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Construction of non separable dyadic compactly supported orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity

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Abstract. By means of simple computations, we construct new classes of non separable QMF's. Some of these QMF's will lead to non separable dyadic compactly supported orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity.

1. Introduction.

In the most general sense, wavelet bases consist of discrete families of functions obtained by dilations and translations of well chosen fundamental functions [8], [9]. In this paper we will focus on compactly supported dyadic orthonormal wavelet bases for $L^2(\mathbb{R}^2)$, they are of the form

$$\{2^j\,\psi_i(2^jx_1-k_i,2^jx_2-l_i):\ j,k_i,l_i\in\mathbb{Z},\ i=1,2,3\}\,.$$

I. Daubechies has constructed compactly supported wavelet bases for $L^2(\mathbb{R})$ of arbitrarily high regularity, generalising the classic Haar basis

[6]. The most commonly used method to construct compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity, is the tensor product method [9]. It leads to the scaling function $\varphi(x_1, x_2) = \varphi_1(x_1)\,\varphi_2(x_2)$ and to the fundamental wavelets

$$\psi_a(x_1, x_2) = \varphi_1(x_1)\,\psi_2(x_2)\,,$$

$$\psi_b(x_1, x_2) = \psi_1(x_1)\,\varphi_2(x_2)\,,$$

and

$$\psi_c(x_1, x_2) = \psi_1(x_1)\,\psi_2(x_2)\,,$$

(where φ_1 (respectively φ_2) is a scaling function for $L^2(\mathbb{R})$ and ψ_1 (respectively ψ_2) is the corresponding fundamental wavelet). The scaling functons and the wavelets that result from the tensor product method are called separable. In this paper, we will also call separable the scaling functions and the wavelets that are the images of separable scaling functions and wavelets by an isometry of $L^2(\mathbb{R}^2)$ of the type $f(x) \longmapsto f(Bx)$ ($B \in SL(2, \mathbb{Z})$).

Let us now give an outline of the present article.

In the second section, by means of simple computations we construct new classes of bidimensional non separable QMF's (Theorems 2.2 and 2.3).

In the third section, we show that some of these QMF's generate non separable, compactly supported, orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity. These wavelets will be constructed by two methods:

The first method consists in perturbing the separable I. Daubechies QMF's (Theorem 3.4). Thus it leads to wavelets that are close to the I. Daubechies separable wavelets with the same number of vanishing moments (for the L^∞ norm).

The second method permits to construct wavelets that are not near to the I. Daubechies separable wavelets (Theorem 3.5).

All the results of the second section and some of the results of the third section may be adapted to multidimensional compactly supported orthonormal wavelets bases for $L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ of dilation matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where for $i = 1, 2$, A_i is a matrix $d_i \times d_i$ such that all the eigenvalues λ of A_i satisfy $|\lambda| > 1$ and $A_i\,\mathbb{Z}^{d_i} \subset \mathbb{Z}^{d_i}$ [2].

2. New classes of non separable QMF's.

A d-dimensional QMF is a trigonometric polynomial on $\mathbb{R}^d$, $M_0(\xi)$ such that

$$\begin{cases} M_0(0) = 1\,, \\ \displaystyle\sum_{s\in\{0,1\}^d} |M_0(\xi+\pi\, s)|^2 = 1\,. \end{cases} \tag{2.1}$$

The conjugate filters are 2^d-1 trigonometric polynomials on $\mathbb{R}^d$,

$$M_1(\xi), \ldots, M_{2^d-1}(\xi)$$

such that the matrix

$$U(\xi) = (M_k(\xi+\pi s))_{s\in\{0,1\}^d, k\in\{0,\ldots,2^d-1\}}\ ,$$

is unitary for all $\xi \in \mathbb{R}^d$.

When $d=1$, we will take $M_1(\xi) = -e^{-i\xi}\,\overline{M_0(\xi+\pi)}$.

If $\varphi(x)$ is a compactly supported scaling function with d variables, there exists a unique d-dimensional QMF $M_0(\xi)$ such that the Fourier transform of $\varphi(x)$ satisfies $\hat{\varphi}(\xi) = M_0(2^{-1}\xi)\,\hat{\varphi}(2^{-1}\xi)$. Thus, we have a one-to-one correspondance between the multiresolution analyses for $L^2(\mathbb{R}^d)$ with a compactly supported scaling function and the d-dimensional QMF's that satisfy the A. Cohen's criterion [3], [4].

A. Cohen's criterion is satisfied when $|M_0(\xi)| > 0$, for all $\xi \in [-\pi/2, \pi/2]^d$.

One can notice that, in general, it is not clear that one may always associate compactly supported wavelets to a multidimensional multiresolution analysis, even if its scaling function is compactly supported [7]. In this paper this difficulty will be solved by ad hoc constructions (Theorems 2.2 and 2.3).

The bidimensional QMF that corresponds to a separable wavelet basis is also said separable. It can be written

$$M(\xi_1,\xi_2) = m_1(a_{11}\,\xi_1 + a_{12}\,\xi_2)\, m_2(a_{21}\,\xi_1 + a_{22}\,\xi_2)\,, \tag{2.2}$$

where (a_{ij}) belongs to $SL(2,\mathbb{Z})$ and $m_1(x), m_2(x)$ are two monodimensional QMF's.

Let us now study a class of bidimensional QMF's , which is rather easy to construct. This class seems to be a natural extension of the class of the separable QMF's for $(a_{ij}) = I_2$.

2.1. The class of the semi separable QMF's.

Theorem 2.1. *Let* $a(x)$, $b(x)$, $m(x)$ *be three monodimensional* QMF*'s and* $\tilde{a}(x), \tilde{b}(x), \tilde{m}(x)$ *their conjugate filters. If* $c(x)$ *is a trigonometric polynomial, we set*

$$c_e(x) = \frac{1}{2}\left(c(x) + c(x+\pi)\right)$$

and

$$c_o(x) = \frac{1}{2}\left(c(x) - c(x+\pi)\right).$$

Then

$$M_0(\xi_1,\xi_2) = a(\xi_1)\, m_e(\xi_2) + b(\xi_1)\, m_o(\xi_2)\,, \tag{2.3}$$

is a bidimensional QMF *called a semi separable* QMF *and its conjugate filters are*

$$\begin{cases} M_1(\xi_1,\xi_2) = a(\xi_1)\,\{\tilde{m}\}_e(\xi_2) + b(\xi_1)\,\{\tilde{m}\}_o(\xi_2)\,, \\ M_2(\xi_1,\xi_2) = \tilde{a}(\xi_1)\, m_e(\xi_2) + \tilde{b}(\xi_1)\, m_o(\xi_2)\,, \\ M_3(\xi_1,\xi_2) = \tilde{a}(\xi_1)\,\{\tilde{m}\}_e(\xi_2) + \tilde{b}(\xi_1)\,\{\tilde{m}\}_o(\xi_2)\,. \end{cases} \tag{2.4}$$

If $a \neq b$ *and if* $m(x)$ *has at least three non vanishing coefficients, then* $M_0(\xi_1,\xi_2)$ *is non separable.*

PROOF. An obvious calculus shows that $M_0(\xi_1,\xi_2)$ is a QMF and that $M_1(\xi_1,\xi_2)$, $M_2(\xi_1,\xi_2)$, $M_3(\xi_1,\xi_2)$ are its conjugate filters.

