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Local limit theorems on some non unimodular groups

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Abstract. Let G_d be the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$, $d \geq 1$ and let us consider the product group $G_{d,N} = G_d \times \mathbb{R}^N$, $N \geq 1$. For a large class of probability measures μ on $G_{d,N}$, one proves that there exists $\rho(\mu) \in \,]0,1]$ such that the sequence of finite measures

$$\Big\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \, \mu^{*n} \Big\}_{n \geq 1}$$

converges weakly to a non-degenerate measure.

Résumé. Soit G_d le produit semi-direct de $\mathbb{R}^{*+}$ et de $\mathbb{R}^d$ et $G_{d,N}$ le groupe produit $G_d \times \mathbb{R}^N$, $N \geq 0$. Pour une large classe de mesures de probabilité sur $G_{d,N}$ nous montrons qu'il existe $\rho(\mu) \in \,]0,1]$ tel que la suite de mesures finies

$$\Big\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \, \mu^{*n} \Big\}_{n \geq 1}$$

converge vaguement vers une mesure non nulle.

1. Introduction.

Fix two integers $d \geq 1$, $N \geq 0$ and choose a norm $\| \cdot \|$ on $\mathbb{R}^d$ and $\mathbb{R}^N$ (when $N \geq 1$). Let $G_{d,N}$ be the connected group $\mathbb{R}^{*+} \times \mathbb{R}^d \times \mathbb{R}^N$

with the composition law

$$\text{for all } g = (a, u, b), \text{ for all } g' = (a', u', b') \in G\,,$$
$$g \cdot g' = (a\,a', a\,u' + u, b + b')\,.$$

We will note $g = (a(g), u(g), b(g))$ (or $g = (a, u, b)$ when there is no ambiguity). The group $(G_{d,N}, \cdot)$ is a non unimodular solvable group with exponential growth and the right Haar measure m_D on $G_{d,N}$ is

$$m_D(da\,du\,db_1 \cdots db_N) = \frac{da\,du\,db_1 \cdots db_N}{a}\,.$$

Note that $G_{d,0}$ is the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$; in particular $G_{1,0}$ is the affine group of the real line.

We consider a probability measure μ on G; we denote by μ^{*n} its n^{th} power of convolution. Under quite general assumptions on μ we show that there exists $\rho(\mu) \in]0, 1[$ such that the sequence

$$\left\{\frac{n^{(N+3)/2}}{\rho(\mu)^n}\,\mu^{*n}\right\}_{n\geq 0}$$

converges weakly to a non-degenerate measure. This problem has already been tackled by Ph. Bougerol in [3] where were established local limit theorems on some solvable groups with exponential growth; in particular, for a class R of probability measures μ on the affine group of the real line (that is $d = 1$ and $N = 0$) he showed that the sequence

$$\left\{\frac{n^{3/2}}{\rho(\mu)^n}\mu^{*n}\right\}_{n\geq 0}$$

converges weakly to a non-degenerate measure. In [7] we extend this result to a quite large class of probability measures; the new ingredient in our proof was the fact that there exists closed connections between this problem and the theory of the fluctuations of a random walk on the real line. In the present paper, we extend this result to the case $N \geq 1$; we first obtain uniform upperbounds in the Local limit theorem for a random walk on $\mathbb{R}^d$ and, secondly, we use a generalisation of the Wiener-Hopf's factorisation due to Ch. Sunyach [9].

This study is also related with the work by N. T. Varopoulos [10], [11] where upperbounds and lowerbounds for the asymptotic behaviour of the convolution powers μ^{*n} of a large class of probability measures are given.

