider the quantity $\inf\{u(x) : |x| \le R/2\}$. If $\lambda, r_0$ are fixed and $R$ becomes large we should expect this quantity to converge to 1 since a Brownian path is unlikely to avoid all the discs centered at points in $\mathbb{Z}^2_\lambda$ over large distances. The following lemma gives an estimate which verifies this intuition:

**Lemma 3.2.** *Suppose* $8\,r_0 < R$ *and* $u(x)$ *is the solution of* (3.6). *Then there is a universal constant* $c > 0$ *such that*

$$\inf_{|x| \le R/2} u(x) > 1 - \frac{c\,\lambda}{R}\,, \tag{3.7}$$

*provided* $\lambda$ *lies in the region,*

$$2\,r_0 < \lambda < \frac{R}{\log\left(\dfrac{R}{r_0}\right)}\,. \tag{3.8}$$

PROOF. Let us take $R_0 = R/4$ in (3.2). Then by Lemma 3.1 we have that

$$u(x) \ge \sum_{a \in \mathbb{Z}^2_\lambda} \frac{v(x-a)}{\sup\Big\{ \sum_{a \in \mathbb{Z}^2_\lambda} v(\delta - a) : |\delta| = r_0 \Big\}}\,, \tag{3.9}$$

provided $|x| \le R/2$.

We have now that

$$\begin{aligned} \sum_{a \in \mathbb{Z}^2_\lambda} v(\delta - a) &\sim \sum_{\substack{n \in \mathbb{Z}^2 \\ 0 < |n| < R/(4\lambda)}} \frac{\log\left(\dfrac{R}{4\,\lambda\,|n|}\right)}{\log\left(\dfrac{R}{4\,r_0}\right)} \\ &\sim \frac{1}{\log\left(\dfrac{R}{4\,r_0}\right)} \int_{|x| < R/(4\lambda)} \log\left(\frac{R}{4\,\lambda\,|x|}\right) dx \\ &= \frac{\dfrac{\pi}{2}\left(\dfrac{R}{4\,\lambda}\right)^2}{\log\left(\dfrac{R}{4\,r_0}\right)}\,. \end{aligned}$$

By virtue of (3.8) we can conclude then that

$$\sum_{a \in \mathbb{Z}^2_\lambda} v(\delta - a) \ge c\,\frac{\pi}{2}\,\frac{\left(\dfrac{R}{4\,\lambda}\right)^2}{\log\left(\dfrac{R}{4\,r_0}\right)}\,, \tag{3.10}$$

for some universal constant $c$.

We estimate the numerator of (3.9) by Taylor expansion. Let $b \in \mathbb{Z}^2_\lambda$ be the nearest lattice point to $x$ and $y = x - b$. Thus $|y| < \lambda/\sqrt{2}$. Hence we have

$$
\begin{aligned}
\sum_{a\in\mathbb{Z}^2_\lambda} v(x-a) &= \sum_{a\in\mathbb{Z}^2_\lambda} v(y-a) \\
&= \sum_{a\in\mathbb{Z}^2_\lambda} v(\delta-a) + \sum_{a\in\mathbb{Z}^2_\lambda} \left(v(y-a) - v(\delta-a)\right),
\end{aligned}
\tag{3.11}
$$

where $|\delta| = r_0$. By Taylor's theorem we have

$$
\sum_{a\in\mathbb{Z}^2_\lambda} (v(y-a) - v(\delta-a)) = \sum_{a\in\mathbb{Z}^2_\lambda} \int_0^1 (y-\delta)\cdot\nabla v\left(\delta - a + t\,(y-\delta)\right) dt\,.
$$

Now if we use the inequality

$$
|\nabla v(x)| \le \frac{\log\left(\dfrac{R}{4\,r_0}\right)}{|x|}\,,
$$

we conclude from (3.10), (3.11) that

$$
1 - u(x) \le \frac{C}{\left(\dfrac{R}{4\,\lambda}\right)^2} \sum_{\substack{n\in\mathbb{Z}^2 \\ 0<|n|<R/(4\lambda)}} \frac{1}{|n|}
\tag{3.12}
$$

where $C$ is a universal constant. The inequality (3.7) follows now by observing that the sum in (3.12) is of order $\lambda/R$.

Next we wish to obtain a three dimensional generalization of Lemma 3.2. First we consider a generalization of the boundary value problem (3.1).

Let $v(x)$ be the solution of the problem

$$
\begin{cases}
\Delta v(x) = \eta\, v(x)\,, & r_0 < |x| < R_0\,, \\
v(x) = 1\,, & |x| = r_0\,, \\
v(x) = 0\,, & |x| = R_0\,.
\end{cases}
\tag{3.13}
$$

The function $v(x)$ is a Brownian motion expectation value. In fact let $X(t)$ be Brownian motion started at a point $x$ and $\tau$ be the exit time from the region $\{y : r_0 < |y| < R_0\}$. Let $\chi$ be the characteristic function,

$$\chi(z) = \begin{cases} 1\,, & \text{if } |z| = r_0\,, \\ 0\,, & \text{if } |z| = R_0\,. \end{cases}$$

Then we have

$$v(x) = E_x[e^{-\eta\tau}\chi(X(\tau))]\,. \tag{3.14}$$

It is well known that the solution of (3.13) exists provided the parameter $\eta$ is larger than the largest eigenvalue of the Dirichlet Laplacian. For $\eta = 0$ the solution of (3.13) is given by (3.2). For $\eta \neq 0$ we have the following:

**Lemma 3.3.** *Let $I_0$ be the modified Bessel function of the first kind defined by the infinite series,*

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}.$$

*Suppose $\eta$ satisfies the condition*

$$I_0(\sqrt{\eta}\, t) \neq 0\,, \qquad r_0 \leq t \leq R_0\,. \tag{3.15}$$

*Then the solution $v$ of* (3.13) *is given by*

$$v(x) = \frac{I_0(\sqrt{\eta}\, r)\displaystyle\int_r^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}}{I_0(\sqrt{\eta}\, r_0)\displaystyle\int_{r_0}^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}}\,, \tag{3.16}$$

*where $r = |x|$.*

PROOF. The problem (3.13) is rotation invariant. Hence $v(x)$ is just a function of $r = |x|$, $v(x) = v(r)$, and satisfies the equation

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\left(r\,\dfrac{dv}{dr}\right) = \eta\, v\,, \\ v(r_0) = 1\,, \\ v(R_0) = 0\,. \end{cases} \tag{3.17}$$

This is a Bessel equation of order zero. It is easy to see that $v(r) = I_0(\sqrt{\eta}\, r)$ is a solution of the equation (3.17), but not the boundary condition. A second solution can be found by the method of reduction of order provided (3.15) holds. It is given by

$$v(r) = I_0(\sqrt{\eta}\, r) \int_r^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)} \,. \tag{3.18}$$

It follows from (3.18) that the function (3.16) satisfies (3.17).

We consider a region in $\mathbb{R}^3$ which has a two dimensional structure. For $0 < 4\, r_0 < R_0$ consider the two cylinders

$$\begin{aligned} S_1 &= \{x = (x_1, x_2, x_3) : \ -R_0 < x_3 < R_0,\ x_1^2 + x_2^2 < R_0^2\}\,, \\ S_2 &= \left\{x = (x_1, x_2, x_3) : \ \frac{-R_0}{2} < x_3 < \frac{R_0}{2},\ x_1^2 + x_2^2 < r_0^2\right\}. \end{aligned} \tag{3.19}$$

The region $\mathcal{D}$ we wish to consider is given by $\mathcal{D} = S_1 \backslash \overline{S}_2$. The boundary $\partial\mathcal{D}$ of $\mathcal{D}$ is evidently the union of $\partial S_1$ and $\partial S_2$. We consider the problem

$$\begin{cases} \Delta v(x) = 0\,, & x \in \mathcal{D}\,, \\ v(x) = 1\,, & x \in \partial S_2\,, \\ v(x) = 0\,, & x \in \partial S_1\,. \end{cases} \tag{3.20}$$

**Lemma 3.4.** *Suppose* $x = (x_1, x_2, x_3) \in \mathcal{D}$, *and* $r^2 = x_1^2 + x_2^2$. *Then there is a universal constant* $c > 0$ *such that*

$$v(x) \geq c\, \frac{\log\left(\dfrac{R_0}{r}\right)}{\log\left(\dfrac{R_0}{r_0}\right)} \,, \tag{3.21}$$

*provided* $|x_3| < R_0/4$.

PROOF. Consider the two dimensional Brownian motion started at $(x_1, x_2)$ and consider all paths which hit the circle $r = r_0$ before hitting the circle $r = R_0$. Let $\tau$ be the hitting time for such paths and suppose $\rho(r,t)$ is the density for $\tau$. Then the function

$$\int_0^\infty e^{-\eta t}\, \rho(r,t)\, dt \tag{3.22}$$

is the solution to the problem (3.13). This follows from the representation (3.14). Next let $X_3(t)$ be Brownian motion started at $x_3 \in \mathbb{R}$ and let $\tau_3$ be the first exit time from the interval $[-R_0, R_0]$. We define $w(x_3,t)$ by

$$w(x_3,t) = P_{x_3}\Big[-\frac{R_0}{2} < X_3(t\wedge\tau_3) < \frac{R_0}{2}\Big]. \tag{3.23}$$

Evidently $w(x_3,t)$ satisfies the heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x_3^2}\,, \qquad -R_0 < x_3 < R_0\,,\ t > 0\,, \tag{3.24}$$

with the boundary conditions

$$w(R_0,t) = w(-R_0,t) = 0\,, \qquad t > 0\,, \tag{3.25}$$

and the initial conditions

$$w(x_3,0) = \begin{cases} 1\,, & -\dfrac{R_0}{2} < x_3 < \dfrac{R_0}{2}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.26}$$

It is clear from the definition (3.23) that the solution $v$ of (3.20) has the representation

$$v(x_1,x_2,x_3) = v(r,x_3) = \int_0^\infty \rho(r,t)\,w(x_3,t)\,dt\,. \tag{3.27}$$

Observe that for any $\alpha > 0$ there is a constant $\gamma_\alpha > 0$ depending only on $\alpha$, such that

$$w(x_3,t) \ge \gamma_\alpha > 0\,, \qquad |x_3| < \frac{R_0}{4}\,,\ 0 < t < \alpha R_0^2\,.$$

Hence, provided $|x_3| < R_0/4$ there is the inequality

$$v(r,x_3) \ge \gamma_\alpha \int_0^{\alpha R_0^2} \rho(r,t)\,dt\,. \tag{3.28}$$

Now from Lemma 3.3 we see there exists an $\varepsilon > 0$ independent of $R_0$ such that

$$\int_0^\infty \exp\Big(\frac{\varepsilon\, t}{R_0^2}\Big)\rho(r,t)\,dt \le C_\varepsilon \frac{\log\Big(\dfrac{R_0}{r}\Big)}{\log\Big(\dfrac{R_0}{r_0}\Big)}\,,$$

for some constant $C_\varepsilon > 0$ depending only on $\varepsilon$. Thus for any $\alpha > 0$ one has the inequality

$$\int_{\alpha R_0^2}^{\infty} \rho(r,t)\,dt \le \exp\left(-\varepsilon\alpha\right) C_\varepsilon \,\frac{\log\left(\dfrac{R_0}{r}\right)}{\log\left(\dfrac{R_0}{r_0}\right)}\,.$$

Choosing $\alpha$ such that $\exp\left(-\varepsilon\,\alpha\right) C_\varepsilon < 1/2$, we conclude that

$$\int_0^{\alpha R_0^2} \rho(r,t)\,dt \ge \frac{1}{2}\,\frac{\log\left(\dfrac{R_0}{r}\right)}{\log\left(\dfrac{R_0}{r_0}\right)}\,. \tag{3.29}$$

The inequality (3.21) follows now from (3.28), (3.29).

**Lemma 3.5.** *Let $v(x) = v(x_1,x_2,x_3) = v(r,x_3)$ be the solution of (3.20). Then there is a universal constant $C > 0$ such that*

$$\left|\frac{\partial v(r,0)}{\partial r}\right| \le \frac{C}{r}\log\left(\frac{R_0}{r_0}\right). \tag{3.30}$$

PROOF. The eigenfunction expansion for the solution to the problem (3.24), (3.25), (3.26) is given by

$$w(x_3,t) = \frac{1}{R_0}\sum_{m=1}^{\infty} \exp\Big(-\frac{\pi^2 m^2}{4R_0^2}\,t\Big)\sin\Big(\frac{\pi\,m}{2R_0}\,(x_3+R_0)\Big)$$
$$\cdot \int_{-R_0/2}^{R_0/2} \sin\Big(\frac{\pi\,m}{2R_0}\,(\zeta+R_0)\Big)\,d\zeta\,.$$

Hence from (3.27) we have

$$v(r,0) = \sum_{m=1}^{\infty} \frac{2}{\pi\,m}\left(1-(-1)^m\right) u\Big(r,\frac{\pi^2 m^2}{4R_0^2}\Big)\sin\Big(\frac{\pi\,m}{4}\Big)\,,$$

where $u(r,\eta)$ is the function given by (3.22). Consequently

$$\frac{\partial v(r,0)}{\partial r} = -\sum_{m=1}^{\infty} a_m(r)\sin\Big(\frac{\pi\,m}{4}\Big)\,, \tag{3.31}$$

where

$$a_m(r) = \frac{-2}{\pi\, m}\,(1-(-1)^m)\,\frac{\partial u}{\partial r}\Big(r, \frac{\pi^2 m^2}{4R_0^2}\Big) \geq 0\,. \tag{3.32}$$

The inequality (3.32) follows from the maximum principle applied to the equation (3.13) which $u(r,\eta)$ satisfies. We shall prove in the appendix that

$$\frac{\partial}{\partial r}\,\frac{\partial}{\partial \eta}\Big(\frac{u(r,\eta)}{\sqrt{\eta}}\Big) > 0\,, \qquad r_0 < r < R_0\ ,\ \eta > 0\,. \tag{3.33}$$

It follows then from (3.33) that $a_m(r)$ is a decreasing function of odd integers $m$. Hence by the alternating series theorem applied to (3.31) we conclude that

$$\Big|\frac{\partial v(r,0)}{\partial r}\Big| \leq a_1(r) + a_3(r)\,.$$

Next we use Lemma 3.3 to estimate $a_1(r)$, $a_3(r)$. From (3.16) we see that

$$\frac{\partial u}{\partial r}\,(r,\eta) = -\frac{\Big(\dfrac{I_0(\sqrt{\eta}\, r)}{r} - \sqrt{\eta}\, I_0'(\sqrt{\eta}\, r)\displaystyle\int_r^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}\Big)}{I_0(\sqrt{\eta}\, r_0)\displaystyle\int_{r_0}^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}}\,.$$

Thus we have

$$\Big|\frac{\partial u}{\partial r}\Big(r, \frac{\pi^2}{4R_0^2}\Big)\Big| \leq \frac{C}{r}\,\frac{1 + \Big(\dfrac{r}{R_0}\Big)^2 \log\Big(\dfrac{R_0}{r}\Big)}{\log\Big(\dfrac{R_0}{r_0}\Big)}\,,$$

for some universal constant $C > 0$. In view of the fact that $z^2\log(1/z) \leq 1/e$, for $0 < z < 1$, we conclude that

$$a_1(r) \leq \frac{C}{r\log\Big(\dfrac{R_0}{r_0}\Big)}\,,$$

for some universal constant $C$. Since a similar inequality holds for $a_3(r)$ the inequality (3.30) follows.

We wish to obtain a three dimensional analogue of Lemma 3.2. For $a = (a_1, a_2) \in \mathbb{R}^2$ let $S_2(a)$ be the cylinder $S_2$ of (3.19) centered at the point $(a_1, a_2, 0) \in \mathbb{R}^3$. Then for $2\,r_0 < \lambda < L$ we define the set $W_{\lambda,L}$ to be

$$W_{\lambda,L} = S_1 \backslash \cup \{S_2(a) :\ a \in \mathbb{Z}^2_\lambda,\ \overline{S_2(a)} \subset S_1\}\,, \tag{3.34}$$

where we take $R_0 = L$ in (3.19). Thus $W_{\lambda,L}$ is a three dimensional analogue of the set (3.5). Consider the Dirichlet problem corresponding to (3.6),

$$\begin{cases} \Delta u(x) = 0\,, & x \in W_{\lambda,L}\ , \\ u(x) = 0\,, & x \in \partial S_1\ , \\ u(x) = 1\,, & x \in \partial S_2(a)\,,\ \overline{S_2(a)} \subset S_1\ ,\ a \in \mathbb{Z}^2_\lambda\ . \end{cases} \tag{3.35}$$

The following lemma generalizes Lemma 3.2.

**Lemma 3.6.** *Suppose* $8\,r_0 < L$ *and* $u(x)$ *is the solution of* (3.35). *Then there is a universal constant* $c > 0$ *such that*

$$\inf_{|x| \le L/4} u(x) > 1 - \frac{c\,\lambda}{L}\ , \tag{3.36}$$

*provided* $\lambda$ *lies in the region*

$$2\,r_0 < \lambda < \frac{L}{\log\left(\dfrac{L}{r_0}\right)}\ .$$

PROOF. First consider $x = (x_1, x_2, 0)$. Let $v(x)$ be the solution of the problem (3.20) with $R_0 = L/4$. Then

$$u(x) \ge \frac{\displaystyle\sum_{a \in \mathbb{Z}^2_\lambda} v(x-a)}{\sup\left\{\displaystyle\sum_{a \in \mathbb{Z}^2_\lambda} v(y-a) :\ y \in \partial S_2\right\}}\ . \tag{3.37}$$

From Lemma 3.4 it follows that

$$\sup\left\{\sum_{a \in \mathbb{Z}^2_\lambda} v(y-a) :\ y \in \partial S_2\right\} \ge c\,\frac{\left(\dfrac{L}{4\,\lambda}\right)^2}{\log\left(\dfrac{L}{4\,r_0}\right)}\ ,$$

for some universal constant $c > 0$. Now we can obtain a lower bound on the numerator in (3.37) by the same argument as in Lemma 3.2, using Lemma 3.5. Hence (3.36) follows for $x$ of the form $x = (x_1, x_2, 0)$, $|x| < L/4$. Finally it is easy to extend these considerations to the case $x_3 \neq 0$, $|x| < L/4$, by observing that $u(x)$ is bounded below by the solution for cylinders centered on the $x_3$ constant plane of length $L/2$. This last situation is just the $x_3 = 0$ case again.

Next let $2 < r_0 < R_0$ and $\mathcal{D} \subset \mathbb{R}^3$ be the cylinder

$$\mathcal{D} = \{x = (x_1, x_3, x_3) : \ -R_0 < x_3 < R_0, \ r^2 = x_1^2 + x_2^2 < R_0^2\} .$$

We define a drift $\mathbf{b} : \mathcal{D} \longrightarrow \mathbb{R}^3$ as follows

$$\begin{aligned} &\mathbf{b}(x_1, x_2, x_3) = 0 , && r_0 < r < R_0 \ , \ -R_0 < x_3 < R_0 \ , \\ &\mathbf{b}(x_1, x_2, x_3) = -\Big(\frac{x_1}{r}, \frac{x_2}{r}, 1\Big) , && r < r_0 \ , \ -R_0 < x_3 < R_0 \ . \end{aligned} \tag{3.38}$$

For $x \in \mathcal{D}$ let $P_x(\mathcal{D})$ be the probability that the Brownian process with drift $\mathbf{b}$, started at $x$, exits $\partial\mathcal{D}$ through the bottom of the cylinder, $\partial\mathcal{D} \cap \{x : x_3 = -R_0\}$. We wish to obtain a lower bound for $P_x(\mathcal{D})$ when $r = r_0$. To obtain this we consider an auxiliary region $\mathcal{D}'$ defined by

$$\mathcal{D}' = \Big\{x : \ r < 1 , \ -R_0 < x_3 < \frac{R_0}{2}\Big\} .$$

Let $Q_x(\mathcal{D}')$ be the probability of exiting the region $\mathcal{D} \backslash \mathcal{D}'$ through the bottom of the cylinder $\partial\mathcal{D} \cap \{x_3 = -R_0\}$ or through $\partial\mathcal{D}'$. Then it is clear that for $x \in \mathcal{D}\backslash\mathcal{D}'$,

$$P_x(\mathcal{D}) \geq Q_x(\mathcal{D}') \inf \{P_y(\mathcal{D}) : \ y \in \partial\mathcal{D}'\} . \tag{3.39}$$

We shall estimate both quantities on the right in (3.39).

**Lemma 3.7.** *Let* $\mathbf{b}'$ *be a drift on* $\mathcal{D}$ *which is the same as* $\mathbf{b}$ *except the* $x_3$ *component is always zero. Let* $Q'_x(\mathcal{D}')$ *be the probability corresponding to* $\mathbf{b}'$. *Then*

$$Q_x(\mathcal{D}') \geq Q'_x(\mathcal{D}') . \tag{3.40}$$

PROOF. Let $u(x) = Q_x(\mathcal{D}')$, $x \in \mathcal{D} \backslash \mathcal{D}'$. Then $u$ is the solution of the Dirichlet problem

$$\begin{cases} -\Delta u(x) - \mathbf{b}(x) \cdot \nabla u(x) = 0\,, & x \in \mathcal{D} \backslash \mathcal{D}'\,, \\ u(x) = 1\,, \quad x \in (\partial\mathcal{D} \cap \{x_3 = -R_0\}) \cup \partial\mathcal{D}'\,, \\ u(x) = 0\,, \quad x \in \partial\mathcal{D} \cap \{x_3 > -R_0\}\,. \end{cases} \tag{3.41}$$

Similarly if $v(x) = Q'_x(\mathcal{D}')$ then $v$ satisfies the equation

$$-\Delta v(x) - \mathbf{b}'(x) \cdot \nabla v(x) = 0\,, \qquad x \in \mathcal{D} \backslash \mathcal{D}'\,,$$

with the same boundary conditions as in (3.41). We shall show later that

$$\frac{\partial v(x)}{\partial x_3} \le 0\,, \qquad x \in \mathcal{D} \backslash \mathcal{D}'\,. \tag{3.42}$$

Thus we have

$$-\Delta(u-v) - \mathbf{b}(x) \cdot \nabla(u-v) = (\mathbf{b}(x) - \mathbf{b}'(x)) \cdot \nabla v(x) \ge 0\,, \qquad x \in \mathcal{D} \backslash \mathcal{D}'\,,$$

in view of (3.42). Since $u - v$ has zero boundary conditions on $\partial\mathcal{D} \cup \partial\mathcal{D}'$ it follows by the maximum principle that

$$u(x) \ge v(x)\,, \qquad x \in \mathcal{D} \backslash \mathcal{D}'\,.$$

This is exactly the inequality (3.40).

To prove (3.42) we use a representation for the function $v(x)$ which is analogous to (3.27). Consider two dimensional Brownian motion with drift $\mathbf{b}(x_1, x_2)$ defined by

$$\mathbf{b}(x_1, x_2) = \begin{cases} 0\,, & r > r_0\,, \\ -\left(\frac{x_1}{r}, \frac{x_2}{r}\right), & r < r_0\,. \end{cases}$$

Suppose the motion starts at $(x_1, x_2)$ and consider only paths which hit the circle $r = 1$ before the circle $r = R_0$. Let $\tau_1$ be the hitting time for such paths and $\rho_1(r, t)$ be the density for $\tau_1$. Similarly let $\tau_2$ be the hitting time for paths which first hit the circle $r = R_0$ and $\rho_2(r, t)$ the density for $\tau_2$.

Next let $X_3(t)$ be Brownian motion started at $x_3 \in \mathbb{R}$ and $\tau_3$ be the first exit time from the interval $[-R_0, R_0]$. Let $w(x_3,t)$ be given by

$$w(x_3,t) = P_{x_3}\Big( -R_0 < X_3(t \wedge \tau_3) < \frac{R_0}{2}\Big),$$

$$h(x_3,t) = P_{x_3}(\tau_3 < t,\ X_3(\tau_3) = -R_0)\,.$$

Then we have the representation,

$$\begin{aligned} v(x_1,x_2,x_3) &= \int_0^\infty \rho_1(r,t)\, w(x_3,t)\, dt \\ &\quad + \int_0^\infty (\rho_1(r,t) + \rho_2(r,t))\, h(x_3,t)\, dt\,. \end{aligned} \tag{3.43}$$

The function $w(x_3,t)$ satisfies the heat equation (3.24) with boundary condition (3.25) and initial condition given by

$$w(x_3,0) = \begin{cases} 1\,, & -R_0 < x_3 < \dfrac{R_0}{2}\,, \\ 0\,, & \dfrac{R_0}{2} < x_3 < R_0\,, \end{cases} \tag{3.44}$$

The function $h(x_3,t)$ satisfies the heat equation (3.24) with boundary conditions

$$h(-R_0,t) = 1\,,\ h(R_0,t) = 0\,, \qquad t > 0\,. \tag{3.45}$$

and initial conditions given by

$$h(x_3,0) = 0\,, \qquad -R_0 < x_3 < R_0\,. \tag{3.46}$$

**Lemma 3.8.** *The function $h(x_3,t)$ is a decreasing function of $x_3$ in the interval $[-R_0,R_0]$.*

PROOF. By the maximum principle one has

$$0 \le h(x_3,t) \le 1\,, \qquad -R_0 < x_3 < R_0\,.$$

Hence if we put $u(x_3,t) = \partial h(x_3,t)/\partial x_3$, then $u(x_3,t)$ satisfies the heat equation with initial and boundary conditions satisfying

$$u(x_3,0) = 0\,, \qquad -R_0 < x_3 < R_0\,,$$

$$u(-R_0,t) \le 0\,,\ u(R_0,t) \le 0\,, \qquad t > 0\,.$$

Again by the maximum principle for the heat equation it follows that

$$u(x_3,t) \le 0\,, \qquad -R_0 < x_3 < R_0\ ,\ t>0\,.$$

Hence $h(x_3,t)$ is a decreasing function of $x_3$.

**Lemma 3.9.** *The function $w(x_3,t)+h(x_3,t)$ is a decreasing function of $x_3$ in the interval $[-R_0,R_0]$.*

PROOF. Putting $u(x_3,t) = w(x_3,t)+h(x_3,t)$, it is easy to see from (3.44), (3.45), (3.46) that $u$ satisfies the heat equation with boundary and initial conditions given by

$$u(x_3,0) = \begin{cases} 1\,, & -R_0 < x_3 < \dfrac{R_0}{2}\,, \\ 0\,, & \dfrac{R_0}{2} < x_3 < R_0\,. \end{cases}$$

$$u(-R_0,t)=1\,,\ u(R_0,t)=0\,, \qquad t>0\,.$$

It follows again by the maximum principle for the heat equation that

$$0 \le u(x_3,t) \le 1\,, \qquad -R_0 < x_3 < R_0\ ,\ t>0\,.$$

Now we apply the same argument as in Lemma 3.8 to complete the proof.

The inequality (3.42) follows easily now from (3.43) and Lemmas 3.8, 3.9.

Next we wish to estimate $Q'_x(\mathcal{D}')$. In view of the fact that the drift $\mathbf{b}'$ does not depend on $x_3$ this is easier to estimate than $Q_x(\mathcal{D}')$. Let us consider the function

$$u(r,\eta) = \int_0^\infty e^{-\eta t}\rho_1(r,t)\,dt\,.$$

Then $u(r,\eta)$ satisfies the equation

$$\begin{cases} \dfrac{d^2u}{dr^2} + \Big(b'(r)+\dfrac{1}{r}\Big)\dfrac{du}{dr} = \eta\, u\,, & 1<r<R_0\,, \\ u(1,\eta)=1\,, & \\ u(R_0,\eta)=0\,. & \end{cases} \tag{3.47}$$

Here $b'(r)$ is given by the magnitude of $\mathbf{b}'$,

$$b'(r) = \begin{cases} 0, & r > r_0 \,, \\ -1\,, & r < r_0 \,. \end{cases}$$

**Lemma 3.10** *Suppose* $2 < r_0 < R_0$, *and* $0 < \eta R_0 \le 1$. *Then there is a universal constant* $C$ *such that*

$$u(r_0, \eta) \ge 1 - C\,\frac{\log r_0}{\log R_0}\,. \tag{3.48}$$

Proof. By the maximum principle the solution of (3.47) is bounded below by the solution of the zero drift problem. Thus from Lemma 3.3 we have the inequality

$$u(r_0,\eta) \ge \frac{I_0(\sqrt{\eta}\, r_0)\displaystyle\int_{r_0}^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}}{I_0(\sqrt{\eta})\displaystyle\int_{1}^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}}$$

$$= \frac{I_0(\sqrt{\eta}\, r_0)}{I_0(\sqrt{\eta})}\left(1 - \frac{\displaystyle\int_{1}^{r_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}}{\displaystyle\int_{1}^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}}\right).$$

Evidently one has

$$\frac{I_0(\sqrt{\eta}\, r_0)}{I_0(\sqrt{\eta})} \ge 1\,, \tag{3.49}$$

$$\int_{1}^{r_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)} \le \log r_0\,. \tag{3.50}$$

If we use the fact that there is a universal constant $C > 0$ such that $I_0(\sqrt{\eta}\, t) \le C$ for $0 \le t \le R_0^{1/2}$ then it is clear that

$$\int_{1}^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)} \ge C_1 \log R_0\,, \tag{3.51}$$

for some universal constant $C_1$. The inequality (3.48) follows now from (3.49), (3.50), (3.51).

**Lemma 3.11.** *Suppose $x = (x_1, x_2, x_3) \in \mathcal{D}\backslash\mathcal{D}'$ with $x_3 \le R_0/4$ and $r_0 = (x_1^2 + x_2^2)^{1/2}$. Then if $2 < r_0 < R_0$ there is a universal constant $C$ such that*

$$Q'_x(\mathcal{D}') \ge 1 - C\,\frac{\log r_0}{\log R_0}\,. \tag{3.52}$$

PROOF. First we show that

$$\int_0^{R_0^{3/2}} \rho_1(r_0, t)\,dt \ge 1 - C_1\,\frac{\log r_0}{\log R_0}\,, \tag{3.53}$$

for some universal constant $C_1$. To see this observe from Lemma 3.10 that

$$\int_0^\infty e^{-t/R_0}\rho_1(r_0, t)\,dt \ge 1 - C_2\,\frac{\log r_0}{\log R_0}\,.$$

Thus

$$\int_0^{R_0^{3/2}} \rho_1(r, t)\,dt + e^{-R_0^{1/2}} \int_{R_0^{3/2}}^\infty \rho_1(r, t)\,dt \ge 1 - C_2\,\frac{\log r_0}{\log R_0}\,. \tag{3.54}$$

Now, if we use the fact that

$$\int_0^\infty \rho_1(r, t)\,dt \le 1\,,$$

we conclude from (3.54) that

$$\begin{aligned}(1 - e^{-R_0^{1/2}}) \int_0^{R_0^{3/2}} &\rho_1(r, t)\,dt \\ &\ge 1 - e^{-R_0^{1/2}} - C_2\,\frac{\log r_0}{\log R_0}\,.\end{aligned} \tag{3.55}$$

The inequality (3.55) clearly implies (3.53).

The result (3.52) follows now from the representation (3.43) by observing from the reflection principle that

$$\begin{aligned}P_{x_3}\Big[X_3(t) < \frac{R_0}{2}\,,\ 0 < t < R_0^{3/2}\Big] \\ = 1 - 2\int_{R_0/2 - x_3}^\infty \frac{1}{\big(4\pi R_0^{3/2}\big)^{1/2}} \exp\Big(-\frac{z^2}{4R_0^{3/2}}\Big)\,dz\,.\end{aligned}$$

Next we wish to obtain a lower bound on $P_y(\mathcal{D})$ for $y \in \partial\mathcal{D}'$. We shall show that if $r_0$ is of order $\log R_0$ then this bound is close to 1.

**Lemma 3.12.** *Let $X_3(t)$ be one dimensional Brownian motion started at $x_3 \in \mathbb{R}$ with constant drift $b(x_3) = -1$, $x_3 \in \mathbb{R}$. Let $\tau_3$ be the exit time from the interval $[-R_0, R_0]$, where $R_0 > 1$. Then there is a universal constant $C > 0$ such that for $x_3$ in the interval $|x_3| \le R_0/2$ there is the inequality*

$$P_{x_3}(\tau_3 < R_0^2\,,\ X_3(\tau_3) = -R_0) \ge 1 - \frac{C}{R_0^{1/2}}\,. \tag{3.56}$$

PROOF. For $\eta > 0$ let $u(x_3, \eta)$ be defined by

$$u(x_3,\eta) = E_{x_3}[e^{-\eta\tau_3}\chi(X_3(\tau_3))]\,, \tag{3.57}$$

where

$$\chi(z) = \begin{cases} 1\,, & \text{if } z < 0\,, \\ 0\,, & \text{if } z \ge 0\,. \end{cases}$$

Then $u(x_3, \eta)$ satisfies the equation

$$\begin{cases} \dfrac{d^2u}{dx_3^2} - \dfrac{du}{dx_3} = \eta\, u\,, & -R_0 < x_3 < R_0\,, \\ u(-R_0,\eta) = 1\,, & \\ u(R_0,\eta) = 0\,. & \end{cases} \tag{3.58}$$

The equation (3.58) can be solved explicitly to yield

$$u(x_3,\eta) = \frac{e^{(x_3+R_0)/2}\sinh\left(\dfrac{(1+4\eta)^{1/2}(R_0-x_3)}{2}\right)}{\sinh\left((1+4\eta)^{1/2}R_0\right)}\,. \tag{3.59}$$

Next we take $\eta = 1/R_0^{3/2}$. Then it is clear from (3.59) that

$$u(x_3,\eta) \ge 1 - \frac{C_1}{R_0^{1/2}}\,, \qquad -\frac{R_0}{2} < x_3 < \frac{R_0}{2}\,. \tag{3.60}$$

Arguing as before we can see from (3.57) that

$$\begin{aligned} P_{x_3}(\tau_3 < R_0^2,\ X_3(\tau_3) = -R_0) \\ \ge (1 - e^{-\eta R_0^2})^{-1}\left(u(x_3,\eta) - e^{-\eta R_0^2}\right). \end{aligned} \tag{3.61}$$

The inequality (3.56) follows now from (3.60), (3.61).

**Lemma 3.13.** *Let $X(t)$ be two dimensional Brownian motion started at $x = (x_1, x_2) \in \mathbb{R}^2$ with drift* $\mathbf{b}$ *defined by*

$$\mathbf{b}(y_1, y_2) = -\frac{(y_1, y_2)}{(y_1^2 + y_2^2)^{1/2}} .$$

*Suppose $(x_1, x_2)$ lies on the unit circle and $\tau$ is the first hitting time on the circle radius $r_0 > 1$. Then for $R_0 > 2$ there exists a universal constant $C > 0$ such that if $r_0 = C \log R_0$ then there is the inequality*

$$P(\tau > R_0^2) \geq 1 - \frac{C}{R_0} . \tag{3.62}$$

PROOF. Let us put

$$u(x) = E_x[e^{-\eta\tau}] , \qquad \eta > 0 ,$$

and let $r = (x_1^2 + x_2^2)^{1/2}$. Then $u(x) = u(r)$ satisfies a boundary value problem,

$$\begin{cases} \dfrac{d^2u}{dr^2} + \Big(\dfrac{1}{r} - 1\Big)\dfrac{du}{dr} = \eta\, u , & 0 < r < r_0 , \\ u(r_0) = 1 . \end{cases}$$

Let $v(r)$ be the solution of the boundary value problem

$$\begin{cases} \dfrac{d^2v}{dr^2} - \dfrac{1}{2}\,\dfrac{dv}{dr} = \eta\, v , & 2 < r < r_0 , \\ v(r_0) = 1 , \\ v'(2) = 0 . \end{cases} \tag{3.63}$$

In view of the fact that $u'(2) \geq 0$ it follows from the maximum principle that

$$u(r) \leq v(r) , \qquad 2 < r < r_0 .$$

Now we have

$$P(\tau < R_0^2) \leq e\, u\Big(1, \frac{1}{R_0^2}\Big) \leq e\, u\Big(2, \frac{1}{R_0^2}\Big) \leq e\, v\Big(2, \frac{1}{R_0^2}\Big) . \tag{3.64}$$

We can estimate the last expression in (3.64) since the solution of (3.63) can be explicitly computed. It is given by

$$(3.65)\qquad \begin{aligned} v(r,\eta) = &\frac{(\alpha-1)\exp\Big(\dfrac{(\alpha+1)\,(r-2)}{4}\Big)}{A} \\ &+ \frac{(\alpha+1)\exp\Big(-\dfrac{(\alpha-1)(r-2)}{4}\Big)}{A} \end{aligned}$$

with

$$A = (\alpha-1)\exp\Big(\frac{(\alpha+1)(r_0-2)}{4}\Big) + (\alpha+1)\exp\Big(-\frac{(\alpha-1)(r_0-2)}{4}\Big)\,,$$

where $\alpha$ is related to $\eta$ by

$$(3.66)\qquad \alpha = (1+16\,\eta)^{1/2}\,.$$

It is easy to see from (3.65), (3.66) that

$$(3.67)\qquad v\Big(2,\frac{1}{R_0^2}\Big) \le \frac{2\alpha}{\alpha-1}\,\exp\Big(-\frac{(\alpha+1)\,(r_0-2)}{4}\Big) \le \frac{C}{R_0}$$

if $r_0 = C\log R_0$ and $C$ is sufficiently large. The inequality (3.62) follows now from (3.64) and (3.67).

**Corollary 3.1.** *There exists a universal constant $C > 0$ such that if $r_0 = C\log R_0$ then for $y \in \partial\mathcal{D}'$ there is the inequality*

$$P_y(\mathcal{D}) \ge 1 - \frac{C}{R_0^{1/2}}\,.$$

Proof. From Lemmas 3.12 and 3.13 there is the inequality

$$P_y(\mathcal{D}) \ge \Big(1 - \frac{C}{R_0^{1/2}}\Big)\Big(1 - \frac{C}{R_0}\Big)\,.$$

Thus we are estimating the probability by restricting to paths which remain in the cylinder $r < r_0$ until they exit. For paths which remain in the cylinder, the components of the Brownian motion in the $x_3$ and $(x_1, x_2)$ directions are independent.

**Corollary 3.2.** *Suppose* $x \in \mathcal{D}\backslash\mathcal{D}'$ *with* $x_3 \le R_0/4$, $r_0 = (x_1^2 + x_2^2)^{1/2}$. *Then there is a universal constant* $C > 0$ *such that for* $r_0 = C \log R_0$, *there is the inequality*

$$P_x(\mathcal{D}) \ge 1 - \frac{C}{(\log R_0)^{1/2}} \,. \tag{3.68}$$

Proof. The inequality (3.68) follows from (3.39) and Lemmas 3.7, 3.11 and Corollary 3.1.

**Lemma 3.14.** *Suppose* $R \ge 2$. *Then there is a drift* $\mathbf{b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *with the following properties*:

a) $\operatorname{supp}(\mathbf{b}) \subset \{x : \ 7R/8 < |x| < 9R/8\}$.

b) $\mathbf{b}(x) \cdot x \le 0$, $x \in \mathbb{R}^3$.

c) $\|\mathbf{b}\|_\infty \le 1$,

$$\int_{\mathbb{R}^3} |\mathbf{b}| \, dx \le CR \, (\log R)^4 \,,$$

*for some universal constant* $C > 0$.

d) *For* $x \in \mathbb{R}^3$ *satisfying* $|x| = R$ *let* $P_x$ *be the probability that the drift process exits the region* $\{y : R/2 < |y| < 2R\}$ *through the outer boundary* $\{y : |y| = 2R\}$. *Then there is a universal constant* $C > 0$ *such that*

$$P_x \le \min\left\{\frac{2}{3}, \frac{C}{(\log R)^{1/2}}\right\}. \tag{3.69}$$

Proof. Let $a \in \mathbb{R}^3$ satisfy $|a| = R$, and $W_{\lambda,L}(a)$ denote the set $W_{\lambda,L}$ of (3.34) rotated and translated such that the origin corresponds to $a$ and the $(x_1, x_2)$ plane to the tangent plane to the sphere $\{x : |x| = R\}$ at the point $a$. We furthermore choose $\lambda, L$ by

$$L = \alpha R \,, \qquad \lambda = \frac{L}{2 \log L} \,, \tag{3.70}$$

where $\alpha$ satisfying $0 < \alpha < 1$ will be chosen independently of $R$.

We define a drift $\mathbf{b}_a(x), x \in \mathbb{R}^3$ as follows: Suppose $S_2$ is one of the cylindrical holes in $W_{\lambda,L}(a)$. Thus $S_2$ has radius $r_0$ and height $L$. Let $(x_1, x_2, x_3)$ be orthogonal coordinates with $x_3$ in direction $a$ and origin at the center of the circle formed by the intersection of $S_2$ with the tangent plane to the sphere $|y| = R$ at $a$. We define $\mathbf{b}_a(x)$ for $x \in S_2$ by (3.38). We similarly define $\mathbf{b}_a(x)$ for $x$ in any cylindrical hole $S_2$ of $W_{\lambda,L}(a)$. Otherwise we set $\mathbf{b}_a(x) = 0$.