It is a bit technical to prove that $M_0(\xi_1,\xi_2)$ is non separable. Suppose that $M_0(\xi_1,\xi_2) = m_1(a_{11}\,\xi_1 + a_{12}\,\xi_2)\, m_2(a_{21}\,\xi_1 + a_{22}\,\xi_2)$ where (a_{ij}) belongs to $SL(2,\mathbb{Z})$ and $m_1(x)$, $m_2(x)$ are two monodimensional QMF's.

Taking $\xi_1 = 0$, we get

$$m(\xi_2) = m_1(a_{12}\,\xi_2)\, m_2(a_{22}\,\xi_2)\,. \tag{$*$}$$

If $a_{12} \equiv a_{22} + 1 \pmod 2$, we will suppose that a_{12} is even and a_{22} is odd, the other case being similiar, then

$$|m_1(a_{12}\,\xi_2)|^2 = |m(\xi_2)|^2 + |m(\xi_2+\pi)|^2 = 1\,,$$

thus we have $a_{12}=0$.

If $a_{12} \equiv a_{22} \pmod 2$, since $(a_{ij}) \in SL(2,\mathbb{Z})$ then a_{12} and a_{22} are odd and

$$\begin{aligned}|m_1(a_{12}\,\xi_2)|^2\,|m_2(a_{22}\,\xi_2)|^2 + |m_1(a_{12}\,\xi_2+\pi)|^2\,|m_2(a_{22}\,\xi_2+\pi)|^2 \\ = |m(\xi_2)|^2 + |m(\xi_2+\pi)|^2 = 1\,,\end{aligned}$$

but this cannot be true since

$$(|m_1(a_{12}\,\xi_2)|^2 + |m_1(a_{12}\,\xi_2+\pi)|^2)\,(m_2(a_{22}\,\xi_2)|^2 + |m_2(a_{22}\,\xi_2+\pi)|^2) = 1\,.$$

Therefore $(*)$ implies that $a_{12}\,a_{22}=0$.

We will only study the case

$$\begin{pmatrix} \varepsilon_1 & 0 \\ a_{21} & \varepsilon_2 \end{pmatrix}, \qquad \varepsilon_i = \pm 1\,,$$

the case

$$\begin{pmatrix} a_{11} & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix}$$

is similar. We have $m(x) = m_2(\varepsilon_2\,x)$, thus

$$M_0(\xi_1,\xi_2) = m_1(\varepsilon_1\,\xi_1)\,m(\varepsilon_2\,a_{21}\,\xi_1 + \xi_2)\,.$$

Taking $\xi_2 = 0$ and then $\xi_2 = \pi$ we get

$$\frac{1}{2}\,(a(\xi_1) + b(\xi_1)) = m_1(\varepsilon_1\,\xi_1)\,m(\varepsilon_2\,a_{21}\,\xi_1)\,,$$

$$\frac{1}{2}\,(a(\xi_1) - b(\xi_1)) = m_1(\varepsilon_1\,\xi_1)\,m(\varepsilon_2\,a_{21}\,\xi_1+\pi)\,,$$

hence

$$a(\xi_1) = 2\,m_1(\varepsilon_1\,\xi_1)\,m_e(\varepsilon_2\,a_{21}\,\xi_1)\,,$$

$$b(\xi_1) = 2\,m_1(\varepsilon_1\,\xi_1)\,m_o(\varepsilon_2\,a_{21}\,\xi_1)\,.$$

Since we must have

$$|a(\xi_1)|^2 + |a(\xi_1+\pi)|^2 = 1\,,$$

$$|b(\xi_1)|^2 + |b(\xi_1+\pi)|^2 = 1\,,$$

it follows that

$$4\,|m_e(a_{21}\,\xi_1)|^2 = 1\,,$$
$$4\,|m_o(a_{21}\,\xi_1)|^2 = 1\,.$$

If $a_{21} \neq 0$ the trigonometric polynomials $m_e(a_{21}\,\xi_1)$ and $m_o(a_{21}\,\xi_1)$ are inversible and thus of the type

$$m_e(a_{21}\,\xi_1) = \frac{1}{2}\,e^{ik\xi_1}$$

and

$$m_o(a_{21}\,\xi_1) = \frac{1}{2}\,e^{il\xi_1}\,.$$

If $a_{21} = 0$ we have $a = b$.

When the QMF's $a(x)$ and $m(x)$ satisfy A. Cohen's criterion and when the norm $\|a - b\|_\infty$ is small enough, the corresponding semi separable QMF satisfies obviously this criterion. However, the constraint on $\|a - b\|_\infty$ does not seem necessary. Indeed, if for example

$$a(x) = m(x) = \frac{1}{2}\,(1 + e^{-ix})$$

and

$$b(x) = \frac{1}{2}\,(1 + e^{-i3x})\,.$$

We have $\|a - b\|_\infty = 1$ but the corresponding semi separable QMF satisfies A. Cohen's criterion [2]. This example also shows that it is even not necessary that both $a(x)$ and $b(x)$ satisfy A. Cohen's for the associated semi separable QMF satisfies it. In [2] we have however established that $m(x)$ must satisfy this criterion.

2.2. Other classes of non separable QMF's.

It is clear that many QMF's are not semi separable, even in the weak sense. This means that they are not of the form

$$a(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_e(c_{21}\,\xi_1 + c_{22}\,\xi_2) + b(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_o(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,,$$

where $a(x)$, $b(x)$, $m(x)$ are three monodimensional QMF's and (c_{ij}) is a matrix of $SL(2,\mathbb{Z})$.

Let $\eta \in]0,1[$ and $\alpha_\eta(x)$, $\beta_\eta(x)$ the two trigonometric polynomials in one variable defined by

$$\alpha_\eta(x) = 1 - \eta\, q(x)\,, \tag{2.5}$$

where $q(x)$ is a trigonometric polynomial vanishing in zero, with values in $[0,1]$ and such that $q \neq 0$, and by

$$|\alpha_\eta(x)|^2 + |\beta_\eta(x)|^2 = 1\,. \tag{2.6}$$

The existence of $\beta_\eta(x)$ is given by the Fejer-Riesz Lemma.

