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Subnormal operators of finite type I. Xia's model and real algebraic curves in $\mathbb{C}^2$

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Abstract. Xia proves in [9] that a pure subnormal operator S is completely determined by its self-commutator $C = S^*S - SS^*$, restricted to the closure M of its range and the operator $\Lambda = (S^*|M)^*$. In [9], [10], [11], he constructs a model for S that involves these two operators and the so-called mosaic, which is a projection-valued function, analytic outside the spectrum of the minimal normal extension of S. He finds all pure subnormals S with $\operatorname{rank} C = 2$. We will give a complete description of pairs of matrices (C, Λ) that correspond to some S for the case of the self-commutator C of arbitrary finite rank. It is given in terms of a topological property of a certain algebraic curve, associated with C and Λ. We also give a new explicit formula for Xia's mosaic.

0. Introduction.

One of the modern approaches to the spectral theory of a nonselfadjoint operator consists in constructing its functional model. The most developed theory of this kind is the Sz.-Nagy-Foiaş theory of Hilbert

space contractions. Recently, several attempts have been made to construct functional models for other classes of operators. This paper concerns some questions that arise in connection with Xia's analytic model of subnormal operators.

A (bounded) linear operator S acting on a (complex) Hilbert space H is called *subnormal* if there exists a larger Hilbert space K, $K \supset H$, and a normal operator $N\colon K \to K$ such that $NH \subset H$ and $S = N|H$. The operator S is called pure if it has no nonzero reducing subspace on which it is normal. We will always assume S to be pure and the normal extension N of S to be minimal; the latter means that there is no subspace K', $H \subset K' \subsetneqq K$, such that $NK' \subset K'$ and $N|K'$ is normal.

This class of operators has been much investigated; we refer to [1] for a background.

It is known that for a subnormal operator S, if we put

$$C \stackrel{\text{def}}{=} S^*S - SS^*\,, \qquad M \stackrel{\text{def}}{=} \operatorname{clos}\operatorname{Range} C\,,$$

then $S^*M \subset M$. In [9], [10], [11], Xia constructs and studies an analytic model of a subnormal operator. He defines two functional model spaces that consist, respectively, of analytic and antianalytic M-valued functions on $\mathbb{C} \setminus \sigma(N)$ and gives formulas for the trancription of S and S^* in each of these two models (here $\sigma(N)$ is the spectrum of N).

One of the consequences of Xia's results is that if we put

$$\Lambda = \big(S^*|M\big)^*\,,$$

then the pair (C, Λ) of operators on M completely determines a pure subnormal operator S. If M is one-dimensional, then C, Λ are, essentially, complex numbers, and the spectrum of S is the closed disk with center in Λ and radius $C^{1/2}$. Therefore, by analogy, Λ and $C^{1/2}$ can be called *the matrix center* and *the matrix radius* of S.

The following question arises: which pairs (C, Λ) can appear in this way? The main result of this paper is a complete answer to this question in the case $\dim M < \infty$. It is given in terms of the algebraic curve

$$\Delta = \big\{(z, w) :\ \det\big(C - (w - \Lambda^*)\,(z - \Lambda)\big) = 0\big\}$$

in $\mathbb{C}P^2$. A crucial topological condition is that Δ has to be separated, that is, that $\Delta \cap \{(z, \bar{z}) : z \in \mathbb{C}\}$ divides each of the (nondegenerate) irreducible components of Δ into two connected components (see Theorem 1 below).

One of the main objects in Xia's model is Xia's mosaic

$$\mu(z) = P_M(N - SP_H)\,(N - z)^{-1}|M\,, \qquad z \in \mathbb{C} \setminus \sigma(N)\,,$$

here N is the minimal normal extension of S and P_W is the orthogonal projection onto a subspace W. We will give an explicit formula for $\mu(z)$ in terms of C, Λ and the curve Δ.

Sections 1-3 are devoted to preliminaries. Main results are formulated in Section 4; in Section 5, proofs are given. Section 6 collects some additional facts and examples. In the subsequent publication [12], we are going to continue the analysis of Xia's model.

The form of the main result resembles some results in the theory of commuting nonselfadjoint operators by Livšic, Vinnikov and others (see [5]). The connection between this theory and the topic of the present paper may exist, but does not seem to be obvious.

1. Xia's results.

We reproduce only those results by Xia that will be necessary for our exposition.

Let S be a pure subnormal operator, and define M, C, Λ as above. We will write $C = C(S)$, $\Lambda = \Lambda(S)$. Let us say that S is *of finite type* if $\dim M < \infty$. Denote by $\mathcal{L}(M)$ the space of bounded linear operators on M. Following Xia [9], define a $\mathcal{L}(M)$-valued measure $e(\,\cdot\,)$ by

$$e(\,\cdot\,) = P_M E(\,\cdot\,)\, P_M\,, \tag{1.1}$$

where $E(\,\cdot\,)$ is the spectral measure of N. Xia shows that $\sigma(N)$ is contained in the set

$$\gamma = \{u \in \mathbb{C} :\ \det\big(C - (\bar{u} - \Lambda^*)\,(u - \Lambda)\big) = 0\} \tag{1.2}$$

and that

$$\big(C - (\overline{u} - \Lambda^*)\,(u - \Lambda)\big)\, de(u) \equiv 0\,. \tag{1.3}$$

He also proves that the values of the function

$$\mu(z) = \int \frac{u - \Lambda}{u - z}\, de(u) = P_M(N - SP_H)\,(N - z)^{-1}|M\,, \quad z \in \mathbb{C}\setminus\gamma \tag{1.4}$$

are parallel projections on M (that is, $\mu(z)^2 \equiv \mu(z)$). We will call this function Xia's mosaic of S. Xia proves that

$$\big[\mu(z),\, C(z-\Lambda)^{-1} + \Lambda^*\big] = 0\,, \qquad z \in \mathbb{C} \setminus \gamma\,, \tag{1.5}$$

where $[A, B] = AB - BA$.

For any non-negative $\mathcal{L}(M)$-valued measure e, we put $\mathcal{L}^2(e)$ to be the space of all measurable M-valued functions f satisfying

$$\|f\|^2 \stackrel{\text{def}}{=} \int_\gamma \big(de(u) f(u), f(u)\big) < \infty\,,$$

factorized by the linear manifold $\{f : \|f\|^2 = 0\}$. It is easy to see that $\mathcal{L}^2(e)$ is a Hilbert space.

The following result is part of [9, Theorems 1 and 2].