From now on, we will suppose that $N \geq 1$ and we set $G = G_{d,N}$. We introduce the following conditions on μ:

Hypothesis G1. *There exists* $\alpha > 0$ *such that*

$$\int_G (e^{\alpha|\log a|} + \|u\|^\alpha + \|b\|^2)\,\mu(da\,du\,db) < +\infty\,.$$

Hypothesis G2. $\int_G \operatorname{Log} a\,\mu(da\,du\,db) = 0$ *and* $\int_G b\,\mu(da\,du\,db) = 0.$

Hypothesis G3. *The support of* μ *is included in* $\mathbb{R}^{*+} \times (\mathbb{R}^+)^d \times \mathbb{R}^N$*, the image of* μ *by the mapping* $(a, u, b) \longmapsto (\operatorname{Log} a, b)$ *is aperiodic in* $\mathbb{R}^{N+1}$ (*see Definition* 2.1) *and there exists* $\beta > 0$ *such that*

$$\int_G \|u\|^{-\beta}\,\mu(da\,du\,db) < +\infty\,.$$

Hypothesis G'3. *The measure* μ *is absolutely continuous with respect to the Haar measure* m_D *on* G *and its density* ϕ_μ *satisfies*

$$\int_{]0,1]\times\mathbb{R}^N} \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a, u, b)\,du}\;\frac{da\,db}{a^\gamma} < +\infty\,.$$

for some γ *and* q *in* $]1, +\infty[$.

We have the

Theorem 1.1. *Let* μ *be a probability measure on* G *satisfying hypotheses* G1, G2 *and* G3 (*or* G'3). *Then, the sequence of finite measures* $\{n^{(N+3)/2}\mu^{*n}\}_{n\geq 0}$ *converges weakly to a non-degenerate Radon measure on* G.

Note that the asymptotic behavior of the sequence $\{\mu^{*n}\}_{n\geq 1}$ does not depend on d.

When μ is not centered, that is

$$\int_G \operatorname{Log} a\,\mu(da\,du\,db) \neq 0$$

or

$$\int_G b \, \mu(da\, du\, db) \neq 0 \, ,$$

we bring back the study to the centered case as in [7]. We introduce the following conditions on μ:

Hypothesis G*1. *There exists $\alpha > 0$ such that*

$$\int_G \left(a^t + \|u\|^\alpha + \exp\left(t\, \|b\|\right)\right) \mu(da\, du\, db) < +\infty$$

for any $t \in \mathbb{R}$.

Hypothesis G*2. *One has*

$$\int_G \operatorname{Log} a \, \mu(da\, du\, db) \neq 0$$

with $\mu\{g \in G: \; a(g) < 1\} > 0$ and $\mu\{g \in G: \; a(g) > 1\} > 0$.

When μ satisfies these two conditions, there exists a unique $(s_0, t_0) \in \mathbb{R} \times \mathbb{R}^N$ such that

$$\int_G a^{s_0} e^{\langle t_0, b\rangle} \mu(da\, du\, db) = \inf_{(s,t)\in\mathbb{R}\times\mathbb{R}^N} \int_G a^s \, e^{\langle t, b\rangle} \, \mu(da\, du\, db) \, .$$

Furthermore,

$$\rho(\mu) = \int_G a^{s_0} e^{\langle t_0, b\rangle} \mu(da\, du\, db)$$

belongs to $]0, 1]$. Note that the probability measure

$$\mu_0(dg) = \frac{1}{\rho(\mu)} \, a(g)^{s_0} \, e^{\langle t_0, b(g)\rangle} \, \mu(dg)$$

satisfies hypotheses G1 and G2. The following condition is the equivalent of Hypothesis G'3 in the non centered case:

Hypothesis G*3. *The measure μ is absolutely continuous with respect to the Haar measure m_D on G and its density ϕ_μ satisfies*

$$\int_{]0,1]\times\mathbb{R}^N} \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a, u, b)\, du} \; \frac{da\, db}{a^\gamma} < +\infty$$

for some $q \in]1, +\infty[$ *and* $\gamma \in]1 - s_0, +\infty[$.

Theorem 1.2. *Let* μ *be a probability measure on* G *satisfying conditions* G*1, G*2 *and* G3 (*or* G*3) *and let*

$$\rho(\mu) = \inf_{(s,t)\in\mathbb{R}\times\mathbb{R}^N} \int_G a^s\, e^{\langle t,b\rangle}\, \mu(da\, du\, db)\,.$$

Then, the sequence of finite measures

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n}\, \mu^{*n} \right\}_{n\geq 1}$$

weakly converges to a non-degenerate Radon measure on G.