Theorem 2.2. *Let $S_0(\xi_1,\xi_2) = a(\xi_1)\, b(\xi_2)$ be a separable* QMF *and let $S_1(\xi_1,\xi_2)$, $S_2(\xi_1,\xi_2)$, $S_3(\xi_1,\xi_2)$ be its conjugate filters ($\tilde{a}(x) = -e^{-ix}\,\overline{a(x+\pi)}$ and $\tilde{b}(x) = -e^{-ix}\,\overline{b(x+\pi)}$ will be the conjugate filters of $a(x)$ and $b(x)$). If $\lambda(\xi_1,\xi_2)$ and $\mu(\xi_1,\xi_2)$ are two trigonometric polynomials π-periodic in ξ_1 and in ξ_2 and such that*

$$\begin{cases} \lambda(0,0) = 1\,, \\ |\lambda(\xi_1,\xi_2)|^2 + |\mu(\xi_1,\xi_2)|^2 = 1\,. \end{cases}$$

Then

$$R_0(\xi_1,\xi_2) = \lambda(\xi_1,\xi_2)\, S_0(\xi_1,\xi_2) + \mu(\xi_1,\xi_2)\, S_1(\xi_1,\xi_2)\,, \tag{2.7}$$

is a QMF *and its conjugate filters are*

$$\begin{cases} R_1(\xi_1,\xi_2) = \overline{\mu(\xi_1,\xi_2)}\, S_0(\xi_1,\xi_2) - \overline{\lambda(\xi_1,\xi_2)}\, S_1(\xi_1,\xi_2)\,, \\ R_2(\xi_1,\xi_2) = \lambda(\xi_1,\xi_2)\, S_2(\xi_1,\xi_2) + \mu(\xi_1,\xi_2)\, S_3(\xi_1,\xi_2)\,, \\ R_3(\xi_1,\xi_2) = \overline{\mu(\xi_1,\xi_2)}\, S_2(\xi_1,\xi_2) - \overline{\lambda(\xi_1,\xi_2)}\, S_3(\xi_1,\xi_2)\,. \end{cases} \tag{2.8}$$

Moreover

i) *If $\lambda(\xi_1,\xi_2) = \alpha_\eta(2\,\xi_1)$ and $\mu(\xi_1,\xi_2) = \beta_\eta(2\,\xi_1)$ (as defined by* (2.5) *and* (2.6)*), the* QMF *$R_0(\xi_1,\xi_2)$ is non separable when $S_1(\xi_1,\xi_2)$ is not the filter $\tilde{a}(\xi_1)\, b(\xi_2)$ and $R_0(\xi_1,\xi_2)$ has zeros of order greater or equal than 2 in $(\pi,0)$, $(0,\pi)$ and (π,π).*

ii) *If $\lambda(\xi_1,\xi_2) = \alpha_\eta(2\,(\xi_1+\xi_2))$ and $\mu(\xi_1,\xi_2) = \beta_\eta(2\,(\xi_1+\xi_2))$ or if $\lambda(\xi_1,\xi_2) = \alpha_\eta(2\,(\xi_1-\xi_2))$ and $\mu(\xi_1,\xi_2) = \beta_\eta(2\,(\xi_1-\xi_2))$, then the* QMF *$R_0(\xi_1,\xi_2)$ is non separable.*

iii) *If* $S_1(\xi_1,\xi_2)=\tilde{a}(\xi_1)\,b(\xi_2)$, $\lambda(\xi_1,\xi_2)=\alpha_\eta(2\,\xi_1)$ *and* $\mu(\xi_1,\xi_2)=\beta_\eta(2\,\xi_1)\,e^{-i2\xi_2}$, *then the* QMF $R_0(\xi_1,\xi_2)$ *is non separable.*

PROOF. One sees immediately that $R_0(\xi_1,\xi_2)$ is a QMF and that $R_1(\xi_1,\xi_2)$, $R_2(\xi_1,\xi_2)$, $R_3(\xi_1,\xi_2)$ are its conjugate filters.

Let us show i).

We will begin by the case where $S_1(\xi_1,\xi_2)=a(\xi_1)\,\tilde{b}(\xi_2)$. Suppose that

$$R_0(\xi_1,\xi_2)=m_1(c_{11}\,\xi_1+c_{12}\,\xi_2)\,m_2(c_{21}\,\xi_1+c_{22}\,\xi_2)\,,$$

where (c_{ij}) belongs to $SL(2,\mathbb{Z})$ and $m_1(x)$, $m_2(x)$ are two monodimensional QMF's. Taking successively $(\xi_1,\xi_2)=(x,0)$, $(0,x)$ and (x,π) where x is an arbitrary real one obtains

$$\text{(a)}\qquad \alpha(2\,x)\,a(x)=m_1(c_{11}\,x)\,m_2(c_{21}\,x)\,,$$

$$\text{(b)}\qquad b(x)=m_1(c_{12}\,x)\,m_2(c_{22}\,x)\,,$$

$$\text{(c)}\qquad \beta(2\,x)\,a(x)=m_1(c_{11}\,x+c_{12}\,\pi)\,m_2(c_{21}\,x+c_{22}\,\pi)\,.$$

Since the product of two QMF's in the same variables is never a QMF (see the proof of the Theorem 2.1), it results from (b) that $c_{12}\,c_{22}=0$. This implies that

$$(c_{ij})=\begin{pmatrix}\varepsilon_1 & 0\\ c_{21} & \varepsilon_2\end{pmatrix}$$

or

$$(c_{ij})=\begin{pmatrix}c_{11} & \varepsilon_1\\ \varepsilon_2 & 0\end{pmatrix}$$

with $\varepsilon_i=\pm1$. We will suppose that we are in the first case, the second case being similar.

We notice that whatever the value of the integer c_{21} may be, one cannot have for all x, $|\alpha(2\,x)|^2=|m_2(c_{21}\,x)|^2$. Indeed, if $c_{21}=0$ then $\beta=0$ and else $\alpha(2\pi/c_{21})=0$. In both cases the hypotheses are contradicted.

It follows from (a) and (c) that

$$\begin{aligned}|a(x)|^2&=|\alpha(2\,x)|^2\,|a(x)|^2+|\beta(2\,x)|^2\,|a(x)|^2\\&=|m_1(\varepsilon_1\,x)|^2\,|m_2(c_{21}\,x)|^2+|m_1(\varepsilon_1\,x)|^2\,|m_2(c_{21}\,x+\pi)|^2\\&=|m_1(\varepsilon_1 x)|^2\,.\end{aligned}$$

Thus by using (a) we obtain the contradiction $|\alpha(2\,x)|^2 = |m_2(c_{21}\,x)|^2$ for all $x \in \mathbb{R}$.