Theorem A (Xia [9])**.** *Let $C, \Lambda \in \mathcal{L}(M)$ and $C > 0$. Suppose that there exists an $\mathcal{L}(M)$-valued positive measure e on a compact subset γ of $\mathbb{C}$ such that*

$$\int \frac{u - \Lambda}{u - z}\, de(u) = 0\,, \tag{1.6}$$

for z in the unbounded component of $\mathbb{C} \setminus \gamma$ and (1.3) holds. Let $\mathcal{D}$ be the set of all $z \in \mathbb{C} \setminus \gamma$ for which (1.6) holds, and $\mathcal{H}$ the closure in $\mathcal{L}^2(e)$ of all linear combinations of functions $(\lambda - (\,\cdot\,))^{-1} m$, $\lambda \in \mathcal{D}$, $m \in M$. Then the operator

$$(\widetilde{S} f)(u) = u\, f(u)\,, \qquad f \in \mathcal{H}\,, \tag{1.7}$$

is pure subnormal,

$$(\widetilde{N} f)(u) = u\, f(u)\,, \qquad f \in \mathcal{L}^2(e)\,, \tag{1.8}$$

is its minimal normal extension, $C = C(\widetilde{S})$, $\Lambda = \Lambda(\widetilde{S})$, and $e(\,\cdot\,)$ is connected with $\widetilde{N}$ in the same way as in formula (1.1). We imbed M into $\mathcal{L}^2(e)$ via the formula $c \longmapsto [c]$, where $[c](z) \equiv c$.

Conversely, if S is a subnormal operator of finite type and $C = C(S)$, $\Lambda = \Lambda(S)$, then the measure $e(\,\cdot\,)$, given by (1.1), enjoys the above properties, and (1.7), (1.8) define operators, unitarily equivalent to S and N, respectively.

The statements $C = C(S)$, $\Lambda = \Lambda(S)$ are not stated explicitly in [9, Theorem 2], but they follow at once from (1.7) and [9, formula (42)].

Theorem A gives a criterion for existence of a pure subnormal S with given matrices $C = C(S)$ and $\Lambda = \Lambda(S)$ in terms of the existence of a $\mathcal{L}(M)$-valued measure with certain properties. Our aim is to give a more explicit criterion.

2. The discriminant curve and its geometry.

Let M be a finite-dimensional Hilbert space and $C > 0$ and Λ operators on M. We associate with C, Λ the polynomial

$$\tau(z,w) = \det\bigl(C - (w - \Lambda^*)\,(z - \Lambda)\bigr) \tag{2.1}$$

and the algebraic curve

$$\Delta = \bigl\{(z,w) \in \mathbb{C}^2 :\ \tau(z,w) = 0\bigr\}\,,$$

which will be called *the discriminant curve* of S. As usual, we pass to homogeneous coordinates $(\zeta, \omega, \vartheta)$ in the complex projective plane $\mathbb{C}P^2$ by putting $z = \zeta\,\vartheta^{-1}$, $w = \omega\,\vartheta^{-1}$ and consider Δ as an algebraic curve in $\mathbb{C}P^2$, defined by the homogeneous polynomial equation $\vartheta^{2\dim M}\tau(\zeta\,\vartheta^{-1}, \omega\,\vartheta^{-1}) = 0$. Since

$$\tau(\overline{w}, \overline{z}) = \overline{\tau(z,w)}\,, \tag{2.2}$$

Δ possesses an antianalytic involution given by

$$\delta = (z,w) \longmapsto \delta^* \stackrel{\text{def}}{=} (\overline{w}, \overline{z})\,.$$

If we substitute $z = x+iy$, $w = x-iy$, then τ becomes a real polynomial in variables x, y. In this sense, Δ is a real algebraic curve. In terms of the coordinates (x,y) in $\mathbb{C}^2$, the map $\delta \longmapsto \delta^*$ is the usual complex conjugation $(x,y) \longmapsto (\overline{x}, \overline{y})$, that is, the reflexion with respect to the linear submanifold $\mathbb{R}^2 = \{x = \overline{x},\ y = \overline{y}\} = \{w = \overline{z}\}$ of real points of $\mathbb{C}^2$. In what follows, only the coordinates (z,w) will be used.

We observe that

a) $\delta \in \Delta$, $w(\delta) = \infty$ implies $z(\delta) \in \sigma(\Lambda)$;

b) $\delta \in \Delta$, $z(\delta) = \infty$ implies $w(\delta) \in \sigma(\Lambda^*)$.

For instance, to prove a), it suffices to rewrite the equation $\tau(z,w) = 0$ as

$$\det\bigl(Cw^{-1} - (1 - w^{-1}\Lambda^*)\,(z - \Lambda)\bigr) = 0$$

and to put here $w^{-1} = 0$.

Let

$$\tau(z,w) = \prod_{j=1}^{T} \tau_j(z,w)^{\alpha_j} \tag{2.3}$$

be the decomposition of τ into irreducible factors [3]; associated is a decomposition

$$\Delta = \bigcup_{j=1}^{T} \Delta_j\ , \tag{2.4}$$

where $\Delta_j = \bigl\{(z,w) : \tau_j(z,w) = 0\bigr\}$. We will call algebraic curves Δ_j the components of Δ.

A component $\widehat{\Delta}_k$ will be called *degenerate* if it has the form $z \equiv$ const or $w \equiv$ const and *nondegenerate* in the opposite case. Let $\widehat{\Delta}_{\rm deg}$ be the union of degenerate components of $\widehat{\Delta}$ and $\widehat{\Delta}_{\rm ndeg}$ the union of nondegenerate components.

Consider the following example. Let S be the shift operator

$$Sf(\,\cdot\,) = (\,\cdot\,)\,f(\,\cdot\,)\,,$$

acting of the Hardy space H^2, equipped with the modified norm $\|f\|_1^2 = \|f\|_{H^2}^2 + a\,|f(0)|^2$, where $a > 0$. It is easy to see that S is simple subnormal and that its discriminant surface is

$$\{z\,w = 1\} \cup \{z = 0\} \cup \{w = 0\}\,.$$

This shows that degenerate surfaces really can appear.

We put

$$\sigma_C(\Lambda) = \bigl\{z \in \mathbb{C} :\ \det(C - (w - \Lambda^*)\,(z - \Lambda)) = 0\,,\ \ \text{for all } w \in \mathbb{C}\bigr\}\,,$$

so that the degenerate components in the decomposition (2.4) are exactly the surfaces $z \equiv \lambda$ and $w \equiv \overline{\lambda}$, $\lambda \in \sigma_C(\Lambda)$. It is immediate that $\sigma_C(\Lambda) \subset \sigma(\Lambda)$ and $\sigma_C(\Lambda) \subset \gamma$.