Next we choose a finite number of points $a_1, \ldots, a_N$ on $\{x : |x| = R\}$ with the properties: 1) For any $x \in \{y : |y| = R\}$ there is an $a_i, 1 \le i \le N$, such that $|x - a_i| < L/4$. 2) None of the holes $S_2$ in the cylinders $W_{\lambda,L}(a_i)$, $1 \le i \le N$, intersect.

Finally we choose $r_0 = \Gamma \log R_0, R_0 = L$, so that Corollary 3.2 holds and define the drift $\mathbf{b}$ by $\mathbf{b} = \sum_{i=1}^N \mathbf{b}_{a_i}$. It is easy to see now that the parameters $\alpha, \Gamma, N$ can be chosen in a universal way so that 1), 2), a), b), c) hold. It remains then to verify d).

To prove d) let $x$ be such that $|x| = R$ and $a_i$ satisfy $|x - a_i| < L/4$. Let $Q_x$ be the probability of hitting one of the cylinders where $\mathbf{b} \ne 0$ before exiting the region $\{y : R(1-\varepsilon) < |x| < R(1+\varepsilon)\}$. Then by Lemma 3.6 and (3.70) there is a constant $C_\varepsilon$ depending on $\varepsilon$ such that

$$Q_x \ge 1 - \frac{C_\varepsilon}{\log R}\ . \tag{3.71}$$

Next, for $y \in \{z : R(1-\varepsilon) < |z| < R(1+\varepsilon), \mathbf{b}(z) \ne 0\}$, let $H_y$ be the probability that the drift process exits the set $\{z : R(1-2\varepsilon) < |z| < R(1+2\varepsilon)\}$ through the inner boundary $\{z : |z| = R(1-2\varepsilon)\}$. Then by Corollary 3.2, $\varepsilon$ can be chosen sufficiently small such that

$$H_y \ge 1 - \frac{C}{(\log R)^{1/2}}\ , \tag{3.72}$$

where the constant $C$ depends only on $\alpha, \Gamma$. Finally, for $y$ satisfying $|y| = R(1-2\varepsilon)$ let $K_y$ be the probability that the drift process exits the set $\{z : R/2 < |z| < R\}$ through the outer boundary $\{z : |z| = R\}$. In view of b) and the maximum principle this probability is less than the corresponding Brownian motion probabilty. Hence one has

$$K_y \le \frac{1 - 4\varepsilon}{1 - 2\varepsilon} < 1\ . \tag{3.73}$$

We use (3.71), (3.72), (3.73) to estimate $P_x$ from above. In fact one clearly has

$$P_x \le (1 - Q_x) + Q_x(1 - \inf_y H_y) + Q_x \sup_y H_y \sup_y K_y \sup_y P_y\ . \tag{3.74}$$

The inequality (3.69) follows now from (3.74) and the previous inequalities since $\varepsilon$ can be chosen in a universal way with $\varepsilon > 0$.

We use Lemma 3.14 to construct a drift on $\mathbb{R}^3$. In fact let $\mathbf{b}_n$ be the drift constructed in Lemma 3.14 with $R = 2^{-n}$, $n = -1, -2, \ldots$ Then we put

$$\mathbf{b} = \sum_{n=-\infty}^{-1} \mathbf{b}_n \,. \tag{3.75}$$

Observe that supp $(\mathbf{b}_n)$ do not overlap for different $n$. Hence from b), d) of Lemma 3.14 we have the inequality

$$\begin{aligned} p_{-n} = \sup_{|x|=2^{-n}} P\,(&\text{drift process started at } x \text{ with drift given by} \\ &(5.75) \text{ exits the region } 2^{-n-1} < |y| < 2^{-n+1} \\ &\text{through the outer boundary}) \\ \le \min\left\{\frac{2}{3}, \frac{C}{|n|^{1/2}}\right\}, \end{aligned} \tag{3.76}$$

for some universal constant $C > 0$, $-n = 1, 2, \ldots$

**Lemma 3.15.** *Let* $\mathbf{b}$ *be the drift given in* (3.75) *and suppose* $a_{n,p}$ *is defined by* (1.4) *and* $n_0, R$ *related by* (1.5). *Then for any constants* $\gamma$, $C_2 > 0$, $0 < \gamma < 1$, *there is the inequality,*

$$\sum_{m=0}^{\infty} \gamma^m \sup_{x \in \Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j,p}(x)\Big) \le K R^{\alpha}\,, \tag{3.77}$$

*for some constants* $K, \alpha$ *depending only on* $\gamma, C_2$ *and* $p$ *satisfying* $1 \le p < 2$.

PROOF. From c) of Lemma 3.14 we see that

$$\sum_{j=0}^{\infty} a_{n_0+j,p}(0) \le C\,, \tag{3.78}$$

where $C$ is a universal constant. This follows because $p < 2$. On the other hand it is easy to see that if $x$ satisfies $2^{-n-1} < |x| < 2^{-n+1}$ then

$$\sum_{j=0}^{\infty} a_{n_0+j,p}(x) \le C\,|n|\,, \tag{3.79}$$

for some universal constant $C$. Hence from (3.78), (3.79), we have

$$\sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j,p}(x)\Big) \le \exp\left(C_2\, C\, |n_0|\right) = R^{\beta}\,,$$

for some $\beta$ depending only on $C_2C$. Hence (3.77) follows.

Our final goal now is to use the inequality (3.76) to prove that the expected time to exit $\Omega_R$, starting at the origin, exceeds $R^{\alpha}$ for any $\alpha$, provided $R$ is sufficiently large. In view of Lemma 3.15 this will show that there is no inequality (1.6) for $p<2$.

**Lemma 3.16.** *Let $S_0, S_1, \dots, S_M$ be a set of concentric spheres with radii $r_0, r_1, \dots, r_M$ satisfying $r_0 < r_1 < \cdots < r_M$. Let $Y(t)$ be a stochastic process with continuous paths which is Brownian motion in the set $\{x : |x| \le r_1\}$. Consider every path of $Y(t)$ as being a random walk on the spheres $S_0, S_1, \dots, S_M$. For $x \in S_0$ let $N_x$ be the number of times this random walk, started at $x$, hits $S_0$ before hitting $S_M$. Let $\tau_x$ be the amount of time taken for the process started at $x$ to reach the sphere $S_M$. Then, if $2\, r_0 < r_1$, there is an inequality*

$$E\left[\tau_x\right] \ge C r_0^2\, E\left[N_x\right], \tag{3.80}$$

*where $C$ is a universal constant.*

Proof. For $z \in S_1$ let $p(z)$ be the probability of the process started at $z$ hitting $S_M$ before $S_0$. For $n = 1, 2, \dots,$ and $x \in S_1$, $y \in S_0$ let $q_n(x,y)$ be the probability density for the process started at $x$ and hitting $S_0$ $n$ times without hitting $S_M$. Thus if $O \subset S_0$ is an open set,

$$\begin{aligned} &P\,(Y \text{ with } Y(0)=x \text{ hits } S_0\ n \text{ times without hitting } S_M \text{ and}\\ &\qquad \text{that on the } n\text{-th hit it lands in the set } O) = \int_O q_n(x,y)\,dy\,. \end{aligned}$$

For $x \in S_0$ let $T_x$ be the first hitting time on $S_1$ for the process $Y$ started at $x$. In view of our assumptions $T_x$ is purely a Brownian motion variable. Then we have the identities

$$\begin{aligned} &P\left(N_x = 1\right) = E\left[p(Y(T_x))\right],\\ &P\left(N_x = m+1\right) = E\Big[\int_{S_0} q_m\left(Y(T_x), y\right) p(Y(T_y))\,dy\Big]\,, \end{aligned} \tag{3.81}$$

with $m = 1, 2, \ldots$ Clearly we also have the relation

$$q_m(x,y) = E\Big[\int_{S_0} q_n(x,z)\, q_{m-n}\,(Y(T_z), y)\, dz\Big], \tag{3.82}$$

with $n = 1, \ldots, m-1$. We shall use the functions $p, q_m$ and the variables $T_y$ to obtain a lower bound on $E\,[\tau_x]$. We do this by bounding $E\,[\tau_x]$ below by the amount of time the path spends in jumping from $S_0$ to $S_1$. Thus

$$\begin{aligned} E\,[\tau_x] \ge{}& E\,[T_x\, p(Y(T_x))] \\ &+ \sum_{m=1}^{\infty} \Big( E\Big[T_x \int_{S_0} q_m(Y(T_x), y)\, p(Y(T_y))\, dy\Big] \\ &\qquad + \sum_{n=1}^{m-1} E\Big[\int_{S_0}\int_{S_0} dy\, dz\, q_n(Y(T_x), y) \\ &\qquad\qquad\qquad \cdot T_y\, q_{m-n}\,(Y(T_y), z)\, p(Y(T_z))\Big] \\ &\qquad + E\Big[\int_{S_0} q_m(Y(T_x), y)\, T_y\, p(Y(T_y))\, dy\Big]\Big)\,. \end{aligned} \tag{3.83}$$

Since $T_y$ is purely a Brownian motion variable and $2\,r_0 < r_1$, there is a universal constant $C > 0$ such that

$$E\,[T_y \mid Y(T_y)] \ge C\, r_0^2\,, \qquad y \in S_1\,. \tag{3.84}$$

Substituting (3.84) into (3.83) and using the identities (3.81), (3.82) yields the inequality (3.80).

**Lemma 3.17.** *Let $S_0, S_1, \ldots, S_M$ be a set of concentric spheres with radii $r_0, r_1, \ldots, r_M$ satisfying $r_0 < r_1 < \cdots < r_M$. For $j = 1, \ldots, M-1$ let $p_j(x,y)$ be nonnegative functions of $x \in S_j$, $y \in S_{j+1}$ satisfying*

$$0 < \int_{S_{j+1}} p_j(x,y)\, dy \le p_j < 1\,, \qquad x \in S_j\,,$$

*for some positive numbers $p_1, \ldots, p_{M-1}$. Suppose now that the $p_j(x,y)$, $j = 1, \ldots, M-1$, are probability density functions for a stochastic process $Y(t)$ with continuous paths in the following sense: for any open set $O \subset S_{j+1}$,*

$$\begin{aligned} &P\,(Y \text{ started at } x \in S_j \text{ exits the region} \\ &\qquad \text{between } S_{j-1} \text{ and } S_{j+1} \text{ through } O) = \int_O p_j(x,y)\, dy\,. \end{aligned}$$

*Let $x \in S_0$ and $N_x$ be the number of times the process hits $S_0$ before hitting $S_M$ when viewed as a random walk on the spheres $S_0, \ldots, S_M$. Then*

$$E[N_x] \geq 1 + \sum_{j=1}^{M-1} \prod_{i=1}^{j} \frac{q_i}{p_i}\,, \tag{3.85}$$

*where $q_i = 1 - p_i$, $i = 1, \ldots, M-1$.*

PROOF. We shall first prove (3.85) in the case $M = 2$. Thus if we put $u(x) = E[N_x]$ it follows that

$$u(x) = \begin{cases} \displaystyle\int_{S_0} q_1(x,y)\,u(y)\,dy\,, & x \in S_1\,, \\ \displaystyle\int_{S_1} p_0(x,y)\,u(y)\,dy + 1\,, & x \in S_0\,, \end{cases} \tag{3.86}$$

where

$$\begin{aligned} &P\,(Y \text{ started at } x \in S_0 \text{ exits the region inside} \\ &\qquad S_1 \text{ through the open set } O \subset S_1) = \int_O p_0(x,y)\,dy\,, \end{aligned} \tag{3.87}$$

$$\begin{aligned} &P\,(Y \text{ started at } x \in S_1 \text{ exits the region between } S_0 \\ &\qquad \text{and } S_2 \text{ through the open set } O \subset S_0) = \int_O q_1(x,y)\,dy\,. \end{aligned} \tag{3.88}$$

Evidently from the definitions (3.87), (3.88) one has

$$\int_{S_1} p_0(x,y)\,dy = 1\,, \qquad x \in S_0\,,$$

$$\int_{S_2} p_1(x,y)\,dy + \int_{S_0} q_1(x,y)\,dy = 1\,, \qquad x \in S_1\,.$$

From (3.86) we have

$$u(x) = \int_{S_1} p_0(x,y) \int_{S_0} q_1(y,z)\,u(z)\,dz\,dy + 1\,, \qquad x \in S_0\,. \tag{3.89}$$

Hence if we put $u_0 = \inf\{u(x) : x \in S_0\}$ then

$$\begin{aligned} u(x) &\geq u_0 \int_{S_1} p_0(x,y) \int_{S_0} q_1(y,z)\,dz\,dy + 1 \\ &= u_0 \int_{S_1} p_0(x,y) \Big(1 - \int_{S_2} p_1(y,z)\,dz\Big)\,dy + 1 \\ &\geq u_0 \int_{S_1} p_0(x,y)\,(1-p_1)\,dy + 1 = u_0\,(1-p_1) + 1\,, \qquad x \in S_0\,. \end{aligned}$$

Taking the infimum on the left in (3.89) we conclude

$$u_0 \geq \frac{1}{p_1}\,. \tag{3.90}$$

This last inequality is just (3.85) for $M = 2$.

To generalize this for $M > 2$ let $P_1(x,y)$ be defined by

$$\begin{aligned} &P\,(Y \text{ started at } x \in S_1 \text{ exits the region between } S_0 \\ &\qquad \text{and } S_M \text{ through the open set } O \subset S_M) = \int_O P_1(x,y)\,dy\,. \end{aligned}$$

From [5, Lemma 6.3] it follows that

$$\int_{S_M} P_1(x,y)\,dy \leq P_1\,, \qquad x \in S_1\,, \tag{3.91}$$

where

$$P_1 = \frac{1}{1 + \displaystyle\sum_{j=1}^{M-1} \prod_{i=1}^{j} \frac{q_i}{p_i}}\,. \tag{3.92}$$

Hence (3.85) follows from (3.90), (3.91), (3.92).

We use Lemmas 3.16 and 3.17 to obtain a lower bound on $u(0)$ where $u$ is the solution of (1.1), (1.2) with $f \equiv 1$ and drift given by (3.75). Let $S_j, j = 0,1,2,\dots$ be spheres centered at the origin with radius $2^j$. Then the probabilities $p_j, j = 1,\dots,M-1$, of Lemma 3.17 satisfy by (3.76) the inequality

$$p_j \leq \min\Big\{\frac{2}{3}, \frac{C}{\sqrt{j}}\Big\}\,, \qquad j = 1,2,\dots$$

Consequently, if $R = 2^{-n_0}$ one has from Lemmas 3.16 and 3.17 the inequality

$$u(0) \geq C\Big(1 + \sum_{j=1}^{|n_0|-1} \prod_{i=1}^{j} \frac{q_i}{p_i}\Big) \geq C \exp\big(C_1 \, |n_0| \log |n_0|\big) ,$$

where $C, C_1$ are universal constants. Thus one has an inequality

$$u(0) \geq C R^{\alpha \log\log R} ,$$

for some $C, \alpha > 0$. In view of Lemma 3.15 the inequality (1.6) does not hold for $R$ sufficiently large.

## 4. Perturbative estimates on the exit probabilities from a spherical shell.

In this section we shall be interested in the drift process with perturbative drift $\mathbf{b}$. For $R_1 < R < R_2$ let $U_{R_1,R_2}$ be the spherical shell

$$U_{R_1,R_2} = \{x \in \mathbb{R}^3 : \; R_1 < |x| < R_2\} .$$

Now suppose we start the process off on the sphere $\{x : |x| = R\}$ with density $f(x)$, $|x| = R$. Some of the paths of the process exit the shell $U_{R_1,R_2}$ through the boundary $\{|x| = R_2\}$ and the others through $\{|x| = R_1\}$. Hence the density $f$ induces densities $f_1$ on $\{|x| = R_1\}$ and $f_2$ on $\{|x| = R_2\}$. We shall be interested in comparing $f_1$, $f_2$ and $f$. To do this we shall need to define norms of these functions. Let $\rho > 0$ and $g$ a measurable function on the sphere $\{|x| = \rho\}$. For $1 \leq q < \infty$ we define the $L^q$ norm of $g$ by

$$\|g\|_q = \Big(\frac{1}{4\pi\rho^2} \int_{|x|=\rho} |g(x)|^q \, dx\Big)^{1/q} .$$

Thus $\|\mathbf{1}\|_q = 1$. For an $L^1$ function $g$ we define $\operatorname{Av} g$ by

$$\operatorname{Av} g = \frac{1}{4\pi\rho^2} \int_{|x|=\rho} g(x) \, dx .$$

It is clear that the functions $f_1, f_2, f$ satisfy

$$\operatorname{Av} f_1 + \operatorname{Av} f_2 = \operatorname{Av} f .$$

We wish to obtain an expression for $f_2$ in terms of $f$ . Let $g(x)$ be a function defined on the sphere $\{|x| = R_2\}$ and $u(x) = Pg(x)$ be defined for $x \in U_{R_1,R_2}$ as the solution of the boundary value problem

$$\begin{cases} \Delta u(x) = 0\,, & R_1 < |x| < R_2\ , \\ u(x) = g(x)\,, & |x| = R_2\ , \\ u(x) = 0\,, & |x| = R_1\ . \end{cases} \tag{4.1}$$

For $x, y \in U_{R_1,R_2}$ let $G_D(x, y)$ be the Dirichlet Green's function and $k_T$ the kernel

$$k_T(x, y) = \mathbf{b}(x) \cdot \nabla_x G_D(x, y)\,, \qquad x, y \in U_{R_1,R_2}\ . \tag{4.2}$$

Suppose $g \in L^q(\{|x| = R_2\})$. Then we define the operator $Q$ by

$$Qg(x) = \int_{U_{R_1,R_2}} G_D(x, y)\,(I-T)^{-1}\,\mathbf{b}\cdot\nabla Pg(y)\,dy\,, \qquad |x| = R\,, \tag{4.3}$$

where $T$ is the operator induced by the kernel $k_T$. The expression (4.3) is purely formal. It takes functions with domain $\{|x| = R_2\}$ to functions with domain $\{|x| = R\}$. Similarly, the operator $P$ defined above takes functions on the sphere $|x| = R_2$ to functions on the sphere $|x| = R$. Hence the formal adjoints $P^*$ and $Q^*$ of $P$ and $Q$ take functions on $|x| = R$ to functions on $|x| = R_2$. We have now the relation

$$f_2 = P^* f + Q^* f\,.$$

Our major goal here will be to show that the operator $Q^*$ is dominated by the operator $P^*$. We shall prove this by showing that $Q$ is dominated by $P$. To do this we shall need various estimates on the Green's function $G_D(x, y)$ and its derivatives. Observe that the Green's function for the shell $U_{R_1,R_2}$ can be obtained from the Green's function for a sphere by the method of images. The estimates we need on $G_D(x, y)$ can easily be derived from this image representation. First we shall consider the simplest of cases $R_1 = 0$, $R_2 = 2R$. We obtain an improvement on Lemma 2.2.

**Lemma 4.1.** *Suppose $R_1 = 0$, $R_2 = 2R$. Let $r, p, q$ satisfy the inequalities $1 < r < p \le 3$, $q > r$,*

$$\frac{1}{q} < \frac{\frac{1}{r} - \frac{1}{p}}{1 - \frac{1}{p}}\ . \tag{4.4}$$

*Then if* $g \in L^q(\{|x| = R_2\})$ *the function* $\mathbf{b} \cdot \nabla P g$ *is in the Morrey space* $M_r^{q_1}(\{|x| < R_2\})$, *where*

$$\frac{1}{q_1} = \frac{1}{3} + \frac{2}{3\,q} \tag{4.5}$$

*and*

$$\|\mathbf{b} \cdot \nabla P g\|_{q_1, r} \le C\, R^{2/q - 1}\, \|\mathbf{b}\|_{3,p}\ \|g\|_q\,. \tag{4.6}$$

PROOF. The idea of the proof here is to use the Harnack inequality. Thus it follows from Harnack that if $g$ is a nonnegative function then there is a universal constant $C$ such that

$$(R_2 - |x|)\,|\nabla P g(x)| \le C P g(x)\,.$$

Hence for any cube $Q$ one has

$$\begin{aligned}
\frac{1}{R_2^r} \int_{Q \cap \{|x| < R_2\}} & (R_2 - |x|)^r\, |\mathbf{b}(x)|^r\, |\nabla P g(x)|^r\, dx \\
&\le \frac{C^r}{R_2^r} \int_{Q \cap \{|x| < R_2\}} |\mathbf{b}(x)|^r\, |P g(x)|^r\, dx \\
&= \frac{C^r}{R_2^r} \int_{Q \cap \{|x| < R_2\}} |\mathbf{b}(x)|^{r(1-\alpha)}\, |\mathbf{b}(x)|^{r\alpha}\, |P g(x)|^r\, dx \\
&\le \frac{C^r}{R_2^r} \Big( \int_Q |\mathbf{b}(x)|^{r(1-\alpha)/(1 - r/q)}\, dx \Big)^{1 - r/q} \\
&\qquad \cdot \Big( \int_{Q \cap \{|x| < R_2\}} |\mathbf{b}(x)|^{q\alpha} |P g(x)|^q\, dx \Big)^{r/q}.
\end{aligned} \tag{4.7}$$

Since $P(1) = 1$ it follows by Jensen that

$$(P g(x))^q \le P g^q(x)\,.$$

Thus

$$\begin{aligned}
\int_{Q \cap \{|x| < R_2\}} & |\mathbf{b}(x)|^{q\alpha}\, |P g(x)|^q\, dx \\
&\le \Big( \sup_{|x| = R_2} C_Q(x) \Big) \int_{|x| = R_2} |g(x)|^q\, dx\,,
\end{aligned} \tag{4.8}$$

where

$$C_Q(x) = \int_{Q \cap \{|y|<R_2\}} |\mathbf{b}(y)|^{q\alpha} P\delta_x(y)\, dy\,,$$

and $\delta_x$ is the Dirac $\delta$ function concentrated at $x$, $|x| = R_2$. We suppose now that $\alpha > 0$ is chosen so that $q\,\alpha < 1$. Then we have

$$C_Q(x) \le C \int_Q \frac{|\mathbf{b}(y)|^{q\alpha}}{|y-x|^2}\, dy \le C \sum_{n=n_1}^{\infty} 2^{2n} \int_{Q_n} |\mathbf{b}(y)|^{q\alpha}\, dy\,,$$

where the $Q_n$ are cubes with side $2^{-n}$ and $n_1$ is chosen so that $|Q| \sim 2^{-3n_1}$. Using the fact that $\mathbf{b} \in M_p^3$ we conclude that

$$C_Q(x) \le C \sum_{n=n_1}^{\infty} 2^{2n}\, |Q_n|^{1-q\alpha/3}\, \|\mathbf{b}\|_{3,p}^{q\alpha} \le C\, |Q|^{(1-q\alpha)/3}\, \|\mathbf{b}\|_{3,p}^{q\alpha}\,,$$

for some universal constant $C$. Hence from (4.7) and (4.8) we conclude that

$$\begin{aligned} &\frac{1}{R_2^r} \int_{Q\cap\{|x|<R_2\}} (R_2 - |x|)^r\, |\mathbf{b}(x)|^r\, |\nabla P g(x)|^r\, dx \\ &\qquad \le \frac{C^r}{R_2^r}\, |Q|^{1-r/q-r(1-\alpha)/3}\, |Q|^{(1-q\alpha)r/3q}\, \|\mathbf{b}\|_{3,p}^r\, R_2^{2r/q}\, \|g\|_q^r \qquad (4.9) \\ &\qquad = C^r R_2^{r(2/q-1)}\, |Q|^{1-r/q_1}\, \|\mathbf{b}\|_{3,p}^r\, \|g\|_q^r\,, \end{aligned}$$

where $q_1$ is given by (4.5) and $\alpha$ must satisfy the inequality

$$\frac{r\,(1-\alpha)}{1-\dfrac{r}{q}} \le\; p\,. \qquad (4.10)$$

The inequality (4.10) taken together with the condition $q\,\alpha < 1$ implies (4.4). The inequality (4.6) is an immediate consequence of (4.9).

REMARK. Observe that (4.5) is the same as (2.5) but (4.4) is an improvement on (2.4).

**Proposition 4.1.** *For $1 < q < \infty$ the operator $Q$ defined by (4.3) is a bounded operator from $L^q(\{|x| = R_2\})$ to $L^q(\{|x| = R\})$ provided $\|\mathbf{b}\|_{3,p} < \varepsilon$ for sufficiently small $\varepsilon$ depending on $p$ and $q$. Furthermore* *the norm of* $Q$, $\|Q\|$ *satisfies an inequality* $\|Q\| \le C\varepsilon$, *where* $C$ *is a universal constant.*

PROOF. We have by Lemma 4.1 and Proposition 2.1 that if $\varepsilon$ is sufficiently small then

$$Qg(x) = \int_{|y|<R_2} G_D(x,y)\, h(y)\, dy\,, \qquad |x| = R\,,$$

where $h$ is in the Morrey space $M_r^{q_1}(\{|x| < R_2\})$ and

$$\|h\|_{q_1,r} \le CR_2^{2/q-1}\, \|\mathbf{b}\|_{3,p}\, \|g\|_q\,, \tag{4.11}$$

for some universal constant $C$. Arguing as in Lemma 2.3 we see that if $m \ge 1$ satisfies the inequality

$$\frac{2}{3} + \frac{1}{q_1 m} > \frac{1}{q_1} + \frac{1}{3\,m}\,, \tag{4.12}$$

then

$$Qg \in L^m(\{|x| = R\})$$

and

$$\|Qg\|_m \le CR_2^{2-3/q_1}\, \|h\|_{q_1,r}\,, \tag{4.13}$$

for some constant $C$. This inequality (4.12) holds provided $m$ satisfies the inequality

$$\frac{1}{m} > \frac{2-q}{2}\,. \tag{4.14}$$

It is easy to see that (4.14) holds with $m = q$ for all $q > 1$. The result now follows from (4.11) and (4.13) by observing that $2/q - 1 = -(2 - 3/q_1)$.

**Corollary 4.1.** *Suppose* $R_1 = 0$, $R_2 = 2R$. *Then for any* $p$, $1 < p \le 3$ *and* $q > 1$ *the following holds: there exists* $\varepsilon, \delta > 0$ *depending only on* $p, q$ *such that if* $\|\mathbf{b}\|_{3,p} < \varepsilon$ *and* $\|f - \mathrm{Av}\, f\|_q \le \delta|\mathrm{Av}\, f|$ *then*

$$\|f_2 - \mathrm{Av}\, f_2\|_q \le \delta|\mathrm{Av}\, f_2|\,.$$

PROOF. By Proposition 4.1 the operator $Q^*$ is a bounded operator from $L^q(\{|x| = R\})$ to $L^q(\{|x| = R_2\})$ and $\|Q^*\| \le C\varepsilon$. We combine this with the fact that there exists $\gamma$, $0 < \gamma < 1$, such that

$$\|P^*(f - \mathrm{Av}\, f)\|_q \le \gamma \|f - \mathrm{Av}\, f\|_q \, . \tag{4.15}$$

The inequality (4.15) follows by the same argument as in [5, Lemma 4.1]. It is clear that

$$\mathrm{Av}\, f = P^*(\mathrm{Av}\, f) = \mathrm{Av} P^* f = \mathrm{Av}\, f_2 \, .$$

Thus

$$\begin{aligned} \|f_2 - \mathrm{Av}\, f_2\|_q &= \|Q^* f - \mathrm{Av}\, Q^* f + P^*(f - \mathrm{Av}\, f)\|_q \\ &\le 2\, C\varepsilon \, \|f\|_q + \gamma \, \|f - \mathrm{Av}\, f\|_q \\ &\le 2\, C\varepsilon \, (1+\delta) \, |\mathrm{Av}\, f| + \gamma \, \delta \, |\mathrm{Av}\, f| \\ &\le \delta \, |\mathrm{Av}\, f| \\ &= \delta \, |\mathrm{Av}\, f_2| \, , \end{aligned}$$

if $\varepsilon$ is chosen so that

$$2\, C\varepsilon \, \frac{1+\delta}{\delta} + \gamma \le 1 \, .$$

The proof is complete.

Next we state an obvious generalization of Corollary 4.1.

**Corollary 4.2.** *Suppose $R_1 = R/2$, $R_2 = 2R$. Then for any $p$, $1 < p \le 3$ and $q > 1$ the following holds: there exist positive constants $c_1, c_2, \varepsilon, \delta$ depending only on $p, q$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ and $\|f - \mathrm{Av}\, f\|_2 \le \delta \, |\mathrm{Av}\, f|$ then*

$$|\mathrm{Av}\, f_1| \ge c_1 \, |\mathrm{Av}\, f| \qquad \textit{and} \qquad \|f_1 - \mathrm{Av}\, f_1\|_q \le \delta \, |\mathrm{Av}\, f_1| \, ,$$

$$|\mathrm{Av}\, f_2| \ge c_2 \, |\mathrm{Av}\, f| \qquad \textit{and} \qquad \|f_2 - \mathrm{Av}\, f_2\|_q \le \delta \, |\mathrm{Av}\, f_2| \, .$$

Proof. We shall just show that $|\mathrm{Av}\, f_2| \geq |\mathrm{Av}\, f|$. Observe that

$$\begin{aligned}\mathrm{Av}\,(P^* f) &= P^*(\mathrm{Av}\, f)\\ &= (\mathrm{Av}\, f)\, P^*(1)\\ &= (\mathrm{Av}\, f)\, P\,[\text{Brownian motion started at } x \text{ with } |x| = R\\ &\qquad\qquad \text{exits } U_{R_1,R_2} \text{ through the boundary } |y| = R_2]\\ &= \frac{\dfrac{1}{R_1} - \dfrac{1}{R}}{\dfrac{1}{R_1} - \dfrac{1}{R_2}}\, \mathrm{Av}\, f\\ &= \frac{2}{3}\, \mathrm{Av}\, f\,.\end{aligned}$$

Hence

$$\begin{aligned}|\mathrm{Av}\, f_2| &= |\mathrm{Av}\,(P^* f) + \mathrm{Av}\,(Q^* f)|\\ &\geq \frac{2\,|\mathrm{Av}\, f|}{3} - C\,\varepsilon\, \|f\|_q\\ &\geq \frac{2\,|\mathrm{Av}\, f|}{3} - C\,\varepsilon\,(1+\delta)\,|\mathrm{Av}\ f|\\ &\geq c_2\,|\mathrm{Av}\, f|\,.\end{aligned}$$

The proof is complete.

In Corollary 4.2 the distances $R - R_1$ and $R_2 - R$ are commensurable. Now we wish to consider the situation when $R - R_1$ is much smaller than $R_2 - R$.

**Lemma 4.2.** *Suppose $R/2 < R_1 < R < R_2 = 2R$. Then if $\mathbf{b} \equiv 0$ there exists a universal constant $c_2 > 0$ and a constant $\gamma$, $0 < \gamma < 1$ such that*

$$|\mathrm{Av}\, f_2| \geq c_2\, |\mathrm{Av}\, f|\, \frac{R - R_1}{R}\,, \tag{4.16}$$

$$\|f_2 - \mathrm{Av}\, f_2\|_q \leq \gamma\, |Av\, f_2|\, \frac{\|f - \mathrm{Av}\, f\|_q}{|\mathrm{Av}\, f|}\,, \tag{4.17}$$

*for any $q$, $1 \leq q \leq \infty$.*

PROOF. Since we are in the Brownian motion case we have $f_2 = P^* f$. The inequality (4.16) follows by the argument in Corollary 4.2. To get the inequality (4.17) we let $k > 0$ be such that $k\,P(1) = 1$. Since $f_2 = P^* f$ it follows that

$$\langle 1, f_2\rangle = \langle 1, P^* f\rangle = \langle P1, f\rangle\,,$$

and so we have

$$\operatorname{Av} f_2 = \frac{1}{k}\operatorname{Av} f\,. \tag{4.18}$$

Using Jensen's inequality and the fact that $P = P^*$ we have that for any $q$, $1 \le q \le \infty$ there is the inequality

$$\|k\,Pg\|_q \le \|g\|_q\,.$$

The inequality (4.17) will follow if we can show a version of the Harnack inequality, namely

$$C\,Pg(x_0) \ge Pg(x) \ge c\,Pg(x_0)\,, \qquad |x| = |x_0| = R\,, \tag{4.19}$$

for universal constants $C,\ c > 0$ and nonnegative functions $g$. In fact we need only repeat the argument of [5, Lemma 4.1] for the operator $k\,P$ and use (4.18).

To see (4.19) we write

$$Pg(x) = E_x[g(X(\tau))] = \int_{|y|=3R/2} \rho_x(y)\,E_y[g(X(\tau))]\,dy\,.$$

Here $\tau$ is the exit time from the shell $U_{R_1,R_2}$ for Brownian motion. The density $\rho_x(y)$ is the density for paths started at $x$, $|x| = R$, which hit the sphere $|y| = 3R/2$ before hitting the sphere $|y| = R_1$. Thus

$$\int_{|y|=3R/2} \rho_x(y)\,dy = \frac{\dfrac{1}{R_1} - \dfrac{1}{R}}{\dfrac{1}{R_1} - \dfrac{2}{3R}}\,.$$

Now by the standard Harnack inequality applied to the shell $U_{R_1,R_2}$ there exist universal constants $C_1,\ c_1 > 0$ such that

$$C_1 Pg(y_0) \ge Pg(y) \ge c_1 Pg(y_0)\,, \qquad |y| = |y_0| = \frac{3R}{2}\,.$$

Hence we have

$$\begin{aligned} Pg(x) &= \int_{|y|=3R/2} \rho_x(y)\, Pg(y)\, dy \\ &\le \int_{|y|=3R/2} \rho_x(y)\, C_1\, Pg(y_0)\, dy \\ &= \int_{|y|=3R/2} \rho_{x_0}(y)\, C_1\, Pg(y_0)\, dy \\ &\le \int_{|y|=3R/2} \rho_{x_0}(y)\, C_1^2\, Pg(y)\, dy \\ &= C_1^2\, Pg(x_0)\,. \end{aligned}$$

Similarly we obtain a lower bound $Pg(x) \ge c_1^2 Pg(x_0)$. Thus (4.19) follows with $C = C_1^2$, $c = c_1^2$.

Next we wish to generalize Lemma 4.2 to the case of nontrivial drift $\mathbf{b}$. To do this we shall need to generalize further the notion of a Morrey space. For $Q$ a dyadic cube intersecting the spherical shell $U_{R_1,R_2}$ let $d(Q)$ be defined by

$$d(Q) = \sup\{d(x, |y| = R_2) : x \in Q\}\,.$$

Observe that $d(Q)$ is *not* the maximum distance from points in $Q$ to the boundary of $U_{R_1,R_2}$, only to the part of the boundary consisting of the sphere $|y| = R_2$. We define the Morrey space $M^q_{r,s}(U_{R_1,R_2})$ where $1 \le r \le q < \infty$ and $s > 0$ by the following: a measurable function $g : U_{R_1,R_2} \longrightarrow \mathbb{C}$ is in $M^q_{r,s}(U_{R_1,R_2})$ if $(R_2 - |x|)^r\, |g(x)|^r$ is integrable on $U_{R_1,R_2}$ and there is a constant $C > 0$ such that

$$\frac{1}{R_2^r} \int_{Q\cap U_{R_1,R_2}} (R_2 - |x|)^r\, |g(x)|^r\, dx \le C^r\, |Q|^{1-r/q} \Big(\frac{R_2}{d(Q)}\Big)^{sr}\,, \tag{4.20}$$

for all cubes $Q \subset \mathbb{R}^3$. The norm of $g$, $\|g\|_{q,r,s}$ is defined as

$$\|g\|_{q,r,s} = \inf\{C :\ (4.20) \text{ holds for all cubes } Q\}\,.$$

**Lemma 4.3.** *Suppose $R/2 < R_1 < R < R_2 = 2R$. Let $r, p, q$ satisfy the inequalities $1 < r < p \le 3$, $q > r$ and (4.4). Then if $g \in L^q(\{|x| = R_2\})$*

*the function* $\mathbf{b}\cdot\nabla Pg$ *is in the Morrey space* $M^3_{r,s}(U_{R_1,R_2})$ *where* $s = 2/q$ *and*

$$\|\mathbf{b}\cdot\nabla Pg\|_{3,r,s} \le CR_2^{-1}\,\|\mathbf{b}\|_{3,p}\,\|g\|_q\,.$$

PROOF. This follows immediately from the argument of Lemma 4.1. The only modification is in estimating the function $C_Q(x)$. It is clear that if $|x| = R_2$ then

$$C_Q(x) \le C\,\frac{|Q|^{1-q\alpha/3}\,\|\mathbf{b}\|^{q\alpha}_{3,p}}{d(Q)^2}\,, \tag{4.21}$$

for some universal constant $C$. Observe that we also have an inequality

$$|\nabla Pg(x)| \le CR_2^{-1}\,\|g\|_q\,, \tag{4.22}$$

provided $R_1 < |x| < 3R/2$. This follows since $Pg(x) = 0$ for $|x| = R_1$. To get the inequality (4.20) we divide the cubes $Q$ into two types, those with $d(Q) < R/2$ and those with $d(Q) \ge R/2$. For the first type we use the estimate (4.21) and the corresponding estimate in Lemma 4.1 to obtain (4.20) with $s = 2/q$. For the second category we use (4.22) and the fact that $\mathbf{b}$ is in $M^3_p$.

**Lemma 4.4.** *Suppose* $R/2 < R_1 < R < R_2 = 2R$. *Then the operator* $T$ *with kernel* $k_T$ *given by* (4.2) *is a bounded operator on the Morrey space* $M^q_{r,s}(U_{R_1,R_2})$ *provided* $1 < r < p$ *and* $1 < q < 3$, $s > 0$. *Furthermore, the norm of* $T$ *is bounded as* $\|T\| \le C\,\|\mathbf{b}\|_{3,p}$ *where the constant* $C$ *depends only on* $r, s, q$.