Let us study now the case where $S_1(\xi_1, \xi_2) = \tilde{a}(\xi_1)\,\tilde{b}(\xi_2)$. As previously, we will suppose that

$$R_0(\xi_1, \xi_2) = m_1(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_2(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,.$$

Taking successively $(\xi_1, \xi_2) = (x, 0)$, $(0, x)$ and (x, π) where x is an arbitrary real, one obtains

$$\text{(a)} \qquad \alpha(2\,x)\,a(x) = m_1(c_{11}\,x)\,m_2(c_{21}\,x)\,,$$

$$\text{(b)} \qquad b(x) = m_1(c_{12}\,x)\,m_2(c_{22}\,x)\,,$$

$$\text{(c)} \qquad \beta(2\,x)\,\tilde{a}(x) = m_1(c_{11}\,x + c_{12}\,\pi)\,m_2(c_{21}\,x + c_{22}\,\pi)\,.$$

It follows from (b), as previously, that $c_{12}\,c_{22} = 0$ and we can suppose that

$$(c_{ij}) = \begin{pmatrix} \varepsilon_1 & 0 \\ c_{21} & \varepsilon_2 \end{pmatrix}$$

with $\varepsilon_i = \pm 1$. (a) and (c) imply then that

$$\alpha(2\,x)\,a(x) + \beta(2\,x)\,\tilde{a}(x) = 2\,m_1(\varepsilon_1\,x)\,m_{2,e}(c_{21}\,x)\,,$$

$$\alpha(2\,x)\,a(x) - \beta(2\,x)\,\tilde{a}(x) = 2\,m_1(\varepsilon_1\,x)\,m_{2,o}(c_{21}\,x)\,,$$

where

$$m_{2,e}(c_{21}\,x) = \frac{1}{2}\,(m_2(c_{21}\,x) + m_2(c_{21}\,x + \pi))$$

and

$$m_{2,o}(c_{21}\,x) = \frac{1}{2}\,(m_2(c_{21}\,x) - m_2(c_{21}\,x + \pi))\,.$$

As

$$|\alpha(2\,x)\,a(x) + \beta(2\,x)\,\tilde{a}(x)|^2 + |\alpha(2\,x)\,a(x+\pi) + \beta(2\,x)\,\tilde{a}(x+\pi)|^2 = 1\,,$$

$$|\alpha(2\,x)\,a(x) - \beta(2\,x)\,\tilde{a}(x)|^2 + |\alpha(2\,x)\,a(x+\pi) - \beta(2\,x)\,\tilde{a}(x+\pi)|^2 = 1\,,$$

we have

$$4\,|m_{2,e}(c_{21}\,x)|^2 = 1\,,$$

$$4\,|m_{2,o}(c_{21}\,x)|^2 = 1\,.$$

If $c_{21} \neq 0$, it follows from the two last equalities that the QMF $m_2(x)$ has only two non vanishing coefficients. This is impossible since $R_0(\xi_1, \xi_2)$ has zeros of order greater or equal than 2 in $(\pi, 0)$, $(0, \pi)$ and (π, π). If $c_{21} = 0$, it follows from (c) that for all x, $\beta(2\,x)\,\tilde{a}(x) = 0$. This is impossible.

Let us show ii).

As the variables ξ_1 and ξ_2 play the same role we will only study the case where $S_1(\xi_1, \xi_2) = e(\xi_1)\,\tilde{b}(\xi_2)$ with $e = \tilde{a}$ or $e = a$. As previously, we will suppose that

$$R_0(\xi_1, \xi_2) = m_1(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_2(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,.$$

Taking successively $(\xi_1, \xi_2) = (x, 0)$, $(0, x)$ and (x, π) one obtains

$$\text{(a)} \qquad \alpha(2\,x)\,a(x) = m_1(c_{11}\,x)\,m_2(c_{21}\,x)\,,$$

$$\text{(b)} \qquad \alpha(2\,x)\,b(x) = m_1(c_{12}\,x)\,m_2(c_{22}\,x)\,, \qquad \text{when } e = \tilde{a}\,,$$

$$\text{(b')} \quad \alpha(2\,x)\,b(x) + \beta(2\,x)\,\tilde{b}(x) = m_1(c_{12}\,x)\,m_2(c_{22}\,x)\,, \quad \text{when } e = a\,,$$

$$\text{(c)} \qquad \beta(2\,x)\,e(x) = m_1(c_{11}\,x + c_{12}\,\pi)\,m_2(c_{21}\,x + c_{22}\,\pi)\,.$$

When $e = \tilde{a}$, it follows from (a) and (b) that c_{11}, c_{12}, c_{21} and c_{22} are all odd. Indeed, suppose for example that c_{11} is even, c_{21} would necessarily be odd and then (a) would imply that

$$\begin{aligned} |\alpha(2\,x)|^2 &= |\alpha(2\,x)|^2\,|a(x)|^2 + |\alpha(2\,x)|^2\,|a(x+\pi)|^2 \\ &= |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x)|^2 + |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x+\pi)|^2 \\ &= |m_1(c_{11}\,x)|^2\,. \end{aligned}$$

But we never have $|\alpha(2\,x)|^2 = |m_1(c_{11}\,x)|^2$ for all x. Thus it results from (c) that

$$\begin{aligned} |\beta(2\,x)|^2 &= |\beta(2\,x)|^2\,|e(x)|^2 + |\beta(2\,x)|^2\,|e(x+\pi)|^2 \\ &= |m_1(c_{11}\,x+\pi)|^2\,|m_2(c_{21}\,x+\pi)|^2 + |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x)|^2\,, \end{aligned}$$

and it results from (a) that

$$|\alpha(2\,x)|^2 = |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x)|^2 + |m_1(c_{11}\,x+\pi)|^2\,|m_2(c_{21}\,x+\pi)|^2\,.$$

This leads to the contradiction $|\alpha|^2 = |\beta|^2$.

When $e = a$, since $\alpha(2\,x)\,b(x) + \beta(2\,x)\,\tilde{b}(x)$ is a QMF, it follows from (b') that $c_{12}\,c_{22} = 0$. So, as previously we can suppose that

$$(c_{ij}) = \begin{pmatrix} \varepsilon_1 & 0 \\ c_{21} & \varepsilon_2 \end{pmatrix}$$

with $\varepsilon_i = \pm 1$. Moreover (a) implies that c_{21} is odd.

It results then from (a) and (c) that

$$\begin{aligned} |a(x)|^2 &= |\alpha(2\,x)|^2\,|a(x)|^2 + |\beta(2\,x)|^2\,|a(x)|^2 \\ &= |m_1(\varepsilon_1\,x)|^2\,|m_2(c_{21}\,x)|^2 + |m_1(\varepsilon_1\,x)|^2\,|m_2(c_{21}\,x + \pi)|^2 \\ &= |m_1(\varepsilon_1\,x)|^2\,. \end{aligned}$$

Thus (a) implies that $|\alpha(2\,x)|^2 = |m_2(c_{21}\,x)|^2$ for all x, which is impossible.

We can prove by the same method that

$$R_0(\xi_1, \xi_2) = \alpha_\eta(2\,(\xi_1 - \xi_2))\,S_0(\xi_1, \xi_2) + \beta_\eta(2\,(\xi_1 - \xi_2))\,S_1(\xi_1, \xi_2)\,,$$

where $S_1(\xi_1, \xi_2)$ is any conjugate filter of $S_0(\xi_1, \xi_2)$, is non separable.

Let us show iii).