A point δ of Δ will be called *regular* if it belongs to only one Δ_j and either

$$\frac{\partial \tau_j}{\partial z}(\delta) \neq 0 \qquad \text{or} \qquad \frac{\partial \tau_j}{\partial w}(\delta) \neq 0\,,$$

and *singular* in all other cases. The set Δ_s of singular points of Δ is finite. Put $\Delta_0 = \Delta \setminus \Delta_s$; then the sets $\Delta_j \cap \Delta_0$ are pairwise disjoint. *The blow-up* $\widehat{\Delta}$ of Δ can be defined as a unique abstract compact Riemann surface that consists of exactly T connected components $\widehat{\Delta}_j$, where each $\widehat{\Delta}_j$ is compact and is obtained by adding a finite number of points to $\Delta_j \cap \Delta_0$. There is a natural projection of $\widehat{\Delta}$ onto Δ which is identical on Δ_0. If $\delta \in \widehat{\Delta}$ and (z, w) is its image on Δ, we will write $\delta \sim (z, w)$. We refer to [3] for the background on the blow-up.

The functions $\delta \longmapsto z(\delta)$, $\delta \longmapsto w(\delta)$ extend to meromorphic functions on $\widehat{\Delta}$. The conjugation $\delta \longmapsto \delta^*$ also extends to $\widehat{\Delta}$.

The function

$$\eta = -\frac{dz}{dw}\,, \tag{2.5}$$

defined initially on regular points $\delta = (z, w) \in \widehat{\Delta}_{\text{ndeg}}$, can be continued to a meromorphic function on $\widehat{\Delta}$. This function will play an important role in the sequel.

It is easy to check, using a) and b), that

$$\begin{aligned} &z(\delta) \longrightarrow \infty \qquad \text{implies} \qquad \eta(\delta) \longrightarrow \infty\,, \\ &w(\delta) \longrightarrow \infty \qquad \text{implies} \qquad \eta(\delta) \longrightarrow 0\,. \end{aligned} \tag{2.6}$$

Since both z-projection and w-projection of each nondegenerate component Δ_j is the whole sphere $\widehat{\mathbb{C}}$, it follows that η is non-constant on each nondegenerate component of Δ.

By (2.2), η has the following symmetry property

$$\eta(\delta^*) = (\overline{\eta}(\delta))^{-1}\,. \tag{2.7}$$

Put

$$\widehat{\Delta}_+ = \{\delta \in \widehat{\Delta}_{\text{ndeg}} : \ |\eta(\delta)| < 1\}\,, \qquad \widehat{\Delta}_- = \{\delta \in \widehat{\Delta}_{\text{ndeg}} : \ |\eta(\delta)| > 1\}\,,$$

then

$$\partial\widehat{\Delta}_+ = \partial\widehat{\Delta}_- = \{\delta \in \widehat{\Delta}_{\text{ndeg}} : \ |\eta(\delta)| = 1\}\,.$$

Let

$$\widehat{\Delta}_{\mathbb{R}} = \{\delta \in \widehat{\Delta}_{\text{ndeg}} : \ \delta = \delta^*\}$$

be the set of real points of $\widehat{\Delta}$, then $\widehat{\Delta}_{\mathbb{R}} \cap \Delta_0 = \Delta_0 \cap \{(z, \overline{z}) : z \in \mathbb{C}\}$. By (2.7),

$$\widehat{\Delta}_{\mathbb{R}} \subset \partial\widehat{\Delta}_+ \, .$$

Definition. *The algebraic curve Δ is called separated if for any nondegenerate component $\widehat{\Delta}_k$ of $\widehat{\Delta}$, $\widehat{\Delta}_{\mathbb{R}} \cap \widehat{\Delta}_k$ separates $\widehat{\Delta}_k$ into at least two connected components.*

Let Δ be separated. Then the set $\widehat{\Delta}_{\mathbb{R}} \cap \widehat{\Delta}_k$ is infinite for each nondegenerate $\widehat{\Delta}_k$ (and contains a continuous curve). In particular, it contains points of Δ_0. It follows that $\left(\widehat{\Delta}_k\right)^* = \widehat{\Delta}_k$ for each nondegenerate component $\widehat{\Delta}_k$. The conjugation transforms degenerate components $z = \text{const}$ into the components $w = \text{const}$, and vice versa. The general theory of Riemann surfaces with antianalytic convolution (see [6]) says that for each nondegenerate $\widehat{\Delta}_k$, $\widehat{\Delta}_{\mathbb{R}} \cap \widehat{\Delta}_k$ separates $\widehat{\Delta}_k$ into exactly two connected components.

Proposition 1. *Δ is separated if and only if $\widehat{\Delta}_{\mathbb{R}} = \partial\widehat{\Delta}_+$.*

PROOF. Clearly, $\partial\widehat{\Delta}_+ \cap \widehat{\Delta}_k$ separates $\widehat{\Delta}_k$ into at least two connected components for all nondegenerate $\widehat{\Delta}_k$; this proves the "if" part.

To prove the converse, suppose that Δ is separated, but $\widehat{\Delta}_{\mathbb{R}} \subsetneqq \partial\widehat{\Delta}_+$. The set $\partial\widehat{\Delta}_+$ has no isolated points. Since both $\partial\widehat{\Delta}_+$ and $\widehat{\Delta}_{\mathbb{R}}$ are closed, $\partial\widehat{\Delta}_+ \setminus \widehat{\Delta}_{\mathbb{R}}$ contains an arc, say, α. Then α is contained in a nondegenerate component $\widehat{\Delta}_k$. Therefore $\widehat{\Delta}_k \setminus \widehat{\Delta}_{\mathbb{R}}$ can be obtained from the connected set $(\widehat{\Delta}_k \setminus \partial\widehat{\Delta}_+) \cup \alpha$ by adding part of its boundary. Hence $\widehat{\Delta}_k \setminus \widehat{\Delta}_{\mathbb{R}}$ is connected, a contradiction.

Suppose Δ is separated. Put

$$\gamma_c = z\,(\partial\widehat{\Delta}_+)\,, \tag{2.8}$$

it is a finite union of piecewise analytic curves. We have that $\gamma_c \subset \gamma$ (see (1.2)), and $\gamma \setminus \gamma_c$ is a finite set.

3. The projection-valued function Q.

Let A be a square matrix. Then $\varphi(A)$ is defined by means of the Riesz-Dunford calculus for any function φ, analytic in a neighbourhood of $\sigma(A)$. It is easy to see that

$$\varphi(A|R) = \varphi(A)|R\,, \tag{3.1}$$

for any invariant subspace R of A.