PROOF. This follows from Corollary 2.1 and the fact that

$$\sum_{n=-\infty}^{n_{Q'}} |\mathbf{b}(x)|\,S_n u(x) \le |\mathbf{b}(x)|\sum_{n=-\infty}^{n_{Q'}} 2^{-n}u_{Q_n}\,\frac{R}{d(Q_n)}\,,$$

where the $Q_n$ are an increasing sequence of dyadic cubes containing the point $x$. We have now from (4.20) that

$$u_{Q_n} \le C\,|Q_n|^{-1/q}\Big(\frac{R_2}{d(Q_n)}\Big)^s\,\|u\|_{q,r,s}\,.$$

Hence,

$$\sum_{n=-\infty}^{n_{Q'}} 2^{-n}\,u_{Q_n}\,\frac{R}{d(Q_n)} \le C\,|Q'|^{1/3-1/q}\Big(\frac{R}{d(Q')}\Big)^{s+1}\|u\|_{q,r,s}\,,$$

for some universal constant $C$, since $q < 3$. Here we have used the fact that $d(Q_n) \geq d(Q')$ since $Q_n \supset Q'$. Thus

$$\frac{1}{R_2^r} \int_{Q' \cap U_{R_1,R_2}} (R_2 - |x|)^r \Big( \sum_{n=-\infty}^{n_{Q'}} |\mathbf{b}(x)|\, S_n u(x) \Big)^r \, dx$$
$$\leq C^r \Big( \int_{Q'} |\mathbf{b}(x)|^r \, dx \Big) \|u\|_{q,r,s}^r \Big( \frac{R}{d(Q')} \Big)^{sr} |Q'|^{r/3 - r/q}$$
$$\leq C^r \, \|\mathbf{b}\|_{3,p}^r \, |Q'|^{1-r/3} \, \|u\|_{q,r,s}^r \Big( \frac{R}{d(Q')} \Big)^{sr} |Q'|^{r/3 - r/q}$$
$$= C^r \, \|\mathbf{b}\|_{3,p}^r \, \|u\|_{q,r,s}^r \Big( \frac{R}{d(Q')} \Big)^{sr} |Q'|^{1 - r/q} .$$

**Proposition 4.2.** *Suppose $R/2 < R_1 < R < R_2 = 2R$. For $1 < q < \infty$ the operator $Q$ defined by (4.3) is a bounded operator from $L^q(\{|x| = R_2\})$ to $L^q(\{|x| = R\})$ provided $\|\mathbf{b}\|_{3,p} < \varepsilon$ for sufficiently small $\varepsilon$ depending on $p$ and $q$. Furthermore, the norm of $Q$, $\|Q\|$ satisfies an inequality $\|Q\| \leq C\varepsilon\,(R - R_1)/R$, where $C$ is a universal constant.*

PROOF. From Lemma 4.3 and Lemma 4.4 we have

$$Qg(x) = \int_{U_{R_1,R_2}} G_D(x,y)\, h(y)\, dy \,, \qquad |x| = R \,,$$

where $h$ is in the Morrey space $M_{r,s}^{q_1}(U_{R_1,R_2})$ for any $1 < r < p$, $r \leq q_1 < 3$, provided (4.4) is satisfied and

$$\text{(4.23)} \qquad \frac{1}{q_1} = \frac{1}{3} - \frac{s}{3} + \frac{2}{3\,q} \,,$$

with $0 \leq s < 2/q$. The norm of $h$ satisfies an inequality

$$\text{(4.24)} \qquad \|h\|_{q_1,r,s} \leq C R_2^{2/q - s - 1} \, \|\mathbf{b}\|_{3,p} \, \|g\|_q \,.$$

We write $Qg(x) = g_1(x) + g_2(x)$, where

$$g_1(x) = \int_{U_{R_1,R_2} \cap \{|y| < 3R/2\}} G_D(x,y)\, h(y)\, dy \,.$$

It follows that for $|x| = R$, there is an inequality

$$
\begin{aligned}
|g_2(x)| &\le \frac{C\,(R-R_1)}{R^3}\int_{U_{R_1,R_2}\cap\{|y|>3R/2\}} (R_2-|y|)\,|h(y)|\,dy \\
&\le \frac{C\,(R-R_1)}{R^2}\,R^{3-3/q_1}\,\|h\|_{q_1,r,s} \qquad (4.25)\\
&= \frac{C\,(R-R_1)}{R}\,\|\mathbf{b}\|_{3,p}\,\|g\|_q\,.
\end{aligned}
$$

Next observe that

$$
|g_1(x)| \le C\,(R-R_1)\int_{U_{R_1,R_2}\cap\{|y|<3R/2\}} \frac{|h(y)|}{|x-y|^2}\,dy\,. \qquad (4.26)
$$

We estimate the integral in a similar way to Lemma 2.3. Thus

$$
\begin{aligned}
\int \frac{|h(y)|}{|x-y|^2}\,dy &= \int \frac{|h(y)|^{r/q}}{|x-y|^{2\alpha/q}}\,\frac{|h(y)|^{1-r/q}}{|x-y|^{2-2\alpha/q}}\,dy \\
&\le \Big(\int \frac{|h(y)|^r}{|x-y|^{2\alpha}}\,dy\Big)^{1/q}\Big(\int \frac{|h(y)|^{q'(1-r/q)}}{|x-y|^{(2-2\alpha/q)q'}}\,dy\Big)^{1/q'}\,,
\end{aligned}
$$

where $1/q+1/q'=1$. We have used here the fact that $r<q$ which is a consequence of (4.4). We can estimate

$$
\int \frac{|h(y)|^{q'(1-r/q)}}{|x-y|^{(2-2\alpha/q)q'}}\,dy \le C\sum_{n=n_0}^{\infty} 2^{n(2-2\alpha/q)q'}\int_{Q_n} |h(y)|^{q'(1-r/q)}\,dy\,,
$$

where $Q_n$ is the cube centered at $x$ with side of length $2^{-n}$ and $2^{-n_0}\sim R$. In view of the fact that $q'(1-r/q)<r$ we have

$$
\int_{Q_n} |h(y)|^{q'(1-r/q)}\,dy \le \|h\|_{q_1,r,s}^{q'(1-r/q)}\,|Q_n|^{1-q'(1-r/q)/q_1}\,.
$$

Hence, provided $\alpha$, $0<\alpha<1$, satisfies the inequality

$$
\Big(2-\frac{2\,\alpha}{q}\Big)q' - 3 + \frac{3\,q'\Big(1-\dfrac{r}{q}\Big)}{q_1} < 0\,, \qquad (4.27)
$$

we have the inequality

$$
\Big(\int \frac{|h(y)|^{q'(1-r/q)}}{|x-y|^{(2-2\alpha/q)q'}}\,dy\Big)^{1/q'} \le C\,\|h\|_{q_1,r,s}^{1-r/q}\,R^{1-(3-2\alpha)/q-3(1-r/q)/q_1}\,.
$$

There exists $\alpha$, $0 < \alpha < 1$ satisfying (3.27) provided

$$\frac{1}{q_1} < \frac{1 - \dfrac{1}{q}}{3\Big(1 - \dfrac{r}{q}\Big)} .$$

Observe that since $r > 1$ the number on the right hand side of the last equation exceeds $1/3$. Since $q_1$ satisfies (4.23) and we can choose $s$ as close as we please to $2/q$ the number $q_1$ may be chosen so that $1/q_1$ is less than any number larger than $1/3$. Hence we can find an $\alpha$ with $0 < \alpha < 1$ such that (4.27) holds. Then

$$\begin{aligned}
&\int_{|x|=R} \Big( \int \frac{|h(y)|}{|x-y|^2}\, dy \Big)^q dx \\
&\qquad \le C^q \, \|h\|_{q_1,r,s}^{q-r} \, R^{q-(3-2\alpha)-3(q-r)/q_1} \int_{|x|=R} \int \frac{|h(y)|^r}{|x-y|^{2\alpha}}\, dy\, dx \\
&\qquad \le C^q \, \|h\|_{q_1,r,s}^{q-r} \, R^{q-(3-2\alpha)-3(q-r)/q_1} R^{2-2\alpha} \, \|h\|_{q_1,r,s}^{r} \, R^{3-3r/q_1} \\
&\qquad = C^q \, \|h\|_{q_1,r,s}^{q} \, R^{qs} \\
&\qquad \le C^q R^{2-q} \, \|\mathbf{b}\|_{3,p}^q \, \|g\|_q^q ,
\end{aligned}$$

for some universal constant $C$ by (4.24). Hence by (4.26) we have

$$\|g_1\|_q \le C \, \frac{R - R_1}{R} \, \|\mathbf{b}\|_{3,p} \, \|g\|_q . \tag{4.28}$$

Putting (4.25) and (4.28) together we conclude that

$$\|Qg\|_q \le C \, \frac{R - R_1}{R} \, \|\mathbf{b}\|_{3,p} \, \|g\|_q ,$$

and hence the result follows.

Next we put Lemma 4.2 and Proposition 4.2 together to obtain an analogue of Corollary 4.2 for the case when $R - R_1$ can be much smaller than $R_2 - R$.

**Corollary 4.3.** *Suppose $R/2 < R_1 < R < R_2 = 2R$. Then for any $p$, $1 < p \le 3$ and $q > 1$ the following holds: there exist positive constants $c, \varepsilon, \delta$ depending only on $p, q$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ and*

$$\|f - \mathrm{Av}\, f\|_q \le \delta \, |\mathrm{Av}\, f| , \tag{4.29}$$

*then*

$$|\mathrm{Av}\, f_2| \ge C\, \frac{R-R_1}{R}\, |\mathrm{Av}\, f| \tag{4.30}$$

*and*

$$\|f_2 - \mathrm{Av}\, f_2\|_q \le \delta\, |\mathrm{Av}\, f_2|\,. \tag{4.31}$$

Proof. We have

$$|\mathrm{Av}\, f_2| = |\mathrm{Av}\,(P^* f) + \mathrm{Av}\,(Q^* f)| \ge |\mathrm{Av}\,(P^* f)| - C\,\varepsilon\, \frac{R-R_1}{R}\, \|f\|_q\,,$$

by Proposition 4.2. Now from the assumption (4.29) we conclude that

$$|\mathrm{Av}\, f_2| \ge |\mathrm{Av}\,(P^* f)| - \frac{C\,\varepsilon\,(R-R_1)}{R}\,(1+\delta)\,|\mathrm{Av}\, f|\,. \tag{4.32}$$

The inequality (4.30) follows now from (4.32) and (4.16) of Lemma 4.2, provided we choose $\varepsilon$ sufficiently small. To get (4.31) observe that

$$\begin{aligned} \frac{\|f_2 - \mathrm{Av}\, f_2\|_q}{|\mathrm{Av}\, f_2|} &\le \frac{\|P^* f - \mathrm{Av}\,(P^* f)\|_q}{|\mathrm{Av}\, f_2|} + \frac{\|Q^* f - \mathrm{Av}\,(Q^* f)\|_q}{|\mathrm{Av}\, f_2|} \\ &\le \gamma\,\delta\, \frac{|\mathrm{Av}\,(P^* f)|}{|\mathrm{Av}\, f_2|} + \frac{2\,C\,\varepsilon\,(R-R_1)}{R}\,(1+\delta)\, \frac{|\mathrm{Av}\, f|}{|\mathrm{Av}\, f_2|}\,, \end{aligned}$$

where we have used (4.17) of Lemma 4.2 and Proposition 4.2 together with (4.29). Now from (4.32) and (4.16) it is clear that for sufficiently small $\varepsilon$ we have

$$\frac{|\mathrm{Av}\,(P^* f)|}{|\mathrm{Av}\, f_2|} < \frac{1}{2} + \frac{1}{2\,\gamma}\,,$$

since $\gamma < 1$. Similarly we see that for sufficiently small $\varepsilon$ there is the inequality

$$\frac{2\,C\,\varepsilon\,(R-R_1)}{R}\,(1+\delta)\, \frac{|\mathrm{Av}\, f|}{|\mathrm{Av}\, f_2|} \le \Big(\frac{1}{2} - \frac{\gamma}{2}\Big)\delta\,.$$

Putting the last three inequalities together we conclude that (4.31) holds.

Observe that, in contrast to Corollary 4.2, we cannot expect the inequality $\|f_1 - \text{Av}\, f_1\|_q \le \delta |\text{Av}\, f_1|$ to hold in the situation of Corollary 4.3. The reason is that if $R - R_1$ is small then Brownian motion has a very small smoothing effect on a smooth density $f$. Thus the fluctuation of $P^* f$ decreases by a small amount proportional to $(R - R_1)/R$. On the other hand the perturbative part $Q^* f$ can generate high frequency modes with norm strictly larger than $(R - R_1)/R$ and hence the relative fluctuation of $f_1$ can be larger than that of $f$. We study this situation further in [3].

## 5. Perturbative estimates on the exit probabilities from a spherical shell with holes.

Consider a set $\mathcal{S} \subset \mathbb{R}^3$ which is a union of disjoint cubes. In this section we shall prove theorems analogous to the theorems of Section 4 for the drift process restricted to paths which do not intersect the set $\mathcal{S}$. To do this we associate with $\mathcal{S}$ a potential function $V_{\mathcal{S}}$ from which we can estimate the probability of hitting the set $\mathcal{S}$.

First we consider the case of Brownian motion $\mathbf{b} \equiv 0$. For each cube $Q$ in $\mathcal{S}$ let $\tilde{Q}$ be the cube concentric with $Q$ but double the size. We define a function $V_Q : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$V_Q(x) = \begin{cases} \dfrac{1}{|Q|^{2/3}}\,, & x \in \tilde{Q}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{5.1}$$

The potential $V_{\mathcal{S}}$ is then defined as

$$V_{\mathcal{S}} = \sum_{Q \subset \mathcal{S}} V_Q\,. \tag{5.2}$$

Now let $X(t)$, $t > 0$, be Brownian motion started at a point $x \in \mathbb{R}^3$. If $X$ hits a cube $Q \subset \mathcal{S}$ then it will spend time of order $|Q|^{2/3}$ in the cube. Thus $\int_0^\infty V_Q(X(t))\,dt$ is of order 1 on paths $X(t)$ which hit $Q$. Hence we expect that the probability of Brownian motion hitting $\mathcal{S}$ can be estimated by the expectation of $\int_0^\infty V_{\mathcal{S}}(X(t))\,dt$ This is in fact the case.

**Proposition 5.1.** *Let $X(t)$ be Brownian motion in $\mathbb{R}^3$. Then there is a universal constant $C > 0$ such that*

$$P_x(X \text{ hits } \mathcal{S}) \le C\, E_x\Big[\int_0^\infty V_{\mathcal{S}}(X(t))\,dt\Big]\,.$$

PROOF. Putting $u(x) = P_x(X \text{ hits } \mathcal{S})$, $x \in \mathbb{R}^3\backslash\mathcal{S}$ it is well known that $u(x)$ is the solution to the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0\,, & x \in \mathbb{R}^3\backslash\mathcal{S}\,, \\ u(x) = 1\,, & x \in \partial\mathcal{S}\,. \end{cases}$$

On the other hand the function

$$w(x) = E_x\Big[\int_0^\infty V_{\mathcal{S}}(X(t))\,dt\Big] = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{V_{\mathcal{S}}(y)}{|x-y|}\,dy$$

satisfies

$$\Delta w(x) = 0\,, \qquad x \in \mathbb{R}^3\backslash\mathcal{S}\,.$$

Suppose $x$ is close to a boundary point of $\mathcal{S}$. Then this point is part of a cube $Q$. Thus

$$\lim_{x\to\partial\mathcal{S}} w(x) \ge \lim_{x\to\partial Q}\frac{1}{4\pi}\int_Q\frac{V_Q(y)}{|x-y|}\,dy \ge c > 0\,,$$

where $c$ is a univeral constant. Consequently we have

$$u(x) \le \frac{w(x)}{c}\,, \qquad x \in \partial\mathcal{S}\,.$$

Hence by the maximum principle we have the inequality

$$u(x) \le \frac{w(x)}{c}\,, \qquad x \in \mathbb{R}^3\backslash\mathcal{S}\,,$$

which proves the result.

We shall use the argument of Proposition 5.1 to prove an analogue of Corollary 4.1.

**Proposition 5.2.** *Suppose $R_1 = 0$, $R_2 = 2R$, $\mathbf{b} \equiv 0$. Let $f$ be a density on the sphere $|x| = R$ and $f_2$ the density induced on $|x| = R_2$ by $f$ propagated along Brownian paths which do not intersect $\mathcal{S}$. Then for any $q$, $1 < q < \infty$, there exists $\delta, \eta > 0$ depending only on $q$ such that if*

$$\|f - \mathrm{Av}\, f\|_q \le \delta\,|\mathrm{Av}\, f|$$

*and*

$$\mathrm{Av}_{|x|=R}\Big(E_x\Big[\int_0^{\tau_{R_2}} V_{\mathcal{S}}(X(t))\,dt\Big]\Big) < \eta\,,$$

*then*

$$\|f_2 - \mathrm{Av}\, f_2\|_q \le \delta |\mathrm{Av}\, f_2| \qquad \textit{and} \qquad |\mathrm{Av}\, f_2| \ge \frac{|\mathrm{Av}\, f|}{2}\,.$$

Proof. We consider the operator from functions $g$ on $|x| = R_2$ to functions on $|x| = R$ given by

$$Ag(x) = E_x[g\,(X(\tau_{R_2}))\,,\; X(t) \in \mathcal{S}\,,\; \text{some } t,\; 0 < t < \tau_{R_2}]\,.$$

Then for any $r, r'$, $1 < r < \infty, 1/r + 1/r' = 1$, we have

$$|Ag(x)|^r \le P_x(X(t) \in \mathcal{S},\; \text{some } t,\; 0 < t < \tau_{R_2})^{r/r'} E_x[|g(X(\tau_{R_2}))|^r]\,,$$

by Holder's inequality. Now by the property of the Poisson kernel we have that

$$E_x[|g(X(\tau_{R_2}))|^r] \le C\,\|g\|_r^r\,,$$

for some universal constant $C$. Hence if $r \ge r'$, we have

$$\|Ag\|_r^r \le C\,\|g\|_r^r\,\mathrm{Av}_{|x|=R}\,P_x(X(t) \in \mathcal{S},\; \text{some } t,\; 0 < t < \tau_{R_2})\,.$$

If $r < r'$ we have by Jensen,

$$\|Ag\|_r^r \le C\,\|g\|_r^r\,(\mathrm{Av}_{|x|=R}\,P_x(X(t) \in \mathcal{S},\; \text{some } t,\; 0 < t < \tau_{R_2}))^{r/r'}\,.$$

Now by the argument of Proposition 5.1 we conclude that

$$\|Ag\|_r \le C\,\|g\|_r\,\eta^{\min\{1/r, 1/r'\}}\,,$$

for some universal constant $C$. Thus the adjoint $A^*$ of $A$ is a bounded operator from $L^q(\{|x| = R\})$ to $L^q(\{|x| = R_2\})$ with norm $\|A\|$ bounded as

$$\|A\| \le C\eta^{\min\{1/q, 1/q'\}}\,,$$

for some constant $C$. Observe next that the densities $f$ and $f_2$ are related by the equation

$$f_2 = P^* f - A^* f\,,$$

where $P$ is the integral operator with Poisson kernel as in Section 4. Hence we have

$$\mathrm{Av}\, f_2 = \mathrm{Av}\, P^* f - \mathrm{Av}(A^* f) = \mathrm{Av}\, f - \mathrm{Av}\,(A^* f)\,.$$

Now since $q > 1$ it follows that

$$\begin{aligned}|\mathrm{Av}\,(A^* f)| &\le \|A^* f\|_q \\ &\le C\,\eta^{\min\{1/q,1/q'\}}\|f\|_q \\ &\le C\,\eta^{\min\{1/q,1/q'\}}\,(1+\delta)\,|Av\,f|\,.\end{aligned}$$

Thus by choosing $\eta$ sufficiently small we have $|\mathrm{Av}\,f_2| \ge |\mathrm{Av}\,f|/2$.

Next observe that there exists $\gamma$, $0 < \gamma < 1$ such that

$$\|P^* f - \mathrm{Av}\,(P^* f)\|_q \le \gamma\,\|f - \mathrm{Av}\,f\|_q\,.$$

Hence

$$\begin{aligned}\|f_2 - \mathrm{Av}\,f_2\|_q &\le \|P^* f - \mathrm{Av}\,(P^*\,f)\|_q + \|A^* f - \mathrm{Av}\,(A^* f)\|_q \\ &\le \gamma\,\delta\,|\mathrm{Av}\,f| + 2\,C\,\eta^{\min\{1/q,1/q'\}}\,(1+\delta)\,|\mathrm{Av}\,f|\,.\end{aligned}$$

It is clear by choosing $\eta > 0$ sufficiently small that the right hand side of the last inequality is less than $\delta\,|\mathrm{Av}\,f_2|$. The result is complete.

REMARK 5.1. Observe that in Proposition 5.2 we have used the fact that if $Q$ is a cube in $\mathcal{S}$ then $V_{\mathcal{S}}(x) \ge |Q|^{-2/3}$ on the double of $Q$, $\tilde{Q}$. The reason is that if $Q$ has a small intersection with $U_{R_1,R_2}$ then $\tilde{Q} \cap U_{R_1,R_2}$ has volume of order $|Q|$. Hence a Brownian path which hits $Q$ makes an order 1 contribution to $\int_0^{\tau_{R_2}} V_{\mathcal{S}}(X(t))\,dt$.

Next we wish to generalize Propositions 5.1, 5.2 to the case of nontrivial drift. First we estimate the probability that the drift process visits a cube $Q$.

**Proposition 5.3.** *Let $Q_m$ be a cube with side of length $2^{-m}$, $m$ an integer, and $P_x(Q_m)$ the probability that the process with drift $\mathbf{b}$ started at $x$ visits $Q_m$ before exiting to $\infty$. Then for any $\alpha < 1$ there exists $\varepsilon > 0$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then*

$$P_x(Q_m) \le \frac{C}{(2^m d(x,Q_m)+1)^\alpha}\,, \tag{5.3}$$

*for some universal constant $C$. Here $d(x,Q_m)$ is the distance from the point $x$ to the cube $Q_m$.*

PROOF. First we consider the solution of a boundary value problem on the shell $U_{R_1,R_2}$ with $R_1 = R/2$ and $R_2 = 2R$. Thus we wish to estimate the solution of

$$\begin{cases} \Delta w(x) + \mathbf{b}(x)\cdot\nabla w(x) = 0\,, & x \in U_{R_1,R_2}\,, \\ w(x) = 0\,, & |x| = R_1\,, \\ w(x) = 1\,, & |x| = R_2\,. \end{cases}$$

Let $w_0$ be the solution when $\mathbf{b} \equiv 0$. Then, in the notation of Section 4, $w_0 = P\,1$. It is easy to see that $w_0$ is given by the formula

$$w_0(x) = \frac{4}{3}\Big(1 - \frac{R}{2\,|x|}\Big)\,.$$

We shall show that $\varepsilon > 0$ can be chosen so that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then there exists a universal constant $C > 0$ such that

$$|w(x) - w_0(x)| \le C\,\|\mathbf{b}\|_{3,p}\,, \qquad x \in U_{R_1,R_2}\,. \tag{5.4}$$

In fact we have

$$w(x) = w_0(x) + Q\,\mathbf{1}(x)\,, \qquad x \in U_{R_1,R_2}\,, \tag{5.5}$$

where $Q$ is the operator (4.3). It is easy to see that if $1 < r < p$, $r \le q < 3$, the function $\mathbf{b}\cdot\nabla w_0$ is in the Morrey space $M_r^q$ and

$$\|\mathbf{b}\cdot\nabla w_0\|_{q,r} \le C R^{3/q-2}\,\|\mathbf{b}\|_{3,p}\,,$$

for some universal constant $C > 0$. It follows then from [5, Theorem 1.2] that for $\varepsilon$ sufficiently small

$$Q\,\mathbf{1}(x) = \int_{U_{R_1,R_2}} G_D(x,y)\,g(y)\,dy\,,$$

where $g \in M_r^q$ and

$$\|g\|_{q,r} \le C R^{3/q-2}\,\|b\|_{3,p}\,,$$

and $C$ is a universal constant. If we take $q > 3/2$ then the inequality of (5.4) follows by standard argument.

To prove the inequality (5.3) let $S_k$, $k = 0, 1, 2, \dots$ be spheres concentric with $Q_m$ and with radius $2^k 2^{-m}$. Thus $S_0$ contains $Q_m$. From (5.4) we can choose $\varepsilon$ sufficiently small such that $w$ satisfies

$$\inf_{|x|=R} w(x) \geq \frac{2^\alpha}{(1+2^\alpha)}, \qquad 0 < R < \infty. \tag{5.6}$$

The inequality (5.3) follows immediately now from (5.6) and [5, Lemma 6.3].

REMARK 5.2. Observe that in the Brownian motion case one can take $\alpha = 1$ in (5.3) but for the case of nontrivial $\mathbf{b}$ one must have $\alpha < 1$. This fact will determine our selection of the function $V_{\mathcal{S}}$ in the case of nontrivial $\mathbf{b}$.

The proof of Proposition 5.3 does not generalize to the situations we are interested in. We shall therefore give a different, more complicated proof of the Proposition which does generalize. Let us consider the region $\Omega_R$ external to the ball of radius $R > 0$ centered at the origin. The Dirichlet Green's function for this region is given by

$$G_D(x,y) = \frac{1}{4\pi}\Big(\frac{1}{|x-y|} - \frac{R}{|y|}\,\frac{1}{|x-\overline{y}|}\Big), \qquad |x|, |y| > R, \tag{5.7}$$

where $\overline{y}$ is the reflection of $y$ in the boundary of $\Omega_R$. We estimate $G_D$ and its gradient $\nabla_x G_D$:

**Lemma 5.1.** a) *There is the inequality*

$$0 \leq G_D(x,y) \leq \frac{1}{4\pi\,|x-y|}, \qquad |x|, |y| > R.$$

b) $|\nabla_x G_D(x,y)| \leq k_1(x,y) + k_2(x,y)$, *where*

$$|k_1(x,y)| \leq \frac{C}{|x-y|^2}, \qquad |x|, |y| > R, \tag{5.8}$$

$$\begin{cases} |k_2(x,y)| \leq \dfrac{C}{|x|\,|y|}, & |y| > 3|x|,\ |x| > R, \\ |k_2(x,y)| = 0, & \textit{otherwise}, \end{cases} \tag{5.9}$$

*and $C$ is a universal constant.*

PROOF. Since a) follows easily from the maximum principle we shall just consider b). We have now

$$\nabla_x G_D(x,y) = \frac{-1}{4\pi}\Big(\frac{x-y}{|x-y|^3} - \frac{R}{|y|}\,\frac{x-\overline{y}}{|x-\overline{y}|^3}\Big)\,.$$

Since $G_D(x,y) \geq 0$ it follows that

$$|\nabla_x G_D(x,y)| \leq \frac{1}{4\pi}\Big(\frac{1}{|x-y|^2} + \frac{1}{|x-y|\,|x-\overline{y}|}\Big)\,.$$

We consider first the case $|y| > 3\,|x|$. It is easy to see that $|x-y| \geq 2\,|y|/3$ and

$$|x-\overline{y}| \geq |x| - |\overline{y}| \geq |x| - \frac{R}{3} \geq \frac{|x|}{2}\,.$$

Hence

$$\frac{1}{|x-y|\,|x-\overline{y}|} \leq \frac{3}{|x|\,|y|}\,.$$

Next consider the situation $R < |y| < 3|x|$. Suppose that $|x| > 2R$. Then

$$|x-\overline{y}| \geq |x| - R \geq \frac{|x|}{2} \geq \frac{|x|+|y|}{8} \geq \frac{|x-y|}{8}\,.$$

In the case $R < |x|,\ |y| < 2R$ it is clear that there exists a universal constant $C_1$ with $|x-\overline{y}| \geq C_1\,|x-y|$. We conclude then that in this situation one has

$$|\nabla_x G_D(x,y)| \leq \frac{C}{|x-y|^2}\,,$$

for some universal constant $C > 0$. The proof is complete.

Next we define Morrey spaces for the region $\Omega_R$ in a similar way to (4.20). Thus for $1 < r \leq q < \infty$ and $s > 0$ we say $g : \Omega_R \longrightarrow \mathbb{C}$ is in the Morrey space $M^q_{r,s}(\Omega_R)$ if

$$\int_{Q\cap\Omega_R} |g(x)|^r\,dx \leq C^r\,|Q|^{1-r/q}\Big(\frac{R}{d(Q)}\Big)^{rs}\,, \tag{5.10}$$

for all cubes $Q$ and constant $C$. Here $d(Q)$ is defined by

$$d(Q) = \sup\,\{|y| : y \in Q \cap \Omega_R\}\,.$$

Evidently one has $d(Q) \geq R$. The norm of $g$, $\|g\|_{q,r,s}$ is then the infimum of all $C$ such that (5.10) holds.

**Lemma 5.2.** *Let $T_1$ be the integral operator on functions with domain $\Omega_R$ which has kernel $|\mathbf{b}(x)|\, k_1(x,y)$ where $\mathbf{b} \in M_p^3$, $1 < p \le 3$ and $k_1$ satisfies* (5.8). *Then for $1 < r < p$, $r \le q < 3$, $s > 0$, $T_1$ is a bounded operator on $M_{r,s}^q(\Omega_R)$ and the norm of $T_1$, $\|T_1\|$ satisfies an inequality $\|T_1\| \le C\|\mathbf{b}\|_{3,p}$ where $C$ depends only on $r,p,q,s$.*

PROOF. Same as for Lemma 4.4.

**Lemma 5.3.** *Let $T_2$ be the integral operator on functions with domain $\Omega_R$ which has kernel $|\mathbf{b}(x)|\, k_2(x,y)$ where $\mathbf{b} \in M_p^3$, $1 < p \le 3$ and $k_2$ satisfies* (5.9). *Then for $1 \le r \le p$, $r \le q$, $s \ge 0$ and $2 < 3/q + s < 3/r$, $T_2$ is a bounded operator on $M_{r,s}^q(\Omega_R)$ and the norm of $T_2$, $\|T_2\|$ satisfies an inequality $\|T_2\| \le C\|\mathbf{b}\|_{3,p}$, where $C$ depends only on $r,p,q,s$.*

PROOF. For $n = 0, \pm 1, \ldots$ let $Q_n$ be the cube centered at the origin with side of length $2^{-n}$. If $u : \Omega_R \longrightarrow \mathbb{C}$ is a locally integrable function we denote by $u_{Q_n}$ the average value of $|u|$ on $Q_n$, whence

$$u_{Q_n} = |Q_n|^{-1} \int_{\Omega_R \cap Q_n} |u(x)|\, dx\,.$$

Hence we have

$$|T_2 u(x)| \le \frac{C\,|\mathbf{b}(x)|}{|x|} \sum_{|x|<2^{-n}} 2^{-2n}\, u_{Q_n}\ ,$$

for some universal constant $C$. Hence for $2^{-m} > R$, we have

$$\begin{aligned}\int_{Q_m \cap \Omega_R} |T_2 u(x)|^r\, dx &\le C^r \sum_{k=m}^{\infty} \int_{Q_k} \Big( |\mathbf{b}(x)|\, 2^k \sum_{n=-\infty}^{k} 2^{-2n}\, u_{Q_n} \Big)^r dx \\ &\le C^r\, \|\mathbf{b}\|_{3,p}^r \sum_{k=m}^{\infty} 2^{k(2r-3)} \Big( \sum_{n=-\infty}^{k} 2^{-2n}\, u_{Q_n} \Big)^r .\end{aligned}$$

Observe next that

$$\begin{aligned}\sum_{n=-\infty}^{k} 2^{-2n}\, u_{Q_n} &\le \|u\|_{q,r,s} \sum_{n=-\infty}^{k} 2^{n(-2+3/q+s)} R^s \\ &\le C_1\, R^s\, \|u\|_{q,r,s}\, 2^{k(-2+3/q+s)}\,,\end{aligned}$$

since $2 < 3/q + s$. Thus we have

$$\int_{Q_m} |T_2 u(x)|^r \, dx \le C^r \, C_1^r \, R^{sr} \sum_{k=m}^{\infty} \|\mathbf{b}\|_{3,p}^r \, \|u\|_{q,r,s}^r \, 2^{k(3r/q+sr-3)}$$
$$\le C_2^r \, R^{sr} \, \|\mathbf{b}\|_{3,p}^r \, \|u\|_{q,r,s}^r \, 2^{m(3r/q+sr-3)} \,,$$

since $3/q + s < 3/r$. Consequently, we have

$$\int_{Q_m} |T_2 u(x)|^r \, dx \le C_2^r \, \|\mathbf{b}\|_{3,p}^r \, \|u\|_{q,r,s}^r \, |Q_m|^{1-r/q} \Big( \frac{R}{d(Q_m)} \Big)^{sr} .$$

We have shown therefore that (5.10) holds for cubes centered at the origin. It is easy now to generalize the previous argument to all cubes.

Proof of Proposition 5.3. Evidently $P_x(Q_m)$ is bounded above by the probability that the drift process started at $x$ hits the ball concentric with $Q_m$ of radius $R = 2^{-m}$. For Brownian motion this probability is given by $w_0(x)$, where

$$\begin{cases} \Delta w_0(x) = 0 \,, & |x| > R \,, \\ w_0(x) = 1 \,, & |x| = R \,. \end{cases}$$

Thus $w_0(x) = R/|x|$, $|x| > R$. For the drift process it is given by $w(x)$, where

$$w(x) = w_0(x) + \int_{\Omega_R} G_D(x,y) \, (I - T)^{-1} \, \mathbf{b} \cdot \nabla w_0(y) \, dy \,. \tag{5.11}$$

Here $G_D$ is the Green's function (5.7) and $T$ is the integral operator with kernel $\mathbf{b}(x) \cdot \nabla_x G_D(x,y)$. We wish to show that the function $\mathbf{b} \cdot \nabla w_0$ is in an appropriate Morrey space $M^q_{r,s}(\Omega_R)$. Evidently one has $|\mathbf{b}(x) \cdot \nabla w_0(x)| \le R \, |\mathbf{b}(x)|/|x|^2$. Now for the cube $Q_n$ with side of length $2^{-n} > R$ centered at the origin one has

$$\int_{Q_n} \Big( \frac{R \, |\mathbf{b}(x)|}{|x|^2} \Big)^r \, dx \le C \sum_{j=n}^{m} \|\mathbf{b}\|_{3,p}^r \, R^r \, 2^{-j(3-3r)} \tag{5.12}$$
$$\le C_1^r \, \|\mathbf{b}\|_{3,p}^r \, R^{3-2r} \,,$$

for some constant $C_1$, provided $1 < r \le p$. On the other hand if $Q$ is a cube such that $d(Q) \gg |Q|^{1/3}$ then we have

$$\begin{aligned}\int_Q \Big(\frac{R\,|\mathbf{b}(x)|}{|x|^2}\Big)^r dx &\le \frac{R^r}{d(Q)^{2r}}\,\|\mathbf{b}\|_{3,p}^r\,|Q|^{1-r/3}\\ &\le R^{r(3/q-2)}\Big(\frac{R}{d(Q)}\Big)^{r(3-3/q)}\|\mathbf{b}\|_{3,p}^r\,|Q|^{1-r/q}\\ &\le R^{r(3/q-2)}\Big(\frac{R}{d(Q)}\Big)^{rs}\|\mathbf{b}\|_{3,p}^r\,|Q|^{1-r/q},\end{aligned}$$

for any $r, s, q$ with $1 \le r \le p$, $q \le 3$, $s \le 3 - 3/q$. Combining this last inequality with (5.12), we see that if $r, s, q$ satisfy the inequalities

$$(5.13)\qquad 1 < r \le p\,, \qquad r \le q \le 3\,, \qquad s \le 3\Big(\frac{1}{r} - \frac{1}{q}\Big)\,,$$

then $\mathbf{b}\cdot\nabla w_0$ is in $M^q_{r,s}(\Omega_R)$ and

$$\|\mathbf{b}\cdot\nabla w_0\|_{q,r,s} \le CR^{3/q-2}\,\|\mathbf{b}\|_{3,p}\,,$$

for some constant $C$ depending only on $q, r, s$.

Observe next that for any $s$, $0 < s < 1$, it is possible to find $r, q$ such that $3/2 < q < 3$ as well as the inequalities (5.13) and the conditions of lemmas 5.2, 5.3 hold. Hence the function

$$g(x) = (I - T)^{-1}\,\mathbf{b}\cdot\nabla w_0$$

is also in $M^q_{r,s}(\Omega_R)$ for sufficiently small $\varepsilon$ and has norm which satisfies

$$\|g\|_{q,r,s} \le CR^{3/q-2}\,\|\mathbf{b}\|_{3,p}\,,$$

for a constant $C$ depending only on $q, r, s$. Now let us suppose that $|x| > 2R$. Then from (5.11) we have

$$(5.14)\qquad \begin{aligned}|w(x) - w_0(x)| &\le \int_{\Omega_R} G_D(x,y)\,|g(y)|\,dy\\ &\le \int_{|x-y|<|x|/2} dy + \int_{|y|<|x|/2} dy\\ &\quad + \int_{\{|x-y|>|x|/2,\,|y|>|x|/2\}} dy\\ &= I_1 + I_2 + I_3\,.\end{aligned}$$

If we take now $2^{-n_0} \sim |x|$ for a suitable integer $n_0$ we have

$$\begin{aligned} I_1 &\le C \sum_{k=n_0}^{\infty} 2^k \int_{|x-y|<2^{-k}} |g(y)|\, dy \\ &\le C \sum_{k=n_0}^{\infty} 2^k R^{3/q-2} \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^s 2^{-3k(1-1/q)} \\ &\le C\, 2^{n_0(3/q-2)} R^{3/q-2} \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^s \\ &= C\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^{s+3/q-2}, \end{aligned}$$

since $q > 3/2$.

On the other hand we have

$$\begin{aligned} I_2 &\le C\, 2^{n_0} \sum_{k=n_0}^{m} \int_{2^{-k}<|y|<2^{-k+1}} |g(y)|\, dy \\ &\le C\, 2^{n_0} \sum_{k=n_0}^{m} R^{3/q-2} \Big(\frac{R}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\ &\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^{s+3/q-2}, \end{aligned}$$

since $s + 3/q > 3$. Finally we have

$$\begin{aligned} I_3 &\le C \sum_{k=-\infty}^{n_0} 2^k \int_{2^{-k}<|x-y|<2^{-k+1}} |g(y)|\, dy \\ &\le C \sum_{k=-\infty}^{n_0} 2^k R^{3/q-2} \Big(\frac{R}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\ &\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^{s+3/q-2}, \end{aligned}$$

provided $s + 3/q - 2 > 0$. Now it is easy to see that we can choose $s, q, r$ appropriately to make $s + 3/q - 2$ as close to 1 as we please. The inequality (5.3) easily follows from this.

Next we consider a cube $Q_m$ with side of length $2^{-m}$ which is contained in the ball $U_{0,R_2}$ of radius $R_2$. For $x \in U_{0,R_2}$ let $P_x(Q_m)$ be given now by

$$P_x(Q_m) = \text{probability that the drift process started at } x \text{ hits } Q_m \text{ before hitting the boundary of } U_{0,R_2}\,.$$

It is easy to estimate this probability in the case of Brownian motion $\mathbf{b} \equiv 0$. In fact by the argument of Proposition 5.1 it is bounded by

$$P_x(Q_m) \le C \int_{Q_m} 2^{2m}\, G_D(x,y)\, dy\,, \tag{5.15}$$

where $G_D$ is the Dirichlet Green's kernel on $U_{0,R_2}$ and $C$ is a universal constant. Since $G_D$ is given explicitly it is easy to estimate the right hand side of (5.15). Let $d(Q_m)$ be defined by

$$d(Q_m) = \sup \{ d(y, \partial U_{0,R_2}) : y \in Q_m \}\,.$$

Then we see from (5.15) that

$$P_x(Q_m) \le \frac{C}{2^m d(x,Q_m) + 1} \min\left\{1, \frac{2^m\, d(Q_m)}{2^m\, d(x,Q_m) + 1}\right\}, \tag{5.16}$$

where $C$ is a universal constant. In view of Proposition 5.3 it would seem that one could generalize (5.16) to the case of nontrivial $\mathbf{b}$ by

$$P_x(Q_m) \le \frac{C_\alpha}{(2^m\, d(x,Q_m) + 1)^\alpha} \min\left\{1, \frac{2^m\, d(Q_m)}{2^m\, d(x,Q_m) + 1}\right\}, \tag{5.17}$$

where $0 < \alpha < 1$ and the constant $C_\alpha$ depends on $\alpha$. We shall prove the inequality (5.17) following the same lines as the second proof of Proposition 5.3.

Let $B_a(\rho)$ be the ball of radius $\rho$ centered at the point $a$. Suppose $a \in U_{0,R_2}$, the ball of radius $R_2$ centered at the origin and the distance from $a$ to $\partial U_{0,R_2}$ is larger than $3\rho$. Let $w_0$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta w_0(x) = 0\,, & x \in U_{0,R_2} \backslash B_a(\rho)\,, \\ w_0(x) = 1\,, & x \in \partial B_a(\rho)\,, \\ w_0(x) = 0\,, & x \in \partial U_{0,R_2}\,. \end{cases} \tag{5.18}$$

**Lemma 5.4.** *There is a universal constant $C > 0$ such that for $x \in U_{0,R_2}\backslash B_a(\rho)$*

$$|\nabla w_0(x)| \leq \frac{C\rho}{|x-a|^2} \min\Big\{1, \frac{d(a, \partial U_{0,R_2})}{|x-a|}\Big\}. \tag{5.19}$$

PROOF. Let $G_D(x,y)$, $x, y \in U_{0,R_2}$ be the Dirichlet Green's function for the ball. Then just as in Proposition 5.1, there exists a universal constant $C$ such that

$$w_0(x) \leq C\rho^{-2} \int_{B_a(\rho)} G_D(x,y)\, dy\,, \qquad x \in U_{0,R_2}\backslash B_a(\rho)\,.$$

It is easy to estimate $w_0(x)$ from the last inequality since we have an explicit formula for $G_D$. Thus there is a universal constant $C > 0$ such that

$$w_0(x) \leq \frac{C\rho}{|x-a|} \min\Big\{1, \frac{d(a, \partial U_{0,R_2})}{|x-a|}\Big\}. \tag{5.20}$$

We obtain the estimate (5.19) from (5.20) and the Harnack principle. First let us consider the case where $\rho < |x-a| < 3\rho/2$. Now the function $w_0$ can be extended in a harmonic way inside the ball $B_a(\rho)$ by using the Kelvin transform [1]. Hence $w_0$ is harmonic in the region $\rho/2 < |x-a| < 7\rho/4$ and $\|w_0\|_\infty \leq C$ for some universal constant $C$. It follows then from the Harnack principle that

$$|\nabla w_0(x)| \leq \frac{C}{\rho}\,, \qquad \rho < |x-a| < \frac{3\rho}{2}\,,$$

for a suitable universal constant $C > 0$.

Next we consider the situation where

$$\frac{3\rho}{2} < |x-a| < \frac{d(a, \partial U_{0,R_2})}{2}\,.$$

Then $w_0$ is harmonic in the ball $|y-x| \leq |x-a|/4$. In fact we have

$$|x-a| \leq |x-y| + |y-a| \leq \frac{|x-a|}{4} + |y-a|\,,$$

whence $|y-a| \geq 3\,|x-a|/4 \geq 9\rho/8 > \rho$. On the other hand

$$|y-a| \leq |x-y| + |x-a| \leq \frac{5\,|x-a|}{4} \leq \frac{5\, d(a, \partial U_{0,R_2})}{8} < d(a, \partial U_{0,R_2})\,.$$

It follows easily now from (5.20) and the inequality $|y-a| \geq 3\,|x-a|/4$ that $|\nabla w_0(x)| \leq C\rho/|x-a|^2$ for some universal constant $C$.