As previously, we will suppose that

$$(*) \qquad R_0(\xi_1, \xi_2) = m_1(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_2(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,.$$

Taking $(\xi_1, \xi_2) = (x, \pi)$ one obtains

$$R_0(x, \pi) = m_1(c_{11}\,x + c_{12}\,\pi)\,m_2(c_{21}\,x + c_{22}\,\pi) = 0\,,$$

and it follows that $c_{11}\,c_{21} = 0$. Thus we may suppose that

$$(c_{ij}) = \begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & c_{22} \end{pmatrix}\,,$$

where $\varepsilon_i = \pm 1$, the other case being similar. Then taking successively in $(*)$, $(\xi_1, \xi_2) = (x, 0)$ and $(0, x)$ one obtains

$$\alpha(2\,x)\,a(x) + \beta(2\,x)\,\tilde{a}(x) = m_2(\varepsilon_2\,x)\,,$$

$$b(x) = m_1(\varepsilon_1\,x)\,m_2(c_{22}\,x)\,.$$

The last equality implies that $c_{22} = 0$ and $b(x) = m_1(\varepsilon_1 x)$. Thus we have

$$R_0(\xi_1, \xi_2) = m_1(\varepsilon_1 \xi_2)\, m_2(\varepsilon_2 \xi_2) = b(\xi_2)\, (\alpha(2\,\xi_1)\, a(\xi_1) + \beta(2\,\xi_1)\, \tilde{a}(\xi_1))\,,$$

and it follows that

$$\begin{aligned} b(\xi_2)\, (\alpha(2\,\xi_1)\, a(\xi_1) &+ \beta(2\,\xi_1)\, \tilde{a}(\xi_1)\, e^{-i2\xi_2}) \\ &= b(\xi_2)\, (\alpha(2\,\xi_1)\, a(\xi_1) + \beta(2\,\xi_1)\, \tilde{a}(\xi_1))\,, \end{aligned}$$

which leads to the contradiction: for all ξ_2, $e^{-i2\xi_2} = 1$.

3. Some of the previous QMF's lead to wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity.

In this section we give two methods for constructing non separable orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity.

In all this section the norm on $\mathbb{R}^2$ will be $|(\xi_1, \xi_2)| = \sup\{|\xi_1|, |\xi_2|\}$.

3.1. The method by perturbing the I. Daubechies QMF's.

Proposition 3.1. *For all $L \geq 1$, let $D_L(\xi_1, \xi_2) = d_L(\xi_1)\, d_L(\xi_2)$ be the separable I. Daubechies* QMF *such that*

$$|d_L(x)|^2 = c_L \int_x^\pi \sin^{2L-1} t\, dt\,.$$

For all $\varepsilon > 0$, one can construct a non separable QMF *$D_{L,\varepsilon}(\xi_1, \xi_2)$ that satisfies*

i) $\|D_{L,\varepsilon} - D_L\|_\infty \leq \varepsilon$,

ii) *$D_{L,\varepsilon}(\xi_1, \xi_2)$ has zeros of order L on $(\pi, 0)$, $(0, \pi)$ and (π, π),*

iii) *the size of $D_{L,\varepsilon}(\xi_1, \xi_2)$ is independent on ε.*

Moreover $D_{L,\varepsilon}(\xi_1, \xi_2)$ may be chosen of the type (2.3) or of the type (2.7). φ_L and $\varphi_{L,\varepsilon}$ will be the scaling functions that correspond to $D_L(\xi_1, \xi_2)$ and $D_{L,\varepsilon}(\xi_1, \xi_2)$.

Proof. See [2, Chapter 3] and see also [1].

It is clear that for $\varepsilon > 0$ small enough $D_{L,\varepsilon}(\xi_1, \xi_2)$ will satisfy the A. Cohen's criterion.

Proposition 3.1 remains valid, if we replace the QMF $D_L(\xi_1, \xi_2)$ by any other separable QMF $a(\xi_1)\, b(\xi_2)$ that has zeros of order L on $(\pi, 0)$, $(0, \pi)$ and (π, π).

We can now state the main result of this subsection.

Theorem 3.2. *The* QMF*'s $D_{L,\varepsilon}(\xi_1, \xi_2)$ generate for $\varepsilon > 0$ small enough non separable orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity. We will say that these wavelets are obtained by perturbing the I. Daubechies* QMF*'s.*

Proof. The critical Sobolev exponent of $f \in L^2(\mathbb{R}^2)$ is by definition

$$\alpha(f) = \sup \Big\{ \alpha : \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \,(1 + |\xi|^{\alpha})^2 \, d\xi < \infty \Big\}.$$

R. Q. Jia has shown in [8] that if $M(\xi)$ is a QMF that satisfies A. Cohen's criterion and that has zeros of order L in $(\pi, 0)$, $(0, \pi)$ and (π, π), then the critical Sobolev exponent of φ the corresponding scaling function is

$$(*) \qquad \alpha(\varphi) = -\log_4 \Big(\rho\Big(\frac{T_M}{\tau_{2L}}\Big)\Big),$$

where $\rho(T_M/\tau_{2L})$ is the spectral radius of the restriction of the transfer operator

$$T_M f(\xi) = \sum_{\nu \in \{0,1\}^2} \Big| M\Big(\frac{\xi}{2} + \pi\,\nu\Big)\Big|^2 f\Big(\frac{\xi}{2} + \pi\,\nu\Big)$$

to the vector space τ_{2L} of the trigonometric polynomials that have a zero of order greater or equal than $2L$ in $(0, 0)$.

Let $\alpha(\varphi_L)$ and $\alpha(\varphi_{L,\varepsilon})$ the critical Sobolev exponent of the scaling functions φ_L and $\varphi_{L,\varepsilon}$ (as defined in the Proposition 3.1). Since the regularity of the I. Daubechies scaling function φ_L can arbitrarily high when L is big enough, we have $\lim_{L\to\infty} \alpha(\varphi_L) = +\infty$. At last, it follows from $(*)$ and from the continuity of the spectral radius, that $\lim_{\varepsilon\to 0} \alpha(\varphi_{L,\varepsilon}) = \alpha(\varphi_L)$, which implies the Theorem 3.2.

We have solved the open theorical problem of establishing the existence of non separable orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitarily high regularity. However the wavelets we have obtained are probabely very similar to the I. Daubechies separable wavelets since $\lim_{\varepsilon\to 0}\varphi_{L,\varepsilon}=\varphi_L$ (for the L^∞ norm) (see [2, Chapter 4]).

3.2. Another method of construction.

The aim of this subsection is to construct non separable orthonormal compactly supported wavelets of arbitrarily high regularity that are not near to the I. Daubechies bidimensional wavelets with the same number of vanishing moments (for the L^∞ norm).

Let us first give a condition ensuring the decrease at infinite of the Fourier transform of a scaling function.