The demonstration of Theorem 2.1 is closely related to the study of the fluctuations of a random walk $(X_1^n, Y_1^n)_{n\geq 0}$ on $\mathbb{R}^{N+1}$. In Section 2, we first state the classical local limit theorem on $\mathbb{R}^{N+1}$ but we add in its statement uniform upperbounds relatively to the starting point of the random walk $(X_1^n, Y_1^n)_{n\geq 0}$. This result is thus very usefull to obtain a precise equivalent in Theorem 2.5 of the joint law of the random walk $(X_1^n, Y_1^n)_{n\geq 0}$ with its first entrance time T_+ in the half space $\mathbb{R}^+ \times \mathbb{R}^N$; a local limit theorem for the process

$$(X_1^n, \max\{0, X_1^1, \ldots, X_1^n\}, Y_1^n)_{n\geq 0}$$

is thus obtained (Theorem 2.6). In Section 3 we give the main steps of the proof of Theorem 1.1.

2. Fluctuations of a random walk on $\mathbb{R}^{N+1}$.

Fix an integer $N \geq 1$ and let $(X_1, Y_1), (X_2, Y_2), \ldots$ be independent $\mathbb{R} \times \mathbb{R}^N$-valued random variables with distribution p defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X_1^n, Y_1^n)_{n\geq 0}$ be the associated random walk on $\mathbb{R}\times\mathbb{R}^N$ starting from $(0,0)$ and defined by $X_1^0 = 0, Y_1^0 = 0$ and $X_1^n = X_1 + \cdots + X_n, Y_1^n = Y_1 + \cdots + Y_n$ for $n \geq 1$; the distribution of the couple (X_1^n, Y_1^n) is the n^{th} power of convolution p^{*n} of the measure p. Denote by $\mathcal{F}_n$ the σ-algebra generated by $(X_1, Y_1), \ldots, (X_n, Y_n), n \geq 1$.

Let us first recall the

Definition 2.1. *Let p be a probability measure on $\mathbb{R}^k$, $k \geq 1$. The measure p is aperiodic on $\mathbb{R}^k$ if there is no closed and proper subgroup H of $\mathbb{R}^k$ and no $\alpha \in \mathbb{R}^k$ such that $p(\alpha + H) = 1$.*

Denote by $\hat{p}$ the characteristic function of p defined by $\hat{p}(u,v) = \mathbb{E}[e^{iuX_1 + i\langle v, Y_1\rangle}]$ for any $(u,v) \in \mathbb{R} \times \mathbb{R}^N$. Recall that the probability measure p is aperiodic if and only if $|\hat{p}(u,v)| < 1$ for $(u,v) \neq (0,0)$.

For any $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}^N$ let $\{T_{\mathcal{A}}^{(k)}\}_{k\geq 0}$ be the the successive times of visit of the random walk $(X_1^n, Y_1^n)_{n\geq 1}$ to the set $\mathcal{A}$; one has $T_{\mathcal{A}}^{(0)} = 0, T_{\mathcal{A}}^{(1)} = \inf\{n \geq 1 : \ (X_1^n, Y_1^n) \in \mathcal{A}\}$ and $T_{\mathcal{A}}^{(k+1)} = \inf\{n \geq T_{\mathcal{A}}^{(k)} + 1 : (X_1^n, Y_1^n) \in \mathcal{A}\}$. Note that the $T_{\mathcal{A}}^{(k)}$ are stopping times with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$. We will associate to $(p, \mathcal{A})$ the transition kernel $P_{\mathcal{A}}$ defined by

$$P_{\mathcal{A}}((x,y), \mathcal{B}) = \int_{\mathbb{R}\times\mathbb{R}^N} \mathbf{1}_{\mathcal{A}^c \cap \mathcal{B}}(x + x', y + y')\, p(dx'dy')\,,$$