Finally we consider the situation $|x-a| > d(a, \partial U_{0,R_2})/2$. Using the Kelvin transformation the function $w_0$ can be extended in a harmonic way to the entire ball $|x-y| \leq |x-a|/4$. Now, using Harnack and the estimate (5.20) we conclude that there is a universal constant $C$ such that

$$|\nabla w_0(x)| \leq C\rho\,\frac{d(a, \partial U_{0,R_2})}{|x-a|^3}\;.$$

All cases of the inequality (5.19) are now covered.

Let $G_{D,1}(x,y)$ be the Dirichlet Green's function for the domain $U_{0,R_2}\backslash B_a(\rho)$. We wish to prove an analogue of Lemma 5.1.

**Lemma 5.5.** a) *Let* $d = d(a, \partial U_{0,R_2})$. *Then there is a universal constant* $C$ *such that*

$$0 \leq G_{D,1}(x,y) \leq \frac{C}{|x-y|}\;\min\Big\{1, \frac{|y-a|+d}{|x-y|}\Big\}. \tag{5.21}$$

b) $|\nabla_x G_{D,1}(x,y)| \leq k_1(x,y) + k_2(x,y)$, *where*

$$|k_1(x,y)| \leq \frac{C}{|x-y|^2}\;\min\Big\{1, \frac{|y-a|+d}{|x-y|}\Big\}, \tag{5.22}$$

$$|k_2(x,y)| \leq \frac{C}{|x-a|\,|y-a|}\;\min\Big\{1, \frac{|x-a|+d}{|y-a|}\Big\}, \tag{5.23.1}$$

*if* $|y-a| > 3\,|x-a|$ *and*

$$|k_2(x,y)| = 0\,, \tag{5.23.2}$$

*otherwise.*

Proof. a) Let $G_D(x,y)$ be the Dirichlet Green's function for the ball $U_{0,R_2}$. Then we have the inequality

$$0 \leq G_D(x,y) \leq \frac{C}{|x-y|}\;\min\Big\{1, \frac{d(y, \partial U_{0,R_2})}{|x-y|}\Big\},$$

for some universal constant $C$. The inequality (5.21) follows now from the fact that

$$0 \leq G_{D,1}(x,y) \leq G_D(x,y)\,, \qquad d(y, \partial U_{0,R_2}) \leq |y-a| + d\,.$$

b) Consider first the situation $|y-a| < 3\,|x-a|$. Then we have

$$|x-y| \le |x-a| + |y-a| \le 4\,|x-a|\,.$$

Consider next the ball $B_x(|x-y|/8)$ centered at $x$ with radius $|x-y|/8$. For $z \in B_x(|x-y|/8)$ we have

$$|z-y| \ge |x-y| - |z-x| \ge \frac{7\,|x-y|}{8}$$

and

$$|z-a| \ge |x-a| - |x-z| \ge |x-a| - \frac{|x-y|}{8} \ge \frac{|x-a|}{2}\,.$$

Hence if $|x-a| > 2\rho$ the ball $B_x(|x-y|/8)$ does not intersect $B_a(\rho)$. Furthermore, the function $u(z) = G_{D,1}(z,y)$ can be extended in a harmonic way by the Kelvin transform to the entire ball $B_x(|x-y|/8)$. From (5.21) it follows that the $L^\infty$ norm of $u$, $\|u\|_\infty$, on this ball satisfies

$$\|u\|_\infty \le \frac{C}{|x-y|}\ \min\Big\{1, \frac{|y-a|+d}{|x-y|}\Big\}\,.$$

The inequality (5.22) follows now from this last inequality by the Harnack principle. To deal with the situation $|x-a| \le 2\rho$ observe that the inequality (5.22) is just the same as $k_1(x,y) \le C/|x-y|^2$.

We get this last inequality by exactly the same argument as before, extending the harmonic function $G_{D,1}(z,y)$ into the ball $B_a(\rho)$ as necessary.

Finally we consider the case $|y-a| > 3\,|x-a|$. As in Lemma 5.1 it follows that $|y-x| > 2\,|y-a|/3$. For $z \in B_x(|x-a|/4)$ we have

$$|y-z| \ge |y-x| - |z-x| \ge \frac{2\,|y-a|}{3} - \frac{|x-a|}{4} \ge \frac{7\,|y-a|}{12}\,.$$

Furthermore, $|z-a| \le 5\,|x-a|/4$. Now consider again the function $u(z) = G_{D,1}(z,y)$ which can be continued in a harmonic way to the entire ball $B_x(|x-a|/4)$. By the symmetry of $G_{D,1}$ it follows from (5.21) that

$$0 \le u(z) \le \frac{C}{|z-y|}\ \min\Big\{1, \frac{|z-a|+d}{|z-y|}\Big\}\,.$$

The inequality (5.23) follows now from this last inequality and the Harnack principle.

Next let $\Omega_\rho$ be the domain

$$\Omega_\rho = \{x \in \mathbb{R}^3 : |x-a| > \rho\}.$$

We define Morrey spaces on $\Omega_\rho$ which generalize (5.10). For $1 < r \le q < \infty$ and $s > 0$ we say that $g : \Omega_\rho \longrightarrow \mathbb{C}$ is in the weighted Morrey space $M^q_{r,s}(\Omega_\rho)$ with weight $w$ if

$$\int_{Q\cap\Omega_\rho} w(x)^r \, |g(x)|^r \, dx \le C^r |Q|^{1-r/q} \Big(\frac{\rho}{d(Q)}\Big)^{rs}, \tag{5.24}$$

for all cubes $Q$ and constant $C$. Here $d(Q) = \sup\{|x-a| : x \in Q\cap\Omega_\rho\}$. The norm of $g$, $\|g\|_{q,r,s}$ is then the infimum of all $C$ such that (5.24) holds.

**Lemma 5.6.** *Let $T_1$ be the integral operator on functions with domain $\Omega_\rho$ which has kernel $|\mathbf{b}(x)|\, k_1(x,y)$ where $\mathbf{b} \in M^3_p$, $1 < p \le 3$ and $k_1$ satisfies* (5.22). *Then for $1 < r < p$, $r \le q < 3$, $s > 0$, $T_1$ is a bounded operator on the weighted Morrey space $M^q_{r,s}(\Omega_\rho)$ with weight $w$ given by*

$$w(x) = \frac{1}{\min\Big\{1, \dfrac{d}{|x-a|}\Big\}}, \qquad x \in \Omega_\rho. \tag{5.25}$$

*The norm $\|T_1\|$ of $T_1$ satisfies an inequality $\|T_1\| \le C\,\|\mathbf{b}\|_{3,p}$, where $C$ depends only on $r,p,q,s$.*

Proof. We proceed in a similar way to the proof of Proposition 2.1. Consider a dyadic decomposition of $\mathbb{R}^3$ into cubes $Q$. For $u : \Omega_\rho \longrightarrow \mathbb{C}$ we define $u_Q$ by

$$u_Q = \frac{d(Q)}{d}\,|Q|^{-1}\int_{\Omega_\rho\cap Q} |u(x)|\,dx, \qquad |Q| < d^3,$$

$$u_Q = |Q|^{-1}\int_{\Omega_\rho\cap Q} w(x)\,|u(x)|\,dx, \qquad |Q| > d^3.$$

Let $n \in \mathbb{Z}$ and $S_n u(x)$ be given by

$$S_n u(x) = 2^{-n}\Big(\frac{d}{d(Q_n)}\Big) u_{Q_n}, \qquad x \in Q_n,$$

where $Q_n$ is the unique dyadic cube with side of length $2^{-n}$ containing $x$. The operator $S$ on functions $u : \Omega_\rho \longrightarrow \mathbb{C}$ is then defined as

$$Su(x) = \sum_{n=-\infty}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,, \qquad x \in \Omega_\rho\,. \tag{5.26}$$

Now we can think of the dyadic decomposition as being centered at some point $\xi \in \mathbb{R}^3$. The operator $S$ of (5.26) should therefore be more accurately written as $S_\xi$. Then, in analogy to (2.2) we have

$$\int_{Q\cap\Omega_\rho} w(x)^r\, |T_1 u(x)|^r\, dx \le \frac{C^r}{|\Lambda|} \int_\Lambda d\xi \int_{Q\cap\Omega_\rho} w(x)^r\, |S_\xi u(x)|^r\, dx\,,$$

where $\Lambda$ is a sufficiently large cube and $C$ is a universal constant. This follows from the inequality (5.22). We can therefore restrict ourselves to showing that $S_\xi$ is a bounded operator on the weighted Morrey space for an arbitrary $\xi$. Let $n_0$ be the smallest integer $n$ such that $2^{-n} < d$. Then we may write $S_\xi = A + B$ where

$$Au(x) = \sum_{n=n_0}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,, \qquad x \in \Omega_\rho\,.$$

Suppose $Q_m$ is a dyadic cube with side of length $2^{-m}$ where $m \ge n_0$. Then $\sup w / \inf w$ is bounded above by a universal constant on $Q_m$. We write $Au(x) = A_1 u(x) + A_2 u(x)$, for $x \in Q_m$ where

$$A_1 u(x) = \sum_{n=m}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,.$$

Then we have

$$\begin{aligned} \int_{Q_m\cap\Omega_\rho} w(x)^r\, |A_1 u(x)|^r\, dx &\le (\sup w)^r \int_{Q_m\cap\Omega_\rho} |A_1 u(x)|^r\, dx \\ &\le (\sup w)^r\, C_1^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q_m\cap\Omega_\rho} |u(x)|^r\, dx \\ &\le C_2^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q_m\cap\Omega_\rho} w(x)^r\, |u(x)|^r\, dx\,, \end{aligned} \tag{5.27}$$

where $C_1$ and $C_2$ are constants depending only on $r < p$. Here we are using the boundedness of the operator $A_1$ as given in [5, Theorem 1.2].

Since $\sup w/\inf w$ is bounded above on the dyadic cube $Q_{n_0}$ with side of length $2^{-n_0}$ which contains $Q_m$ we have

$$\begin{aligned}
|A_2u(x)| &\le |\mathbf{b}(x)| \sum_{n=n_0}^{m} 2^{2n} \int_{Q_n\cap\Omega_\rho} |u(y)|\,dy \\
&\le \frac{C\,|\mathbf{b}(x)|}{\sup w} \sum_{n=n_0}^{m} 2^{2n} \int_{Q_n\cap\Omega_\rho} w(y)\,|u(y)|\,dy \\
&\le \frac{C\,|\mathbf{b}(x)|}{\sup w} \sum_{n=n_0}^{m} 2^{n(3/r-1)} \Big( \int_{Q_n\cap\Omega_\rho} w(y)^r\,|u(y)|^r\,dy \Big)^{1/r} \\
&\le \frac{C\,|\mathbf{b}(x)|}{\sup w} \sum_{n=n_0}^{m} 2^{n(3/r-1)}\, C\, 2^{-3n(1/r-1/q)} \Big( \frac{\rho}{d(Q_n)} \Big)^{s} \\
&\le \frac{C_1\,|\mathbf{b}(x)|}{\sup w} |Q_m|^{1/3-1/q} \Big( \frac{\rho}{d(Q_m)} \Big)^{s}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_{Q_m\cap\Omega_\rho} w(x)^r\,|A_2u(x)|^r\,dx \\
&\qquad \le C_2^r\,|Q_m|^{r/3-r/q} \Big( \frac{\rho}{d(Q_m)} \Big)^{rs} \int_{Q_m} |\mathbf{b}(x)|^r\,dx \\
&\qquad \le C_3^r\,\|\mathbf{b}\|_{3,p}^r\,|Q_m|^{1-r/q} \Big( \frac{\rho}{d(Q_m)} \Big)^{rs}.
\end{aligned} \tag{5.28}$$

If we put this last inequality together with (5.27) we conclude that

$$\int_{Q_m\cap\Omega_\rho} w(x)^r\,|Au(x)|^r\,dx \le C_4^r\,\|\mathbf{b}\|_{3,p}^r\,|Q_m|^{1-r/q} \Big( \frac{\rho}{d(Q_m)} \Big)^{rs}.$$

Suppose next that $m < n_0$. Then we have

$$\begin{aligned}
&\int_{Q_m\cap\Omega_\rho} w(x)^r\,|Au(x)|^r\,dx \\
&\qquad = \sum_{Q_{n_0}\subset Q_m} \int_{Q_{n_0}} w(x)^r\,|Au(x)|^r\,dx \\
&\qquad \underset{\text{by (5.27)}}{\le} \sum_{Q_{n_0}\subset Q_m} C^r\,\|\mathbf{b}\|_{3,p}^r \int_{Q_{n_0}} w(x)^r\,|u(x)|^r\,dx
\end{aligned}$$

$$= C^r \, \|\mathbf{b}\|_{3,p}^r \int_{Q_m} w(x)^r \, |u(x)|^r \, dx \, .$$

We conclude therefore that if $m > n_0$ then the inequality (5.28) holds. Therefore the operator $A$ is bounded on the weighted Morrey space and $\|A\| \le C \, \|\mathbf{b}\|_{3,p}$ for some constant $C$ depending only on $r, p, q, s$.

Next we turn to the operator $B$. To bound it we follow the same strategy as in Lemma 2.1 and Corollary 2.1. Observe that $Bu(x)$ is constant for $x \in Q_{n_0}$ where $Q_{n_0}$ is an arbitrary dyadic cube with side of length $2^{-n_0}$. We can bound $Bu(x)$ by

$$|Bu(x)| \le |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{2n} \, \frac{d}{d(Q_n)} \int_{\Omega_\rho \cap Q_n} w(y) \, |u(y)| \, dy \, ,$$

where the $Q_n$ are the unique dyadic cubes with side of length $2^{-n}$ containing $Q_{n_0}$. Hence we have

$$\begin{aligned}
|Bu(x)| &\le |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{2n} \, \frac{d}{d(Q_n)} \, |Q_n|^{1-1/r} \Big( \int_{\Omega_\rho \cap Q_n} w(y)^r \, |u(y)|^r \, dy \Big)^{1/r} \\
&\le |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{n(3/r-1)} \, \frac{d}{d(Q_n)} \, C \, |Q_n|^{1/r-1/q} \Big( \frac{\rho}{d(Q_n)} \Big)^s \\
&= C \, |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{n(3/q-1)} \, \frac{d}{d(Q_n)} \Big( \frac{\rho}{d(Q_n)} \Big)^s \\
&\le C_1 \, |\mathbf{b}(x)| \, 2^{n_0(3/q-1)} \, \frac{d}{d(Q_{n_0})} \Big( \frac{\rho}{d(Q_{n_0})} \Big)^s \, .
\end{aligned}$$

Let $Q_m$ be a dyadic cube with $m > n_0$. Then if $Q_m \subset Q_{n_0}$ we have

$$\begin{aligned}
\int_{Q_m \cap \Omega_\rho} w(x)^r \, |Bu(x)|^r \, dx &\le \max \Big\{ 1, \frac{d(Q_m)}{d} \Big\}^r \\
&\quad \cdot \Big( C_1 \, 2^{n_0(3/q-1)} \, \frac{d}{d(Q_{n_0})} \Big( \frac{\rho}{d(Q_{n_0})} \Big)^s \Big)^r \\
&\quad \cdot \int_{Q_m} |\mathbf{b}(x)|^r \, dx \\
&\le C_1^r \, \|\mathbf{b}\|_{3,p}^r \, |Q_m|^{1-r/q} \Big( \frac{\rho}{d(Q_m)} \Big)^{rs} \, ,
\end{aligned}$$

since

$$d(Q_{n_0}) \geq d(Q_m)\,, \qquad |Q_{n_0}| \geq |Q_m|\,.$$

Next we consider dyadic cubes $Q_m$ with $m \leq n_0$. Putting $Q' = Q_m$, one can easily verify the analogue of Lemma 2.1. Thus there are constants $\varepsilon, C > 0$, depending only on $r$ and $p$ such that

$$|Q|^{1/3+\varepsilon}\, u_Q \leq |Q'|^{1/3+\varepsilon}\, u_{Q'}\ ,$$

for all dyadic subcubes $Q$ of $Q'$ with $|Q| \geq 2^{-3n_0}$ implies the inequality

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big( \sum_{n=n_{Q'}}^{n_0} |\mathbf{b}(x)|\, S_n u(x)\Big)^r\, dx \leq C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'|\, u_{Q'}^r\ .$$

Now the analogue of Corollary 2.1 yields

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big( \sum_{n=n_{Q'}}^{n_0} |\mathbf{b}(x)|\, S_n u(x)\Big)^r\, dx \leq C^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q'\cap\Omega_\rho} w(x)^r\, |u(x)|^r\, dx\,,$$

for some constant $C$ depending only on $r, p$. We conclude therefore that

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big( \sum_{n=n_{Q'}}^{n_0} |\mathbf{b}(x)|\, S_n u(x)\Big)^r\, dx \leq C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'|^{1-r/q} \Big(\frac{\rho}{d(Q')}\Big)^{rs}\,,$$

by virtue of the fact that $u$ is in the weighted Morrey space. Finally we see just as in Lemma 4.4 that

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big( \sum_{n=-\infty}^{n_{Q'}-1} |\mathbf{b}(x)|\, S_n u(x)\Big)^r\, dx \leq C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'|^{1-r/q} \Big(\frac{\rho}{d(Q')}\Big)^{rs}\,.$$

Hence the operator $B$ is bounded on the weighted Morrey space. Since the operator $A$ is also bounded it follows that $T_1$ is bounded.

**Lemma 5.7.** *Let $T_2$ be the integral operator on functions with domain $\Omega_\rho$ which has kernel $|\mathbf{b}(x)|\, k_2(x,y)$ where $\mathbf{b} \in M_p^3$ and $k_2$ satisfies (5.23). Then for $1 \le r \le p$, $r \le q$, $s \ge 0$ and $2 < 3/q + s < 3/r$, $T_2$ is a bounded operator on the weighted Morrey space $M_{r,s}^q(\Omega_\rho)$ with weight $w$ given by (5.25). The norm $\|T_2\|$ of $T_2$ satisfies an inequality $\|T_2\| \le C\, \|\mathbf{b}\|_{3,p}$ where $C$ depends only on $r, p, q, s$.*

Proof. We follow the same lines as the proof of Lemma 5.3. Thus for $n = 0, \pm 1, \ldots$, let $Q_n$ be the cube centered at $a$ with side of length $2^{-n}$ and assume that the integer $n_0$ satisfies $2^{-n_0} \sim d$. Then if $|x-a| < d$ we have the inequality

$$|T_2 u(x)| \le \frac{C\,|\mathbf{b}(x)|}{|x-a|} \sum_{|x-a|<2^{-n}<d} 2^{-2n}\, u_{Q_n} + \frac{C\,|\mathbf{b}(x)|\, d}{|x-a|} \sum_{n=-\infty}^{n_0} 2^{-n}\, u_{Q_n}\ ,$$

where $C$ is a constant and $u_{Q_n}$ is an average of $u$ on $Q_n$ given by

$$u_{Q_n} = |Q_n|^{-1} \int_{Q_n \cap \{|x-a|>2^{-n-2}\}} |u(x)|\, dx\,.$$

Thus if $m > n_0$ we have

$$\begin{aligned} \int_{Q_m\cap\Omega_\rho} & w(x)^r\, |T_2 u(x)|^r\, dx \\ & \le \int_{Q_m\cap\Omega_\rho} |T_2 u(x)|^r\, dx \\ & \le C^r \sum_{k=m}^{\infty} \int_{Q_k} \Big( |\mathbf{b}(x)|\, 2^k \sum_{n=n_0}^{k} 2^{-2n}\, u_{Q_n} \Big)^r dx \\ & \quad + C^r \sum_{k=m}^{\infty} \int_{Q_k} \Big( |\mathbf{b}(x)|\, 2^k \sum_{n=-\infty}^{n_0} d\, 2^{-n}\, u_{Q_n} \Big)^r dx\,. \end{aligned} \tag{5.29}$$

Arguing as in Lemma 5.3 we see that

$$\begin{aligned} \sum_{k=m}^{\infty} \int_{Q_k} & \Big( |\mathbf{b}(x)|\, 2^k \sum_{n=n_0}^{k} 2^{-2n}\, u_{Q_n} \Big)^r dx \\ & \le C_1^r\, \rho^{sr}\, \|\mathbf{b}\|_{3,p}^r\, \|u\|_{q,r,s}^r\, 2^{m(3r/q+sr-3)}\,, \end{aligned}$$

since we are assuming $3/q+s<3/r$. To bound the second term on the right in (5.29) we estimate

$$\begin{aligned}\sum_{n=-\infty}^{n_0} d\,2^{-n}\,u_{Q_n} &\le \|u\|_{q,r,s}\sum_{n=-\infty}^{n_0} d^2\,2^{n(3/q+s)}\,\rho^s\\ &\le C_1\,\rho^s\,\|u\|_{q,r,s}\,d^2\,2^{n_0(3/q+s)}\\ &= C_1\,\rho^s\,\|u\|_{q,r,s}\,d^{2-3/q-s}\,,\end{aligned}$$

since $0<3/q+s$.

Hence

$$\begin{aligned}&\sum_{k=m}^{\infty}\int_{Q_k}\Big(|\mathbf{b}(x)|\,2^k\sum_{n=-\infty}^{n_0} d\,2^{-n}\,u_{Q_n}\Big)^r\,dx\\ &\qquad\le C_1^r\,\rho^{sr}\,\|u\|_{q,r,s}^r\,d^{(2-3/q-s)r}\sum_{k=m}^{\infty}\|\mathbf{b}\|_{3,p}^r\,2^{k(2r-3)}\\ &\qquad\le C_1^r\,\rho^{sr}\,\|\mathbf{b}\|_{3,p}^r\,\|u\|_{q,r,s}^r\,2^{m(3r/q+sr-3)}\,,\end{aligned}$$

since $2<3/q+s$. We conclude then that if $m>n_0$ there is the estimate

$$\begin{aligned}&\int_{Q_m\cap\Omega_\rho} w(x)^r\,|T_2u(x)|^r\,dx\\ &\qquad\le C_2^r\,\|\mathbf{b}\|_{3,p}^r\,\|u\|_{q,r,s}^r\,|Q_m|^{1-r/q}\Big(\frac{\rho}{d(Q_m)}\Big)^{sr}\,.\end{aligned}\tag{5.30}$$

Next we consider the case $m\le n_0$. Observe that if $|x-a|>d$ then

$$|T_2u(x)|\le C\,|\mathbf{b}(x)|\sum_{|x-a|<2^{-n}} 2^{-n}\,u_{Q_n}\ .$$

Hence we have for $k\le n_0$

$$\begin{aligned}&\int_{Q_k\cap\{|x-a|>2^{-k-2}\}} w(x)^r\,|T_2u(x)|^r\,dx\\ &\qquad\le C^r\Big(\sum_{n=-\infty}^{k+2} d^{-1}\,2^{-n-k}\,u_{Q_n}\Big)^r\int_{Q_k}|\mathbf{b}(x)|^r\,dx.\end{aligned}$$

We have now

$$\sum_{n=-\infty}^{k+2} d^{-1}\, 2^{-n-k}\, u_{Q_n} \le \sum_{n=-\infty}^{k+2} 2^{-k}\, \|u\|_{q,r,s}\, |Q_n|^{-1/q} \Big(\frac{\rho}{d(Q_n)}\Big)^s$$
$$\le C\, \rho^s\, \|u\|_{q,r,s}\, 2^{k(-1+3/q+s)}\,.$$

Combining the last two inequalities we conclude

$$\int_{Q_k \cap \{|x-a|>2^{-k-2}\}} w(x)^r\, |T_2 u(x)|^r\, dx$$
$$\le C^r\, \rho^{sr}\, \|\mathbf{b}\|_{3,p}^r\, \|u\|_{q,r,s}^r\, 2^{k(3r/q+sr-3)}\,.$$

Now by summing this last inequality over $k$, $m \le k \le n_0$ and using the fact that (5.30) holds with $m = n_0$ we conclude that (5.30) continues to hold for $m < n_0$.

We have shown that $g = T_2 u$ satisfies the inequality (5.24) provided $d(Q) \sim |Q|^{1/3}$. The inequality (5.24) for cubes $Q$ with $d(Q) > |Q|^{1/3}$ follows by similar argument.

**Proposition 5.4.** *Let $Q_m$ be a cube with side of length $2^{-m}$, $m$ an integer, which is contained in the ball $U_{0,R_2}$ of radius $R_2$. For $x \in U_{0,R_2}$ let $P_x(Q_m)$ be the probability that the drift process started at $x$ hits $Q_m$ before hitting the boundary of $U_{0,R_2}$. Then for any $\alpha < 1$ there exists $\varepsilon > 0$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then the inequality* (5.17) *holds where the constant $C_\alpha$ depends only on $\alpha$.*

Proof. We follow the same argument as the second proof of Proposition 5.3. We can choose a point $a \in Q_m$ such that the ball $B_a(\rho)$ of radius $\rho \sim 2^{-m}$ centered at $a$ is a distance larger than $3\rho$ from $\partial U_{0,R_2}$. Let $v(x)$ be the probability of the drift process started at $x \in U_{0,R_2}$ of hitting $B_a(\rho)$ before $\partial U_{0,R_2}$. Then we have

$$v(x) = w_0(x) + \int_\Omega G_{D,1}(x,y)\, (I-T)^{-1}\, \mathbf{b}\cdot\nabla w_0(y)\, dy\,,$$

where $\Omega = U_{0,R_2} \backslash B_a(\rho)$ and $G_{D,1}$ is the Dirichlet Green's function on $\Omega$. The function $w_0$ is given by (5.18) and $T$ is the integral operator with kernel $\mathbf{b}(x)\cdot\nabla_x G_{D,1}(x,y)$, $x, y \in \Omega$.

We wish to show that $\mathbf{b}\cdot\nabla w_0$ is in a weighted Morrey space $M^q_{r,s}$ with weight given by (5.25), where $d = d(Q_m)$. It is an immediate consequence of Lemma 5.4 that this is so provided $r, q, s$ satisfy (5.13) and that

$$\|\mathbf{b}\cdot\nabla w_0\|_{q,r,s} \le C\rho^{3/q-2}\,\|\mathbf{b}\|_{3,p}\ .$$

Now $T = T_1 + T_2$ where $T_1$ and $T_2$ satisfy the conditions of lemmas 5.6, 5.7 respectively. Since the conditions in these lemmas on $r, p, q, s$ are exactly the same as in lemmas 5.2, 5.3, we have that

$$|v(x) - w_0(x)| \le \int_\Omega G_{D,1}(x,y)\,|g(y)|\,dy\,, \tag{5.31}$$

where $g$ is in the weighted Morrey space $M^q_{r,s}$,

$$\|g\|_{q,r,s} \le C\rho^{3/q-2}\,\|\mathbf{b}\|_{3,p}\ ,$$

and $3/2 < q < 3$, $0 < s < 1$, as well as the inequalities (5.13) hold.

We need then to estimate the integral on the right in (5.31). If $d(x, Q_m) \le d(Q_m)$ then the inequality (5.17) is the same as (5.3). Hence we may argue directly as in the second proof of Proposition 5.3. The estimates on the integrals $I_1, I_2, I_3$ in (5.14) are exactly as previously, since the weight function for our Morrey space is always greater than 1. Hence we may consider the situation when $d(x, Q_m) > d(Q_m)$. We write

$$\begin{aligned}\int_\Omega G_{D,1}(x,y)\,|g(y)|\,dy &= \int_{|x-y|<|x-a|/2} + \int_{|y-a|<|x-a|/2} \\ &\quad + \int_{\{|x-y|>|x-a|/2,\ |y-a|>|x-a|/2\}} \\ &= I_1 + I_2 + I_3\ .\end{aligned}$$

Then from Lemma 5.5 we have if $|x-a| \sim 2^{-n_1}$.

$$\begin{aligned}I_1 &\le C\sum_{k=n_1}^\infty 2^k \int_{|x-y|<2^{-k}} |g(y)|\,dy \\ &\le C\sum_{k=n_1}^\infty 2^k\,\rho^{3/q-2}\,\|\mathbf{b}\|_{3,p}\left(\frac{\rho}{2^{-n_1}}\right)^s 2^{-3k(1-1/q)}\frac{d}{2^{-n_1}} \\ &\le C\,\|\mathbf{b}\|_{3,p}\left(\frac{\rho}{2^{-n_1}}\right)^{s+3/q-2}\frac{d}{2^{-n_1}}\ ,\end{aligned}$$

as in the estimate of $I_1$ in the proof of Proposition 5.3. Similarly we can estimate $I_3$ as

$$\begin{aligned}
I_3 &\le C \sum_{k=-\infty}^{n_1} 2^k \int_{2^{-k}<|y-a|<2^{-k+1}} |g(y)|\, dy \\
&\le C \sum_{k=-\infty}^{n_1} 2^{2k}\, d \int_{2^{-k}<|y-a|<2^{-k+1}} w(y)\, |g(y)|\, dy \\
&\le C \sum_{k=-\infty}^{n_1} 2^{2k}\, d\, \rho^{3/q-2} \Big(\frac{\rho}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\
&\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}}\ ,
\end{aligned}$$

provided $s+3/q-1>0$.

Next we write $I_2$ as a sum,

$$I_2 = \int_{|y-a|<d} + \int_{d<|y-a|<|x-a|/2} = I_4 + I_5\ .$$

We can estimate $I_4$ from Lemma 5.5 as

$$\begin{aligned}
I_4 &\le C\, \frac{d}{2^{-2n_1}} \int_{|y-a|<d} |g(y)|\, dy \\
&\le C\, \frac{d}{2^{-2n_1}}\, \rho^{3/q-2}\, \|\mathbf{b}\|_{3,p}\, d^{3-3/q} \Big(\frac{\rho}{d}\Big)^s \\
&= C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}} \Big(\frac{d}{2^{-n_1}}\Big)^{3-s-3/q} \\
&\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}}\ ,
\end{aligned}$$

since $d<2^{-n_1}$.

Finally, from Lemma 5.5 we have

$$\begin{aligned}
I_5 &\le C \sum_{k=n_1}^{\infty} \frac{d}{2^{-2n_1}} \int_{2^{-k}<|y-a|<2^{-k+1}} w(y)\, |g(y)|\, dy \\
&\le C \sum_{k=n_1}^{\infty} \frac{d}{2^{-2n_1}}\, \rho^{3/q-2} \Big(\frac{\rho}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\
&\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}}\ ,
\end{aligned}$$

since $s + 3/q < 3$.

We conclude therefore that there is a constant $C$ such that

$$\int_\Omega G_{D,1}(x,y)\,|g(y)|\,dy \le C\,\|\mathbf{b}\|_{3,p}\Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2}\frac{d}{2^{-n_1}}\,.$$

The result follows now from this last inequality just as in the proof of Proposition 5.3.

We can use Proposition 5.4 to generalize Proposition 5.1 to the case of nontrivial drift $\mathbf{b}$. First we need to modify the definition of $V_{\mathcal{S}}$ in (5.1), (5.2). For any $Q$ such that $Q \cap U_{0,R_2} \ne \varnothing$ we define a potential function $V_{Q,\eta} : U_{0,R_2} \longrightarrow \mathbb{R}$ which depends on a parameter $\eta > 0$ by

$$V_{Q,\eta}(x) = \begin{cases} |Q|^{-2/3}\Big(\dfrac{R_2}{|Q|^{1/3}}\Big)^{\eta}, & x \in \tilde{Q}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

With this new definition of $V_{Q,\eta}$ the potential $V_{\mathcal{S},\eta}$ is defined exactly as in (5.2). Thus

$$V_{\mathcal{S},\eta} = \sum_{Q\subset\mathcal{S}} V_{Q,\eta}\,.$$

**Proposition 5.5.** *Let $X(t)$ be Brownian motion in $\mathbb{R}^3$ and $X_{\mathbf{b}}(t)$ be the drift process with drift $\mathbf{b}$. Suppose $\mathcal{S}$ is a union of cubes with sides of length $\le R_2$. Then for any $\eta > 0$ there exists $\varepsilon > 0$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then*

$$P_x(X_{\mathbf{b}} \text{ hits } \mathcal{S} \text{ before exiting } U_{0,R_2}) \le C E_x\Big[\int_0^\tau V_{\mathcal{S},\eta}(X(t))\,dt\Big]\,,$$

*where $|x| \le R_2/2$. Here $\tau$ is the first exit time out of the region $U_{0,R_2}$ and $C$ is a constant depending only on $\eta, \varepsilon$.*

PROOF. It is sufficient for us to assume that $\mathcal{S}$ consists of a single cube $Q$ with side $\le R_2$ which intersects $U_{0,R_2}$. In that case $\tilde{Q} \cap U_{0,R_2}$ contains a cube $Q_m$ with side of length $2^{-m}$ which has the same order of magnitude as the length of $Q$. In view of Proposition 5.4 it will be sufficient for us to show that

$$E_x\Big[\int_0^\tau \chi_{Q_m}(X(t))\,dt\Big]$$
$$\ge c_\eta\Big(\frac{2^{-m}}{R_2}\Big)^{\eta}\frac{2^{-2m}}{(2^m d(x,Q_m)+1)^{\alpha}}\,\min\Big\{1, \frac{2^m d(Q_m)}{2^m d(x,Q_m)+1}\Big\}\,, \tag{5.32}$$

for some $\alpha$, $0 < \alpha < 1$ and constant $c_\eta$ depending only on $\eta$. Now the left hand side of the above inequality is just

$$\int_{Q_m} G_D(x,y)\,dy\,,$$

where $G_D$ is the Dirichlet Green's function on the ball $U_{0,R_2}$. It is easy to see from the explicit formula for $G_D$ that if $|x| \le R_2/2$ then

$$\int_{Q_m} G_D(x,y)\,dy \ge \frac{c\,2^{-2m}}{2^m d(x,Q_m)+1}\ \min\Big\{1, \frac{2^m d(Q_m)}{2^m d(x,Q_m)+1}\Big\}\,,$$

for some universal constant $c > 0$. Thus the inequality (5.32) holds provided $\alpha \ge 1-\eta$.

Next we generalize Proposition 5.2 to the case of nontrivial $\mathbf{b}$.

**Proposition 5.6.** *Suppose $R_1 = 0$, $R_2 = 2R$, and suppose $\mathcal{S}$ consists of cubes of length $\le R_2$ Let $f$ be a density on the sphere $|x| = R$ and $f_2$ the density on $|x| = R_2$ by $f$ propagated by the process with drift $\mathbf{b}$ along paths which do not intersect $\mathcal{S}$. Let $\eta > 0$, $1 < q < \infty$, $1 < p \le 3$. Then there exist $\varepsilon, \delta, \xi > 0$ depending only on $\eta, p, q$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$, $\|f - \mathrm{Av}\, f\|_q \le \delta\,|\mathrm{Av}\, f|$ and*

$$\mathrm{Av}_{|x|=R}\, E_x\Big[\int_0^{\tau_{R_2}} V_{\mathcal{S},\eta}(X(t))\,dt\Big] < \xi\,,$$

*then*

$$\|f_2 - \mathrm{Av}\, f_2\|_q \le \delta\,|\mathrm{Av}\, f_2| \qquad \textit{and} \qquad |\mathrm{Av}\, f_2| \ge \frac{|\mathrm{Av}\, f|}{2}\,.$$

PROOF. We proceed as in Proposition 5.2. Letting $q'$ satisfy $1/q + 1/q' = 1$, we need to show that the operator $A$ defined by

$$Ag(x) = E_x[g(X_{\mathbf{b}}(\tau_{R_2}));\ X_{\mathbf{b}}(t) \in \mathcal{S},\ \text{some } t,\ 0 < t < \tau_{R_2}]$$

which maps functions on $|x| = R_2$ to functions on $|x| = R$ satisfies an inequality

$$\|Ag\|_{q'} \le \gamma(\xi)\,\|g\|_{q'}\,,$$

where $\gamma(\xi) \longrightarrow 0$ for $\xi \longrightarrow 0$. To prove this let $1 < r < q'$. Then it is sufficient to show that

$$E_x[|g(X_{\mathbf{b}}(\tau_{R_2}))|^r] \le C\,\|g\|_{q'}^r\,, \qquad |x| = R\,, \tag{5.33}$$

for some constant $C$ depending only on $r, q, p, \varepsilon$. Now we can write

$$E_x[|g(X_{\mathbf{b}}(\tau_{R_2}))|^r] = \langle \rho_x, |g|^r \rangle ,$$

where $\rho_x$ is the density of the drift process started at $x$, $|x| = R$, on the sphere $|y| = R_2$. Arguing similarly to the proof of [5, Lemma 4.3] and using Corollary 4.1 we see that for any $s$, $1 < s < \infty$, we can choose $\varepsilon > 0$ sufficiently small so that $\rho_x$ is $s$ integrable on $|y| = R_2$ and $\|\rho_x\|_s \le C$ where $C$ is a universal constant. Now we obtain the inequality (5.33) by choosing $s$ to satisfy $1/s + r/q' = 1$ and applying Holder's inequality.

## 6. Auxiliary perturbative estimates.

In this section we shall prove a perturbative theorem which will be needed in the induction argument of Section 7. The theorem is similar in spirit to the results of sections 5 and 6 and our proof will depend on these. Let $\Omega_R$ be the ball of radius $R$ in $\mathbb{R}^3$ centered at the origin and suppose $a_1, a_2$ are points which satisfy $|a_1| = |a_2| = R/2$, $|a_1 - a_2| = R$. Thus $a_1$ and $a_2$ lie on a diameter of $\Omega_R$ at a distance $R/2$ from the center. Let $B_{r_1}$ be a ball of radius $r_1 \ge 10R$ such that $a_1 \in \partial B_{r_1}$ and the outward normal to $\partial B_{r_1}$ at $a_1$ makes an angle less than $\pi/100$ with the vector $a_2 - a_1$. Similarly, let $B_{r_2}$ be a ball of radius $r_2 \ge 10R$ such that $a_2 \in \partial B_{r_2}$ and the outward normal to $\partial B_{r_2}$ at $a_2$ makes an angle less that $\pi/100$ with the vector $a_2 - a_1$. We shall be interested in the surfaces $D_1 = B(a_1, R/4) \cap \partial B_{r_1}$ and $D_2 = B(a_2, R/4) \cap \partial B_{r_2}$.

Next suppose we have a vector field $\mathbf{b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and a dyadic decomposition of $\mathbb{R}^3$ into cubes $Q$. For $n_0$ an integer and $\varepsilon > 0$ let $\mathcal{S}$ be the set of all dyadic cubes $Q_n$ with side of length $2^{-n}$, $n \ge n_0$, such that

$$\int_{Q_n} |\mathbf{b}(x)|^p \, dx \ge \varepsilon^p \, |Q_n|^{1-p/3} . \tag{6.1}$$

For $Q \in \mathcal{S}$ and $\eta > 0$ define $V_{Q,\eta} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$V_{Q,\eta}(x) = \begin{cases} |Q|^{-2/3}\Big(\dfrac{2^{-n_0}}{|Q|^{1/3}}\Big)^{\eta} , & x \in \tilde{Q} , \\ 0 , & \text{otherwise} , \end{cases}$$

where $\tilde{Q}$ is the double of $Q$. The potential $V_\eta$ is then given by

$$V_\eta = \sum_{Q \in \mathcal{S}} V_{Q,\eta} \,. \tag{6.2}$$

Observe that the potential $V_\eta$ defined here is a particular case of the potential $V_{\mathcal{S},\eta}$ of Section 5. Suppose $\rho_1$ is a density on the surface $D_1$. Then $\mathrm{Av}_{D_1}\,\rho_1$ is the average of $\rho_1$ on $D_1$ and $\|\rho_1\|_{D_1,q}$, $1 < q < \infty$, is the $L^q$ norm of $\rho_1$ normalized so that $\|\mathbf{1}\|_{D_1,q} = 1$. The theorem we wish to prove is as follows:

**Theorem 6.1.** *Let $R = 2^{-n}$, $n \ge n_0$, and $\rho_1$ be a density on $D_1$. Suppose $f \in M_r^t(\mathbb{R}^3)$ with $1 < r \le t$, $r < p$, $3/2 < t < 3$. Let $\rho_2$ be the density induced on $D_2$ by the paths of the drift process $X_{\mathbf{b}}(t)$ which start on $D_1$, avoid the cubes $Q \in \mathcal{S}$ with $|Q| \le 2^{-3n}$, exit the region $\Omega_R \cap B_{r_2}$ through $D_2$, and satisfy the inequality*

$$\int_0^\tau |f|\,(X_{\mathbf{b}}(t))\,dt \le C_1\, R^{2-3/t}\, \|f\|_{t,r} \,,$$

*where $C_1$ is a constant. Let $0 < \eta' < \eta$ and suppose*

$$\frac{1}{R}\int_{\Omega_R} V_\eta(x)\,dx \le \xi\, 2^{\eta'(n-n_0)} \,, \tag{6.3}$$

*where $\xi > 0$ is a constant. Then there exists a constant $\alpha > 1$ depending only on $\eta'$ such that if $1 < q < \infty$ and $C_1$ is sufficiently large, $\xi$ sufficiently small, one can find constants $C_2, c_2$ such that*

$$\|\rho_1\|_{D_1,q} \le C_2\, \alpha^{n-n_0}\, \mathrm{Av}_{D_1}\rho_1 \tag{6.4}$$

*implies that*

$$\|\rho_2\|_{D_2,q} \le C_2\, \mathrm{Av}_{D_2}\rho_2 \,, \qquad \mathrm{Av}_{D_2}\rho_2 \ge c_2\, \mathrm{Av}_{D_1}\rho_1 \,.$$

*The constants $\xi$, $C_1$, $C_2$, $c_2$ can be chosen independent of $R$.*

REMARK. Theorem 6.1 is rather like the results we have already proven. In fact, if we take $C_1 = \infty$, $\xi = 0$, we are in the situation studied in Section 4. The case $C_1 = \infty$, $\xi > 0$, $n = n_0$, is the situation studied in Section 5. Observe that since the regions $D_1$, $D_2$ are not spheres the results of sections 4 and 5 do not immediately yield a proof of Theorem 6.1 in the above mentioned cases.