For $\lambda \in \sigma(A)$, we put

$$\Pi_\lambda(A) = \chi_\lambda(A)\,,$$

where χ_λ is a locally constant function on a neighbourhood of $\sigma(A)$ such that $\chi_\lambda \equiv 1$ in a neighbourhood of λ and $\chi_\lambda \equiv 0$ in a neighbourhood of $\sigma(A) \setminus \{\lambda\}$. We put $\Pi_\lambda(A) = 0$ if $\lambda \notin \sigma(A)$. The operator $\Pi_\lambda(A)$ is a parallel projection; it is called the Riesz projection corresponding to the eigenvalue λ of A.

Let

$$\widehat{\Delta}' = \widehat{\Delta}_{\mathrm{ndeg}} \cup \bigcup_{w_0 \in \sigma_C(\Lambda)} \{(z,w) :\ w \equiv \overline{w}_0\}$$

be the algebraic curve obtained from $\widehat{\Delta}$ by excluding from it the "vertical" degenerate components $z \equiv z_0$. For $z \notin \sigma(\Lambda)$, a point (z,w) is in Δ if and only if w belongs to $\sigma(C(z-\Lambda)^{-1} + \Lambda^*)$. Therefore for any $\delta = (z,w) \in \Delta_0 \setminus z^{-1}(\sigma(\Lambda))$,

$$Q(\delta) \stackrel{\mathrm{def}}{=} \Pi_w(C\,(z-\Lambda)^{-1} + \Lambda^*) \tag{3.2}$$

is a non-zero parallel projection in M. The function Q is a projection-valued meromorphic function on $\widehat{\Delta}'$. The well-known properties of the functional calculus imply that

i) $Q(\delta_1)\,Q(\delta_2) = 0$ if $\delta_1,\ \delta_2 \in \Delta_0$, $z(\delta_1) = z(\delta_2) \notin \sigma(\Lambda)$, $\delta_1 \neq \delta_2$,

ii) $\sum_{z(\delta)=z_0} Q(\delta) = I$ for any z_0 such that $z^{-1}(z_0) \subset \Delta_0$.

It follows from (1.5) that

$$\big[Q\big((z,w)\big),\, \mu(z)\big] = 0\,, \qquad \text{for } (z,w) \in \Delta_0 \setminus z^{-1}\big(\sigma(\Lambda)\big)\,. \tag{3.3}$$

4. Main results.

Theorem 1. *Let M be a finite-dimensional Hilbert space and C, Λ operators on M with $C > 0$. Define Δ, $\Delta_\pm$, Q as above, and put*

$$\mu(z) = \sum_{w:(z,w)\in\Delta_+} Q((z,w))\,, \qquad z \in \mathbb{C} \setminus \big(\sigma(\Lambda) \cup \gamma \cup z(\Delta_s)\big)\,. \tag{4.1}$$

Then there exists a subnormal operator S satisfying $C = C(S)$ and $\Lambda = \Lambda(S)$ if and only if the following conditions hold:

i) *Δ is separated.*

ii) *There exists a positive $\mathcal{L}(M)$-valued measure $de(\,\cdot\,)$ such that*

$$(\Lambda - z)^{-1}(1 - \mu(z)) = \int \frac{de(u)}{u - z}\,, \qquad z \in \mathbb{C} \setminus \big(\sigma(\Lambda) \cup \gamma \cup z(\Delta_s)\big) \tag{4.2}$$

and

$$\big(C - (\bar{u} - \Lambda^*)\,(u - \Lambda)\big)de(u) \equiv 0\,. \tag{4.3}$$

If i), ii) *hold, then the measure $de(\,\cdot\,)$ is connected with the operator S by the formula* (1.1), *and μ is Xia's mosaic of S.*

It follows, in particular, that (4.1) expresses the mosaic of any subnormal operator S of finite type in terms of matrices $C = C(S)$, $\Lambda = \Lambda(S)$.

By (1.4), the set of singularities of the function $(\Lambda - z)^{-1}(1 - \mu(z))$ is contained in the set $\sigma(\Lambda) \cup \gamma$, which has zero area. By the Hartogs-Rosenthal theorem (see [4]), $e(\cdot)$ is uniquely determined by (4.2), whenever it exists.

The next Theorem 2 is a more detailed version of Theorem 1. Before formulating it, we need to introduce a few more notions.

Definition. *The pair (C, Λ) will be called non-exceptional if there exists a finite subset Z of $\mathbb{C}$ such that for $z \in \mathbb{C} \setminus Z$, all Jordan blocks of the matrix $C(z - \Lambda)^{-1} + \Lambda^*$ corresponding to eigenvalues w with $\overline{w} \notin \sigma_C(\Lambda)$ are simple.*

In fact, the author does not know whether exceptional pairs (C, Λ) exist.

Suppose that Δ is separated, that is, $\partial\widehat{\Delta}_+ = \widehat{\Delta}_{\mathbb{R}}$. Let $\gamma_{\rm cns}$ be the set of all nonsingular points of the curve γ_c (see (2.8)), then $\gamma_c \setminus \gamma_{\rm cns}$ is finite. If $(z,\overline{z}) \in \Delta$ and $\tau'_z(z,\overline{z}) \neq 0$, $\tau'_w(z,\overline{z}) \neq 0$, then $(z,\overline{z}) \in \Delta_0$ and $z \in \gamma_{\rm cns}$.

Let us orient the curve γ_c according to the positive orientation of $\partial\widehat{\Delta}_+$ as a boundary of $\widehat{\Delta}_+$. There is a continuous function $\xi : \gamma_{\rm cns} \longrightarrow \mathbb{C}$ with $|\xi| \equiv 1$ such that $dz = i\,\xi(z)\,|dz|$ on $\gamma_{\rm cns}$. Then $\eta((z,\overline{z})) \equiv \xi(z)^2$, $z \in \gamma_{\rm cns}$.

Theorem 2. *In the above Theorem* 1*, conditions* i)-ii) *can be replaced by the following conditions.*

i') Δ *is separated.*

ii') *The pair* (C,Λ) *is non-exceptional.*

iii') *The matrix-valued measure* $\xi(z)\,(z-\Lambda)^{-1}Q((z,\overline{z}))\,|dz|\,\big|_{\gamma_c}$ *is positive and finite.*

iv') *There exists a finite subset* R *of* $\mathbb{C}$ *such that a representation*

$$(\Lambda - z)^{-1}(1-\mu(z)) = \frac{1}{2\pi}\int_{\gamma_c} \frac{(u-\Lambda)^{-1}Q((u,\overline{u}))}{u-z}\,\xi(u)\,|du| + \sum_{\zeta\in R}\frac{A_\zeta}{\zeta - z}\,, \tag{4.4}$$

holds for some non-negative matrices A_ζ, $\zeta \in R$.

v') $\big(C - (\bar{\zeta} - \Lambda^*)\,(\zeta - \Lambda)\big)A_\zeta = 0$ *for all* $\zeta \in R$.