Theorem 3.3. *Given two reals $\delta\in]0,1[$ and $C\geq(2\pi)^{-1}$, there exists an exponent $\alpha=\alpha(\delta,C)>0$ having the following property. If $M(\xi_1,\xi_2)$ is a* QMF *that satisfies for some integer $N\geq 1$*

a) $|M(\xi_1,\xi_2)|\leq\delta^N$ *when* $\xi_1\in[2\pi/3,4\pi/3]$ *or* $\xi_2\in[2\pi/3,4\pi/3]$,

b) $|M(\xi_1,\xi_2)|\leq C^N|(\xi_1-s_1\pi,\xi_2-s_2\pi)|^N$ *for all* ξ_1, ξ_2 *for all* $(s_1,s_2)\in\{0,1\}^2$ *and* $(s_1,s_2)\neq(0,0)$,

c) $|M(\xi_1,\xi_2)|=|M(-\xi_1,-\xi_2)|$ *for all* ξ_1, ξ_2.

Then φ the scaling function that corresponds to $M(\xi_1,\xi_2)$, satisfies $\hat{\varphi}(\xi_1,\xi_2)=O\left(|(\xi_1,\xi_2)|^{-\alpha N}\right)$.

To prove the Theorem 3.3 we need the following lemma.

Lemma 3.4. *Given two reals $\delta\in]0,1[$ and $C\geq 1$, there exists an exponent $\alpha=\alpha(\delta,C)>0$ having the following property. If $f(s,t)$ is a continuous function from $\mathbb{R}^2$ to $[0,1]$, 1-periodic in s and t, which satisfies*

i) $0\leq f(s,t)\leq\delta<1$, *when* $s\in[1/3,2/3]$ *or* $t\in[1/3,2/3]$,

ii) $f(s,t)\leq C\,|(s-\nu_1/2,t-\nu_2/2)|$, *for all* s,t, *for all* $(\nu_1,\nu_2)\in\{0,1\}^2$ *and* $(\nu_1,\nu_2)\neq(0,0)$,

iii) $f(s,t)=f(-s,-t)$, *for all* s,t.

Then if $j \geq 1$ and if (s,t) satisfies $1/4 \leq |(s,t)| \leq 1/2$ we have the inequality

$$f(s,t)\, f(2\,s, 2\,t) \cdots f(2^j\,s, 2^j\,t) \leq 2^{-\alpha j}\,.$$

PROOF. First we set $h(s,t) = f(s,t)\, f(2\,s, 2\,t)$. The function $f(s,t)$ satisfies the property ii) and the inequality i) when $s \in [1/6, 5/6]$ or when $t \in [1/6, 5/6]$. Moreover this function being with values in $[0,1]$ we have

$$\prod_{k=0}^{j-1} h(2^k\,s, 2^k\,t) \geq \Big(\prod_{k=0}^{j} f(2^k\,s, 2^k\,t)\Big)^2\,,$$

thus it is sufficient to show that for some $\beta = \beta(\delta, C) > 0$ for all $j \geq 1$

$$\prod_{k=0}^{j-1} h(2^k\,s, 2^k\,t) \leq 2^{-\beta j}\,.$$

Consider now (s,t) satisfying $|(s,t)| = |s| \in [1/4, 1/2]$. Because of the periodicity of $f(s,t)$ and because of iii), one can suppose that $(s,t) \in [1/4, 1/2] \times [0,1]$. It follows that

$$s = \frac{1}{4} + \frac{\alpha_3}{8} + \cdots\,, \qquad t = \frac{\beta_1}{2} + \frac{\beta_2}{4} + \cdots\,,$$

where α_j, $\beta_j \in \{0,1\}$.

We then define $q_1, \ldots, q_r$ the transition indices of the finite vectorial sequence $(\alpha_0, \beta_0), \ldots, (\alpha_{j+1}, \beta_{j+1})$ where $(\alpha_0, \beta_0) = (0,0)$ as follows: r will be the number of the indices q that satisfy, $0 \leq q \leq j-1$ and $(\alpha_{q+1}, \beta_{q+1}) \neq (\alpha_{q+2}, \beta_{q+2})$, $q_1 = 0$ and for all l, $2 \leq l \leq r$,

$$q_l = \min\{n :\ q_{l-1} < n \leq j-1 \text{ and } (\alpha_{n+1}, \beta_{n+1}) \neq (\alpha_{n+2}, \beta_{n+2})\}\,.$$

For $m = 1, \ldots, r$ we set $l_m = q_{m+1} - q_m$ ($q_{r+1} = j$ by convention), thus we have $\sum_{m=1}^{r} l_m = j$.

We have introduced the transition indices in order to get the inequalities

$$(M_1) \qquad h(2^{q_m}\,s, 2^{q_m}\,t) \leq \delta\,,$$

$$(M_2) \qquad h(2^{q_m}\,s, 2^{q_m}\,t) \leq C\, 2^{-l_m}\,.$$

We have $2^{q_m} s \in [1/6, 5/6]$ or $2^{q_m} t \in [1/6, 5/6]$ therefore i) implies (M_1). To prove (M_2) we will suppose that $\alpha_{q_m+1} \neq \alpha_{q_m+2}$, we then have $|2^{q_m} s - 1/2| \leq 2^{-(l_m+1)}$. So, if $\beta_{q_m+1} = \beta_{q_m+2}$ it follows that $|2^{q_m} t - \beta_{q_m+1}| \leq 2^{-(l_m+1)}$ and else that $|2^{q_m} t - 1/2| \leq 2^{-(l_m+1)}$, in the both cases ii) implies (M_2).

At last, for all $j \geq 1$ we have

$$h(s,t)\, h(2\,s, 2\,t) \cdots h(2^{j-1}\,s, 2^{j-1}\,t) \leq h(2^{q_1}\,s, 2^{q_1}\,t) \cdots h(2^{q_r}\,s, 2^{q_r}\,t)\,.$$

So, if A and β are two reals such that $2^A \geq C$, $2^{-\beta A} \geq \delta$ and $2^{A(1-\beta)} \geq C$ (for example $A = \log_2 C + \log_2(1/\delta)$ and $\beta = \log_2(1/\delta)/A$) then we will have

$$(*) \qquad h(2^{q_m}\,s, 2^{q_m}\,t) \leq 2^{-\beta l_m}\,,$$

indeed, when $l_m < A$ (M_1) implies that $h(2^{q_m} s, 2^{q_m} t) \leq 2^{-\beta A} < 2^{-\beta l_m}$ and when $l_m \geq A$ (M_2) implies that $h(2^{q_m} s, 2^{q_m} t) \leq C\, 2^{-l_m} \leq 2^{-\beta l_m}$.