We shall prove Theorem 6.1 in a series of steps starting from the simplest situation. We first consider the case of Brownian motion where $\mathbf{b} \equiv 0$.

**Lemma 6.1.** *Let $\rho_1$ be a density on $D_1$ with $Av_{D_1}\rho_1 < \infty$ and $\rho_2$ the density induced on $D_2$ by Brownian paths started on $D_1$ which exit $\Omega_R \cap B_{r_2}$ through $D_2$. Then there exists universal constants $c_2$, $C_2$ such that for $1 \le q \le \infty$,*

$$\|\rho_2\|_{D_2,q} \le C_2\, \mathrm{Av}_{D_2}\rho_2 \ , \qquad Av_{D_2}\rho_2 \ge c_2\, \mathrm{Av}_{D_1}\rho_1 \ .$$

PROOF. Suppose $g$ is a function defined on $D_2$ and let $u(x) = Pg(x)$, $x \in \Omega_R \cap B_{r_2}$ be given by the solution of the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0\,, & x \in \Omega_R \cap B_{r_2}\ , \\ u(x) = g(x)\,, & x \in D_2\ , \\ u(x) = 0\,, & x \in \partial(\Omega_r \cap B_{r_2})\backslash D_2\ . \end{cases} \tag{6.5}$$

Thus $P$ defines a mapping of functions on $D_2$ to functions on $D_1$. Let $P^*$ be the adjoint of $P$ defined by

$$\langle f, Pg\rangle_{D_1} = \langle P^* f, g\rangle_{D_2}\ ,$$

where $\langle\cdot\,,\cdot\rangle_{D_1}$, $\langle\cdot\,,\cdot\rangle_{D_2}$ are the standard inner products on $L^2(D_1)$ and $L^2(D_2)$ normalized so that $\|\mathbf{1}\|_{D_1,2} = \|\mathbf{1}\|_{D_2,2} = 1$. Then $\rho_1$ and $\rho_2$ are related by the equation $\rho_2 = P^*\rho_1$. We have therefore that

$$\mathrm{Av}_{D_2}\rho_2 = \langle \rho_2, 1\rangle_{D_2} = \langle P^*\rho_1, 1\rangle_{D_2} = \langle \rho_1, P1\rangle_{D_1}\ .$$

Thus to show that $\mathrm{Av}_{D_2}\rho_2 \ge c_2\,\mathrm{Av}_{D_1}\rho_1$ it is sufficient to prove that $P\mathbf{1}(x) \ge c_2 > 0$ for all $x \in D_1$. Hence we need to prove that there is a universal constant $c_2 > 0$ such that

$$P_x(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2) \ge c_2\ , \qquad x \in D_1\ . \tag{6.6}$$

To see this let $B_1, B_2, \ldots, B_N$ be balls with radius $\sim R$ having the following properties:

a) $B_i \subset \Omega_R \cap B_{r_2}$, $1 \le i \le N-1$, $B_N$ is centered at $a_2$, $B_1$ is centered at $x$.

b) $|\partial B_i \cap B_{i+1}| \sim R^2$, $1 \le i \le N-1$.

Now for $1 \le i \le N$ let $S_i$ be the sets

$$S_i = \{y \in \partial B_i : \; y \in B_{i+1}, \; d(y, \partial B_{i+1}) > c\, R\}\,, \qquad 1 \le i \le N-1\,,$$

$$S_N = \partial B_N \cap (\mathbb{R}^3 \backslash \Omega_R \cap B_{r_2})\,.$$

It is clear from a), b) that we may choose $c > 0$ such that $|S_i| \sim R^2$, $1 \le i \le N$. Next define $p_0, \ldots, p_{N-1}$ by

$$p_0 = P\,(BM \text{ started at } x \text{ exits } B_1 \text{ through } S_1)\,,$$

$$p_i = \inf_{y \in S_i} P\,(BM \text{ started at } y \in S_i \text{ exits } B_{i+1} \text{ through } S_{i+1})\,,$$

with $1 \le i \le N-1$. It is clear from the Poisson formula that there is a constant $c > 0$ such that $p_i \ge c$, $0 \le i \le N-1$. Hence we have

$$P_x(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2) \ge p_0\, p_1 \cdots p_{N-1} \ge c^N\,.$$

Since we can choose $N$ to be an absolute constant the inequality (6.6) follows.

Next, to show that $\|\rho_2\|_{D_2,q} \le C_2 \operatorname{Av}_{D_2} \rho_2$, we can prove that

$$|\langle \rho_2, f \rangle_{D_2}| \le C_2 \operatorname{Av}_{D_2} \rho_2 \, \|f\|_{D_2,q'}\,,$$

where $1/q + 1/q' = 1$. Since $\langle \rho_2, f \rangle_{D_2} = \langle \rho_1, Pf \rangle_{D_1}$ and we have already proved that $\operatorname{Av}_{D_2} \rho_2 \ge c_2 \operatorname{Av}_{D_1} \rho_1$, it is sufficient to show that

$$\|Pf\|_{D_1,\infty} \le C\, \|f\|_{D_2,q'}\,, \tag{6.7}$$

for some universal constant $C$. We can prove this last inequality by observing that $|Pf(x)| \le \overline{P}\,|f|(x)$, where $\overline{P}$ is the Poisson kernel for the ball $B_{r_2}$.

**Lemma 6.2.** *Let $\rho_1$ be a density on $D_1$ with $\operatorname{Av}_{D_1} \rho_1 < \infty$ and $\rho_2$ the density induced on $D_2$ by Brownian paths $X(t)$ started on $D_1$ which exit $\Omega_R \cap B_{r_2}$ through $D_2$ and satisfy*

$$\int_0^\tau |f|(X(t))\, dt \le C_1\, R^{2-3/t}\, \|f\|_{t,r}\,, \qquad 1 \le r \le t\,, \; t > \frac{3}{2}\,.$$

*Then there exist universal constants $c_2, C_2$ such that for $1 < q < \infty$ and sufficiently large $C_1$, depending only on $r, t$, one has the inequalities*

$$\|\rho_2\|_{D_2,q} \le C_2 \operatorname{Av}_{D_2}\rho_2 \ , \qquad \operatorname{Av}_{D_2}\rho_2 \ge c_2 \operatorname{Av}_{D_1}\rho_1 \ .$$

PROOF. Suppose $g$ is a function defined on $D_2$ and extend $g$ to $\partial(\Omega_R \cap B_{r_2})$ by setting $g$ to be zero on the rest of the boundary. Then $Pg(x)$ is defined for $x \in \Omega_R \cap B_{r_2}$ by

$$Pg(x) = E_x\Big[g(X(\tau))H\Big(C_1\, R^{2-3/t}\, \|f\|_{t,r} - \int_0^\tau |f|(X(t))\, dt\Big)\Big] \ ,$$

where $H$ is the Heaviside function $H(z) = 1,\ z > 0,\ H(z) = 0,\ z \le 0$. Then just as in Lemma 6.1 we have $\rho_2 = P^*\rho_1$. It is furthermore clear that the inequality (6.7) continues to hold. Hence we need only prove that $\operatorname{Av}_{D_2}\rho_2 \ge c_2 \operatorname{Av}_{D_1}\rho_1$. This follows if we can show that

$$\begin{aligned} &P_x\Big(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2 \\ &\quad \text{and } \int_0^\tau |f|\,(X(t))\, dt \le C_1\, R^{2-3/t}\|f\|_{t,r}\Big) \ge c_2 \ , \quad x \in D_1 \ . \end{aligned} \tag{6.8}$$

Evidently from the Chebyshev inequality the left hand side of the previous inequality is bounded below by

$$P_x(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2) - \frac{E_x\Big[\displaystyle\int_0^\tau |f|\,(X(t))\, dt\Big]}{C_1\, R^{2-3/t}\, \|f\|_{t,r}} \ .$$

If we use now the fact that

$$E_x\Big[\int_0^\tau |f|(X(t))\, dt\Big] \le \frac{1}{4\pi}\int_{\Omega_R} \frac{|f(y)|}{|x-y|}\, dy \le K R^{2-3/t}\, \|f\|_{t,r} \ ,$$

for some constant $K$ depending on $t$, then it is clear that (6.8) holds and hence the result.

**Lemma 6.3.** *Let $\mathcal{S}$ be a set of dyadic cubes and suppose $V_\eta$ is defined by* (6.2). *Let $\rho_1$ be a density on $D_1$ and $\rho_2$ the density induced on $D_2$ as in Theorem* 6.1. *Then if $\mathbf{b} \equiv 0$ the conclusion of Theorem* 6.1 *holds.*

PROOF. As in Lemma 6.2 we may confine ourselves to proving that $\mathrm{Av}_{D_2}\rho_2 \geq c_2\,\mathrm{Av}_{D_1}\rho_1$. Thus we need to show

$$\int_{D_1} \rho_1(x)\,P_x\Big(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2, \text{ avoids cubes } Q \in \mathcal{S} \text{ with } |Q| \leq 2^{-3n} \text{ and } \int_0^\tau |f|(X(t))\,dt \leq C_1\,R^{2-3/t}\|f\|_{t,r}\Big)\,d\mu(x) \geq c_2\,\mathrm{Av}_{D_1}\,\rho_1\ ,$$

where $\mu$ is the surface measure on $D_1$ normalized so that $\mu(D_1) = 1$. From Lemmas 6.1, 6.2 it will be sufficient to show that

$$\int_{D_1} \rho_1(x)\,P_x\Big(X(t) \text{ hits } \bigcup_{\substack{Q\in\mathcal{S}\\ |Q|\leq 2^{-3n}}} Q \text{ before exiting } \Omega_R \cap B_{r_2}\Big)\,d\mu(x) \leq \gamma\,\mathrm{Av}_{D_1}\rho_1\ , \tag{6.9}$$

where $\gamma$ is a number which can be chosen arbitrarily small depending on $\xi$. Let $Q_m$ be a cube in $\mathcal{S}$ with side of length $2^{-m}$, $m \geq n$. In view of the inequality (6.3) $m$ must satisfy the inequality

$$2^{(1-\eta)(m-n)} > \xi^{-1}\,2^{(\eta-\eta')(n-n_0)}\,, \tag{6.10}$$

whence $m - n$ is larger than a constant times $n - n_0$ plus a constant which may be made arbitrarily large depending on $\xi$. Let $d(x, Q_m)$ be defined by $d(x, Q_m) = 2^{-m}$ if $x \in Q_m$, $d(x, Q_m) =$ distance from $x$ to the center of $Q_m$ if $x \notin Q_m$. Then as in Section 5 we have

$$\begin{aligned}\int_{D_1} \rho_1(x)\,P_x(X(t) \text{ hits } Q_m \text{ before exiting } \Omega_R \cap B_{r_2})\,d\mu(x) &\leq C\int_{D_1} \frac{2^{-m}\,\rho_1(x)}{d(x, Q_m)}\,d\mu(x)\\ &\leq C\Big(\int_{D_1} \frac{2^{-mq'}}{d(x, Q_m)^{q'}}\,d\mu(x)\Big)^{1/q'}\|\rho_1\|_{D_1,q'}\ ,\end{aligned}$$

where $1/q + 1/q' = 1$. We have now that

$$\int_{D_1} \frac{2^{-mq'}}{d(x, Q_m)^{q'}}\,d\mu(x) \leq C\,2^{-(m-n)}\,, \qquad 1 \leq q' \leq \infty\,,$$

for some constant $C$. Hence by the assumption (6.4) we conclude that there is a constant $C$ such that

$$\int_{D_1} \rho_1(x)\, P_x(X(t) \text{ hits } Q_m \text{ before exiting } \Omega_R \cap B_{r_2})\, d\mu(x) \\ \leq C\, 2^{-(m-n)/q'}\, \alpha^{n-n_0}\, \mathrm{Av}_{D_1}\, \rho_1 \ .$$

It is clear now from (6.10) that if $\alpha$ satisfied the inequality

$$1 < \alpha < 2^{(\eta-\eta')/q'(1-\eta)}\,, \tag{6.11}$$

then for any $\gamma > 0$, $\xi$ can be chosen sufficiently small so that

$$\int_{D_1} \rho_1(x)\, P_x(X(t) \text{ hits } Q_m \text{ before exiting } \Omega_R \cap B_{r_2})\, d\mu(x) \\ \leq \gamma\, \mathrm{Av}_{D_1}\, \rho_1 \ .$$

Suppose now that $m$ satisfies (6.10) and $N_m$ is the number of cubes $Q_m$ in $\mathcal{S}$ with side of length $2^{-m}$. Then from (6.3) it follows that

$$N_m \leq \xi\, 2^{(1-\eta)(m-n)-(\eta-\eta')(n-n_0)}\,.$$

Let $g_m$ be the function defined by

$$g_m(x) = P_x\Big(X(t) \text{ hits } \bigcup_{Q_m\in\mathcal{S}} Q_m \text{ before exiting } \Omega_R \cap B_{r_2}\Big)\,.$$

Then we have

$$\|g_m\|_{D_1,1} \leq \sum_{Q_m\in\mathcal{S}} C \int_{D_1} \frac{2^{-m}\, d\mu(x)}{d(x, Q_m)} \\ \leq C N_m\, 2^{-(m-n)} \leq C\, \xi\, 2^{-\eta(m-n)-(\eta-\eta')(n-n_0)}\,.$$

Now, using the obvious fact that $g_m(x) \leq 1$, we have that

$$\begin{aligned}\int_{D_1} \rho_1(x)\, g_m(x)\, d\mu(x) &\leq \|g_m\|_{D_1,q'}\, \|\rho_1\|_{D_1,q} \\ &\leq \|g_m\|_{D_1,1}^{1/q'}\, \|\rho_1\|_{D_1,q} \\ &\leq C\, \xi^{1/q'}\, 2^{-\eta(m-n)/q'-(\eta-\eta')(n-n_0)/q'} \\ &\quad \cdot \alpha^{n-n_0}\, \mathrm{Av}_{D_1} \rho_1 \ .\end{aligned}$$

Letting $m_0$ be the minimum integer $m$ such that (6.10) holds we conclude that

$$\begin{aligned}\int_{D_1} \rho_1(x)\, P_x\Big(X(t) \text{ hits } \bigcup_{\substack{Q\in\mathcal{S}\\ |Q|\le 2^{-3n}}} Q \text{ before exiting } \Omega_R\cap B_{r_2}\Big)\, d\mu(x)\\ \le \sum_{m=m_0}^{\infty} C\,\xi^{1/q'}\, 2^{-\eta(m-n)/q'-(\eta-\eta')(n-n_0)/q'}\alpha^{n-n_0}\,\mathrm{Av}_{D_1}\rho_1\\ \le C'\,\xi^{1/q'}\, 2^{-\eta(m_0-n)/q'-(\eta-\eta')(n-n_0)/q'}\alpha^{n-n_0}\,\mathrm{Av}_{D_1}\rho_1\,.\end{aligned}$$

If we use the inequality (6.11) we have that

$$2^{-\eta(m_0-n)/q'-(\eta-\eta')(n-n_0)/q'}\alpha^{n-n_0}\,,$$

$$(2^{-(1-\eta)(m_0-n)+(\eta-\eta')(n-n_0)})^{\eta/q'(1-\eta)} < \xi^{\eta/q'(1-\eta)}\,,$$

from the definition of $m_0$ and (6.10). The inequality (6.9) immediately follows from this.

Next we wish to consider the case of nontrivial drift $\mathbf{b}$ with $\xi = 0$.

**Lemma 6.4.** *Let $\rho_1$ be a density on $D_1$ with $\mathrm{Av}_{D_1}\rho_1 < \infty$ and $\rho_3$ the density induced on the sphere $\partial B(a_1, R/3)$ by paths of the drift process $X_{\mathbf{b}}(t)$ started on $D_1$. Suppose that $\mathbf{b} \in M_p^3$ and $\|\mathbf{b}\|_{3,p} < \varepsilon$. Then for any $q$, $1 < q < \infty$, and sufficiently small $\varepsilon$, depending only on $p, q$ one has, with $D_3 = \partial B(a, R/3)$,*

$$\mathrm{Av}_{D_1}\rho_1 = \mathrm{Av}_{D_3}\rho_3\,, \qquad \|\rho_3\|_{D_3,q} \le C_3\,\mathrm{Av}_{D_3}\rho_3\,,$$

*where $C_3$ depends only on $p, q, \varepsilon$.*

PROOF. For $y \in D_1$ let $\delta_y$ be the Dirac $\delta$ function concentrated at $y$. Then it follows from Corollary 4.1 that if $h_y$ is the density induced on $D_3$ by $\delta_y$ then $\|h_y\|_{D_3,q} \le C_3$, for some constant $C_3$, provided $\varepsilon$ is sufficiently small. Since

$$\rho_1 = \int_{D_1} \rho(y)\,\delta_y\,d\mu(y)\,,$$

it follows that

$$\begin{aligned}\|\rho_3\|_{D_3,q} &= \Big\| \int_{D_1} \rho(y)\, h_y \, d\mu(y) \Big\|_{D_3,q} \\ &\le \int_{D_1} \rho(y)\, \|h_y\|_{D_3,q}\, d\mu(y) \\ &\le C_3 \, \mathrm{Av}_{D_1} \rho_1 \ .\end{aligned}$$

The fact that $\mathrm{Av}_{D_1}\rho_1 = \mathrm{Av}_{D_3}\rho_3$ follows simply from the observation that $u(x) \equiv 1$ is a solution of the equation $\Delta u(x) + \mathbf{b}(x) \cdot \nabla u(x) = 0$.

Next let $G_D(x,y)$ be the Dirichlet kernel for $-\Delta$ on the domain $\Omega_R \cap B_{r_2}$. As in Section 4 we shall be concerned with the integral operator $T$ on functions with domain $\Omega_R \cap B_{r_2}$ which has kernel $k_T$ given by

$$k_T(x,y) = \mathbf{b}(x) \cdot \nabla_x G_D(x,y)\,, \qquad x, y \in \Omega_R \cap B_{r_2}\ . \tag{6.12}$$

**Lemma 6.5.** *There is a universal constant $C$ such that*

$$|\nabla_x G_D(x,y)| \le \frac{C}{|x-y|^2}\,, \qquad x,y \in \Omega_R \cap B_{r_2}\ . \tag{6.13}$$

PROOF. Let $u(x) = G_D(x,y)$. We shall show that there is a universal constant $C$ such that

$$u(x) \le \frac{C}{|x-y|}\ \min\Big\{1, d\Big(x, \frac{\partial(\Omega_R \cap B_{r_2})}{|x-y|}\Big)\Big\}\,, \tag{6.14}$$

where $x \in \Omega_R \cap B_{r_2} \backslash \{y\}$. The estimate (6.13) follows from (6.14) by using the fact that $u$ is harmonic in $\Omega_R \cap B_{r_2} - \{y\}$ and the Poisson formula. One can easily prove (6.14) by constructing a barrier function. Thus let us suppose that $d(x, \partial(\Omega_R \cap B_{r_2})) < |x-y|/4$ and that $x_0$ is the nearest point on $\partial(\Omega_R \cap B_{r_2})$ to $x$. Let $x_1$ be the point $x_1 = x_0 + c\,|x-y|\,(x_0-x)/|x_0-x|$ where $c > 0$. Let $U_x = \{z : |z-x_1| > c\,|x-y|\}$. Then it is clear that we may choose $c < 1/8$ in a universal way so that $\Omega_R \cap B_{r_2} \subset U_x$ and $|z-x_1| > |x-y|/4$ if $|z-y| = |x-y|/4$. Let $v(z)$ be the function

$$v(z) = 1 - \frac{|x_0 - x_1|}{|z-x_1|}\,, \qquad z \in U_x\ .$$

Thus $v$ is harmonic in the region $\Omega_R \cap B_{r_2} \backslash B(y, |x-y|/4)$ and satisfies the boundary conditions

$$v(z) \geq 0\,, \qquad z \in \partial(\Omega_R \cap B_{r_2})\,,$$
$$v(z) \geq 1 - 4\,c > \frac{1}{2}\,, \qquad z \in \partial B\Big(y, \frac{1}{4}\,|x-y|\Big)\,.$$

On the other hand $u(z)$ is also harmonic in the region

$$\Omega_R \cap B_{r_2} \backslash B\Big(y, \frac{|x-y|}{4}\Big)$$

and satisfies the boundary conditions

$$u(z) = 0\,, \qquad z \in \partial(\Omega_R \cap B_{r_2})\,,$$
$$u(z) \leq \frac{C}{|x-y|}\,, \qquad z \in \partial B\Big(y, \frac{1}{4}\,|x-y|\Big)\,,$$

where $C$ is a universal constant. It follows then by the maximum principle that

$$u(z) \leq 2\,C\,\frac{v(z)}{|x-y|}\,, \qquad z \in \Omega_R \cap B_{r_2} \backslash B\Big(y, \frac{1}{4}\,|x-y|\Big)\,. \tag{6.15}$$

Observing now that

$$\begin{aligned} v(x) &= 1 - \frac{|x_0 - x_1|}{|x - x_1|} \\ &= 1 - \frac{|x_0 - x_1|}{|x - x_0| + |x_0 - x_1|} \\ &= \frac{|x - x_0|}{|x - x_0| + |x_0 - x_1|} \\ &\leq \frac{|x - x_0|}{|x_0 - x_1|} \\ &= \frac{|x - x_0|}{c\,|x - y|}\,, \end{aligned}$$

the inequality (6.14) follows from (6.15) on setting $z = x$.

The estimates in Lemma 6.5 can be improved when $y$ is close to $\partial(\Omega_R \cap B_{r_2})$ but a distance from $\partial\Omega_R \cap \partial B_{r_2}$ In fact one can see this by using the Kelvin transform just as in the proof of Lemma 5.5. In particular we have an estimate similar to Proposition 2.1 for $\nabla_x G_D(x,y)$ when $y$ is close to $D_2$.

**Lemma 6.6.** *There exist universal constants, $c, C$ such that if $d(y, D_2) < cR$ then*

$$|\nabla_x G_D(x,y)| \le \frac{C}{|x-y|^2} \min\Big\{1, \frac{d(y,\partial B_{r_2})}{|x-y|}\Big\}, \qquad x, y \in \Omega_R \cap B_{r_2} .$$

PROOF. Suppose $d(y, D_2) < cR$. Then if $c$ is sufficiently small one can choose $\gamma$, $0 < \gamma < 1/2$, in a universal way such that the harmonic function $u(z) = \nabla_x G_D(x, z)$ extends to the entire ball $B(y, \gamma|x-y|)$. This follows by using the Kelvin transform. Furthermore, by Lemma 6.5 there is a universal constant $C$ such that

$$\sup_{z \in B(y,\gamma|x-y|)} |u(z)| \le \frac{C}{|x-y|^2} . \tag{6.16}$$

Let $y_0$ be the closest point on $\partial B_{r_2}$ to $y$ and suppose that $|y - y_0| < \gamma\,|x-y|/2$. Then from the Poisson integral formula and (6.16) one has that $|\nabla u(z)| \le C_1/|x-y|^3$ for all $z$ on the line segment joining $y$ to $y_0$, where $C_1$ is a constant. Since $u(y_0) = 0$ if follows from this that $|u(y)| \le C_1\, d(y, \partial B_{r_2})/|x-y|^3$. The result easily follows.

We use Lemmas 6.5 and 6.6 to show that the operator $T$ with kernel $k_T$ given by (6.12) is a bounded operator on a weighted Morrey space. Let $\lambda > 0$ be a parameter and define the weight function $w_\lambda$ on $\Omega_R \cap B_{r_2}$ by

$$w_\lambda(y) = \begin{cases} \dfrac{d(y,\partial B_{r_2})}{R}\,, & \text{if } d(y, D_2) \le \lambda R\,, \\ 1\,, & \text{if } d(y, D_2) \ge 2\lambda R\,, \end{cases}$$

and

$$w_\lambda(y) = \Big(2 - \frac{d(y,D_2)}{\lambda R}\Big)\frac{d(y,\partial B_{r_2})}{R} + \Big(\frac{d(y,D_2)}{\lambda R} - 1\Big)\,,$$

if $\lambda R < d(y, D_2) < 2\,\lambda\, R$.

**Lemma 6.7.** *Let $Q$ be an arbitrary cube which intersects $\Omega_R \cap B_{r_2}$ and suppose*

$$d(Q) = \sup \{d(x, \partial B_{r_2}) : \ x \in Q\} .$$

*Then there exists a constant $C_\lambda$ depending only on $\lambda$ such that*

$$\frac{d(y, \partial B_{r_2})}{d(Q)} \le \frac{C_\lambda \, \omega_\lambda(y)}{\|\omega_\lambda\|_{\infty,Q}} , \qquad y \in Q \cap \Omega_R \cap B_{r_2} ,$$

*where $\|\omega_\lambda\|_{\infty,Q}$ denotes the $L^\infty$ norm of $\omega_\lambda$ on $Q$.*

PROOF. Suppose $|Q|^{1/3} \ge cR$ for some constant $c > 0$. Hence there are constants $C_1, C_2, C_3$ such that $d(Q) \ge C_1 R$, $\|\omega_\lambda\|_{\infty,Q} \le C_2$, $w_\lambda(y) \ge C_3 \, d(y, \partial B_{r_2})/R$. The inequality (6.17) clearly follows from this and so we may assume from here on that $|Q|^{1/3} \le cR$ where $c > 0$ is an arbitrarily small universal constant.

Next suppose that for all $y \in Q$ one has $d(y, D_2) \le \lambda R$. In view of the definition of $w_\lambda(y)$ for $d(y, D_2) \le \lambda R$ the inequality (6.17) immediately follows. Similarly (6.17) follows if for all $y \in Q$ one has $d(y, D_2) \ge 2\,\lambda\, R$. Hence we may assume that there exists $y \in Q$ such that $\lambda R < d(y, D_2) < 2\,\lambda\, R$. We put $\gamma = d(y, D_2)/\lambda R - 1$, whence $0 \le \gamma \le 1$. Let $\delta = |Q|^{1/3}/\lambda R$. Then if $|Q|^{1/3} \le cR$ and $c > 0$ is small we have $0 < \delta < 1$. One has the inequalities

$$\|w_\lambda\|_{\infty,Q} \le (1 - \gamma + \delta) \, \frac{d(Q)}{R} + \gamma + \delta ,$$

$$w_\lambda(y) \ge (1 - \gamma - \delta) \, \frac{d(y, \partial B_{r_2})}{R} + \gamma - \delta .$$

Suppose now that $2\,\delta < \gamma < 1 - 2\,\delta$. Then

$$\frac{w_\lambda(y)}{\|w_\lambda\|_{\infty,Q}} \ge \frac{\dfrac{(1-\gamma)\, d(y, \partial B_{r_2})}{2R} + \dfrac{\gamma}{2}}{\dfrac{3\,(1-\gamma)\, d(Q)}{2R} + \dfrac{3\,\gamma}{2}} \ge \frac{1}{3} \, \frac{d(y, \partial B_{r_2})}{d(Q)} ,$$

since $d(y, \partial B_{r_2}) \le d(Q)$. Next suppose $0 < \gamma < 2\,\delta$. Since $\delta \le C\, d(Q)/R$ for some constant $C > 0$ we have that

$$\|w_\lambda\|_{\infty,Q} \le (1 + 3C) \, \frac{d(Q)}{R} .$$

On the other hand one also has $w_\lambda(y) \ge d(y, \partial B_{r_2})/(2R)$ if $\delta$ is sufficiently small. Hence (6.17) holds again. Finally, for $1-2\,\delta < \gamma < 1$ one has $w_\lambda(y) \ge 1/2$ for sufficiently small $\delta$ and hence (6.17) holds in this case also.

For $1 \le r \le q < \infty$ we define the weighted Morrey space $M^q_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ as follows: a measurable function $g : \Omega_R \cap B_{r_2} \longrightarrow \mathbb{C}$ is in $M^q_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ if $w_\lambda(y)^r\,|g(y)|^r$ is integrable on $\Omega_R \cap B_{r_2}$ and there is a constant $C$ such that

$$\int_{Q\cap\Omega_R\cap B_{r_2}} w_\lambda(y)^r\,|g(y)|^r\,dy \le C^r\,|Q|^{1-r/q}\,, \tag{6.18}$$

for all cubes $Q \subset \mathbb{R}^3$. The norm of $g$, $\|g\|_{q,r,w_\lambda}$ is defined as

$$\|g\|_{q,r,w_\lambda} = \inf\{C : \ (6.18) \text{ holds for all cubes } Q\}\,.$$

**Lemma 6.8.** *Suppose* $\mathbf{b} \in M^3_p$, $1 < p \le 3$, *and* $r, q$ *satisfy* $1 < r < p$, $r \le q < 3$. *Then there exists a universal constant* $\lambda > 0$ *such that the operator* $T$ *with kernel* $k_T$ *given by* (6.12) *is a bounded operator on the space* $M^q_{r,w_\lambda}(\Omega_R \cap B_{r_2})$. *The norm of* $T$ *satisfies the inequality* $\|T\| \le C\,\|\mathbf{b}\|_{3,p}$, *where the constant* $C$ *depends only on* $r, p, q$.

PROOF. We follow the same lines as the proof of Proposition 2.1. Define an integer $n_0$ by $2^{-n_0-1} < 8R \le 2^{-n_0}$ and let $Q_0(\xi)$ be the cube centered at $\xi$ with side of length $2^{-n_0}$. It is clear that for $\xi \in \Omega_R \cap B_{r_2}$ then $\Omega_R \cap B_{r_2} \subset Q_0(\xi)$. We define an operator $T_K$ on functions $u : \Omega_R \cap B_{r_2} \longrightarrow \mathbb{C}$ which have the property that $w_\lambda(x)\,u(x)$ is integrable. To do this we decompose $K$ into a dyadic decomposition of cubes $Q_n$ with sides of length $2^{-n}$, $n \ge n_0$. For any dyadic cube $Q \subset K$ with volume $|Q|$ let $u_Q$ be defined by

$$u_Q = |Q|^{-1} \int_{Q\cap\Omega_R\cap B_{r_2}} w_\lambda(x)\,u(x)\,dx\,.$$

For $n \ge n_0$ define the operator $S_n$ by

$$S_n u(x) = 2^{-n}\,\frac{u_{Q_n}}{\|w_\lambda\|_{\infty,Q_n}}\,, \qquad x \in Q_n\,.$$

The operator $T_K$ is then given by

$$T_K u(x) = \sum_{n=n_0}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,, \qquad x \in \Omega_R \cap B_{r_2}\,.$$

It follows now from Lemmas 6.5, 6.6, 6.7 and Jensen's inequality that one can choose $\lambda$ in a universal way such that for every cube $Q$

$$\begin{aligned}\int_{Q\cap\Omega_R\cap B_{r_2}} & w_\lambda(x)^r\, |Tu(x)|^r\, dx \\ & \le \frac{C^r}{|\Omega_R \cap B_{r_2}|} \int_{\Omega_R\cap B_{r_2}} d\xi \int_{Q\cap\Omega_R\cap B_{r_2}} w_\lambda(x)^r\, |T_{Q_0(\xi)} u(x)|^r\, dx\,,\end{aligned}$$

for some universal constant $C$. Hence it is sufficient to prove the result of the lemma for the operator $T_K$.

Next we have the analogue of Lemma 2.1. Thus let $Q' \subset K$ be an arbitrary dyadic subcube of $K$ with side of length $2^{-n_{Q'}}$. Suppose $r, p$ satisfy the inequality $1 \le r < p$. Then there are constants $\varepsilon$, $C > 0$ depending only on $r, p$ such that $|Q|^{1/3+\varepsilon}\, u_Q \le |Q'|^{1/3+\varepsilon}\, u_{Q'}$ for all dyadic subcubes $Q$ of $Q'$ implies the inequality

$$\int_{Q'} w_\lambda(x)^r \Big(\sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\Big)^r dx \le C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'| u_{Q'}^r\,.$$

The analogue of Corollary 2.1 follows from this last inequality. Thus we have for any dyadic subcube $Q' \subset K$,

$$\int_{Q'} w_\lambda(x)^r \Big(\sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\Big)^r dx \le C^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q'} w_\lambda(x)^r\, |u(x)|^r\, dx\,.$$

To complete the proof of the lemma we need to show that for any dyadic subcube $Q' \subset K$ one has

$$\int_{Q'} w_\lambda(x)^r \Big(\sum_{n=n_0}^{n_{Q'}-1} |\mathbf{b}(x)|\, S_n u(x)\Big)^r dx \le C^r\, \|\mathbf{b}\|_{3,p}^r\, \|u\|_{q,r,w_\lambda}^r\, |Q'|^{1-r/q}\,,$$

for some constant $C$. This inequality is clear.

Next we prove the analogue of Lemma 4.1.

**Lemma 6.9.** *Suppose $g$ is a function defined on $D_2$ and let $Pg(x)$, $x \in \Omega_R \cap B_{r_2}$, be the function given by the solution of the Dirichlet problem* (6.5)*. Let $r, p, q, q_1$ be as in Lemma* 4.1*. Then if $g \in L^q(D_2)$ the function $\mathbf{b} \cdot \nabla Pg$ is in the Morrey space $M^{q_1}_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ for some universal $\lambda > 0$ and*

$$\|\mathbf{b} \cdot \nabla Pg\|_{q_1,r,w_\lambda} \le CR^{2/q-1} \|\mathbf{b}\|_{3,p} \, \|g\|_{D_2,q} \, .$$

PROOF. The inequality will follow just as in Lemma 4.1 if we can show that

$$(6.19) \quad w_\lambda(x) \, |\nabla Pg(x)| \le CR^{-1}(P|g|(x) + \|g\|_{D_2,1}) \, , \quad x \in \Omega_R \cap B_{r_2} \, ,$$

where $C$ is a constant depending only on $\lambda$. To prove (6.19) first consider the case where $d(x, D_2) < \lambda R$. Since the Harnack principle implies that

$$d(x, \partial B_{r_2}) \, |\nabla Pg(x)| \le CP \, |g|(x) \, ,$$

the inequality follows. Next suppose $d(x, D_2) > \gamma R$. If $d(x, \partial(\Omega_R \cap B_{r_2})) > cR$ for an arbitrary constant $c > 0$ then the Harnack principle again implies that

$$w_\lambda(x) \, |\nabla Pg(x)| \le C_1 \, |\nabla Pg(x)| \le C_2 \, R^{-1} \, \|g\|_{D_2,1} \, ,$$

where $C_2$ depends on $c$. Hence we may assume that $d(x, D_2) > \lambda R$ and $d(x, \partial(\Omega_R \cap B_{r_2})) < cR$ where $c > 0$ can be arbitrarily small. We proceed now as in the argument of Lemma 6.5. Thus let $x_0$ be the nearest point on $\partial(\Omega_R \cap B_{r_2})$ to $x$ and $x_1 = x_0 + \gamma R \, (x_0 - x)/|x_0 - x|$, where $\gamma$ is to be chosen depending on $\lambda, c$. Let $U_x = \{z : |z - x_1| > \gamma R\}$. Then it is clear that we may choose $\gamma$ sufficiently small so that $\Omega_R \cap B_{r_2} \subset U_x$ and $|z - x_1| > 3 \gamma R$ if $z \in D_2$. Next let $v(z) = 1 - |x_0 - x_1|/|z - x_1|$, $z \in U_x$ and $W$ be the region $W = \{z \in \Omega_R \cap B_{r_2} : |z - x_1| < 2 \gamma R\}$. Evidently the functions $P|g|(z)$ and $v(z)$ are harmonic in $W$ and there is a constant $C$ depending on $\gamma$ such that

$$P|g|(z) \le C \, \|g\|_{D_2,1} \, v(z) \, , \qquad z \in \partial W \, .$$

Hence by the maximum principle this last inequality holds for all $z \in W$. For $c$ sufficiently small $x \in W$ and hence there is a constant $C$ such that

$$P|g|(x) \le C \, \|g\|_{D_2,1} \, \frac{d(x, \partial(\Omega_R \cap B_{r_2}))}{R} \, .$$

Using the Harnack principle we immediately conclude that

$$|\nabla Pg(x)| \le C\,\frac{\|g\|_{D_2,1}}{R}\,,$$

for some constant $C$. Hence (6.19) holds in all cases.

**Lemma 6.10.** *Let $\rho_3$ be a density on $D_3 = \partial B(a_1, R/3)$ which satisfies $\|\rho_3\|_{D_3,q} \le C_3\,\mathrm{Av}_{D_3}\rho_3$. Let $\rho_2$ be the density induced on $D_2$ by the paths of the drift process $X_{\mathbf{b}}(t)$ which start on $D_3$ with density $\rho_3$ and exit the region $\Omega_R \cap B_{r_2}$ through $D_2$. Then if $\mathbf{b} \in M_p^3$, $\|\mathbf{b}\|_{3,p} < \varepsilon$ and $\varepsilon$ is sufficiently small there are constants $C_2, c_2$ such that*

$$\|\rho_2\|_{D_2,q} \le C_2\,\mathrm{Av}_{D_2}\rho_2\,, \qquad Av_{D_2}\rho_2 \ge c_2\,\mathrm{Av}_{D_3}\rho_3\,.$$

PROOF. Let $g \in L^q(D_2)$. We consider the operator $Q$ analogous to (3.3), defined by

$$Qg(x) = \int_{\Omega_R \cap B_{r_2}} G_D(x,y)\,(I-T)^{-1}\,\mathbf{b}\cdot\nabla Pg(y)\,dy\,, \qquad x \in D_3\,.$$

Then $\rho_2 = P^*\rho_3 + Q^*\rho_3$. We shall show just as in Proposition 4.1 that for any $q$, $1 < q < \infty$, and $\varepsilon$ sufficiently small $Q$ is a bounded operator from $L^q(D_2)$ to $L^q(D_3)$ and $\|Q\| \le C\,\|\mathbf{b}\|_{3,p}$. The result follows from this by the same argument as in Section 4.

To prove that $Q$ is bounded we use Lemmas 6.8, 6.9. Thus if $\varepsilon$ is sufficiently small the function

$$h(y) = (I-T)^{-1}\,\mathbf{b}\cdot\nabla Pg(y)\,, \qquad y \in \Omega_R \cap B_{r_2}\,,$$

is in the weighted Morrey space $M^{q_1}_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ where $q_1$ is given by (3.5) and $\lambda > 0$ is universal. Furthermore by Lemma 6.9 there is the bound

$$\|h\|_{q_1,r,w_\lambda} \le CR^{2/q-1}\|\mathbf{b}\|_{3,p}\,\|g\|_{D_2,q}\,.$$

For $\gamma > 0$ let $W_1 = \{y \in \Omega_R \cap B_{r_2} : d(y, \partial B_{r_2}) > \gamma R\}$ and $W_2 = \Omega_R \cap B_{r_2} \backslash W_1$. It is clear that for $\gamma$ and $\lambda$ sufficiently small there is a constant $C$ such that

$$G_D(x,y) \le C\,\frac{w_\lambda(y)}{|x-y|}\,, \qquad y \in W_1\,,\ x \in D_3\,,$$

$$G_D(x,y) \le C\,\frac{w_\lambda(y)}{R}\,, \qquad y \in W_2\,,\ x \in D_3\,.$$

Hence we have

$$|Qg(x)| \le C \int_{W_1} \frac{w_\lambda(y)\,|h(y)|}{|x-y|}\,dy + \frac{C}{R}\int_{W_2} w_\lambda(y)\,|h(y)|\,dy\,, \qquad x \in D_3\ .$$

Now we argue exactly as in Proposition 4.1 to see that $\|Q\| \le C\,\|\mathbf{b}\|_{3,p}$.