If i')-v') *hold, the measure* $de(\,\cdot\,)$ *that corresponds to the (unique) subnormal operator* S *such that* $C = C(S)$, $\Lambda = \Lambda(S)$ *is given by*

$$de(u) = \frac{1}{2\pi}\,(u-\Lambda)^{-1}Q((u,\overline{u}))\,\xi(u)\,|du|\,\big|_{\gamma_c} + \sum_{\zeta\in R} A_\zeta\,\delta_\zeta(u)\,, \tag{4.5}$$

where δ_ζ *is the delta-measure concentrated in* ζ.

In fact, the difference between the left-hand side and the integral in the right-hand side in (4.4) is always a rational matrix function. So iv') is only a restriction on the form of this function.

We remark that if i'), ii') hold, then, by [10, formula (57)], the matrix $\xi(u)\,(u-\Lambda)^{-1}Q((u,\overline{u}))$ is self-adjoint for $u \in \gamma_c$. It seems that Xia uses ii') implicitly in some of his arguments. In Section 6 below, we

give an example of a non-exceptional pair (C, Λ) such that this matrix fails to be positive on certain arcs of γ_c. It would be interesting to know whether i') and ii') imply iii').

Theorem 2 and the above Theorem A by Xia permit one to construct the operator S from matrices $C = C(S)$, $\Lambda = \Lambda(S)$ (whenever it is possible). In [12], we will discuss this construction in detail.

5. Proofs of Theorems 1 and 2.

Lemma 1. *Let S be a subnormal operator of finite type, and put $C = C(S)$, $\Lambda = \Lambda(S)$. Let Δ be the discriminant surface of S and μ its mosaic. Let U be an open connected set contained in $\widehat{\Delta} \setminus \widehat{\Delta}_{\mathbb{R}}$, and let $\delta_0, \delta_1 \in U$, with $z(\delta_0), z(\delta_1) \in \mathbb{C} \setminus \gamma$.*

If $z(\delta_0) \in \mathbb{C} \setminus \sigma(S)$, then

$$\mu(z(\delta_1))\, Q(\delta_1) = 0\,. \tag{5.1}$$

If $\overline{w(\delta_0)} \in \mathbb{C} \setminus \sigma(S)$, then

$$\mu(z(\delta_1))\, Q(\delta_1) = Q(\delta_1)\,. \tag{5.2}$$

PROOF. We remind that for a domain G with piecewise smooth boundary, the Smirnov class $E^p(G)$ consists of functions f analytic in G such that

$$\sup_n \int_{\partial G_n} |f(z)|^p\, |dz| < \infty\,,$$

for some increasing sequence $\{G_n\}$ of domains with smooth boundaries such that $\cup G_n = G$; here $0 < p < \infty$. We refer to [2], [6] for basic properties of Smirnov classes. The Cauchy integral of any finite measure supported in $\mathbb{C} \setminus G$ belongs to $E^p(G)$ for any $p < 1$. So it follows from (1.4) that for each $p < 1$ and each connected component Ω of $\mathbb{C} \setminus \gamma$, μ belongs to $E^p(\Omega \longrightarrow \mathcal{L}(M))$. By (1.4) and the Plemelj "jump" formula [7], the interior and exterior boundary values μ_i, μ_e of μ satisfy

$$\mu_i(z) - \mu_e(z) = 2\pi i\,(z - \Lambda)\,\frac{de(z)}{|dz|}\,\frac{|dz|}{dz}\,,$$

almost everywhere on γ_c with respect to the arc length measure. Here $de(z)/|dz|$ is the Radon-Nikodim density of the absolutely continuous part of $de(z)$ with respect to $|dz|$. By (1.3), it follows that

$$\big(C(z - \Lambda)^{-1} + \Lambda^* - \overline{z}\big)\big(\mu_i(z) - \mu_e(z)\big) = 0\,, \tag{5.3}$$

almost everywhere on γ_c. We may assume that $U \setminus z^{-1}(\gamma_c)$ consists of a finite number of connected components $\Omega_0, \ldots, \Omega_k$, that $\partial\Omega_j$ and $\partial\Omega_{j+1}$ have a common arc for $j = 0, \ldots, k-1$ and $\delta_0 \in \Omega_0$, $\delta_1 \in \Omega_k$. Take any Ω_j and any $\delta = (z, w) \in \partial\Omega_j \cap \partial\Omega_{j+1} \cap \Delta_0$. Then we have $w \neq \overline{z}$ by the hypothesis.

Let ψ_δ be an analytic function in a neighbourhood of $\sigma(C(z - \Lambda)^{-1} + \Lambda^*)$ such that $\psi_\delta(u) = (u - \overline{z})^{-1}$ on a small neighbourhood of w and $\psi_\delta(u) = 0$ outside this neighbourhood. Putting $\Psi(\delta) = \psi_\delta(C(z - \Lambda)^{-1} + \Lambda^*)$, we obtain from the Riesz-Dunford calculus that

$$\Psi(\delta)\big(C\,(z - \Lambda)^{-1} + \Lambda^* - \bar{z}\big) = Q(\delta)\,,$$

so that (5.3) and (3.3) give

$$(\mu_i(z) - \mu_e(z))\,Q(\delta) = Q(\delta)\,(\mu_i(z) - \mu_e(z)) = 0\,, \tag{5.4}$$

almost everywhere on γ_c. Consider first the case $z(\delta_0) \in \mathbb{C} \setminus \sigma(S)$. Put

$$\varphi(\delta) = \mu(z(\delta))\,Q(\delta)\,, \qquad \delta \in U\,.$$

Since $\mu(z) \equiv 0$ in a neighbourhood of $z(\delta_0)$, it follows that $\varphi \equiv 0$ in Ω_0. Then (5.4) implies that $\varphi|\Omega_1$ has zero boundary values on $\partial\Omega_1 \cap \partial\Omega_0$. By the Privalov uniqueness theorem [7], $\varphi|\Omega_1 \equiv 0$. Continuing in the same way, we see that $\varphi \equiv 0$ in U, and this implies (5.1).