Since $\sum_1^r l_m = j$ it results from $(*)$ that

$$h(2^{q_1}\,s, 2^{q_1}\,t) \cdots h(2^{q_r}\,s, 2^{q_r}\,t) \leq 2^{-\beta j}\,.$$

PROOF (OF THE THEOREM 3.3). If $M(\xi_1, \xi_2)$ is a QMF satisfying (a), (b) and (c) the function $f(s,t) = |M(2\pi\, s, 2\pi\, t)|^{1/N}$ satisfies the conditions i), ii) and iii) of the Lemma 3.4. Let $(\xi_1, \xi_2) \in \mathbb{R}^2$ such that $|(\xi_1, \xi_2)| \geq 2\pi$ and let $j \geq 1$ the integer such that $2^j \pi \leq |(\xi_1, \xi_2)| \leq 2^{j+1}\,\pi$. Thus, if

$$(s,t) = \frac{1}{2^{j+2}\pi}\,(\xi_1, \xi_2)\,,$$

we have $1/4 \leq |(s,t)| \leq 1/2$ and it results from the Lemma 3.4 that

$$\begin{aligned} |\hat{\varphi}_M(\xi_1, \xi_2)| &\leq \big(f(s,t)\, f(2\,s, 2\,t) \cdots f(2^j\,s, 2^j\,t)\big)^N \\ &\leq 2^{-\alpha N j} \\ &\leq C(\alpha, N)\, |(\xi_1, \xi_2)|^{-\alpha N}\,. \end{aligned}$$

From now on, our aim will be to construct a sequence of non separable QMF's $\{A_L(\xi_1, \xi_2)\}_{L \geq 1}$ such that for all L big enough,

i) $A_L(\xi_1, \xi_2)$ satisfies the conditions (a), (b) and (c) of the Theorem 3.3,

ii) $A_L(\xi_1, \xi_2)$ satisfies A. Cohen's criterion,

iii) $A_L(\xi_1, \xi_2)$ is not near to the I. Daubechies QMF $D_L(\xi_1, \xi_2)$ as defined in the Proposition 3.1. More precisely we will have

$$\liminf_{L\to\infty} \|A_L - D_L\|_\infty \geq \frac{1}{4}\,.$$

Let $A_{L,\eta}(\xi_1, \xi_2)$ be a QMF of the form

$$A_{L,\eta}(\xi_1,\xi_2) = d_L(\xi_2)\,(\alpha_\eta(2\,\xi_1)\,d_L(\xi_1) + \beta_\eta(2\,\xi_1)\,\tilde{d}_L(\xi_1)\,e^{-i2\xi_2})\,, \tag{3.1}$$

where,

- $\eta \in\,]0,1[\,$,
- $d_L(x)$ is the monodimensional I. Daubechies QMF such that

$$|d_L(x)|^2 = c_L \int_x^\pi \sin^{2L-1} t\,dt$$

and $\tilde{d}_L(x) = -e^{-ix}\,\overline{d_L(x+\pi)}$ is its conjugate filter,

- $\alpha_\eta(x) = 1 - \eta\,q(x)$ and $\beta_\eta(x)$ are the trigonometric polynomials as defined by (2.5) and (2.6).

Let us first give some useful properties of the QMF $d_L(x)$.

Proposition 3.5. *The monodimensional I. Daubechies* QMF $d_L(x)$ *satisfies:*

i) *for all real* $\alpha \in\,]0,\pi/4[\,$, *one can find a real* $\delta \in\,]0,1[$ *such that for all* L *big enough, for all* $x \in [\pi/2+\alpha, 3\pi/2-\alpha]$, $|d_L(x)| \leq \delta^L$,

ii) *there exists a real* $C \geq (2\pi)^{-1}$ *such that for all* $x \in \mathbb{R}$, $|d_L(x)| \leq C^L\,|x-\pi|^L$,

iii) *for all* $x \in\,]-\pi/2, \pi/2[\,$,

$$\lim_{L\to\infty} |d_L(x)| = 1 \qquad \textit{and} \qquad \lim_{L\to\infty} |\tilde{d}_L(x)| = 0\,.$$

PROOF. The function $|d_L(x)|$ being even one can suppose that $x \in [0,\pi]$. One can notice that $c_L = O(\sqrt{L})$. For all $\alpha > 0$, we have for all $x \in [\pi/2+\alpha, \pi]$, $|d_L(x)|^2 \leq |\pi/2-\alpha|\,c_L\,|\sin^{2L-1}(\pi/2+\alpha)|$ which implies i).

We have obviously ii) since

$$|d_L(x)|^2 \le c_L \int_x^\pi (\pi - t)^{2L-1}\, dt \le C' \, |x - \pi|^{2L} .$$

iii) is a consequence of i) and of $|d_L(x)|^2 + |d_L(x + \pi)|^2 = 1$.

The following proposition will permit us to make an appropriate choice of the real η that occurs in (3.1).

Proposition 3.6. *If* $\eta_0 = 1/4$ *then for all* $L \ge 1$ *the* QMF $A_{L,\eta_0}(\xi_1, \xi_2)$, *as defined by* (3.1) *satisfies A. Cohen's criterion.*

Proof. Let us show that for all $(\xi_1, \xi_2) \in [-\pi/2, \pi/2]^2$ and for all $L \ge 1$ we have $|A_{L,\eta_0}(\xi_1, \xi_2)| > 0$. As $|d_L(\xi_2)| > 0$ it is sufficient to show that

$$|\alpha_{\eta_0}(2\,\xi_1)\, d_L(\xi_1) + \beta_{\eta_0}(2\,\xi_1)\, \tilde{d}_L(\xi_1)\, e^{-i2\xi_2}| > 0 .$$

We have

$$|\alpha_{\eta_0}(2\,\xi_1)\, d_L(\xi_1) + \beta_{\eta_0}(2\,\xi_1)\, \tilde{d}_L(\xi_1)\, e^{-i2\xi_2}| \ge \frac{\sqrt{2}}{2} \left(|\alpha_{\eta_0}(2\,\xi_1)| - |\beta_{\eta_0}(2\,\xi_1)| \right) .$$

At last, since $|\alpha_{\eta_0}(2\,\xi_1)| \ge 3/4$ and $|\alpha_{\eta_0}(2\,\xi_1)|^2 + |\beta_{\eta_0}(2\,\xi_1)|^2 = 1$, it follows that $|\alpha_{\eta_0}(2\,\xi_1)| > |\beta_{\eta_0}(2\,\xi_1)|$.

The following lemma will permit us to make an appropriate choice of the trigonometric polynomial $q(x)$ that occurs in (3.1).

Lemma 3.7. *For all reals* δ *and* α *satisfying* $\delta \in]0, 1[$ *and* $\alpha \in]0, \pi/6[$ *there exists* $\{q_L(x)\}_{L \ge 1}$ *a sequence of trigonometric polynomials in one variable with values in* $[0, 1]$ *and with real coefficients, having the following properties*:

i) $\|q_L\|_\infty = 1$,

ii) $q_L(2\,x) \le \delta^L$ *for all* $x \in [0, \pi] - [\pi/3, 2\pi/3]$,

iii) $q_L(2\,x)$ *converges uniformly to* 1 *on* $[\pi/3 + \alpha, 2\pi/3 - \alpha]$,

iv) *there exists a real* $C \ge 1$ *such that* $q_L(2\,x) \le C^{2L}\, |x|^{2L}$ *for all* x.