PROOF OF THEOREM 6.1. If $\xi = 0$ and $C_1 = \infty$ the result is a consequence of Lemmas 6.4 and 6.10. Hence it is sufficient for us to prove that for $\xi > 0$ small and $C_1 < \infty$ large then $\mathrm{Av}_{D_2}\rho_2 \ge c\,\mathrm{Av}_{D_1}\rho_1$ for some constant $c > 0$. For $C_1 < \infty$ we argue as in Lemma 6.2 using [5, Theorem 1.1]. For $\xi > 0$ we argue as in Lemma 6.3 and use Proposition 5.3.

## 7. Nonperturbative estimates on the exit probabilities from a spherical shell.

In this section we shall generalize Corollary 4.2 to the nonperturbative case. The main tool we use to do this is the following nonperturbative version of Theorem 6.1:

**Theorem 7.1.** *Let $R = 2^{-n}$, $n$ an integer, $n \ge n_0$, and $\rho_1$ be a density on $D_1$. Suppose $f \in M_r^t(\mathbb{R}^3)$ with $1 < r \le t$, $r < p$, $3/2 < t < 3$. Let $\overline{\rho}_2$ be the density induced on $D_2$ by the paths of the drift process $X_{\mathbf{b}}(t)$ which start on $D_1$, exit the region $\Omega_R \cap B_{r_2}$ through $D_2$, and satisfy the inequality*

$$\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt \le C_1\,R^{2-3/t}\,\|f\|_{t,r}\ ,$$

*where $C_1$ is a constant. Then for $\eta > 0$, $1 < q < \infty$ and $C_1$ sufficiently large there exist constants $\alpha > 1$, $\beta$, $C_2$, $c_2 > 0$ such that*

$$\|\rho_1\|_{D_1,q} \le C_2\,\alpha^{n-n_0}\,\mathrm{Av}_{D_1}\rho_1$$

*implies that there is a function $\rho_2$ on $D_2$ such that $\overline{\rho}_2(x) \ge \rho_2(x) \ge 0$, $x \in D_2$, and*

$$\|\rho_2\|_{D_2,q} \le C_2\,\mathrm{Av}_{D_2}\rho_2\ ,$$

$$\mathrm{Av}_{D_2}\rho_2 \ge c_2\,\mathrm{Av}_{D_1}\rho_1 \exp\Big(-\frac{\beta}{R}\int_{\Omega_R} V_\eta(x)\,dx\Big)\,.$$

REMARK. Theorem 6.1 implies Theorem 7.1 when (6.3) holds by taking $\beta > 0$. We can prove Theorem 7.1 under the assumption that $\mathbf{b} \in L^\infty$ since none of the constants depend on $\mathbf{b}$. In that case when $R = 2^{-n}$ and $n$ is sufficiently large we are in the perturbative case and the theorem follows again from Theorem 6.1.

We shall prove Theorem 7.1 by induction. In particular we will prove that if $m$ is an integer, $m \geq n_0$ and if Theorem 7.1 holds for $R = 2^{-n}$, $n > m$, then it also holds for $R = 2^{-m}$. The key fact in reducing the $R = 2^{-m}$ case to the case $R = 2^{-n}$, $n > m$, is the following:

**Lemma 7.1.** *For $x \in D_1$, $z \in D_2$, let $\Gamma_{x,z,k}$ be the cylinder whose axis is the line joining $x$ to $z$ and with radius $2^{-k}$. Let $V : \Omega_R \longrightarrow \mathbb{R}$ be a nonnegative potential. Then there is a universal constant $C$ such that*

$$\int_{D_1} d\mu(x) \int_{D_2} d\mu(z) \int_{\Gamma_{x,z,k} \cap \Omega_R} V(y)\, dy \leq C\Big(\frac{2^{-2k}}{R^2}\Big) \int_{\Omega_R} V(y)\, dy\,,$$

*where $d\mu$ denotes the normalized euclidean measures on $D_1$, $D_2$.*

PROOF. Let $\chi_{x,z,k}$ be the characteristic function of the set $\Gamma_{x,z,k} \cap \Omega_R$. For any $y \in \Omega_R$ either $|y - a_1| \geq R/2$ or $|y - a_2| \geq R/2$. Suppose $|y - a_1| \geq R/2$. Then there is a universal constant $C$ such that

$$\int_{D_2} \chi_{x,z,k}(y)\, d\mu(z) \leq C\Big(\frac{2^{-2k}}{R^2}\Big)\,, \qquad x \in D_1\ .$$

Similarly if $|y - a_2| \geq R/2$ we have

$$\int_{D_1} \chi_{x,z,k}(y)\, d\mu(x) \leq C\Big(\frac{2^{-2k}}{R^2}\Big)\,, \qquad z \in D_2\ .$$

The lemma follows easily from these last two inequalities.

**Lemma 7.2.** *For $x \in D_1$ and $\delta > 0$ let $D_x = \{y \in D_1 : |y - x| < \delta\}$. Suppose $\gamma, q > 1$ and $\|f\|_{D_1,q} \leq \mathcal{K}\, \mathrm{Av}_{D_1} f$. Let $G$ be the set*

$$G = \{x \in D_1 :\ d(x, \partial D_1) > 2\,\delta,\ \|f\|_{D_x,q} \leq \mathcal{K}\,\gamma\, \mathrm{Av}_{D_x} f\}\,.$$

*Then there is a universal constant $C$ such that*

$$\int_G \mathrm{Av}_{D_x} f\, d\mu(x) \geq \Big(1 - \frac{1}{\gamma} - C\Big(\frac{\delta}{R}\Big)^{1/q'} \mathcal{K}\Big) \mathrm{Av}_{D_1} f\ .$$

PROOF. We have

$$\begin{aligned}\mathrm{Av}_{D_1} f &= \frac{1}{|D_1|}\int_{D_1} f(y)\,dy \\ &= \frac{1}{|D_1|}\int_{D_1} f(y)\,\frac{1}{|D_1\cap B(y,\delta)|}\,dy\int_{D_1}\chi_{D_x}(y)\,dx\,,\end{aligned}$$

where $\chi_{D_x}$ is the characteristic function of $D_x$. Letting $H_i = \{x \in D_1 : d(x, \partial D_1) > i\,\delta\}$, $i = 1, 2, \ldots$, we can rewrite this last expression as

$$\begin{aligned}\mathrm{Av}_{D_1} f = {}& \frac{1}{|D_1|}\int_{D_1\backslash H_1} f(y)\,\frac{1}{|D_1\cap B(y,\delta)|}\,dy\int_{D_1}\chi_{D_x}(y)\,dx \\ &+ \frac{1}{|D_1|}\int_{H_1} f(y)\,\frac{dy}{|D_x|}\int_{D_1}\chi_{D_x}(y)\,dx\,.\end{aligned}\tag{7.1}$$

Next observe that

$$\begin{aligned}&\frac{1}{|D_1|}\int_{H_1} f(y)\,\frac{dy}{|D_x|}\int_{D_1}\chi_{D_x}(y)\,dx \\ &\qquad = \frac{1}{|D_1|}\int_{H_1} f(y)\,\frac{dy}{|D_x|}\int_{D_1\backslash H_2}\chi_{D_x}(y)\,dx \\ &\qquad\quad + \frac{1}{|D_1|}\int_{H_2}\mathrm{Av}_{D_x} f\,dx\,.\end{aligned}\tag{7.2}$$

We can bound the first term in (7.1) as

$$\begin{aligned}\frac{1}{|D_1|}\int_{D_1\backslash H_1} f(y)\,dy &\le \frac{|D_1\backslash H_1|^{1/q'}}{|D_1|^{1/q'}}\Big(\frac{1}{|D_1|}\int_{D_1} f(y)^q\,dy\Big)^{1/q} \\ &\le \Big(\frac{|D_1\backslash H_1|}{|D_1|}\Big)^{1/q'}\|f\|_{D_1,q} \\ &\le \frac{1}{2}\,C\Big(\frac{\delta}{R}\Big)^{1/q'}\mathcal{K}\,\mathrm{Av}_{D_1} f\,,\end{aligned}$$

for some universal constant $C$. Similarly we can bound the first term in (7.2) by

$$\frac{1}{|D_1|}\int_{D_1\backslash H_3} f(y)\,dy \le \frac{1}{2}\,C\Big(\frac{\delta}{R}\Big)^{1/q'}\mathcal{K}\,\mathrm{Av}\,f\,.$$

We conclude from these last two inequalities that

$$\frac{1}{|D_1|}\int_{H_2} \mathrm{Av}_{D_x} f\,dx \geq \Big(1 - C\Big(\frac{\delta}{R}\Big)^{1/q'}\mathcal{K}\Big)\mathrm{Av}\,f\,.$$

Next observe that

$$\begin{aligned}
\frac{1}{|D_1|}\int_{H_2\setminus G} \mathrm{Av}_{D_x} f\,dx &\leq \frac{1}{|D_1|\mathcal{K}\,\gamma}\int_{H_2\setminus G} \|f\|_{D_x,q}\,dx\\
&= \frac{1}{|D_1|\mathcal{K}\,\gamma}\int_{H_2\setminus G}\Big(\frac{1}{|D_x|}\int_{D_x} f(y)^q\,dy\Big)^{1/q} dx\\
&\leq \frac{|H_2\setminus G|^{1/q'}}{|D_1|\mathcal{K}\,\gamma}\Big(\int_{H_2\setminus G}\frac{dx}{|D_x|}\int_{D_x} f(y)^q\,dy\Big)^{1/q}\\
&\leq \Big(\frac{|H_2\setminus G|}{|D_1|}\Big)^{1/q'}\frac{1}{\mathcal{K}\,\gamma}\Big(\frac{1}{|D_1|}\int_{D_1} f(y)^q\,dy\Big)^{1/q}\\
&= \Big(\frac{|H_2\setminus G|}{|D_1|}\Big)^{1/q'}\frac{1}{\mathcal{K}\,\gamma}\,\|f\|_{D_1,q}\\
&\leq \frac{1}{\gamma}\,\mathrm{Av}_{D_1} f\,.
\end{aligned}$$

The lemma follows from this last inequality and (7.3).

Let us assume now that Theorem 7.1 holds for $R = 2^{-n}$ with $n > m$, $m \geq n_0$, and consider the case $R = 2^{-m}$. If (6.3) holds the theorem is correct so we shall assume that (6.3) is violated. Put $k_0 = m$ and define an integer $k_1 > k_0$ by

$$2^{k_1-k_0} \sim 2^{\lambda_0}\Big(\frac{1}{R}\int_{\Omega_R} V_\eta(y)\,dy\Big)^{1/3}, \tag{7.4}$$

where $\lambda_0 \geq 0$ is a fixed integer to be chosen later. Since we are assuming that

$$\frac{1}{R}\int_{\Omega_R} V_\eta(y)\,dy \geq \xi\,2^{\eta'(m-n_0)} \tag{7.5}$$

and $m \geq n_0$ we should choose $\lambda_0$ to satisfy $2^{\lambda_0}\xi^{1/3} \geq 2$ to ensure $k_1 > k_0$.

**Proposition 7.1.** *Suppose that Theorem* 7.1 *holds for* $n > m \geq n_0$ *and that for every* $z \in D_2$ *the following inequality holds*

$$\frac{1}{2^{-k_1}} \int_{B(z,2^{-k_1})} V_\eta(y)\, dy \leq \xi\, 2^{\eta'(k_1-n_0)}\,. \tag{7.6}$$

*Then Theorem* 7.1 *holds for* $R = 2^{-m}$.

Proof. From Lemma 7.1 we have that

$$\begin{aligned} \int_{D_1} d\mu(x) \int_{D_2} d\mu(z)\, \frac{1}{2^{-k_1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} & V_\eta(y)\, dy \\ & \leq C\, 2^{-(k_1-k_0)}\, \frac{1}{R} \int_{\Omega_R} V_\eta(y)\, dy \\ & \leq 2^{-\lambda_0}\, 2^{2(k_1-k_0-\lambda_0)}\,. \end{aligned} \tag{7.7}$$

Next for $x \in D_1$ and $f$ a function on $D_1$ let $Av_{x,k_1} f$ be the average of $f$ on the set $D_1 \cap B(x, 2^{-k_1-4})$ and $\|f\|_{x,k_1,q}$ be the corresponding $L^q$ norm normalized so that $\|\mathbf{1}\|_{x,k_1,q} = 1$. Let $\tilde{D}_1$ be the set of $x \in D_1$ which satisfy the following properties:

a) $d(x, \partial D_1) > 2^{-k_1}$ ,

b) $\|\rho_1\|_{x,k_1,q} \leq C_2\, \alpha^{k_1+4-n_0}\, \mathrm{Av}_{x,k_1} \rho_1$ ,

c) $\displaystyle\int_{D_2} \frac{d\mu(z)}{2^{-k_1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\, dy \leq 2^{-\lambda_0/2}\, 2^{5\,(k_1-k_0-\lambda_0)/2}$ .

In view of Lemma 7.2 we have that

$$\begin{aligned} \int_{\tilde{D}_1} & \mathrm{Av}_{x,k_1} \rho_1\, d\mu(x) \\ & \geq \Big(1 - \frac{1}{\alpha^{k_1+4-k_0}} - C\, 2^{-(k_1-k_0)/q'}\, \alpha^{k_0-n_0}\, C_2 \\ & \qquad - C\, (2^{-\lambda_0/2} 2^{-(k_1-k_0-\lambda_0)/2})^{1/q'}\, \alpha^{k_0-n_0}\, C_2\Big) \mathrm{Av}_{D_1} \rho_1\,. \end{aligned} \tag{7.8}$$

Observe that the last term in the previous expression is a consequence of the restriction c). In fact, in view of (7.7) one has

$$\frac{\mathrm{meas}\,\{x \in D_1 :\ \text{c) is violated}\}}{|D_1|} \leq 2^{-\lambda_0/2}\, 2^{-(k_1-k_0-\lambda_0)/2}\,.$$

Now from (7.4), (7.5) it follows that

$$2^{-(k_1-k_0)/q'}\alpha^{k_0-n_0} \le 2^{-\lambda_0/q'}\,\xi^{-1/3}\,2^{-\eta'(k_0-n_0)/3}\,\alpha^{k_0-n_0}\,.$$

Hence if we choose $\alpha < 2^{\eta'/3}$ and $\lambda_0$ sufficiently large depending on $\xi$, $C_2$ we can have

$$\int_{\tilde D_1} \mathrm{Av}_{x,k_1}\rho_1\,d\mu(x) \ge \frac{1}{2}\,\mathrm{Av}_{D_1}\rho_1\,. \tag{7.9}$$

For $x \in \tilde D_1$ we define a subset $\tilde D_2 \subset D_2$ as the set of $z \in D_2$ which satisfy

d) $d(z,\partial D_2) > 2^{-k_1}$,

e) $\dfrac{1}{2^{-k_1}}\displaystyle\int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\,dy \le 2^{-\lambda_0/4}\,2^{11(k_1-k_0-\lambda_0)/4}\,.$

From c), e) and the Chebyshev inequality we have that

$$\frac{|\tilde D_2|}{|D_2|} > 1 - 2^{-\lambda_0/4}\,2^{-(k_1-k_0-\lambda_0)/4} - C\,2^{-(k_1-k_0)}\,, \tag{7.10}$$

for some universal constant $C$. Evidently the set $\tilde D_2$ depends on $x \in \tilde D_1$.

Let $x \in \tilde D_1$, $z \in \tilde D_2$. Then we can use the induction hypothesis to propagate the density $\rho_1$ restricted to $D_1 \cap B(x, 2^{-k_1-4})$ through the cylinder $\Gamma_{x,z,k_1}$. To implement it we choose points $x_0, x_1, \ldots, x_N$ with the property that $x_0 = x$, $x_N = z$, $|x_i - x_{i+1}| = 2^{-k_1-2}$, $0 \le i \le N-1$, such that the balls centered at $(x_i + x_{i+1})/2$ with radius $2^{-(k_1+2)}$ are contained in $\Gamma_{x,z,k_1}$. Finally we insist that $N \le C\,2^{k_1-k_0}$ for some universal constant $C$.

Consider the ball $B_0$ centered at $(x_0+x_1)/2$ with radius $2^{-(k_1+2)}$. Letting $D_x = D_1 \cap B(x, 2^{-k_1-4})$ then from b) and the induction hypothesis $\rho_1$ restricted to $D_x$ can be propagated to a density $\rho_1^{(1)}$ on $D_{x_1} = \partial B_r \cap B(x_1, 2^{-k_1-4})$ where $B_r$ is a ball of radius $r \ge 10\,2^{-k_1-2}$ such that $x_1 \in \partial B_r$. Furthermore $\rho_1^{(1)}$ satisfies the conditions

$$\|\rho_1^{(1)}\|_{D_{x_1},q} \le C_2\,\mathrm{Av}_{D_{x_1}}\rho_1^{(1)}\,,$$

$$\mathrm{Av}_{D_{x_1}}\rho_1^{(1)} \ge c_2\,\mathrm{Av}_{D_x}\rho_1 \exp\Big(\frac{-\beta}{2^{-k_1-2}}\int_{B_0} V_\eta(x)\,dx\Big)\,.$$

In view of the above inequalities and the induction assumption we may propagate $\rho_1^{(1)}$ to a density $\rho_1^{(2)}$ on $D_{x_2} = \partial B_r \cap B(x_2, 2^{-k_1-4})$ and continue to do this until we obtain a density $\rho_1^{(N-1)}$ on $D_{x_{N-1}} = \partial B_r \cap B(x_{N-1}, 2^{-k_1-4})$ with the properties

$$\|\rho_1^{(N-1)}\|_{D_{x_{N-1}},q} \le C_2 \operatorname{Av}_{D_{x_{N-1}}} \rho_1^{(N-1)} , \tag{7.11}$$

$$\operatorname{Av}_{D_{x_{N-1}}} \rho_1^{(N-1)} \ge c_2^{N-1} \exp\Big(\frac{-\beta}{2^{-k_1-1}} \int_{\Gamma_{x,z,k_1}} V_\eta(y)\, dy\Big) \operatorname{Av}_{D_x} \rho_1 . \tag{7.12}$$

In the inequalities (7.11), (7.12) the constants $C_2$, $c_2$, $\beta$ are from Theorem 7.1. They are therefore part of the induction hypothesis. To ensure that these constants continue to hold on the next level up we use the assumption (7.6). Hence in propagating $\rho_1^{(N-1)}$ to $\rho_1^{(N)}$ we may use the perturbative Theorem 6.1. Let us denote the constants $C_2, c_2$ in Theorem 6.1 by $C_{2,\text{perturb}}$ and $c_{2,\text{perturb}}$ to distinguish them from the corresponding constants $C_2, c_2$ in Theorem 7.1. It is clear that by choosing $\lambda_0$ large enough we have

$$C_2 \le C_{2,\text{perturb}}\, \alpha^{k_1+2-n_0} . \tag{7.13}$$

Hence by Theorem 6.1 $\rho_1^{(N-1)}$ propagates to a density $\rho_1^{(N)}$ on $D_{x_N} = D_2 \cap B(z, 2^{-k_1-4})$ which has the properties

$$\|\rho_1^{(N)}\|_{D_{x_N},q} \le C_{2,\text{perturb}} \operatorname{Av}_{D_{x_N}} \rho_1^{(N)} , \tag{7.14}$$

$$\operatorname{Av}_{D_{x_N}} \rho_1^{(N)} \ge c_{2,\text{perturb}}\, c_2^{N-1} \cdot \exp\Big(\frac{-\beta}{2^{-k_1-1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\, dy\Big) \operatorname{Av}_{D_x} \rho_1 . \tag{7.15}$$

Evidently we can assume $c_2 < 1$ and $c_2 < c_{2,\text{perturb}}$. Hence the inequality (7.15) yields

$$\begin{aligned}\operatorname{Av}_{D_{x_N}} \rho_1^{(N)} &\ge \exp\Big( - N \log\Big(\frac{1}{c_2}\Big) - \frac{\beta}{2^{-k_1-1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\, dy\Big) \\ &\quad \cdot \operatorname{Av}_{D_x} \rho_1 \\ &\ge \exp\Big( - C\, 2^{k_1-k_0} \log\Big(\frac{1}{c_2}\Big) - 2\beta 2^{-\lambda_0/4}\, 2^{11(k_1-k_0-\lambda_0)/4}\Big) \\ &\quad \cdot \operatorname{Av}_{D_x} \rho_1 ,\end{aligned}$$

upon using e) and the fact that $N \le C\,2^{k_1-k_0}$. Observe now that

$$
\begin{aligned}
&C\,2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big)+2\,\beta\,2^{-\lambda_0/4}\,2^{11(k_1-k_0-\lambda_0)/4}\\
&\qquad=\beta\,2^{3(k_1-k_0-\lambda_0)}\\
&\qquad\quad\cdot\Bigg(\frac{C\,2^{\lambda_0}\log\Big(\dfrac{1}{c_2}\Big)}{\beta}\,2^{-2(k_1-k_0-\lambda_0)}+2^{1-\lambda_0/4}\,2^{-(k_1-k_0-\lambda_0)/4}\Bigg)\,.
\end{aligned}
$$

In view of the assumption (7.5) we can choose $\lambda_0$ dependent only on $\xi$ such that

$$2^{1-\lambda_0/4}\,2^{-(k_1-k_0-\lambda_0)/4}<\frac{1}{4}\,.$$

With this choice of $\lambda_0$ and arbitrary $c_2$, $0<c_2<1$ we can choose $\beta>0$ such that

$$\beta^{-1}\,C\,2^{\lambda_0}\log\Big(\frac{1}{c_2}\Big)2^{-2(k_1-k_0-\lambda_0)}<\frac{1}{4}\,.$$

Hence it follows that

$$\operatorname{Av}_{D_{x_N}}\rho_1^{(N)}\ge\exp\Big(-\frac{\beta}{2R}\int_{\Omega_R}V_\eta(y)\,dy\Big)\operatorname{Av}_{D_x}\rho_1\,. \tag{7.16}$$

We wish to define the density $\rho_2$ on $D_2$. For $x\in\tilde D_1$ let

$$\gamma(k_1)=|D_1\cap B(x,2^{-k_1-4})|\,.$$

Evidently $\gamma(k_1)$ is independent of $x$ and $\gamma(k_1)\sim 2^{-2k_1}$. Also

$$\rho_1(y)\ge\int_{D_1}\gamma(k_1)^{-1}\,\rho_1(y)\,\chi_{D_x}(y)\,dx\,,\qquad y\in D_1\,, \tag{7.17}$$

where $\chi_{D_x}$ is the characteristic function of $D_x=D_1\cap B(x,2^{-k_1-4})$. For $x\in\tilde D_1$, $z\in\tilde D_2$ let $\rho_1^{x,z}$ be the density $\rho_1^{(N)}$ defined above. Thus

$$\rho_1^{x,z}(y)=\begin{cases}\rho_1^{(N)}(y)\,, & y\in D_{x_N}\,,\\ 0\,, & y\in D_2\backslash D_{x_N}\,.\end{cases}$$

It follows then from (7.17) that the density $\overline{\rho}_2$ induced on $D_2$ by $\rho_1$ as in Theorem 7.1 satisfies

$$\overline{\rho}_2(y)\ge\int_{\tilde D_1}\frac{dx}{\gamma(k_1)\,|\tilde D_2|}\int_{\tilde D_2}\rho_1^{x,z}(y)\,dz\,,\qquad y\in D_2\,. \tag{7.18}$$

From (7.16) and the above we have that

$$\mathrm{Av}_{D_2}\overline{\rho}_2 \ge \exp\Big(\frac{-\beta}{2R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\frac{1}{|D_2|}\int_{\tilde{D}_2} \mathrm{Av}_{D_x}\rho_1\,dx\,.$$

Now if we use the inequality (7.9) we conclude that

$$\mathrm{Av}_{D_2}\overline{\rho}_2 \ge c_2\exp\Big(\frac{-\beta}{R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\mathrm{Av}_{D_1}\rho_1\ ,$$

provided $c_2$ is sufficiently small. This last inequality is consistent with the lower bound on $\mathrm{Av}_{D_2}\rho_2$ in Theorem 7.1.

It seems reasonable from the previous argument that we shall define $\rho_2$ by the right hand side of (7.18). We need to be more subtle than this in order to keep control of $\|\rho_2\|_{D_2,q}$ as required by Theorem 7.1. We accomplish this by insisting that the integral of $\rho_1^{x,z}$ is independent of $z\in\tilde{D}_2$. In view of (7.16) we may insist that

$$\text{(7.19)}\quad \mathrm{Av}_{D_z}\rho_1^{x,z} = \exp\Big(-\frac{\beta}{2R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\mathrm{Av}_{D_x}\rho_1\ ,\qquad z\in\tilde{D}_2\ .$$

Then the density $\rho_2$ is defined like the right hand side of (7.18) by

$$\text{(7.20)}\qquad \rho_2 = \int_{\tilde{D}_1}\frac{dx}{\gamma(k_1)\,|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}\,dz\,.$$

Evidently $\mathrm{Av}_{D_2}\rho_2$ satisfies the lower bound of Theorem 7.1. To estimate $\|\rho_2\|_{D_2,q}$ we use the Minkowski inequality. Thus

$$\text{(7.21)}\qquad \|\rho_2\|_{D_2,q} \le \int_{\tilde{D}_1}\frac{dx}{\gamma(k_1)}\Big\|\frac{1}{|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}\,dz\Big\|_{D_2,q}\,.$$

Now we have

$$\Big\|\frac{1}{|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}\,dz\Big\|_{D_2,q}^q = \frac{1}{|D_2|}\int_{D_2}\Big(\frac{1}{|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}(y)\,dz\Big)^q\,dy\,.$$

Observe next that $\rho_1^{x,z}(y)=0$ if $|z-y|>2^{-k_1}$, whence

$$\Big(\int_{\tilde{D}_2}\rho_1^{x,z}(y)\,dz\Big)^q \le C^q\,2^{-2k_1(q-1)}\int_{\tilde{D}_2}\rho_1^{x,z}(y)^q\,dz\,,$$

for some universal constant $C$. Hence

$$
\begin{aligned}
\Big\| \frac{1}{|\tilde{D}_2|} \int_{\tilde{D}_2} \rho_1^{x,z}\, dz \Big\|_{D_2,q}^q &\le \frac{C^q}{|D_2|} \int_{D_2} \frac{2^{-2k_1(q-1)}}{|\tilde{D}_2|^q}\, dy \int_{\tilde{D}_2} \rho_1^{x,z}(y)^q\, dz \\
&= \frac{C^q\, 2^{-2k_1(q-1)}}{|D_2|\, |\tilde{D}_2|^q} \int_{\tilde{D}_2} dz \int_{D_2} \rho_1^{x,z}(y)^q\, dy \\
&\le \frac{C^q\, 2^{-2k_1(q-1)}}{|D_2|\, |\tilde{D}_2|^q} \\
&\quad \cdot \int_{\tilde{D}_2} 2^{-2k_1}\, C_{2,\mathrm{perturb}}^q \Big( \mathrm{Av}_{D_z} \rho_1^{x,z} \Big)^q\, dz\,.
\end{aligned} \tag{7.22}
$$

If we use now this last inequality together with (7.19) and (7.21) we can conclude that

$$
\|\rho_2\|_{D_2,q} \le C\, C_{2,\mathrm{perturb}} \exp\Big( -\frac{\beta}{2R} \int_{\Omega_R} V_\eta(y)\, dy \Big) \frac{\int_{\tilde{D}_1} \mathrm{Av}_{D_x} \rho_1\, dx}{(|D_2|\, |\tilde{D}_2|^{q-1})^{1/q}}\,.
$$

It follows from (7.10) that we can choose $\lambda_0$ sufficiently large depending only on $\xi$ so that $(|D_2|\, |\tilde{D}_2|^{q-1})^{1/q} \ge |D_2|/2$. In view of (7.19) we have that

$$
\mathrm{Av}_{D_2} \rho_2 \ge \exp\Big( -\frac{\beta}{2R} \int_{\Omega_R} V_\eta(y)\, dy \Big) \frac{1}{|D_2|} \int_{\tilde{D}_1} \mathrm{Av}_{D_x} \rho_1\, dx\,.
$$

We conclude therefore that

$$
\|\rho_2\|_{D_2,q} \le C\ C_{2,\mathrm{perturb}}\, \mathrm{Av}_{D_2} \rho_2\,, \tag{7.23}
$$

where $C$ is a universal constant. Theorem 7.1 follows then if we have $C\, C_{2,\mathrm{perturb}} \le C_2$. This inequality is consistent with the inequality (7.13) provided we choose $\lambda_0$ large enough.

REMARK 7.2. The assumption (7.6) is only used in concluding (7.23). If we did not assume (7.6) then the constant in (7.23) would be $C\, C_2$ and we obviously cannot conclude that $C\, C_2 \le C_2$ if $C > 1$.

PROOF OF THEOREM 7.1. The idea is to extend the argument of Proposition 7.1 until a perturbative situation holds at the boundary of $D_2$. This will require introduction of further cylindrical decompositions until we are in a situation where (7.6) holds. We begin as in Proposition 7.1 by defining $k_0 = m$, $k_1$ by (7.4) and assume that (7.5) holds. To simplify notation we shall refer to the set $D_2$ from here on in this proof as $E_1$, and the density $\rho_1$ on $D_1$ as $\rho$.

The set $\tilde{D}_1$ is defined exactly as in Proposition 7.1 by a), b), c) following (7.7). For $x_1 \in \tilde{D}_1$ we define a subset $\tilde{E}_1 \subset E_1$ which depends on $x_1$ similarly to the set $\tilde{D}_2$ of Proposition 7.1. Thus we define it by the conditions d), e) following (7.9) but we also impose the requirement (7.6). Thus $z_1 \in \tilde{E}_1$ if

d$^{(1)}$) $d(z_1, \partial E_1) > 2^{-k_1}$,

e$^{(1)}$) $\frac{1}{2^{-k_1}} \int_{\Gamma_{x_1,z_1,k_1} \cap \Omega_R} V_\eta(y)\, dy \le 2^{-\lambda_0/4}\, 2^{11(k_1-k_0-\lambda_0)/4}$,

f$^{(1)}$) $\frac{1}{2^{-k_1}} \int_{B(z_1,2^{-k_1})} V_\eta(y)\, dy \le \xi\, 2^{\eta'(k_1-n_0)}$.

The set $\tilde{F}_1 \subset E_1$ is defined as the set of $z_1 \in E_1$ for which d$^{(1)}$), e$^{(1)}$) above hold but not f$^{(1)}$). The inequality (7.10) yields therefore the inequality

$$\frac{|\tilde{E}_1 \cup \tilde{F}_1|}{|E_1|} \ge 1 - 2^{-\lambda_0/4}\, 2^{-(k_1-k_0-\lambda_0)/4} - C\, 2^{-(k_1-k_0)}. \tag{7.24}$$

Evidently if $\lambda_0$ is sufficiently large depending on $\xi$ the right hand side of the above inequality is strictly positive.

Now for $x_1 \in \tilde{D}_1$, $z_1 \in \tilde{E}_1$ we can as in Proposition 7.1 propagate the density $\rho$ restricted to $D_1 \cap B(x_1, 2^{-k_1-4})$ to a density $\rho_{x_1,z_1}$ on $E_1 \cap B(z_1, 2^{-k_1-4})$ whose average value and fluctuation we can control exactly as in Proposition 7.1. Next suppose $z_1 \in \tilde{F}_1$. Then we may use the induction hypothesis to propagate $\rho$ restricted to $E_1 \cap B(z_1, 2^{-k_1-4})$ to a density $\rho_{x_1,z_1}$ which is concentrated on a set $D_2 = \partial B_r \cap B(\overline{x}, 2^{-k_1-4})$ and $\overline{x}$ has the property that $B(\overline{x}, 2^{-k_1-4}) \subset B(z_1, 2^{-k_1-1}) \cap \Omega_R$ but has no intersection with $B(z_1, 2^{-k_1-2})$. The density $\rho_{x_1,z_1}$ on $D_2$ corresponds to $\rho_1^{(N-1)}$ in Proposition 7.1 and can be controlled by the inequalities (7.11) and (7.12).

For $z_1 \in \tilde{F}_1$ we define $k_2$ by

$$2^{k_2-k_1} \sim 2^{\lambda_0} \Big( \frac{1}{2^{-k_1}} \int_{B(z_1,2^{-k_1})} V_\eta(y)\, dy \Big)^{1/3}.$$

Thus $k_2$ has the same relationship to $k_1$ as $k_1$ has to $k_0$, but now it depends on the variable $z_1 \in \tilde{F}_1$. Let $E_2 = B(z_1, 2^{-k_1-4}) \cap E_1$ and define $\tilde{D}_2$ in analogy with $\tilde{D}_1$. Thus $\tilde{D}_2 \subset D_2$ and $x_2 \in \tilde{D}_2$ if

a$^{(2)}$) $d(x_2, \partial D_2) > 2^{-k_2}$ ,

b$^{(2)}$) $\|\rho_{x_1,z_1}\|_{x_2,k_2,q} \le C_2\, \alpha^{k_2+4-n_0}\, \mathrm{Av}_{x_2,k_2}\rho_{x_1,z_1}$ ,

c$^{(2)}$)

$$\frac{1}{|E_2|}\int_{E_2} \frac{dz_2}{2^{-k_2}} \int_{\Gamma_{x_2,z_2,k_2}\cap B(z_1,2^{-k_1})} V_\eta(y)\, dy \le 2^{-\lambda_0/2}\, 2^{5(k_2-k_1-\lambda_0)/2} .$$

By (7.11) we have that $\|\rho_{x_1,z_1}\|_{D_2,q} \le C_2\, Av_{D_2}\rho_{x_1,z_1}$. In analogy to the derivation of (7.8) we have that

$$\begin{aligned}\frac{1}{|D_2|}\int_{\tilde{D}_2} \mathrm{Av}_{x_2,k_2}\rho_{x_1,z_1}\, dx_2 \ge{}& \mathrm{Av}_{D_2}\rho_{x_1,z_1} \\ &\cdot \Big(1 - \frac{1}{\alpha^{k_2+4-n_0}} - C\, 2^{-(k_2-k_1)/q'}\, C_2 \\ &\quad - C\, 2^{-(k_2-k_1)/2q'}\, C_2\Big) .\end{aligned} \tag{7.25}$$

For $x_2 \in \tilde{D}_2$ we define a subset $\tilde{E}_2 \subset E_2$ in analogy to $E_1$. Thus $z_2 \in \tilde{E}_2$ if

d$^{(2)}$) $d(z_2, \partial E_2) > 2^{-k_2}$ ,

e$^{(2)}$) $\displaystyle \frac{1}{2^{-k_2}} \int_{\Gamma_{x_2,z_2,k_2}\cap B(z_1,2^{-k_1})} V_\eta(y)\, dy \le 2^{-\lambda_0/4}\, 2^{11(k_2-k_1-\lambda_0)/4}$ ,

f$^{(2)}$) $\displaystyle \frac{1}{2^{-k_2}} \int_{B(z_2,2^{-k_2})} V_\eta(y)\, dy \le \xi\, 2^{\eta'(k_2-n_0)}$ .

The subset $\tilde{F}_2 \subset E_2$ is the set of $z_2 \in E_2$ for which d$^{(2)}$) and e$^{(2)}$) hold but not f$^{(2)}$). In analogy with (7.24) we have the inequality

$$\frac{|\tilde{E}_2 \cup \tilde{F}_2|}{|E_2|} \ge 1 - 2^{-\lambda_0/4}\, 2^{-(k_2-k_1-\lambda_0)/4} - C\, 2^{-(k_2-k_1)} .$$

For $x_2 \in \tilde{D}_2$, $z_2 \in \tilde{E}_2$ we use Proposition 7.1 to propagate the density $\rho_{x_1,z_1}$ restricted to $D_2 \cap B(x_2, 2^{-k_2-4})$ to a density on $E_2 \cap B(z_2, 2^{-k_2-4})$

whose average value and fluctuation we can control. This density is denoted by $\rho_{x_1,z_1,x_2,z_2}$. Just as previously if $z_2 \in \tilde{F}_2$ we use the induction hypothesis to propagate $\rho_{x_1,z_1}$ restricted to $D_2 \cap B(x_2, 2^{-k_2-4})$ to a density $\rho_{x_1,z_1,x_2,z_2}$ concentrated on a set $D_3 = B(\overline{x}, 2^{-k_2-4}) \cap \partial B_r$. The point $\overline{x}$ is to be chosen similarly to before. Thus we require that $B(\overline{x}, 2^{-k_2-4})$ is contained in $B(z_2, 2^{-k_2-1}) \cap \Omega_R$ but has no intersection with $B(z_2, 2^{-k_2-2})$.