Now assume that $w(\delta_0) \in \mathbb{C} \setminus \sigma(S)$ and let us prove (5.2). Our arguments are motivated by the proof of [10, Lemma 7.8]. Xia proves in [9] that the function

$$S(z, w) = \int \frac{de(u)}{(u - z)\,(\overline{u} - w)}\,, \tag{5.5}$$

defined for $z, \overline{w} \in \mathbb{C} \setminus \gamma$, for $(z, w) \notin \Delta$ has a representation

$$\begin{aligned} S(z, w) = &-\big(C - (w - \Lambda^*)\,(z - \Lambda)\big)^{-1}(1 - \mu(z)) \\ &+ \mu(\overline{w})^*\big(C - (w - \Lambda^*)\,(z - \Lambda)\big)^{-1}\,. \end{aligned} \tag{5.6}$$

It follows that if $(z, w) \in \mathbb{C}^2 \setminus \Delta$ is such that $\mu(\overline{w}) = 0$, then

$$(C\,(z - \Lambda)^{-1} + \Lambda^* - w)\,(z - \Lambda)\,S(z, w) = -(1 - \mu(z))\,. \tag{5.7}$$

By continuity, we can assert that this equality also holds for $(z, w) \in \Delta$ if $z \in \mathbb{C} \setminus (\gamma \cup \sigma(\Lambda))$, $\bar{w} \in \mathbb{C} \setminus \gamma$ and $\mu(\overline{w}) = 0$. In particular, (5.7) holds

por points (z,w), $(z,w) \in \Delta_0$, in a neighbourhood of δ_0. By (1.5), the operator $C\,(z-\Lambda)^{-1}+\Lambda^*$ has an invariant subspace $(1-\mu(z))M$. It follows from (3.1) that

$$\Pi_0\big(C\,(z-\Lambda)^{-1}+\Lambda^*-w\big|(1-\mu(z))\,M\big)=(1-\mu(z))\,Q((z,w))\,. \tag{5.8}$$

On the other hand, by (5.7) and (1.5),

$$(C\,(z-\Lambda)^{-1}+\Lambda^*-w)\,(1-\mu(z))\,(z-\Lambda)\,S(z,w)=-(1-\mu(z))\,,$$

which shows that $C\,(z-\Lambda)^{-1}+\Lambda^*-w\big|(1-\mu(z))M$ is invertible. By (5.8),

$$(1-\mu(z))\,Q((z,w))\equiv 0\,,$$

for points $(z,w)\in\Delta_0$ in a neighbourhood of δ_0. Putting $\varphi(\delta)=(1-\mu(z(\delta))\,Q(\delta)$, $\delta\in U$ and proceeding as above, we obtain (5.2) in the same way as (5.1). The proof of the Lemma is complete.

PROOF OF THEOREM 1. SUFFICIENCY. This follows from the above Theorem A by Xia. Indeed, (4.1) and (2.6) imply that $\mu(z)\equiv 0$ in the unbounded component of $\mathbb{C}\setminus(\gamma\cup\sigma(\Lambda))$. Therefore, letting $z\to\infty$ in (4.2) we get $e(\mathbb{C})=I$. Now one gets that (4.2) is equivalent to

$$\mu(z)=\int\frac{u-\Lambda}{u-z}\,de(u)\,,\qquad z\in\mathbb{C}\setminus\gamma\,.$$

So all the hypotheses of Theorem A are satisfied. In the model for S, given by this theorem,

$$P_M f=\int de(u)\,f(u)\,,\qquad f\in\mathcal{L}^2(e)\,. \tag{5.9}$$

Formulas (5.9) and (1.8) imply that representation (1.1) of e holds.

NECESSITY. Now we start with a subnormal operator S of finite type and put $C=C(S)$, $\Lambda=\Lambda(S)$. If Δ is not separated, then there is a nondegenerate component $\widehat{\Delta}_k$ of $\widehat{\Delta}$ such that $\widehat{\Delta}_k\setminus\widehat{\Delta}_{\mathbb{R}}$ is connected. Take $U=\widehat{\Delta}_k\setminus\widehat{\Delta}_{\mathbb{R}}$; then U has points δ_0 with $z(\delta_0)\in\mathbb{C}\setminus\sigma(S)$ and points δ_0 with $\overline{w(\delta_0)}\in\mathbb{C}\setminus\sigma(S)$. We conclude from Lemma 1 that

$$0=\mu(z(\delta))\,Q(\delta)=Q(\delta)\,,$$

for all δ in U, which is a contradiction.

So we may assume Δ to be separated. For any nondegenerate component $\widehat{\Delta}_l$ of $\widehat{\Delta}$, $\widehat{\Delta}_l \cap \widehat{\Delta}_+$ is the connected component of $\widehat{\Delta}_l \setminus \widehat{\Delta}_{\mathbb{R}}$ containing (all) points $\delta_0 \sim (z, \infty)$, where $z \in \sigma(\Lambda)$. This follows from (2.6) and the fact that $\widehat{\Delta}_l \setminus \widehat{\Delta}_{\mathbb{R}}$ has only two connected components. Moreover, $\widehat{\Delta}_l \cap \widehat{\Delta}_+$ contains neighbourhoods of the points of the above type. A similar fact about $\widehat{\Delta}_l \cap \widehat{\Delta}_-$ takes place. By applying Lemma 1 to connected sets $\widehat{\Delta}_l \cap \widehat{\Delta}_+$ and $\widehat{\Delta}_l \cap \widehat{\Delta}_-$, we see that for $\delta \in \widehat{\Delta}_l$,

$$\mu(z(\delta))\, Q(\delta) = \begin{cases} 0\,, & \delta \in \widehat{\Delta}_-\,, \\ Q(\delta)\,, & \delta \in \widehat{\Delta}_+\,. \end{cases} \tag{5.10}$$

Now let $\widehat{\Delta}_l$ be a degenerate component of $\widehat{\Delta}'$: $\widehat{\Delta}_l = \{w \equiv w_0\}$. Then there is a point $\delta_0 \in \widehat{\Delta}_l$ with $\delta_0 \sim (\infty, w_0)$. We conclude from Lemma 1 that $\mu(z(\delta))\, Q(\delta) \equiv 0$ on $\widehat{\Delta}_l$ in this case. Therefore for $z \notin \sigma(\Lambda) \cup z(\Delta_s)$

$$\mu(z) = \mu(z) \sum_{w:(z,w)\in\widehat{\Delta}'} Q((z,w)) = \sum_{w:(z,w)\in\Delta_+} Q((z,w))\,.$$

So the righthand part of (4.1) coincides with Xia's mosaic of S. Let $e(\,\cdot\,)$ be defined by (1.1), then (4.3) follows from Theorem A, and (4.2) from (1.4).