PROOF. Consider T an even, π-periodic, C^1 function with values on $[0,1]$ such that

a) $T(0)=0$ and for all $x\in[0,\pi/3]$, $0\le T'(x)<\beta$ (where $\beta\,\pi/3<\sqrt{\delta}\,$),

b) for all $x\in[\pi/3+\alpha,\pi/2]$, $T(x)=1$,

c) for all $x\in[\pi/2,\pi]$, $T(x)=T(\pi-x)$.

Let $K_N(x)$ be the Fejer kernel, $K_N(x)$ is the trigonometric polynomial

$$K_N(x)=\frac{1}{2\pi\,(N+1)}\Big|\sum_{k=0}^{N}e^{ikx}\Big|^2=\frac{1}{2\pi\,(N+1)}\,\frac{\sin^2\Big(\dfrac{(N+1)}{2}\,x\Big)}{\sin^2\Big(\dfrac{x}{2}\Big)}\,.$$

For every function $f\in L^2[0,2\pi]$,

$$K_N*f(x)=\int_{-\pi}^{\pi}K_N(x-y)\,f(y)\,dy$$

will be the convolution product of K_N and f. Let $Q_N(x)=K_N*T(x)-K_N*T(0)$ and

$$R_N(x)=\frac{Q_N}{\|Q_N\|_\infty}\,(x)\,.$$

Since T is even and π-periodic the trigonometric polynomial R_N is with real coefficients and π-periodic.

The sequences $\{R_N\}$ and $\{R'_N\}$ converge uniformly to the functions T and T'. Thus it follows from a) that:

- There exists $C\ge 1$ such that for all x, for all N, $|R_N(x)|\le C\,|x|$.
- For all $N\ge N_0$ and for all $x\in[0,\pi]-[\pi/3,2\pi/3]$, $|R_N(x)|\le\sqrt{\delta}$.

At last, one can extract a sequence $\{R_{N_L}\}_{L\ge1}$ satisfying $N_1\ge N_0$ and $\|T-R_{N_L}\|_\infty\le e^{-L}$. We will take $q_{_L}(2\,x)=|R_{N_L}(x)|^{2L}$.

Definition 3.8. *$A_L(\xi_1,\xi_2)$ will be a* QMF *of the type* (3.1) *such that $\eta=1/4$ and $q(x)=q_{_L}(x)$, where $q_{_L}(x)$ is the trigonometric polynomial we have constructed in the Lemma* 3.7.

We can now state the main result of this subsection.

Theorem 3.9. *The* QMF*'s* $\{A_L(\xi_1,\xi_2)\}_{L\geq 1}$ *generate non separable orthonormal compactly supported wavelet bases for* $L^2(\mathbb{R}^2)$ *of arbitrarily high regularity. Moreover these wavelets are not near to the separable I. Daubechies wavelets with the same number of vanishing moments since* $\liminf_{L\to\infty}\|A_L - D_L\|_\infty \geq 1/4$.

Proof. It follows from the Proposition 3.5, the Lemma 3.7 and the inequality

$$|A_L(\xi_1,\xi_2)| \leq |d_L(\xi_1)|\,|d_L(\xi_2)| + \sqrt{\frac{q_L(2\,\xi_1)}{2}}\,|\tilde{d}_L(\xi_1)|\,|d_L(\xi_2)|\,,$$

that, for L big enough the QMF $A_L(\xi_1,\xi_2)$ satisfies the conditions a) and b) of the Theorem 3.3. This QMF also satisfies the condition c) of the same theorem since its coefficients are reals.

Let us show that $\liminf_{L\to\infty}\|A_L - D_L\|_\infty \geq 1/4$.

We have

$$\begin{aligned}\frac{1}{4}\,q_L(2\,\xi_1)\,&|d_L(\xi_1)|\,|d_L(\xi_2)| - |\beta_{1/4}(2\,\xi_1)|\,|\tilde{d}_L(\xi_1)|\,|d_L(\xi_2)| \\ &\leq |A_L(\xi_1,\xi_2) - D_L(\xi_1,\xi_2)| \\ &\leq \frac{1}{4}\,q_L(2\,\xi_1)\,|d_L(\xi_1)|\,|d_L(\xi_2)| + |\beta_{1/4}(2\,\xi_1)|\,|\tilde{d}_L(\xi_1)|\,|d_L(\xi_2)|\,.\end{aligned}$$

It follows from the Propositions 3.5.iii) and from the Lemma 3.7.iii) that for all $(\xi_1,\xi_2)\in[\pi/3+\alpha,\pi/2\,[\,\times\,]-\pi/2,\pi/2\,[$

$$\lim_{L\to\infty}|A_L(\xi_1,\xi_2) - D_L(\xi_1,\xi_2)| = \frac{1}{4}\,,$$

therefore

$$\liminf_{L\to\infty}\|A_L - D_L\|_\infty \geq \frac{1}{4}\,.$$

4. Conclusion.

Some of the techniques we have used to construct non separable, dyadic, compactly supported, orthonormal, wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity, may be adapted to other types of wavelet bases.

In [2] we have constructed non separable, dyadic, compactly supported, biorthogonal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity by perturbing separable biorthogonal filters.

We have found recently a method for constructing QMF's that generate compactly supported, orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of dilation matrix

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(R is a rotation of $\pi/4$ and a dilation of $\sqrt{2}$). This method is inspired from the Theorem 2.2 Let $\alpha(x)$ and $\beta(x)$ two trigonometric polynomials in one variable such that $\alpha(0) = 1$ and $|\alpha(x)|^2 + |\beta(x)|^2 + 1$. Let $m(x)$ be a monodimensional QMF (*i.e.* $m(0) = 1$ and $|m(x)|^2 + |m(x+\pi)|^2 = 1$) and $\tilde{m}(x)$ its conjugate filter ($\tilde{m}(x) = -e^{-ix}\,\overline{m(x+\pi)}$). If $P(\xi_1, \xi_2)$ is one of the trigonometric polynomials

$$u(\xi_1, \xi_2) = \alpha(2\,\xi_1)\, m(\xi_2) + \beta(2\,\xi_1)\, \tilde{m}(\xi_2)\,,$$

$$v(\xi_1, \xi_2) = \alpha(\xi_1 + \xi_2)\, m(\xi_2) + \beta(\xi_1 + \xi_2)\, \tilde{m}(\xi_2)\,,$$

$$w(\xi_1, \xi_2) = \alpha(\xi_1 - \xi_2)\, m(\xi_2) + \beta(\xi_1 - \xi_2)\, \tilde{m}(\xi_2)\,,$$

then we have

$$\begin{cases} P(0,0) = 1\,, \\ |P(\xi_1, \xi_2)|^2 + |P(\xi_1 + \pi, \xi_2 + \pi)|^2\,. \end{cases}$$

This means that when $P(\xi_1, \xi_2)$ satisfies A. Cohen's criterion it generates a compactly supported, orthonormal wavelet basis for $L^2(\mathbb{R}^2)$ of dilation matrix R. We do not know yet whether the regularity of such wavelets could be made arbitrarily high.

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