Evidently we may continue this process by induction. Thus we obtain densities $\rho_{x_1,z_1}, \rho_{x_1,z_1,x_2,z_2}, \dots, \rho_{x_1,z_1,x_2,z_2,\dots,x_r,z_r,\dots}$, where $\rho_{x_1,z_1}$ is defined for $x_1 \in \tilde{D}_1 \subset D_1$, $z_1 \in \tilde{E}_1(x_1) \subset E_1$. The function $\rho_{x_1,z_1,x_2,z_2}$ is defined for $x_1 \in \tilde{D}_1 \subset D_1$, $z_1 \in \tilde{F}_1(x_1) \subset E_1, x_2 \in \tilde{D}_2(x_1, z_1) \subset D_2(x_1, z_1)$, $z_2 \in \tilde{E}_2(x_1, z_1, x_2) \subset E_2(z_1)$. Here we have shown the dependence of the sets $\tilde{E}_1$, $\tilde{E}_2$ etc. on the variables $x_1, x_2, z_1, z_2$. More generally the density $\rho_{x_1,z_1,\dots x_r,z_r}$ is defined for $x_1 \in \tilde{D}_1 \subset D_1$, $z_1 \in \tilde{F}_1(x_1) \subset E_1, \dots,$

$$x_{r-1} \in \tilde{D}_{r-1}(x_1, z_1, \dots, x_{r-2}, z_{r-2}) \subset D_{r-1}(x_1, z_1, \dots, x_{r-2}, z_{r-2})\,,$$

$$z_{r-1} \in \tilde{F}_{r-1}(x_1, z_1, \dots, x_{r-2}, z_{r-2}, x_{r-1}) \subset E_{r-1}(z_{r-2})\,,$$

$$x_r \in \tilde{D}_r(x_1, z_1, \dots, x_{r-1}, z_{r-1}) \subset D_r(x_1, z_1, \dots, x_{r-1}, z_{r-1})\,,$$

$$z_r \in \tilde{E}_r(x_1, z_1, \dots, x_{r-1}, z_{r-1}, x_r) \subset E_r(z_{r-1})\,.$$

Letting $\overline{\rho}$ be the density $\rho$ propagated to $E_1$, it is clear by analogy with (7.18) that we have

$$\begin{aligned}
\overline{\rho} \ge & \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{E}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|}\, \rho_{x_1,z_1} \\
& + \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|} \int_{\tilde{D}_2} \frac{dx_2}{\gamma(k_2)} \\
& \qquad \cdot \int_{\tilde{E}_2} \frac{dz_2}{|\tilde{E}_2 \cup \tilde{F}_2|}\, \rho_{x_1,z_1,x_2,z_2} \\
& + \cdots + \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|} \cdots \int_{\tilde{D}_r} \frac{dx_r}{\gamma(k_r)} \\
& \qquad \cdot \int_{\tilde{E}_r} \frac{dz_r}{|\tilde{E}_r \cup \tilde{F}_r|}\, \rho_{x_1,z_1,\dots,x_r,z_r} \\
& + \cdots,
\end{aligned} \tag{7.26}$$

where $\gamma(k) \sim 2^{-2k}$. Observe that the previous sum is finite. In fact $k_r$ is defined by

$$2^{k_r - k_{r-1}} \sim 2^{\lambda_0} \Big( \frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \Big)^{1/3}, \tag{7.27}$$

and one also has the inequality

$$\frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \geq \xi\, 2^{\eta'(k_{r-1} - n_0)} .$$

Since we may assume $\mathbf{b} \in L^\infty(\mathbb{R}^3)$ this last inequality cannot hold for arbitrarily large $k_{r-1}$, whence $r$ is bounded since $k_r \geq k_{r-1} + 1$. The last two inequalities imply that

$$2^{k_r - k_{r-1}} \geq 2^{\lambda_0} (\xi\, 2^{\eta'(k_{r-1} - n_0)})^{1/3} ,$$

and hence

$$k_r - n_0 \geq \Big(1 + \frac{\eta'}{3}\Big)(k_{r-1} - n_0) + 1 , \tag{7.28}$$

if we choose $\lambda_0$ to satisfy $2^{\lambda_0} \xi^{1/3} > 2$. Thus $k_r - n_0$ is increasing exponentially fast as a function of $r$. Next we shall show that the difference $k_r - k_{r-1}$ actually *decreases.* To see this observe that

$$\begin{aligned}
&\frac{1}{2^{-k_r}} \int_{B(z_r, 2^{-k_r})} V_\eta(y)\, dy \\
&\quad \leq \frac{1}{2^{-k_r}} \int_{\Gamma_{x_r, z_r, k_r} \cap B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \\
&\quad \leq 2^{-\lambda_0/4} \Big( \frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \Big)^{11/12} \\
&\quad \leq 2^{-\lambda_0/4} (\xi\, 2^{\eta'(k_{r-1} - n_0)})^{-1/12} \frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy .
\end{aligned}$$

Here we have used the definition (7.27) of $k_r$ and the condition e$^{(r)}$) corresponding to e$^{(2)}$). Hence from (7.27) we have the inequality

$$2^{3(k_{r+1} - k_r - \lambda_0)} \leq 2^{-\lambda_0/4} (\xi\, 2^{\eta'(k_{r-1} - n_0)})^{-1/12}\, 2^{3(k_r - k_{r-1} - \lambda_0)} .$$

It follows that we may choose $\lambda_0$ large enough depending only on $\xi$ such that

$$2^{k_r-k_{r-1}} \le \frac{1}{2^r}\, 2^{k_1-k_0}. \tag{7.29}$$

We shall use (7.26), (7.28), (7.29) to get a lower bound on $Av_{E_1}\overline{\rho}$. Suppose $x_1 \in \tilde{D}_1$, $z_1 \in \tilde{F}_1$, $x_2 \in \tilde{D}_2$, $z_2 \in \tilde{F}_2 \cdots x_{r-1} \in \tilde{D}_{r-1}$, $z_{r-1} \in \tilde{F}_{r-1}$, $x_r \in \tilde{D}_r$. Then in analogy to (7.12) we have from the induction hypothesis that if $z_r \in \tilde{F}_r$,

$$\begin{aligned}\int_{D_{r+1}} & \rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy \\ &\ge \int_{D_r\cap B(x_r,2^{-k_r-4})} \rho_{x_1,z_1,\ldots x_{r-1},z_{r-1}}(y)\,dy \\ &\quad\cdot \exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big) \\ &\qquad - \frac{\beta}{2^{-k_r}}\int_{\Gamma_{x_r,z_r,k_r}\cap B(z_{r-1},2^{-k_{r-1}})} V_\eta(y)\,dy\Big).\end{aligned}$$

Using now the condition e$^{(r)}$) corresponding to e$^{(1)}$) we conclude

$$\begin{aligned}\int_{D_{r+1}} & \rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy \\ &\ge \int_{D_r\cap B(x_r,2^{-k_r-4})} \rho_{x_1,z_1,\ldots x_{r-1},z_{r-1}}(y)\,dy \\ &\quad\cdot \exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big) - \beta\,2^{-\lambda_0/4}\,2^{11(k_r-k_{r-1}-\lambda_0)/4}\Big).\end{aligned} \tag{7.30}$$

Similarly if $z_r \in \tilde{E}_r$ then

$$\begin{aligned}\int_{E_1} & \rho_{x_1,z_1,\cdots,x_r,z_r}(y)\,dy \\ &\ge \int_{D_r\cap B(x_r,2^{-k_r-4})} \rho_{x_1,z_1,\ldots x_{r-1},z_{r-1}}(y)\,dy \\ &\quad\cdot \exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big) - \beta\,2^{-\lambda_0/4}\,2^{11(k_r-k_{r-1}-\lambda_0)/4}\Big).\end{aligned}$$

Consequently we have that

$$\begin{aligned}\int_{\tilde D_r}\frac{dx_r}{\gamma(k_r)}\Big(\int_{\tilde F_r}\frac{dz_r}{|\tilde E_r\cup\tilde F_r|}\int_{\tilde D_{r+1}}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\\ +\int_{\tilde E_r}\frac{dz_r}{|\tilde E_r\cup\tilde F_r|}\int_{E_1}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\Big)\\ \ge\int_{\tilde D_r}\frac{dx_r}{\gamma(k_r)}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\dots,x_{r-1},z_{r-1}}(y)\,dy\\ \cdot\exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big)-\beta\,2^{-\lambda_0/4}\,2^{11(k_r-k_{r-1}-\lambda_0)/4}\Big).\end{aligned}$$

Next observe that in analogy to (7.25) we have

$$\begin{aligned}\int_{\tilde D_r}\frac{dx_r}{\gamma(k_r)}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\dots x_{r-1},z_{r-1}}(y)\,dy\\ \ge\int_{D_r}\rho_{x_1,z_1,\dots,x_{r-1},z_{r-1}}(y)\,dy\\ \cdot\Big(1-\frac{1}{\alpha^{k_r+4-n_0}}-C\,2^{-(k_r-k_{r-1})/q'}C_2-C\,2^{-(k_r-k_{r-1})/2q'}C_2\Big).\end{aligned}$$

It is clear from (7.28) that there is a constant $a>1$ such that

$$\frac{1}{\alpha^{k_r+4-n_0}}+C\,2^{-(k_r-k_{r-1})/q'}C_2+C\,2^{-(k_r-k_{r-1})/2q'}C_2<\frac{1}{a^r}\,, \tag{7.31}$$

$r=1,2,\dots$. From the last three inequalities and (7.29) we conclude that

$$\begin{aligned}\int_{\tilde D_r}\frac{dx_r}{\gamma(k_r)}\Big(\int_{\tilde F_r}\frac{dz_r}{|\tilde E_r\cup\tilde F_r|}\int_{D_{r+1}}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\\ +\int_{\tilde E_r}\frac{dz_r}{|\tilde E_r\cup\tilde F_r|}\int_{E_1}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\Big)\\ \ge\Big(1-\frac{1}{a^r}\Big)\int_{D_r}\rho_{x_1,z_1,\dots,x_{r-1},z_{r-1}}(y)\,dy\\ \cdot\exp\Big(-\frac{C}{2^r}\,2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big)-\beta\,2^{-\lambda_0/4}\,\frac{C^{11/4}}{2^{11r/4}}\,2^{11(k_1-k_0-\lambda_0)/4}\Big),\end{aligned}$$

for some constant $C$ depending only on $\xi$. We may apply the previous inequality inductively to (7.26) to obtain

$$\int_{E_1} \overline{\rho}(y)\,dy \geq \prod_{r=1}^{\infty}\Big(1-\frac{1}{a^r}\Big)\int_{D_1}\rho(y)dy$$
$$\cdot \exp\Big(-C\,2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big)\sum_{r=1}^{\infty}\frac{1}{2^r}$$
$$-\beta\,2^{-\lambda_0/4}\,C^{11/4}\,2^{11(k_1-k_0-\lambda_0)/4}\sum_{r=1}^{\infty}\frac{1}{2^{11r/4}}\Big).$$

Now we argue exactly as in Proposition 7.1 to verify that $Av_{E_1}\overline{\rho}$ is bounded below as the induction hypothesis requires.

Just as in Proposition 7.1 we cannot define $\rho_2$ by the right hand side of (7.26) since we cannot then control the fluctuation of $\rho_2$ in terms of its average value. We proceed as in Proposition 7.1 by generalizing (7.19). Thus we prescribe the averages of the densities $\rho_{x_1,z_1}, \rho_{x_1,z_1,x_2,z_2,\ldots}$. First we modify (7.19) by insisting that

$$\begin{aligned}\int_{E_1}\rho_{x_1,z_1}(y)\,dy &= e^{-\eta_1}\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\,, \quad z_1\in\tilde{E}_1\,,\\ \int_{D_2}\rho_{x_1,z_1}(y)\,dy &= e^{-\eta_1}\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\,, \quad z_1\in\tilde{F}_1\,,\end{aligned} \tag{7.32}$$

where $\eta_1$ is a constant which satisfies

$$\eta_1 \geq C\,2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big)+\beta\,2^{-\lambda_0/4}\,2^{11(k_1-k_0-\lambda_0)/4}\,.$$

In view of (7.30) this is clearly possible. More generally, let $x_1\in\tilde{D}_1$, $z_1\in\tilde{F}_1$, $x_2\in\tilde{D}_2$, $z_2\in\tilde{F}_2,\ldots,x_r\in\tilde{D}_r$ Then from (7.30) we can insist that

$$\begin{aligned}&\int_{E_1}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\\ &\qquad = e^{-\eta_r}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\ldots,x_{r-1},z_{r-1}}(y)\,dy\,, \quad z_r\in\tilde{E}_r\,,\\ &\int_{D_{r+1}}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\\ &\qquad = e^{-\eta_r}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\ldots,x_{r-1},z_{r-1}}(y)\,dy\,, \quad z_r\in\tilde{F}_r\,,\end{aligned} \tag{7.33}$$

where $\eta_r$ is a constant satisfying the inequality

$$\eta_r \geq \frac{C}{2^r}\, 2^{k_1-k_0} \log\left(\frac{1}{c_2}\right) + \beta\, 2^{-\lambda_0/4}\, \frac{C^{11/4}}{2^{11r/4}}\, 2^{11(k_1-k_0-\lambda_0)/4},$$

for some constant $C$ depending only on $\xi$. Now we define $\rho_2$ by altering the right hand side of (7.26) to

$$\begin{aligned}\rho_2 = &\int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{E}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|}\, \rho_{x_1,z_1} \exp\Big(-\sum_{j=2}^{\infty} \eta_j\Big) \\ &+ \cdots + \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|} \cdots \int_{\tilde{D}_r} \frac{dx_r}{\gamma(k_r)} \\ &\qquad \cdot \int_{\tilde{E}_r} \frac{dz_r}{|\tilde{E}_r \cup \tilde{F}_r|}\, \rho_{x_1,z_1,\ldots,x_r,z_r} \exp\Big(-\sum_{j=r+1}^{\infty} \eta_j\Big) \\ &+ \cdots .\end{aligned}$$

We consider the problem of estimating $\|\rho_2\|_{E_1,q}$ by writing $\rho_2$ as

$$\rho_2 = \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{E}_1 \cup \tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|}\, \psi_{x_1,z_1} ,$$

in analogy with the representation (7.20) of proposition 7.1. We argue now as in Proposition 7.1, using the Minkowski inequality to obtain the bound

$$\|\rho_2\|_{E_1,q} \leq \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \Big\| \int_{\tilde{E}_1 \cup \tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|}\, \psi_{x_1,z_1} \Big\|_{E_1,q} . \tag{7.34}$$

Since $\psi_{x_1,z_1}(y) = 0$ if $|z_1 - y| > 2^{-k_1}$ we have just as in (7.22) the inequality

$$\begin{aligned}&\Big\| \int_{\tilde{E}_1 \cup \tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|}\, \psi_{x_1,z_1} \Big\|_{E_1,q} \\ &\qquad \leq \frac{C\, 2^{-2k_1/q'}}{|\tilde{E}_1 \cup \tilde{F}_1|} \Big( \int_{\tilde{E}_1 \cup \tilde{F}_1} \|\psi_{x_1,z_1}\|_q^q\, dz_1 \Big)^{1/q} ,\end{aligned} \tag{7.35}$$

where $1/q + 1/q' = 1$, $C$ is a universal constant and $\|\psi_{x_1,z_1}\|_q$ is the *unnormalized* $L^q$ norm on $E_1$. Now

$$\psi_{x_1,z_1} = \exp\Big(-\sum_{j=2}^{\infty} \eta_j\Big) \rho_{x_1,z_1} , \qquad z_1 \in \tilde{E}_1 ,$$

whence

$$
\begin{aligned}
\|\psi_{x_1,z_1}\|_q^q &= \exp\Big(-q\sum_{j=2}^{\infty}\eta_j\Big)\|\rho_{x_1,z_1}\|_q^q \\
&\le \exp\Big(-q\sum_{j=2}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\mathrm{perturb}}^q\Big(\int_{E_1}\rho_{x_1,z_1}(y)\,dy\Big)^q \\
&= \exp\Big(-q\sum_{j=1}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\mathrm{perturb}}^q \\
&\quad\cdot\Big(\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\Big)^q,
\end{aligned}
$$

by (7.32) where $C$ is a universal constant. Let us assume now that for $z_1 \in \tilde{F}_1$ there is a universal constant $C$ such that

$$
\begin{aligned}
\|\psi_{x_1,z_1}\|_q^q &\le \exp\Big(-q\sum_{j=2}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\mathrm{perturb}}^q \\
&\quad\cdot\Big(\int_{D_2}\rho_{x_1,z_1}(y)\,dy\Big)^q \\
&= \exp\Big(-q\sum_{j=1}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\mathrm{perturb}}^q \\
&\quad\cdot\Big(\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\Big)^q.
\end{aligned}
\tag{7.36}
$$

Then we have that

$$
\begin{aligned}
&\Big(\int_{\tilde{E}_1\cup\tilde{F}_1}\|\psi_{x_1,z_1}\|_q^q\,dz_1\Big)^{1/q} \\
&\qquad\le |\tilde{E}_1\cup\tilde{F}_1|^{1/q}\exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big) \\
&\qquad\quad\cdot 2^{2k_1/q'}\,C\,C_{2,\mathrm{perturb}}\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\,.
\end{aligned}
$$

It follows now from (7.34), that

$$
\|\rho_2\|_q \le \int_{\tilde{D}_1}\frac{1}{|\tilde{E}_1\cup\tilde{F}_1|^{1/q'}}\exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big)\frac{dx_1}{\gamma(k_1)}
$$

$$\begin{aligned}
&\cdot C\, C_{2,\text{perturb}} \int_{D_1 \cap B(x_1, 2^{-k_1-4})} \rho(y)\, dy \\
&\le \frac{2\, C\, C_{2,\text{perturb}}}{|D_1|^{1/q'}} \exp\Big(-\sum_{j=1}^{\infty} \eta_j\Big) \int_{D_1} \rho(y)\, dy\,,
\end{aligned} \tag{7.37}$$

by making the ratio of $|D_1|$ to $|\tilde{E}_1 \cup \tilde{F}_1|$ close to unity. This can be arranged in view of d$^{(1)}$), e$^{(1)}$) by choosing $\lambda_0$ large. It follows from this last inequality that

$$\|\rho_2\|_{E_1,q} \le C\, C_{2,\text{perturb}} \exp\Big(-\sum_{j=1}^{\infty} \eta_j\Big) \mathrm{Av}_{D_1} \rho\,, \tag{7.38}$$

where $C$ is a universal constant.

We shall show now that the inequality (7.38) holds in general. To do this we shall prove by induction that (7.36) holds. Thus for $x_1 \in \tilde{D}_1$, $z_1 \in \tilde{F}_1, \ldots, x_r \in \tilde{D}_r$, $z_r \in \tilde{E}_r$,

$$\psi_{x_1,z_1,\ldots,x_r,z_r} = \exp\Big(-\sum_{j=r+1}^{\infty} \eta_j\Big) \rho_{x_1,z_1,\ldots,x_r,z_r}\,.$$

For $x_1 \in \tilde{D}_1$, $z_1 \in \tilde{F}_1, \ldots, x_r \in \tilde{D}_r$, $z_r \in \tilde{F}_r$,

$$\begin{aligned}
\psi_{x_1,z_1,\ldots,x_r,z_r} = &\int_{\tilde{D}_{r+1}} \frac{dx_{r+1}}{\gamma(k_{r+1})} \\
&\cdot \int_{\tilde{E}_{r+1} \cup \tilde{F}_{r+1}} \frac{dz_{r+1}}{|\tilde{E}_{r+1} \cup \tilde{F}_{r+1}|} \psi_{x_1,z_1,\ldots,x_{r+1},z_{r+1}}\,.
\end{aligned} \tag{7.39}$$

We now make the inductive assumption that for $z_{r+1} \in \tilde{F}_{r+1}$,

$$\begin{aligned}
\|\psi_{x_1,z_1,\ldots x_{r+1},z_{r+1}}\|_q^q \le &\exp\Big(-q \sum_{j=r+2}^{\infty} \eta_j\Big) \gamma(k_{r+1})^{-(q-1)}\, C^q\, C_{2,\text{perturb}}^q \\
&\cdot \Big(\int_{D_{r+2}} \rho_{x_1,z_1,\ldots,x_{r+1},z_{r+1}}(y)\, dy\Big)^q \Big(1 + \frac{1}{a^{r+1}}\Big)\,,
\end{aligned} \tag{7.40}$$

where $a > 1$ is some number to be specified and $C$ is universal. To verify the assumption for $\psi_{x_1,z_1,\ldots,x_r,z_r}$, $z_r \in \tilde{F}_r$, we argue as before using (7.39). Thus

$$
\begin{aligned}
\|\psi_{x_1,z_1,\dots,x_r,z_r}\|_q^q \le \Big(\int_{\tilde{D}_{r+1}} \frac{dx_{r+1}}{\gamma(k_{r+1})}
&\cdot \Big\| \int_{\tilde{E}_{r+1}\cup\tilde{F}_{r+1}} \frac{dz_{r+1}}{|\tilde{E}_{r+1}\cup\tilde{F}_{r+1}|}\,\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\Big\|_q\Big)^q \\
\le \Big(\int_{\tilde{D}_{r+1}} \frac{dx_{r+1}}{\gamma(k_{r+1})^{1/q}\,|\tilde{E}_{r+1}\cup\tilde{F}_{r+1}|}
&\cdot \Big(\int_{\tilde{E}_{r+1}\cup\tilde{F}_{r+1}} \|\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q\,dz_{r+1}\Big)^{1/q}\Big)^q.
\end{aligned}
$$

Now for $z_{r+1}\in\tilde{E}_{r+1}$ we have

$$
\begin{aligned}
\|\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q &\le \exp\Big(-q\sum_{j=r+2}^{\infty}\eta_j\Big)\|\rho_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q \\
&\le \exp\Big(-q\sum_{j=r+2}^{\infty}\eta_j\Big)\gamma(k_{r+1})^{-(q-1)}\,C_{2,\text{perturb}}^q \\
&\quad\cdot\Big(\int_{E_1}\rho_{x_1,z_1,\dots,x_{r+1},z_{r+1}}(y)\,dy\Big)^q \\
&= \exp\Big(-q\sum_{j=r+1}^{\infty}\eta_j\Big)\gamma(k_{r+1})^{-(q-1)}\,C_{2,\text{perturb}}^q \\
&\quad\cdot\Big(\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\Big)^q,
\end{aligned}
$$

by (7.33) and Theorem 6.1. From the induction assumption (7.40) and (7.33) we have that if $z_{r+1}\in\tilde{F}_{r+1}$ then

$$
\begin{aligned}
\|\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q &\le \exp\Big(-q\sum_{j=r+1}^{\infty}\eta_j\Big)\gamma(k_{r+1})^{-(q-1)}\,C^q\,C_{2,\text{perturb}}^q \\
&\quad\cdot\Big(\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\Big)^q \\
&\quad\cdot\Big(1+\frac{1}{a^{r+1}}\Big).
\end{aligned}
$$

It follows then from these last three inequalities that

$$\|\psi_{x_1,z_1,\ldots,x_r,z_r}\|_q^q \le \exp\Big(-q\sum_{j=r+1}^{\infty}\eta_j\Big)C^q\,C^q_{2,\mathrm{perturb}}\Big(1+\frac{1}{a^{r+1}}\Big) \cdot\Big(\int_{\tilde D_{r+1}}\frac{dx_{r+1}}{\gamma(k_{r+1})\,|\tilde E_{r+1}\cup\tilde F_{r+1}|^{1/q'}} \cdot\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\Big)^q .$$

We have now that

$$\int_{\tilde D_{r+1}}\frac{dx_{r+1}}{\gamma(k_{r+1})\,|\tilde E_{r+1}\cup\tilde F_{r+1}|^{1/q'}} \cdot\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy \le C_r\,\gamma(k_r)^{-1/q'}\int_{D_{r+1}}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\,,$$

where we need

$$C_r \ge \Big(\frac{\gamma(k_r)}{|\tilde E_{r+1}\cup\tilde F_{r+1}|}\Big)^{1/q'}, \qquad x_{r+1}\in\tilde D_{r+1}\,.$$

From d$^{(r+1)}$), e$^{(r+1)}$) it is clear we can choose $a>1$ so that

$$\Big(\frac{|E_{r+1}|}{|\tilde E_{r+1}\cup\tilde F_{r+1}|}\Big)^{q-1} \le \frac{1+\dfrac{1}{a^r}}{1+\dfrac{1}{a^{r+1}}} = C_r^q\,.$$

We conclude that

$$\|\psi_{x_1,z_1,\ldots,x_r,z_r}\|_q^q \le \exp\Big(-q\sum_{j=r+1}^{\infty}\eta_j\Big)\gamma(k_r)^{-(q-1)}C^q\,C^q_{2,\mathrm{perturb}} \cdot\Big(\int_{D_{r+1}}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\Big)^q\Big(1+\frac{1}{a^r}\Big),$$

Thus we have verified the induction hypothesis (7.40) at the next level down. Setting $r=1$ in this last inequality yields (7.36).

The proof of the theorem will be complete if we can show that

$$\mathrm{Av}_{E_1}\rho_2 \geq c\exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big)\mathrm{Av}_{D_1}\rho\,, \tag{7.41}$$

where the constant $c$ is independent of the constant $C_2$ in the statement of Theorem 7.1. We can prove this in an exactly similar way to the proof of (7.37). Our induction assumption here is that for $z_{r+1}\in\tilde{F}_{r+1}$,

$$\int_{E_1}\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}(y)\,dy$$
$$\geq \exp\Big(-\sum_{j=r+2}^{\infty}\eta_j\Big)\Big(1-\frac{1}{a^{r+1}}\Big)\int_{D_{r+1}}\rho_{x_1,z_1,\dots,x_{r+1},z_{r+1}}(y)\,dy\,.$$

If we use now (7.39), (7.33) and (7.31) we can verify the induction hypothesis on the next level down. The fluctuation bound on $\rho_2$ in Theorem 7.1 follows from (7.37) and (7.41) if we choose the constant $C_2$ in the induction assumption sufficiently large.

We will use Theorem 7.1 to obtain estimates on the exit probability from a spherical shell. To do this we use the function $a_{\varepsilon,n,s,p}(x)$ defined in (1.7) in terms of the number of nonperturbative cubes inside the sphere of radius $2^{-n}$ centered at $x$. Let us suppose now that $a_{\varepsilon,n-1,s,p}(0)\gg 1$ so that we are in the nonperturbative situation and $a_{\varepsilon,n-1,s,p}(0)\sim 2^{n_0-n}$, $n_0>n$. If we define $V_\eta$ by (6.2) then we have:

**Lemma 7.3.** *Suppose $0<\eta<s-2$. Then there is a constant $C_\eta$ depending only on $\eta$ such that*

$$\frac{1}{2^{-n}}\int_{B(0,2^{-n})}V_\eta(x)\,dx \leq C_\eta\, a_{\varepsilon,n-1,s,p}(0)^2\,.$$

Proof. Clearly there is a universal constant $C$ such that

$$\int_{B(0,2^{-n})}V_\eta(x)\,dx \leq \sum_{m=n_0}^{\infty}2^{\eta(m-n_0)}\,2^{-m}N_m\ ,$$

where $N_m$ is the number of dyadic cubes with side of length $2^{-m}$ contained in the ball $B(0,2^{-n+1})$ such that (6.1) holds. In view of the fact that

$$N_m\leq 2^{(m-n)(3-s)}\,a_{\varepsilon,n-1,s,p}(0)^s\sim 2^{(m-n)(3-s)}\,2^{(n_0-n)s}\ ,$$

it follows that

$$\int_{B(0,2^{-n})} V_\eta(x)\,dx \le 2^{-n}\,2^{2(n_0-n)} \sum_{m=n_0}^{\infty} 2^{(m-n_0)(\eta+2-s)}$$
$$\le C_\eta\,2^{-n}\,a_{\varepsilon,n-1,s,p}(0)^s,$$

since $\eta < s-2$.

**Theorem 7.2.** *Let $f$ be a density on the sphere $|x| = 2^{-n}$. Suppose the drift process started on $|x| = 2^{-n}$ with density $f$ induces a density $\overline{f}_2$ on the sphere $|x| = 2^{-n+1}$ when it exits the spherical shell $\{2^{-n-1} < |y| < 2^{-n+1}\}$. Then there exist constants $C_1$, $C_2$ such that if $1 < q < \infty$ and $\|f\|_q \le C_1\,Av\,f$ there is a density $f_2$ on $|x| = 2^{-n+1}$ with $0 \le f_2 \le \overline{f}_2$ such that $\|f_2\|_q \le C_1\,\mathrm{Av}\,f_2$ and*

$$\mathrm{Av}\,f_2 \ge \exp\left(-C_2\,a_{\varepsilon,n-2,s,p}(0)\right)\mathrm{Av}\,f\,,$$

*provided $a_{\varepsilon,n-2,s,p}(0) \ge 1$. The $L^q$ norm here is normalized so that $\|\mathbf{1}\|_q = 1$.*

Proof. For $|x| = 2^{-n}$, $|z| = 2^{-n+1}$ we consider cylinders $\Gamma_{x,z,k}$ with $k$ defined by

$$2^{k-n} \sim 2^{\lambda_0}\,a_{\varepsilon,n-2,s,p}(0)\,.$$

Defining $n_0$ by $a_{\varepsilon,n-2,s,p}(0) \sim 2^{n_0-n}$, it follows that $k = \lambda_0 + n_0$. Letting $D_1 = \{|x| = 2^{-n}\}$, $E_1 = \{|x| = 2^{-n+1}\}$ it follows from Lemma 7.1, 7.3 that

$$\frac{1}{|D_1|}\int_{D_1}\frac{dx}{|E_1|}\int_{E_1}\frac{dz}{2^{-k}}\int_{\Gamma_{x,z,k}\cap B(0,2^{-n+2})} V_\eta(y)\,dy$$
$$\le C\,2^{n-k}\Big(\frac{1}{2^{-n}}\int_{B(0,2^{-n+2})} V_\eta(y)\,dy\Big)$$
$$\le C\,2^{n-k}\,2^{2(n_0-n)} = C\,2^{-2\lambda_0}\,2^{k-n}\,.$$

We follow now the lines of the proof of Proposition 7.1 and use Theorem 7.1 to propagate the drift process through the cylinder. Let $D_1'$ be the set of $x \in D_1$ such that

$$\frac{1}{|E_1|}\int_{E_1}\frac{dz}{2^{-k}}\int_{\Gamma_{x,z,k}\cap B(0,2^{-n+2})} V_\eta(y)\,dy \le 2^{-\lambda_0/2}\,2^{(k-n-\lambda_0)}\,. \tag{7.42}$$

It follows by Chebyshev from the last two inequalities that

$$\frac{|D_1'|}{|D_1|} > 1 - C\,2^{-\lambda_0/2}\,.$$

Next we have by the argument of Lemma 7.2 that

$$\int_{D_1\backslash D_1'} \frac{dx}{2^{-2k}} \int_{D_1\cap B(x,2^{-k-4})} f(y)\,dy \le (C\,2^{-\lambda_0/2})^{1/q'}\,C_1\,\mathrm{Av}_{D_1} f\,.$$

For $x \in D_1$ let $\|f\|_{x,k}$ be the $L^q$ norm on $D_1 \cap B(x, 2^{-k-4})$ normalized so that $\|\mathbf{1}\|_{x,k} = 1$. Let $D_1''$ be the set of $x \in D_1$ such that

$$\|f\|_{x,k} \le C_{2,\mathrm{thm}}\,\alpha^{k-n_0}\,\mathrm{Av}_{x,k} f\,,$$

where $C_{2,\mathrm{thm}}$ denotes the constant $C_2$ of Theorem 7.1 and $\alpha$ is the same constant as in the statement of the theorem. We choose $\lambda_0$ sufficiently large so that $2\,C_1 < C_{2,\mathrm{thm}}\alpha^{k-n_0}$. Since $\lambda_0 = k - n_0$ this is certainly possible. Setting $\tilde{D}_1 = D_1' \cap D_1''$, we conclude on taking $\gamma = 1/2$ in Lemma 7.2 that

$$\int_{\tilde{D}_1} \frac{dx}{2^{-2k}} \int_{D_1\cap B(x,2^{-k-4})} f(y)\,dy \ge \frac{1}{4}\,\mathrm{Av}_{D_1} f\,,$$

provided $\lambda_0$ is sufficiently large.

Next for $x \in \tilde{D}_1$ let $\tilde{E}_1$ be the set of $z \in E_1$ such that

$$\frac{1}{2^{-k}} \int_{\Gamma_{x,z,k}\cap B(0,2^{-n+2})} V_\eta(y)\,dy \le 2^{-\lambda_0/4}\,2^{(k-n-\lambda_0)}\,.$$

It follows from (7.42) that

$$\frac{|\tilde{E}_1|}{|E_1|} > 1 - 2^{-\lambda_0/4}\,.$$

Now we use Theorem 7.1 to propagate the density $f$ restricted to $D_1 \cap B(x, 2^{-k-4})$ through the cylinder $\Gamma_{x,z,k}$ to $E_1 \cap B(z, 2^{-k-4})$. Let $f_{x,z}$ be this propagated density. In view of (7.43) we can arrange for this density to satisfy

$$\int_{E_1\cap B(z,2^{-k-4})} f_{x,z}(y)\,dy = e^{-\eta} \int_{D_1\cap B(x,2^{-k-4})} f(y)\,dy\,,$$

where $\eta = C\,2^{k-n}$ for some constant $C$. Theorem 7.1 also yields an estimate on the fluctuation of $f_{x,z}$. Thus

$$\|f_{x,z}\|_q \le C_{2,\mathrm{thm}}\,\mathrm{Av}\,f_{x,z}\ .$$

The propagated density $f_2$ is defined now by

$$f_2 = \int_{\tilde D_1} \frac{dx}{2^{-2k}} \int_{\tilde E_1} f_{x,z}\,dz\ .$$

We can argue now exactly as in Proposition 7.1 to conclude that

$$\mathrm{Av}_{E_1} f_2 \ge \exp\left(-C\,2^{k-n}\right) \mathrm{Av}_{D_1} f\ ,$$

$$\|f_2\|_q \le 2\,C_{2,\mathrm{thm}}\,\mathrm{Av}_{E_1} f_2\ .$$

The result follows by taking $C_1 = 2\,C_{2,\mathrm{thm}}$.

## 8. Proof of Theorem 1.3.

Here we follow closely the argument of [5, Section 6]. In fact we shall repeat the entire argument of [5, Section 6] with the function $a_{n,p}(x)$ given in (1.4) replaced by the function $a_{\varepsilon,n,s,p}(x)$, $s > 2$ defined in (1.7). Our first lemma is identical to [5, Lemma 6.1]. In the following we shall denote the function $a_{\varepsilon,n,s,p}$ simply by $a_n$.

**Lemma 8.1.** *Let $Q_0$ be a cube containing $\Omega_R$ with side of length $2^{-n_0} \sim R$. Suppose for some integer $m \ge 0$, the drift $\mathbf{b}$ satisfies the inequality*

$$\int_Q |\mathbf{b}|^p\,dx \le \varepsilon^p\,|Q|^{1-p/3}\,, \tag{8.1}$$

*on all dyadic subcubes $Q \subset Q_0$ with side of length $2^{-n}$, $n \ge m + n_0$. Let $u$ be the solution of the Dirichlet problem* (1.1), (1.2). *Then if $\varepsilon$ is sufficiently small, depending on $p > 1$, $s > 2$, there exist constants $C_1$ depending only on $p, q, r$ and $C_2$ depending only on $p > 1$, $s > 2$, such that*

$$\|u\|_\infty \le C_1 R^{2-3/q}\,\|f\|_{q,r}\,\sup_{x\in\Omega_R}\,\exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big)\,.$$

PROOF. We consider the function $\xi(x)$, $x \in \Omega_R$, given by

$$\xi(x) = E_x\Big[\exp\Big( -\frac{1}{\mu}\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt\Big)\Big],$$

where $\tau$ is the time for the drift process $X_{\mathbf{b}}(t)$ to exit $Q_0$. By (8.1) the ball $B(x, 2^{-m})$ is perturbative for the drift $\mathbf{b}$. We need now an obvious generalization of Theorem 7.2. Thus let $\rho_n$ be a density on the sphere $|x-y| = 2^{-n}$ and $\overline{\rho}_{n-1}$ be the density induced on the sphere $|x-y| = 2^{-n+1}$ by paths of the drift process which satisfy

$$\int_0^{\tau_{n-1}} |f|(X_{\mathbf{b}}(t))\,dt \le C\,2^{-n(2-3/q-\delta)}\,2^{-n_0\delta}\,\|f\|_{q,r}\,, \tag{8.2}$$

where $\tau_{n-1}$ is the first hitting time on $|y-x| = 2^{-n+1}$, $C$ is a positive constant and $0 < \delta < 2 - 3/q$. Suppose now that $a_{n-2}(x) \ge \eta > 0$. It follows from Section 7 that for any $t$, $1 < t < \infty$, $C$ can be chosen sufficiently large so that $\|\rho_n\|_t \le C\text{Av}\,\rho_n$ implies that $\overline{\rho}_{n-1} \ge \rho_{n-1} \ge 0$, $\|\rho_{n-1}\|_t \le C\text{Av}\,\rho_{n-1}$ and

$$\text{Av}\,\rho_{n-1} \ge \text{Av}\,\rho_n \exp\left(-K_\eta a_{n-2}(x)\right), \tag{8.3}$$

where $K_\eta$ is a constant depending on $\eta$. If $a_{n-2}(x) \le \eta$ and $\eta$ is sufficiently small then we are in the perturbative situation described in Section 5. Now $\rho_{n-1}$ is the density induced on $|x-y| = 2^{-n+1}$ by paths which avoid nonperturbative cubes and such that (8.2) holds. By examining the proofs of Proposition 5.2, Proposition 5.6 and Lemma 7.3 we see that on choosing q sufficiently close to 2, depending on $s > 2$, one has

$$\text{Av}\,\rho_{n-1} \ge (1 - \nu\,2^{-(n-n_0)\delta})\,(1 - C_1\,a_{n-2}(x))\,\text{Av}\,\rho_n\,, \tag{8.4}$$

where the constant $\nu > 0$ can be made arbitrarily small and $C_1$ is independent of $\eta$. It follows from (8.2), (8.3), (8.4) that

$$\xi(x) \ge \frac{3}{4}\exp\Big( -\frac{1}{\mu}\sum_{n=n_0}^{\infty} C\,2^{-n(2-3/q-\delta)}\,2^{-n_0\delta}\,\|f\|_{q,r}\Big)$$
$$\cdot \exp\Big( -K\sum_{j=0}^{m} a_{n_0+j}(x)\Big).$$

If we choose now $\mu \sim 2^{-n_0(2-3/q)}\,\|f\|_{q,r}$ we can conclude that

$$\xi(x) \geq \frac{1}{2}\exp\Big(-K\sum_{j=0}^{m} a_{n_0+j}(x)\Big)\,.$$

The result follows from this last inequality and [4, Lemma 5.1].

Next we consider the analogue of [5, Lemma 6.2].

**Lemma 8.2.** *For $n \in \mathbb{Z}$ let $\Omega_n$ be the spherical shell $\Omega_n = \{x \in \mathbb{R}^3 : 2^{-n-1} < |x| < 2^{-n+1}\}$. For $x \in \Omega_n$ let $P_x$ be the probability that the drift process started at $x$ exits $\Omega_n$ through the sphere $|y| = 2^{-n+1}$. Let $\delta$ be a number satisfying $0 < \delta < 2/3$. Then if $|x| = 2^{-n}$ there is a constant $C$ depending only on $\delta < 2/3$, $p > 1$, $s > 2$ and $\varepsilon$ such that*

$$P_x \geq \delta \exp\left(-C\, a_{n-2}(0)\right).$$

Proof. Observe that if $\mathbf{b} \equiv 0$ then

$$P_x = \frac{4}{3}\Big(1 - \frac{2^{-n-1}}{|x|}\Big)\,.$$

Hence if $|x| = 2^{-n}$ then $P_x = 2/3$. It follows that for fixed $x_0$ with $|x_0| = 2^{-n}$ then

$$P_x \geq \frac{1}{2}\Big(\delta + \frac{2}{3}\Big)\,, \tag{8.5}$$

for $x$ in the set

$$B = \Big\{x : |x - x_0| < \frac{2^{-n}(2 - 3\,\delta)}{3\,(2-\delta)}\Big\}\,. \tag{8.6}$$

Consider next the case when $\mathbf{b} \not\equiv 0$, and let us first assume that we are in the perturbative case so that $a_{n-2}(0) < \eta$ and $\eta$ is small. For $x \in \mathbb{R}^3$, $m, k$ integers with $k \geq m$ let $N_{m,k}(x)$ be the number of dyadic cubes with side of length $2^{-k}$ contained in the ball $\{y : |x-y| < 2^{-n}\}$ which satisfy (6.1). Then from the definition of $a_{n-2}(0)$ we have that

$$N_{n-2,m}(0) \leq \eta^s\, 2^{(m+2-n)(3-s)}\,, \qquad m \geq n-2\,. \tag{8.7}$$

Let $X(t)$ be an arbitrary continuous path with $X(0) = x_0$, $X(t) \in B$, $t < \tau$, and $X(\tau) \in \partial B$. Let $s'$ satisfy $2 < s' < s$, We claim that there are constants $C_1, \beta$ depending only on $s, s'$, such that $C_1 > 0$, $0 < \beta < 1$, and a point $x = X(t)$ for some $t$, $0 \le t \le \tau$, satisfying

$$(8.8.) \qquad N_{m,k}(x) \le C_1\, \eta^s \beta^{m-n}\, 2^{(k-m)(3-s')}\,, \qquad k \ge m \ge n\,.$$

To prove this inequality we assume its negation and obtain a contradiction. Thus for each $x$ on the path $X$ there exists integers $m(x)$, $k(x)$ such that (8.8) is violated when $m = m(x)$, $k = k(x)$. Now the balls $B(x, 2^{-m(x)})$ form an open cover for the compact set $X$. Hence there exists a finite subcover $\Gamma = \{D_j : 1 \le j \le N\}$ for some integer $N$. For each integer $m \ge n$, let $\Gamma_m$ be the subset of $\Gamma$ consisting of balls with radius $2^{-m}$. Let $D$ be an arbitrary ball and $\tilde{D}$ the ball concentric with $D$ but with three times the radius. Then there exists a subset $\tilde{\Gamma}_m \subset \Gamma_m$ of disjoint balls such that

$$\bigcup_{D\in\Gamma_m} D \subset \bigcup_{D\in\tilde{\Gamma}_m} \tilde{D}\,.$$

For $k \ge m$ let $\tilde{\Gamma}_{m,k}$ be the subset of $\tilde{\Gamma}_m$ consisting of balls $D = B(x, 2^{-m})$ such that $k(x) = k$. Since the balls in $\Gamma_{m,k}$ are disjoint it follows from (8.8), (8.7) that

$$|\tilde{\Gamma}_{m,k}|\, C_1\, \eta^s \beta^{m-n}\, 2^{(k-m)(3-s')} \le \eta^s\, 2^{(k+2-n)(3-s)}\,,$$

whence

$$|\tilde{\Gamma}_m| \le \sum_{k=m}^{\infty} |\tilde{\Gamma}_{m,k}| \le C\, C_1^{-1} \beta^{-(m-n)}\, 2^{(m-n)(3-s)}\,,$$

for some constant $C$ depending on $s' < s$. We choose $\beta < 1$ now so that

$$\frac{2^{3-s}}{\beta} < 2\,.$$

This is possible since $s > 2$. It is clear that for any point $x$ on the path $X(t)$, $0 \le t \le \tau$, one must have the inequality

$$|x - x_0| \le 6 \sum_{m=n}^{\infty} 2^{-m}\, |\tilde{\Gamma}_m| \le A\, \frac{2^{-n}}{C_1}\,,$$

for some constant $A$ depending on $s, s'$. Since $X(\tau)$ lies on the boundary of the ball $B$ in (8.6) this last inequality is violated for $x = X(\tau)$ provided $C_1$ is chosen sufficiently large. Hence we have a contradiction.

We may therefore assume that an $x = X(t)$ exists such that (8.8) holds. Now from Section 5 it follows that the Brownian particle at $x$ can be propagated to the sphere of radius $2^{-n}$ centered at $x$ with a loss of density which can be made arbitrarily small as $\eta \longrightarrow 0$. The density on the sphere of radius $2^{-n}$ is approximately uniform. Again from Section 5 the probability of exiting the outer sphere $\{|y| = 2^{-n+1}\}$ starting from $\partial B(x, 2^{-n})$ with approximately uniform density can be made arbitrarily close to the probability for Brownian motion $\mathbf{b} \equiv 0$ by choosing $\eta$ sufficiently small. In view of (8.5) the result of the lemma follows if $a_{n-2}(0) < \eta$ and $\eta$ is sufficiently small.