PROOF OF THEOREM 2. First we remark that conditions i')-v') of Theorem 2 imply conditions i), ii) of Theorem 1. Indeed, if i')-v') hold, then the measure de, defined by (4.5), is finite, positive-valued, and (4.2) holds. Condition v') implies (4.2) for the discrete part of $de(\,\cdot\,)$. Since the pair (C, Λ) is non-exceptional, one has

$$(C\,(z - \Lambda)^{-1} + \Lambda^*)\, Q((z,w)) = 0$$

identically for $(z,w) \in \widehat{\Delta}_{\mathrm{ndeg}}$. This and v') give (4.3).

Conversely, let us suppose that i), ii) hold, so that $C = C(S)$, $\Lambda = \Lambda(S)$ for an operator S of finite type. First we observe that (4.1) and (1.4) imply

$$(\Lambda - z)^{-1} \sum_{(z,w)\in\widehat{\Delta}'\setminus\widehat{\Delta}_+} Q((z,w)) = (\Lambda - z)^{-1}(1 - \mu(z)) = \int \frac{de(u)}{u - z}\,.$$

It follows that there exists a finite subset F of γ_c such that

$$de\big|_{\gamma_c\setminus F} = \frac{1}{2\pi i}\,(u-\Lambda)^{-1}Q((u,\overline{u}))\,du = \frac{1}{2\pi}\,\xi(u)\,(u-\Lambda)^{-1}Q((u,\overline{u}))\,|du|\,.$$

Therefore iii') holds.

Put $R = F\cup\sigma_C(\Lambda)$. Then there exist positive matrices A_ζ, $\zeta\in R$, such that (4.5) holds. By (4.3),

$$(C\,(u-\Lambda)^{-1}+\Lambda^*-\overline{u})\,Q((u,\overline{u}))\equiv 0\,,$$

for all $u\in\gamma_c\setminus R$. By the definition of Q, the matrix $C\,(u-\Lambda)^{-1}+\Lambda^*$ for these u has no non-trivial Jordan blocks corresponding to eigenvalue $\overline{u}$.

Fix any nondegenerate component Δ_k of Δ. Then, since $\Delta_{\mathbb{R}}\cap\Delta_k$ contains an arc, there are infinitely many points $(z,w)\in\Delta_k$ such that all Jordan blocks of $C(z-\Lambda)^{-1}+\Lambda^*$ that correspond to the eigenvalue w are trivial. From a simple algebraic argument one sees that this property holds for all but a finite number of points (z,w) in Δ_k. Thus (C,Λ) is not exceptional. We conclude that all properties i')-v') take place.

6. Some additional results.

Proposition 2. *Let $C>0$ and Λ be two operators on a finite-dimensional space M. Then there exists an operator S with $C=C(S)$, $\Lambda=\Lambda(S)$ if and ony if there exist a two-sided sequence of spaces $\{M_n\}_{n\in\mathbb{Z}}$ and operators $\Lambda_n\in\mathcal{L}(M_n)$, $R_n: M_{n-1}\to M_n$ such that*

1) $M_0=M$,
2) $\operatorname{Range} R_n = M_n$ *for* $n>0$ *and* $\operatorname{Range} R_n^* = M_{n-1}$ *for* $n\leqslant 0$,
3) $R_{n+1}^*\,R_{n+1} = R_n\,R_n^* + \Lambda_n\,\Lambda_n^* - \Lambda_n^*\,\Lambda_n$,
4) $R_{n+1}^*\,\Lambda_{n+1} = \Lambda_n\,R_{n+1}^*$;
5) $\Lambda_0=\Lambda$, *and* $C = R_0\,R_0^*$,
6) $\sup_{n\in\mathbb{Z}}\|\Lambda_n\|<\infty$ *and* $\sup_{n\in\mathbb{Z}}\|R_n\|<\infty$.

For any such operators, put

$$K=\bigoplus_{n=-\infty}^{\infty} M_n\,,\qquad H=\bigoplus_{n=0}^{\infty} M_n\,,$$

and define N, S, S' by the following two-diagonal block matrix

$$N = \left(\begin{array}{c|c} S'^* & 0 \\ \hline 0 & S \end{array}\right)$$

$$= \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & & & \\ & 0 & R_{-2} & \Lambda_{-2} & 0 & & & & \\ & & 0 & R_{-1} & \Lambda_{-1} & 0 & & & \\ & & & 0 & R_0 & \Lambda_0 & 0 & & \\ & & & & 0 & R_1 & \Lambda_1 & 0 & \\ & & & & & 0 & R_2 & \Lambda_2 & 0 \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$= \left(\begin{array}{ccccc|cccc} \ddots & \ddots & \ddots & \ddots & & & & & \\ & 0 & R_{-2} & \Lambda_{-2} & 0 & & & & \\ & & 0 & R_{-1} & \Lambda_{-1} & 0 & & & \\ \hline & & & 0 & R_0 & \Lambda_0 & 0 & & \\ & & & & 0 & R_1 & \Lambda_1 & 0 & \\ & & & & & 0 & R_2 & \Lambda_2 & 0 \\ & & & & & & \ddots & \ddots & \ddots \end{array}\right),$$

so that $S = N|H$. Then S is pure subnormal, N is its minimal normal extension, and $C(S) = C$, $\Lambda(S) = \Lambda$.

This proposition may be known to specialists. A similar fact about hyponormal operators is contained in [8]. Therefore we omit the proof, and make only the following observations.

If N, S, S' are defined in the above way, then S' is also subnormal; it is called dual to S. Conditions 3), 4) comprise to the equality $N^*N = NN^*$, and 5) follows from the definition of $C(S)$ and $\Lambda(S)$. Without loss of generality, one can assume that $M_n \subset M$ for all $n \in \mathbb{Z}$, and that $R_n = \widetilde{R}_n|M_n$, with $\widetilde{R}_n \in \mathcal{L}(M)$ and $\widetilde{R}_n^* = \widetilde{R}_n \geqslant 0$. Then 3), 4) permit one to define $\widetilde{R}_n$, Λ_n by forward and backward inductive processes in a unique way. Namely, if $n \geqslant 0$ and $(\widetilde{R}_n, \Lambda_n)$ have been determined, then $\widetilde{R}_{n+1}^2$ is defined by 3) and $\Lambda_{n+1}\colon M_{n+1} \longrightarrow M_{n+1}$ is uniquely determined from 4), because $\operatorname{Ker}\big(\widetilde{R}_{n+1}|M_{n+1}\big) = 0$. On each inductive step, either 3) or 4) may fail to produce $\widetilde{R}_{n+1}, \Lambda_{n+1}$ (for instance, if the $\widetilde{R}_{n+1}^2$ obtained fails to be non-negative). One has

similar inductive definitions for $n < 0$. The subnormal S exists if and only if this two-sided inductive process fails nowhere and additionally, 6) holds.