Next we turn to the nonperturbative case. We can assume now that there exists $\eta > 0$ and $a_{n-2}(0) \geq \eta$. Let $n_1$ be the unique integer such that

$$2^{n_1+1} > \frac{a_{n-2}(0)}{\eta} \geq 2^{n_1} .$$

Hence, analogously to (8.7) we have

$$N_{n-2,m}(0) \leq \eta^s \, 2^s \, 2^{sn_1} \, 2^{(m+2-n)(3-s)} , \qquad m \geq n-2 . \tag{8.9}$$

We shall show, in analogy to (8.8), that there exists $x = X(t)$ for some $t$, $0 \leq t \leq \tau$, satisfying

$$N_{m,k}(x) \leq C_1 \, \eta^s \, 2^{s' n_1} \beta^{m-n} \, 2^{(k-m)(3-s')} , \tag{8.10}$$

with $k \geq m + n_1$, $m \geq n$. To see this we argue exactly as in the perturbative case. Thus from (8.10), (8.9) the cardinality of the set $\tilde{\Gamma}_{m,k}$ satisfies

$$|\tilde{\Gamma}_{m,k}| \, C_1 \, \eta^s \, 2^{s' n_1} \beta^{m-n} 2^{(k-m)(3-s')} \leq \eta^s \, 2^s \, 2^{sn_1} \, 2^{(k+2-n)(3-s)} ,$$

whence

$$|\tilde{\Gamma}_m| \leq \sum_{k=m+n_1}^{\infty} |\tilde{\Gamma}_{m,k}| \leq C \, C_1^{-1} \beta^{-(m-n)} \, 2^{(m-n)(3-s)} ,$$

for some constant $C$ depending on $s' < s$.

Now for $x$ which satisfies (8.10) we see from the argument of Lemma 7.3 and Theorem 7.2 that the Brownian particle at $x$ can be propagated to the sphere of radius $2^{-n}$ centered at $x$ with a decrease in density by a factor

$$\exp\Big(-A\sum_{m=n}^{\infty}\beta^{m-n}\,2^{n_1}\Big),$$

for some constant $A$. Now this density on the sphere of radius $2^{-n}$ centered at $x$ can be propagated to the outer sphere $\{|y|=2^{-n+1}\}$ with a further decrease in density by at most a factor

$$\exp\left(-A'\,2^{n_1}\right),$$

for some constant $A'$. Hence the total decrease in density from $x_0$ to the outer sphere is by a factor

$$\exp\left(-A''\,2^{n_1}\right),$$

for some constant $A''$.

The proof of the theorem follows now exactly as in [5, Section 6]. To prove (1.9) we need to prove the analogue of [5, Proposition 6.6].

**Proposition 8.1.** *Suppose $\eta>0$. Then there exists a constant $C$ depending on $\eta,\rho$ and a universal constant $c$ such that*

$$\sum_{n=-\infty}^{\infty}a_n(x)\,H\big(a_n(x)-\eta\big)\le C\,N_{c\,\varepsilon}(\mathbf{b}),$$

*where $H(t)$ is the Heaviside function,*

$$H(t)=\begin{cases}1, & \textit{if } t>0,\\ 0, & \textit{if } t<0.\end{cases}$$

Proof. We have

$$\sum_{n=-\infty}^{\infty}a_n(x)\,H(a_n(x)-\eta)\le\frac{1}{\eta^{s-1}}\sum_{n=-\infty}^{\infty}a_n(x)^s.$$

Letting $N_{m,n}(x)$ be the number of non perturbative dyadic cubes with side of length $2^{-m}$, $m\ge n$, contained in the ball $|x-y|<2^{-n}$ we have from the definition of $a_n(x)$ that

$$\sum_{n=-\infty}^{\infty}a_n(x)^s\le\sum_{n=-\infty}^{\infty}\sum_{m=n}^{\infty}\frac{N_{m,n}(x)}{2^{(m-n)(3-s)}}=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{m}\frac{N_{m,n}(x)}{2^{(m-n)(3-s)}}.$$

Let $N_m$ be the number of nonperturbative cubes with side of length $2^{-m}$ in $\mathbb{R}^3$. Then it is clear there is a constant $C$ such that

$$\sum_{n=-\infty}^{m} \frac{N_{m,n}(x)}{2^{(m-n)(3-s)}} \leq C\, N_m \,.$$

It follows that

$$\sum_{n=-\infty}^{\infty} a_n(x)^s \leq C \sum_{m=-\infty}^{\infty} N_m = N\,,$$

where $N$ is the total number of nonperturbative cubes in $\mathbb{R}^3$. Finally, we use the result of Fefferman [9] that $N \leq C\, N_{c\,\varepsilon}(\mathbf{b})$ for suitable universal constants $C$, $c$.

## Appendix A. Brownian motion confined to a cylinder.

In this section we give a new proof of [4, Theorem 1.1.a)]. To do this we shall use a result concerning Brownian motion confined to a cylinder. For $\lambda > 0$ let $D_\lambda$ be the disc of radius $\lambda$ in $\mathbb{R}^2$,

$$D_\lambda = \{x = (x_1, x_2) : r^2 = x_1^2 + x_2^2 < \lambda^2\}\,.$$

Then for $m > 0$ the set

$$D_\lambda \times (-m\lambda, m\lambda) = \{x = (x_1, x_2, x_3) : (x_1, x_2) \in D_\lambda,\ x_3 \in (-m\lambda, m\lambda)\}$$

is a cylinder in $\mathbb{R}^3$. We are interested in studying Brownian motion confined to the cylinder when $m \gg 1$. In particular let $X(t)$ be Brownian motion in $\mathbb{R}^3$ started at the origin and $\tau$ be the first exit time from the cylinder. We shall consider Brownian motion under the constraint that $|X(\tau)_3| = m\lambda$. Thus the paths must exit the cylinder through one of the discs $D_\lambda \times \{m\lambda\}$ or $D_\lambda \times \{-m\lambda\}$. For $m \gg 1$ this is an unlikely event. Hence Brownian motion under this constraint behaves very differently to the standard Brownian motion. In fact it appears to behave ballistically on length scales much larger than $\lambda$. As a consequence one has

$$\text{(A.1)} \qquad E\,[\tau : |X(\tau)_3| = m\lambda] \sim m\lambda^2\,, \qquad m \gg 1\,.$$

One of our main goals here will be to prove (A.1). We shall need something more general to prove [4, Theorem 1.1.a)]. In fact we have the following:

**Theorem A.1.** *Let $V$ be a nonnegative potential on $D_\lambda \times (-m\lambda, m\lambda)$. Then if $m \geq 1$ there is a universal constant $C$ such that if $X(0)$ is uniformly distributed on a crosssection $D_\lambda \times \{\xi\}$,*

$$E\Big[\int_0^\tau V(X(t))\,dt : |X(\tau)_3| = m\lambda\Big] \leq \frac{C}{\lambda}\int_{D_\lambda\times(-m\lambda,m\lambda)} V(y)\,dy\,.$$

We recall [4, Theorem 1.1.a)]. Thus let $V$ be a nonnegative potential on the ball $\Omega_R$ and for $n = 0, \pm 1, \pm 2, \ldots, x \in \Omega_R$ let $a_n(x)$ be the functions

$$a_n(x) = 2^n \int_{|x-y|<2^{-n}} V(y)\,dy\,.$$

[4, Theorem 1.1.a)] is then given by:

**Theorem A.2.** *For $x \in \Omega_R$ and Brownian motion $X(t)$ started at $x$, let $\tau$ be the first exit time from $\Omega_R$. Then there is a universal constant $C > 0$ such that*

$$E_x\Big[\exp\Big(-\int_0^\tau V(X(t))\,dt\Big)\Big] \geq \exp\Big(-C\sum_{n=n_0}^\infty \min\big\{a_n(x), a_n(x)^{1/2}\big\}\Big)\,,$$

*$x \in \Omega_R$, where $n_0$ is the unique integer which satisfies the inequality $2R < 2^{-n_0} \leq 4R$.*

PROOF. We define a subset $\mathcal{S}$ of Brownian paths started at $x$. For $n \geq n_0$ define $m_n$, $\lambda_n$ by

$$m_n = a_n(x)^{1/2}\,, \qquad m_n\lambda_n = 2^{-n}\,.$$

For a Brownian path $X(t)$ started at $x$ let $\tau_n$ be the first hitting time on the sphere $|x - y| = 2^{-n}$. Thus $\tau_{n+1} < \tau_n$. The set $\mathcal{S}$ is then all Brownian paths started at $x$ which for $\tau_{n+1} < t < \tau_n$ are contained within the cylinder centered at $X(\tau_{n+1})$ with axis given by the vector $X(\tau_{n+1}) - x$ and with radius $\lambda_n$, $n \geq n_0$. It is clear that there is a universal constant $c > 0$ such that if $m_n < c$ there is no constraint on the Brownian path for $\tau_{n+1} < t < \tau_n$.

We have now from Jensen's inequality that

$$
\begin{aligned}
&E_x\Big[\exp\Big(-\int_0^\tau V(X(t))\,dt\Big)\Big] \\
&\qquad \ge E_x\Big[\exp\Big(-\int_0^\tau V(X(t))\,dt\Big),\,\mathcal{S}\Big] \\
&\qquad \ge P(\mathcal{S})\exp\Big(-E\Big[\int_0^\tau V(X(t))\,dt:\mathcal{S}\Big]\Big)\,.
\end{aligned}
\tag{A.2}
$$

It is obvious from the proof of Theorem A.1 that

$$
P(\mathcal{S}) \ge \exp\Big(-C\sum_{n=n_0}^{\infty} m_n\,H(m_n-c)\Big)\,,
\tag{A.3}
$$

where $C, c$ are universal constants and $H$ is the Heaviside function, $H(t)=1$, $t>0$, $H(t)=0$, $t<0$. We can write now

$$
E\Big[\int_0^\tau V(X(t))\,dt:\mathcal{S}\Big] = \sum_{n=n_0}^{\infty} E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt:\mathcal{S}\Big]\,.
$$

By symmetry $X(\tau_{n+1})$ is uniformly distributed on the sphere $|y-x|=2^{-n-1}$. Hence if $m_n<c$ one has

$$
\begin{aligned}
&E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt:\mathcal{S}\Big] \\
&\qquad \le \frac{1}{4\pi\,2^{-2n-2}}\int_{|x-y|=2^{-n-1}}\frac{dy}{4\pi}\int_{|x-z|<2^{-n}}\frac{V(z)\,dz}{|y-z|} \\
&\qquad \le K a_n(x)\,,
\end{aligned}
\tag{A.4}
$$

for some universal constant $K$. For $m_n>c$ we use Theorem A.1 . Thus

$$
\begin{aligned}
&E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt:\mathcal{S}\Big] \\
&\qquad \le \frac{1}{4\pi\,2^{-2n-2}}\int_{|x-y|=2^{-n-1}}\frac{C\,dy}{\lambda_n}\int_{\Gamma_{y,\lambda_n}\cap\{|x-z|<2^{-n}\}} V(z)\,dz\,,
\end{aligned}
$$

where $\Gamma_{y,\lambda_n}$ is the cylinder centered at $y$ with axis $y-x$ and radius $\lambda_n$. Arguing now as in Lemma 7.1 we have that

$$
\begin{aligned}
&\frac{1}{4\pi\,2^{-2n-2}}\int_{|x-y|=2^{-n-1}} dy\int_{\Gamma_{y,\lambda_n}\cap\{|x-z|<2^{-n}\}} V(z)\,dz \\
&\qquad \le C\,(\lambda_n 2^n)^2\int_{\{|x-z|<2^{-n}\}} V(y)\,dy\,,
\end{aligned}
$$

for some universal constant $C$. Hence by the previous two inequalities we have

$$E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt : \mathcal{S}\Big] \le C\lambda_n\, 2^n\, a_n(x) = Ca_n(x)^{1/2}\,, \tag{A.5}$$

for some universal constant $C$. The result follows now from (A.2), (A.3), (A.4), (A.5).

REMARK. It is possible to prove Theorem A.2 after the fashion of the proof of Proposition 7.1, avoiding the use of the Jensen inequality in (A.2) and Theorem A.1. This would on a technical level be a simpler proof. Our main purpose here is to draw a comparison between the proof of Theorem A.2 above and the proof in [4]. In the latter proof Jensen's inequality was combined with restricting to Brownian paths under a time constraint. In the former, Jensen is combined with restricting to Brownian paths under a topological constraint. Thus in some sense time constraints on Brownian paths are equivalent to topological constraints.

Next we turn our attention to the proof of Theorem A.1. First we shall prove (A.1). In order to do this we need to examine the behavior of 2-dimensional Brownian motion on $D_\lambda$ at large time.

**Lemma A.1.** *For $x, y \in D_\lambda$, $t > 0$, let $G_D(x,y,t)$ be the Green's function for the heat equation $D_\lambda$ with Dirichlet boundary conditions. Then there is a universal constant $C > 0$ such that*

$$\int_{|y|<\lambda} G_D(x,y,t)\,dy \le C\int_{|y|<\lambda/2} G_D(x,y,t)\,dy\,, \tag{A.6}$$

*for all $x \in D_\lambda$, $t \ge \lambda^2$.*

PROOF. It follows easily from the semi-group property of $G_D$ that it will be sufficient to prove (A.6) when $t = \lambda^2$. Evidently one has

$$\int_{|y|<\lambda} G_D(x,y,\lambda^2)\,dy = P_x(\tau_\lambda > \lambda^2)\,, \tag{A.7}$$

where $\tau_\lambda$ is the first exit time from $D_\lambda$ of 2-dimensional Brownian motion $Y(t)$ started at $x \in D_\lambda$. By the Chebyshev inequality we have that

$$P_x(\tau_\lambda > \lambda^2) \le \lambda^{-2}\,E_x[\tau_\lambda] = \lambda^{-2}\,u(x)\,, \tag{A.8}$$

where $u(x)$ satisfies the equation

$$\begin{cases} -\Delta u(x) = 1\,, & |x| < \lambda\,, \\ u(x) = 0\,, & |x| = \lambda\,. \end{cases}$$

It is easy to see that the solution of this equation is given by

$$u(x) = \frac{1}{4}\,(\lambda^2 - r^2)\,, \qquad r = |x|\,. \tag{A.9}$$

Hence (A.7), (A.8), (A.9) yield an upper bound on the left hand side of (A.6).

Next we look for a lower bound on the right hand side of (A.6). Let $\alpha$ satisfy $0 < \alpha < 1$. We shall show that there is a positive constant $C_\alpha$ depending only on $\alpha$ such that

$$\int_{|y|<\lambda/2} G_D(x, y, \lambda^2)\, dy \ge C_\alpha\,, \qquad |x| \le \alpha\lambda\,. \tag{A.10}$$

To see this let $G(z, w, t)$ be the heat kernel in $\mathbb{R}^2$,

$$G(z, w, t) = \frac{1}{4\pi t} \exp\Big( - \frac{|z-w|^2}{4\,t}\Big)\,.$$

Then for $|z|$, $|w| < \varepsilon\lambda$, $\varepsilon > 0$ there is a density $\rho(t, z')$, $0 < t < \varepsilon\lambda^2$, $|z'| = \lambda$ such that

$$G(z, w, \varepsilon\lambda^2) = G_D(z, w, \varepsilon\lambda^2) + \int_{|z'|=\lambda} dz' \int_0^{\varepsilon\lambda^2} \rho(t, z')\, G(z', w, t)\, dt\,.$$

The density $\rho(t, z')$ evidently satisfies the inequality

$$\int_{|z'|=\lambda} dz' \int_0^{\varepsilon\lambda^2} \rho(t, z')\, dt \le 1\,.$$

Suppose now that $|z|$, $|w| \le \alpha\lambda$, $|z - w| < (1 - \alpha)\lambda/2$. Then it is clear that for $\varepsilon$ sufficiently small, depending only on $\alpha$ one has

$$G(z', w, t) \le \frac{1}{2}\, G(z, w, \varepsilon\lambda^2)\,, \qquad |z'| = \lambda\,,\ 0 < t < \varepsilon\lambda^2\,.$$

Hence from the last three inequalities we have that

$$G_D(z,w,\varepsilon\lambda^2) \ge \frac{1}{2}\, G(z,w,\varepsilon\lambda^2)\,, \quad |z|\,,\ |w| \le \alpha\lambda\,,\ |z-w| < \frac{(1-\alpha)\,\lambda}{2}\,,$$

provided $\varepsilon > 0$ is sufficiently small. The inequality (A.10) follows from this last inequality by constructing paths from $x$ to $|y| < \lambda/2$ in time steps of length $\varepsilon\lambda^2$ and using the semi-group property of $G_D$.

In view of the fact that the left hand side of (A.6) is bounded above by 1, the inequality (A.6) follows for $t = \lambda^2$ and all $x$ satisfying $|x| \le \alpha\lambda$, $\alpha < 1$, from (A.10). Our main problem then is to deal with the case $|x| \longrightarrow \lambda$ since the right hand side of (A.10) converges to zero as $\alpha \longrightarrow 1$. Let $U_\alpha$ be the set

$$U_\alpha = \{y \in D_\lambda : \lambda\alpha < |y| < \lambda\}\,.$$

Then for $x \in U_\alpha$ we have

$$\begin{aligned}
&\int_{|y|<\lambda/2} G_D(x,y,\lambda^2)\,dy \\
&\qquad = P_x\Big[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{y : |y| = \lambda\alpha\}\,, \\
&\qquad\qquad |Y(t)| < \lambda\,,\ 0 < t < \lambda^2\,,\ |Y(\lambda^2)| \le \frac{\lambda}{2}\Big] \\
&\qquad \ge P_x\Big[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{|y| = \alpha\lambda\} \\
&\qquad\qquad \text{in time } < \frac{\lambda^2}{2}\Big] \\
&\qquad\quad \cdot \inf_{\substack{|y|=\lambda\alpha \\ 0<s<\lambda^2/2}} P_y\Big[|Y(t)| < \lambda\,,\ 0 < t < \lambda^2 - s\,,\ |Y(\lambda^2 - s)| \le \frac{\lambda}{2}\Big]\,.
\end{aligned} \tag{A.11}$$

It is clear from what we have just done that

$$\inf_{\substack{|y|=\lambda\alpha \\ 0<s<\lambda^2/2}} P_y\Big[|Y(t)| < \lambda\,,\ 0 < t < \lambda^2 - s\,,\ |Y(\lambda^2 - s)| \le \frac{\lambda}{2}\Big] \ge c_\alpha > 0\,,$$

where $c_\alpha$ is a constant depending only on $\alpha < 1$. Thus we are left to estimate the first probability in the final expression of (A.11).

We do this by using the inequality

$$P_x\Big[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{|y| = \alpha\lambda\} \text{ in time } < \frac{\lambda^2}{2}\Big]$$
$$\geq P_x[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{|y| = \alpha\lambda\}] - \frac{2}{\lambda^2}\, E_x[\tau]\,,$$

where $\tau$ is the first exit time from $U_\alpha$. If we put $w(r) = E_x[\tau]$, $|x| = r$, then $w$ satisfies the boundary value problem

$$\begin{cases} \dfrac{-d^2 w}{dr^2} - \dfrac{1}{r}\,\dfrac{dw}{dr} = 1\,, & \alpha\lambda < r < \lambda\,, \\ w(\alpha\lambda) = w(\lambda) = 0\,. \end{cases}$$

The solution of this boundary value problem is given by

$$w(r) = \frac{1}{4}\,(\lambda^2 - r^2) - \frac{1}{4}\,\lambda^2\,(1 - \alpha^2)\,\frac{\log\left(\dfrac{\lambda}{r}\right)}{\log\left(\dfrac{1}{\alpha}\right)}\,.$$

If we put $v(r)$ to be the probability that Brownian motion started at $x$, $|x| = r$, exits $U_\alpha$ through the boundary $\{y : |y| = \alpha\lambda\}$ then $v$ satisfies the boundary value problem,

$$\begin{cases} \dfrac{-d^2 v}{dr^2} - \dfrac{1}{r}\,\dfrac{dv}{dr} = 0\,, & \alpha\lambda < r < \lambda\,, \\ v(\alpha\lambda) = 1\,, & v(\lambda) = 0\,. \end{cases}$$

The solution of this last boundary value problem is given by the formula

$$v(r) = \frac{\log\left(\dfrac{\lambda}{r}\right)}{\log\left(\dfrac{1}{\alpha}\right)}\,.$$

Now consider the expression,

$$v(r) - \frac{2}{\lambda^2}\,w(r) = \frac{1}{2}\,(3 - \alpha^2)\,\frac{\log\left(\dfrac{\lambda}{r}\right)}{\log\left(\dfrac{1}{\alpha}\right)} - \frac{1}{2}\Big(1 - \frac{r^2}{\lambda^2}\Big)\,.$$

It is clear that if $\alpha$ is sufficiently close to 1 there is a constant $k_\alpha > 0$ such that

$$v(r) - \frac{2}{\lambda^2}\, w(r) \ge \frac{k_\alpha}{4}\Big(1 - \frac{r^2}{\lambda^2}\Big) = k_\alpha\, \frac{u(r)}{\lambda^2}\,.$$

Thus by (A.7), (A.8), (A.9) we conclude that (A.6) holds with $t = \lambda^2$ if $|x| > \alpha\lambda$ with constant $C = (k_\alpha c_\alpha)^{-1}$ provided $\alpha$ is sufficiently close to 1.

We have proved therefore that (A.6) holds for all $x \in D_\lambda$ and $t = \lambda^2$. The result follows.

**Lemma A.2.** *Let $\kappa_0 > 0$ be the minimum eigenvalue of $-\Delta$ on the unit disc with Dirichlet boundary conditions. Let $G_D(x, y, t)$ be the Dirichlet Green's function for the heat equation on $D_\lambda$. Then for any $\alpha, 0 < \alpha < 1$, there exist positive constants $c_\alpha, C_\alpha$ depending only on $\alpha$ such that*

$$c_\alpha \exp\Big( - \frac{\kappa_0\, t}{\lambda^2}\Big) \le \int_{D_\lambda} G_D(x, y, t)\, dy \le C_\alpha \exp\Big( - \frac{\kappa_0\, t}{\lambda^2}\Big), \tag{A.12}$$

*with $t > 0$, provided $|x| \le \alpha\lambda$.*

Proof. Let $\varphi_0(x)$ be the eigenfunction on the unit disc corresponding to the eigenvalue $\kappa_0$ of $-\Delta$. Then $\varphi_0(x)$ is a positive $C^\infty$ function for $|x| < 1$, and continuous on $|x| \le 1$ with $\varphi_0(x) = 0$, $|x| = 1$. By scaling we have that

$$\int_{D_\lambda} G_D(x, y, t)\, \varphi_0\Big(\frac{y}{\lambda}\Big)\, dy = \exp\Big( - \frac{\kappa_0\, t}{\lambda^2}\Big) \varphi_0\Big(\frac{x}{\lambda}\Big)\,. \tag{A.13}$$

Hence it follows that

$$\int_{D_\lambda} G_D(x, y, t)\, dy \ge \exp\Big( - \frac{\kappa_0\, t}{\lambda^2}\Big) \frac{\varphi_0\Big(\dfrac{x}{\lambda}\Big)}{\|\varphi_0\|_\infty}\,.$$

Now the first inequality in (A.12) follows from this last inequality by taking

$$c_\alpha = \inf\Big\{ \frac{\varphi_0(z)}{\|\varphi_0\|_\infty} : |z| < \alpha \Big\} > 0\,.$$

We use Lemma A.1 to prove the upper bound in (A.12). Thus from (A.13) we have

$$\begin{aligned} \exp\Big( - \frac{\kappa_0\, t}{\lambda^2}\Big) \varphi_0\Big(\frac{x}{\lambda}\Big) &\ge \inf\Big\{ \varphi_0\Big(\frac{y}{\lambda}\Big) : |y| \le \frac{\lambda}{2}\Big\} \int_{|y|<\lambda/2} G_D(x, y, t)\, dy \\ &\ge c \int_{|y|<\lambda} G_D(x, y, t)\, dy\,, \end{aligned}$$

for some universal constant $c > 0$ provided $t \geq \lambda^2$. The upper bound in (A.12) clearly follows from this last inequality provided $t \geq \lambda^2$. The inequality for $t \leq \lambda^2$ is trivial by choosing $C_\alpha$ to satisfy $C_\alpha \exp(-\kappa_0) \geq 1$.

REMARK. The inequality (A.12) has already been proved in a much more general context [2], [8], for divergence form operators in domains with Lipschitz boundary.

Next we wish to prove the formula (A.1).

**Proposition A.1.** *Let $\tau$ be the time taken for 3-dimensional Brownian motion $X(t) = (X_1(t), X_2(t), X_3(t))$ started at the origin to exit the cylinder $D_\lambda \times (-m\lambda, m\lambda)$. There are universal constants $C, c > 0$ such that*

$$c\, m\lambda^2 \leq E\,[\tau : |X_3(\tau)| = m\lambda] \leq Cm\lambda^2\,,$$

*provided $m \geq 1$.*

Consider one-dimensional Brownian motion $X_3(t)$ starting at the origin and let $\tau_1$ be the first hitting time on the boundary of the interval $[-m\lambda, m\lambda]$, and $\rho(t)$, $t \geq 0$, be the probability density for $\tau_1$. Next consider 2-dimensional Brownian motion starting at the origin and let $\tau_2$ be the first hitting time on the boundary of $D_\lambda$. Then

$$E\,[\tau : |X_3(\tau)| = m\lambda] = \frac{\displaystyle\int_0^\infty P(\tau_2 > t)\, t\, \rho(t)\, dt}{\displaystyle\int_0^\infty P(\tau_2 > t)\, \rho(t)\, dt}\,. \tag{A.14}$$

Now from Lemma A.2 it follows that there are universal constants $C, c > 0$ such that

$$c\, I_m \leq E\,[\tau : |X_3(\tau)| = m\lambda] \leq CI_m\,, \tag{A.15}$$

where

$$I_m = \frac{\displaystyle\int_0^\infty \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big) t\, \rho(t)\, dt}{\displaystyle\int_0^\infty \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big) \rho(t)\, dt}\,.$$

We have now that for $\eta > 0$,

$$\int_0^\infty e^{-\eta t}\rho(t)\,dt = E_0\left[\exp\left(-\eta\tau_1\right)\right] = \frac{1}{\cosh\left(\sqrt{\eta}\, m\lambda\right)}\,.$$

Differentiating this last expression with respect to $\eta$ we obtain

$$\int_0^\infty e^{-\eta t}\, t\,\rho(t)\,dt = \frac{m\lambda\sinh\left(\sqrt{\eta}\, m\lambda\right)}{2\sqrt{\eta}\,\cosh^2(\sqrt{\eta}\, m\lambda)}\,.$$

If we take now $\eta = \kappa_0/\lambda^2$ and we use the last two formulas it follows that

$$I_m = \frac{m\lambda^2}{2\sqrt{\kappa_0}}\tanh\left(m\sqrt{\kappa_0}\right).$$

The result follows from this last formula and (A.15).

Proof of Theorem A.1. Let $G_m(\xi,\zeta,t)$ be the Dirichlet Green's function of the interval $[-m\lambda, m\lambda]$. Then if $G_D$ is the Green's function for $D_\lambda$ as in Lemma A.2 it follows that the Dirichlet Green's function for the cylinder $D_\lambda\times(-m\lambda,m\lambda)$ is given by

$$G_D(x,y,t)\,G_m(\xi,\zeta,t)\,,\qquad x,y\in D_\lambda\,,\ -m\lambda<\xi\,,\ \zeta<m\lambda\,,\ t>0\,.$$

For $x\in D_\lambda, \xi\in(-m\lambda,m\lambda)$ let $u(x,\xi)$ be the probability that Brownian motion started at $(x,\xi)$ exits $D_\lambda\times(-m\lambda,m\lambda)$ through $D_\lambda\times\{m\lambda\}$ or $D_\lambda\times\{-m\lambda\}$. If we define $w(\xi)$ by

$$w(\xi) = \frac{1}{|D_\lambda|}\int_{D_\lambda} u(x,\xi)\,dx\,,\qquad \xi\in(-m\lambda,m\lambda)\,,$$

it follows from the argument of Proposition A.1 that there are positive universal constants $C, c > 0$ such that

$$c\,E_\xi\Big[\exp\Big(-\frac{\kappa_0\tau_1}{\lambda^2}\Big)\Big] \le w(\xi) \le CE_\xi\Big[\exp\Big(-\frac{\kappa_0\tau_1}{\lambda^2}\Big)\Big]\,, \tag{A.16}$$

where $\tau_1$ is the first exit time of Brownian motion started at $\xi$ from the interval $(-m\lambda, m\lambda)$. Furthermore there is the identity

$$E_\xi\Big[\exp\Big(-\frac{\kappa_0\tau_1}{\lambda^2}\Big)\Big] = \frac{\cosh\Big(\frac{\xi\sqrt{\kappa_0}}{\lambda}\Big)}{\cosh m\sqrt{\kappa_0}}\,. \tag{A.17}$$

We have now that if $X(t)$ denotes Brownian motion started uniformly on the cross section $D_\lambda \times \{\xi\}$, then

$$
\begin{aligned}
E\Big[\int_0^\tau V(X(t))\,dt : |X(\tau)_3| = m\lambda\Big]& \\
= \frac{1}{|D_\lambda|}\int_{D_\lambda} dx \int_0^\infty dt \int_{-m\lambda}^{m\lambda}\int_{D_\lambda} & G_D(x,y,t)\,G_m(\xi,\zeta,t) \\
&\cdot u(y,\zeta)\,V(y,\zeta)\,\frac{dy\,d\zeta}{w(\xi)}\,.
\end{aligned}
\tag{A.18}
$$

It follows from Lemma A.2 that

$$
\begin{aligned}
\int_{D_\lambda} dx \int_0^\infty G_D(x,y,t)\,G_m(\xi,\zeta,t)\,dt &\le C\int_0^\infty \exp\Big(-\frac{\kappa_0\,t}{\lambda^2}\Big) G_m(\xi,\zeta,t)\,dt \\
&= CG(\xi,\zeta)\,,
\end{aligned}
\tag{A.19}
$$

where $G$ is the Green's function which satisfies

$$
\begin{cases}
\Big(-\dfrac{d^2}{d\xi^2} + \dfrac{\kappa_0}{\lambda^2}\Big) G(\xi,\zeta) = \delta\,(\xi - \zeta)\,, & -m\lambda < \xi < m\lambda\,, \\
G(\xi,\zeta) = 0\,, & |\xi| = m\lambda\,.
\end{cases}
$$

We can solve this boundary value problem to obtain the explicit formula

$$
G(\xi,\zeta) = \lambda \sinh\Big(\frac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda - \xi)\Big)\,\frac{\sinh\Big(\dfrac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda + \zeta)\Big)}{\sqrt{\kappa_0}\,\sinh\,(2\,m\sqrt{\kappa_0})}\,,
$$

if $\xi > \zeta$, and

$$
G(\xi,\zeta) = \lambda \sinh\Big(\frac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda + \xi)\Big)\,\frac{\sinh\Big(\dfrac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda - \zeta)\Big)}{\sqrt{\kappa_0}\,\sinh\,(2\,m\sqrt{\kappa_0})}\,,
$$

if $\xi < \zeta$. It follows now from this last identity and (A.16), (A.17) that there is a universal constant $C$ such that

$$
G(\xi,\zeta)\,w(\zeta) \le C\,\lambda\,w(\xi)\,, \qquad -m\lambda < \xi\,,\ \zeta < m\lambda\,.
$$

Hence from this last inequality and (A.18), (A.19) we have that

$$
E\Big[\int_0^\tau V(X(t))\,dt : |X(\tau)_3| = m\lambda\Big] \le \frac{C}{\lambda}\int_{-m\lambda}^{m\lambda}\int_{D_\lambda} \frac{u(y,\zeta)}{w(\zeta)}\,V(y,\zeta)\,dy\,d\zeta\,,
$$

for some universal constant $C$.

It is obvious from Lemma A.2 that there is a universal constant $C$ such that $u(y,\zeta) \le C\,w(\zeta)$, $y \in D_\lambda$, $\zeta \in (-m\lambda, m\lambda)$. The result follows from this and the last inequality.

## Appendix B. A differential inequality.

Here we prove the inequality (5.33). Consider the solution $u(r,\eta)$ to the Sturm-Liouville problem,

$$\text{(B.1)} \qquad \begin{cases} \rho(r)\,\dfrac{d^2u}{dr^2} + \rho'(r)\,\dfrac{du}{dr} = \eta\,\rho(r)\,u\,, & 1 < r < R\,, \\ u(1) = 1\,, & \\ u(R) = 0\,, & \end{cases}$$

where $\rho(r) > 0, \rho'(r) > 0$, $1 \le r \le R$. We shall show that

$$\text{(B.2)} \qquad \frac{\partial}{\partial r}\frac{\partial}{\partial \eta}\Big(\frac{u(r,\eta)}{\sqrt{\eta}}\Big) > 0\,, \qquad 1 < r < R\,,\ \eta > 0\,.$$

This implies the inequality (5.33) on taking $\rho(r) = r$. The inequality (B.2) is sharp in the sense that the power of $\eta$, *i.e.* $\eta^{1/2}$ in the denominator cannot be improved. To see this consider for $\alpha > 0$, the function

$$w_\alpha(r) = \frac{\partial}{\partial \eta}\Big(\frac{u}{\eta^\alpha}\Big) = \frac{1}{\eta^\alpha}\Big(\frac{\partial u}{\partial \eta} - \alpha\,\frac{u}{\eta}\Big)\,.$$

Now it follows easily from the maximum principle that the function $u$ decreases as a function of $r$. The function $w_0(r) = \partial u/\partial \eta$ satisfies the boundary conditions $w_0(1) = w_0(R) = 0$. It follows from the maximum principle again that $w_0(r) < 0$, $1 < r < R$. Hence there exists a minimum $\alpha_0 > 0$ such that

$$\frac{dw_\alpha}{dr} \ge 0\,, \qquad 1 < r < R\,,\ \alpha \ge \alpha_0\,.$$

We can explicitly compute $\alpha_0$ in the exactly solvable case when $\rho \equiv 1$. Thus for $\rho \equiv 1$, we have

$$u(r,\eta) = \frac{\sinh\sqrt{\eta}\,(R-r)}{\sinh\sqrt{\eta}\,(R-1)}\,,$$

whence

$$\frac{1}{\sqrt{\eta}}\,\frac{\partial u}{\partial r}=\frac{-\cosh\sqrt{\eta}\,(R-r)}{\sinh\sqrt{\eta}\,(R-1)}\,,$$

$$\frac{\partial}{\partial\eta}\Big(\frac{1}{\sqrt{\eta}}\,\frac{\partial u}{\partial r}\Big)=\frac{1}{2\sqrt{\eta}}\,\frac{(r-1)+(R-r)\cosh\sqrt{\eta}\,(r-1)}{\sinh^2\sqrt{\eta}(R-1)}\,.$$

Hence if $0<\alpha<1/2$ we have

$$\begin{aligned}\frac{dw_\alpha}{dr}=\frac{1}{2\,\eta^\alpha}\Big(&\frac{(r-1)\cosh\sqrt{\eta}\,(R-1)\cosh\sqrt{\eta}\,(R-r)}{\sinh^2\sqrt{\eta}\,(R-1)}\\&+\frac{(R-r)\cosh\sqrt{\eta}\,(r-1)}{\sinh^2\sqrt{\eta}\,(R-1)}\Big)\\&-\frac{\Big(\dfrac{1}{2}-\alpha\Big)}{\eta^{\alpha+1/2}}\,\frac{\cosh\sqrt{\eta}\,(R-r)}{\sinh\sqrt{\eta}\,(R-1)}\,.\end{aligned}$$

It is clear from this last identity that by choosing $r=1$ and $R$ large we will have $dw_\alpha/dr<0$ at $r=1$ for any $\alpha<1/2$. Thus $\alpha_0=1/2$ is optimal in this case.

To prove (B.2) we first construct the Dirichlet Green's function for (B.1). Let $v(r)$ be the solution of the equation (B.1) with the boundary conditions $v(1)=0$, $v(R)=1$. Then the Dirichlet Green's function $G(r,r')$, $1<r$, $r'<R$, can be written as

$$G(r,r')=\begin{cases}c(r')\,u(r)\,v(r')\,, & 1<r'<r\,,\\ c(r')\,u(r')\,v(r)\,, & r<r'<R\,.\end{cases}\tag{B.3}$$

The constant $c(r')$ is determined by the jump discontinuity of $\partial G/\partial r$ at $r=r'$,

$$\lim_{r\to r'+}\frac{\partial G}{\partial r}(r,r')-\lim_{r\to r'-}\frac{\partial G}{\partial r}(r,r')=\frac{1}{\rho(r')}\,.$$

Thus if $W(r')$ denotes the Wronskian,

$$W(r')=u'(r')\,v(r')-u(r')\,v'(r')\,,\qquad 1<r'<R\,,$$

we have that

$$c(r')=\frac{1}{\rho(r')\,W(r')}\,.$$

Now using the fact that

$$W(r') = -\frac{\rho(1)\, v'(1)}{\rho(r')}\,, \qquad 1 < r' < R\,,$$

we conclude that

$$c(r') = -\frac{1}{\rho(1)\, v'(1)}\,, \qquad 1 < r' < R\,. \tag{B.4}$$

It is clear that the function $w(r) = w_{1/2}(r)$ satisfies the equation

$$\begin{cases} \rho(r)\, \dfrac{d^2 w}{dr^2} + \rho'(r)\, \dfrac{dw}{dr} = \eta\, \rho(r)\, w + \dfrac{\rho\, u}{\sqrt{\eta}}\,, \\ w(1) = -\dfrac{1}{2\,\eta^{3/2}}\,, \\ w(R) = 0\,. \end{cases}$$

Hence $w$ has the representation

$$w(r) = \frac{-u(r)}{2\,\eta^{3/2}} + \int_0^R G(r, r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr'\,.$$

Using the formulas (B.3), (B.4) we have then

$$\begin{aligned} w(r) = \frac{-u(r)}{2\,\eta^{3/2}} &- \frac{1}{\rho(1)\, v'(1)} \int_0^r u(r)\, v(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr' \\ &- \frac{1}{\rho(1)\, v'(1)} \int_r^R v(r)\, u(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr'\,. \end{aligned}$$

On differentiating the above identity we have then

$$\begin{aligned} \frac{dw}{dr} = \frac{-u'(r)}{2\,\eta^{3/2}} &- \frac{u'(r)}{\rho(1)\, v'(1)} \int_0^r v(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr' \\ &- \frac{v'(r)}{\rho(1)\, v'(1)} \int_r^R u(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr'\,. \end{aligned}$$

It follows that $dw/dr \geq 0$, $1 < r < R$, if we can show that

$$2\,\eta\, v'(r) \int_r^R \rho(r')\, u(r')^2\, dr' \leq -\rho(1)\, v'(1)\, u'(r)\,, \qquad 1 < r < R\,. \tag{B.5}$$

We have now that

$$\text{(B.6)}\qquad \begin{aligned} \eta \int_r^R \rho(r')\, u(r')^2\, dr' &= \int_r^R u\, \frac{d}{dr'}\Big(\rho(r')\, \frac{du}{dr'}\Big)\, dr' \\ &= -\int_r^R \rho(r')\, u'(r')^2\, dr' - \rho(r)\, u(r)\, u'(r)\,. \end{aligned}$$

Next we show that

$$\text{(B.7)}\qquad \eta\, u(r')^2 \le u'(r')^2\,, \qquad 1 < r' < R\,.$$

To see that (B.7) holds observe that

$$\begin{aligned} \frac{d}{dr'}\,(\eta\, u(r')^2 - u'(r')^2) &= 2\, u'(r')\, (\eta u(r') - u''(r')) \\ &= 2\, u'(r')\, \frac{\rho'(r')}{\rho(r')}\, u'(r') \ge 0\,, \qquad 1 < r' < R\,. \end{aligned}$$

We conclude from this that

$$\eta\, u(r')^2 - u'(r')^2 \le \eta\, u(R)^2 - u'(R)^2 = -u'(R)^2 \le 0\,,$$

whence (B.7) follows.

From (B.6), (B.7) it follows that

$$2\,\eta \int_r^R \rho(r')\, u(r')^2\, dr' \le -\rho(r)\, u(r)\, u'(r)\,.$$

Hence the left hand side of (B.5) is bounded above by

$$-v'(r)\, \rho(r)\, u(r)\, u'(r)\,.$$

Thus (B.5) holds if we can show

$$\rho(1)\, v'(1) \ge v'(r)\, \rho(r)\, u(r)\,, \qquad 1 < r < R\,,$$

since $u'(r) < 0$, $1 < r < R$. This last inequality follows from the fact that

$$\begin{aligned} v'(r)\, \rho(r)\, u(r) &= \rho(r)\, (u'(r)\, v(r) - W(r)) \\ &= \rho(r) \Big(u'(r)\, v(r) + \rho(1)\, \frac{v'(1)}{\rho(r)}\Big) \\ &\le \rho(1)\, v'(1)\,, \end{aligned}$$

since $u'(r) < 0$, $v(r) > 0$, $1 < r < R$.

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Joseph G. Conlon* and Peder A. Olsen†
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
conlon@math.lsa.umich.edu

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