For a point $\delta \in \widehat{\Delta}$, define the index $\tilde{\varkappa}(\delta)$ by $\tilde{\varkappa}(\delta) = \alpha_j$ if $\delta \in \widehat{\Delta}_j$; here $\widehat{\Delta}_j$ are the components of $\widehat{\Delta}$ and α_j the corresponding multiplicities, see (2.3), (2.4). Let $\zeta \in \mathbb{C} \setminus \gamma$ be such that $\widehat{\Delta}$ as a Riemann surface over the z-plane has no branching points that project into ζ. Define the index $\varkappa(\zeta)$ at ζ by

$$\varkappa(\zeta) = \sum \left\{ \tilde{\varkappa}(\delta) : \ z(\delta) = \zeta, \ \delta \in \widehat{\Delta}_+ \right\}. \tag{6.1}$$

Proposition 3. *For any $z \in \mathbb{C} \setminus \gamma$,*

$$\begin{aligned} \dim \operatorname{Ker} (S^* - \overline{z}\, I) = (&\textit{the number of positive eigenvalues} \\ &\textit{of } C - (\overline{z} - \Lambda^*)\,(z - \Lambda)) \,. \end{aligned} \tag{6.2}$$

If z is not a projection of a branching point of $\widehat{\Delta}$, then the above two quantities also coincide with $\varkappa(z)$.

PROOF. We have $(\partial/\partial w)\, \tau_j(z,w) \neq 0$ for $(z,w) \in \Delta_j$, except for a finite number of points (z,w), by virtue of the irreducibility of τ_j. Therefore $\det\bigl(C\,(z-\Lambda)^{-1} + \Lambda^* - \omega\bigr)$ has an α_j-th order zero at $\omega = w$ for $(z,w) \in \Delta_j$ for all but a finite number of points (z,w). For these points (z,w), $\operatorname{rank} Q\bigl((z,w)\bigr) = \tilde{\varkappa}\bigl((z,w)\bigr)$, and we conclude that

$$\dim \operatorname{Ker} (S^* - \overline{z}\, I) = \operatorname{rank} \mu(z) = \operatorname{rank} \sum_{(z,w) \in \widehat{\Delta}_+} Q((z,w)) = \varkappa(z) \,,$$

the first equality is from Xia's work [9, p. 277].

Let $z \in \mathbb{C} \setminus \gamma$, then by substituting $w = \overline{z}$ in (5.6), one gets

$$S(z,\overline{z}) = -A(1 - \mu(z)) + \mu(z)^* A \,,$$

where $A = \bigl(C - (\overline{z} - \Lambda^*)\,(z - \Lambda)\bigr)^{-1}$. Therefore

$$\mu(z)^* S(z,\overline{z})\, \mu(z) = \mu(z)^* A \mu(z) \,.$$

Set $n = \dim M$, and let k be the number of the positive eigenvalues of A (which equals to the right hand part of (6.2)). Since $S(z,\overline{z}) > 0$ by

(5.5), we obtain that the quadratic form (Ax, x) on M is positive on $\mu(z)M$. It follows that $\dim \mu(z)M \leqslant k$.

Similarly,

$$(1-\mu(z)^*)\,S(z,\overline{z})\,(1-\mu(z)) = -(1-\mu(z)^*)\,A(1-\mu(z))\,,$$

gives $\dim\,(1-\mu(z))\,M \leqslant n-k$. These two inequalities imply (6.2).

REMARK. If S, S' are subnormals of finite type and their essential spectra coincide, then the nondegenerate parts of their discriminant curves Δ, Δ' also coincide (we do not assert anything here about the corresponding multiplicities α_j).

Indeed, let γ_c, γ, etc. correspond to S and γ_c', γ', etc. to S'. Two subarcs of $\partial\widehat{\Delta}_+$ cannot project into the same arc in the z-plane. It follows from the last statement of Proposition 3 that $\gamma_c \subset \sigma_{\rm ess}(S) \subset \sigma(N) \subset \gamma$. Since $\gamma \setminus \gamma_c$, $\gamma' \setminus \gamma_c'$ are finite and $\sigma_{\rm ess}(S) = \sigma_{\rm ess}(S')$, we conclude that $\gamma_c = \gamma_c'$. Take any nondegenerate component Δ_j of Δ. By Theorem 1, there is an arc β of γ_c such that Δ_j contains $\beta^{\#} = \{(z,\overline{z}) : z \in \beta\}$. Since $\beta \subset \gamma_c'$, $\beta^{\#} \subset \Delta_j \cap \Delta'$. By standard algebraic geometry, this implies $\Delta_j \subset \Delta'$.

If $\sigma_{\rm ess}(S) = \sigma_{\rm ess}(S')$ and, moreover,

$$\dim \operatorname{Ker}(S^* - \overline{\lambda}) = \dim \operatorname{Ker}(S'^* - \bar{\lambda})\,,$$

for $\lambda \notin \sigma_{\rm ess}(S)$, then Proposition 3 and (6.1) imply that nondegenerate parts of Δ, Δ' coincide, and the multiplicities α_j of nondegenerate components also coincide.

AN EXAMPLE. The choice of the orientation of γ_c, made before Theorem 2, does not guarantee automatically that the matrix-valued function $\xi(z)\,(z-\Lambda)^{-1}Q((z,\overline{z}))$ is non-negative on γ_c. To see this, set $C = \left(\begin{smallmatrix} 26 & 5 \\ 5 & 15 \end{smallmatrix}\right) > 0$ and $\Lambda = \left(\begin{smallmatrix} 1 & -5 \\ 5 & 0 \end{smallmatrix}\right)$. The polynomial (2.1) takes the form

$$\tau(z,w) = (w+z-z\,w)\,(-10-z\,w) + (5z-5w)^2\,.$$

Since $\tau_z'(0,0) = \tau_w'(0,0) = -10 \neq 0$, $z=0$ is a nonsingular point of γ_c. The implicit function $w = w(z)$, whose graph near $(0,0)$ is given by equation $\tau(z,w) = 0$, has the form $w = -z + 9z^2 + \overline{\overline{o}}(z^2)$. In

particular, $|w'(z)| > 1$ for negative z with small $|z|$ and $|w'(z)| < 1$ for small positive z. Comparing with our choice of the orientation of γ_c, we conclude that $\xi((0,0)) = 1$. But from (3.2) one calculates that $Q((0,0)) = \begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix}$, so that the matrix $\xi((0,0))\,(0-\Lambda)^{-1}Q((0,0)) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ fails to be non-negative.

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