for any Borel set $\mathcal{B}$ in $\mathbb{R} \times \mathbb{R}^N$; note that for any $k \geq 1$ one has $P_{\mathcal{A}}^k((0,0), \mathcal{B}) = \mathbb{E}[[T_{\mathcal{A}} > k]; (X_1^k, Y_1^k) \in \mathcal{B}]$. In order to simplify the notations we will set $T_- = T_{\mathbb{R}^- \times \mathbb{R}^N}, P_- = P_{\mathbb{R}^- \times \mathbb{R}^N}$ and $T_-^{(k)} = T_{\mathbb{R}^- \times \mathbb{R}^N}^{(k)}$; similar notations will hold, with obvious modifications, when $\mathcal{A} = \mathbb{R}^{*-} \times \mathbb{R}^N, \mathbb{R}^+ \times \mathbb{R}^N$ and $\mathbb{R}^{*+} \times \mathbb{R}^N$.

Troughout this paragraph, for any $k \geq 1$, we denote by λ_k the Lebesgue measure on $\mathbb{R}^k$. Furthermore, for any $\delta > 0$, $\mathcal{H}_\delta(\mathbb{R}^k)$ is the space of $\mathbb{C}$-valued functions φ on $\mathbb{R}^k$ such that

$$\sup_{x\in\mathbb{R}^k} (1 + \|x\|^\delta)^k\, |\varphi(x)| < +\infty\,.$$

2.1. Preliminaries.

The local limit theorem gives the asymptotic behaviour of the sequence $\{p^{*n}(\varphi)\}_{n\geq 1}$ for any continuous function φ with compact support on $\mathbb{R}^{N+1}$; we state it here and we precise some uniform upperbound for the sequence $\{p^{*n}(\varphi)\}_{n\geq 1}$ when φ belongs to $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ with $\delta > 4$.

Theorem 2.2. *Assume that*:

i) *the common distribution p of the variables (X_n, Y_n), $n \geq 1$, is aperiodic on $\mathbb{R}^{N+1}$,*

ii) $\mathbb{E}[|X_1|^2 + \|Y_1\|^2] < +\infty$ *and* $\mathbb{E}[X_1] = 0$, $\mathbb{E}[Y_1] = 0$.

Then:

i) *for any continuous function* φ *with compact support on* $\mathbb{R}^{N+1}$ *one has*

$$\lim_{n\to+\infty} n^{(N+1)/2}\, \mathbb{E}[\varphi(X_1^n, Y_1^n)] = \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \int_{\mathbb{R}^{N+1}} \varphi(x,y)\, \lambda_1(dx)\, \lambda_N(dy)\,,$$

where $|C|$ *denotes the determinant of the positive definite quadratic form*

$$C(u,v) = \mathbb{E}[(u\, X_1 + \langle v, Y_1\rangle)^2]\,.$$

ii) *For any function* φ *in* $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ *with* $\delta > 4$, *the sequence* $\{n^{(N+1)/2}\, \mathbb{E}[\varphi(x + X_1^n, y + Y_1^n)]\}_{n\geq 1}$ *is bounded uniformly in* $(x,y) \in \mathbb{R}\times\mathbb{R}^N$.

PROOF. The first assumption is the classical local limit theorem. To obtain the second claim, fix a non negative function ϕ whose Fourier transform has a compact support $K(\hat{\phi})$. Recall that

$$\hat{p}(u,v) = 1 - \frac{1}{2}\, C(u,v)\, (1 + \varepsilon(u,v))$$

with $\lim_{(u,v)\to(0,0)} \varepsilon(u,v) = 0$; so there exists $\delta > 0$ such that for $|u| + \|v\| < \delta$ one has

$$|\hat{p}(u,v)| \leq 1 - \frac{1}{4}\, C(u,v) \leq e^{-C(u,v)/4}\,.$$

On the other hand, by the aperiodicity of p there exists $\rho = \rho(p, K(\hat{\phi}))$ such that $|\hat{p}(u,v)| \leq \rho$ as soon as (u,v) belongs to $K(\hat{\phi})$ and $|u| + \|v\| \geq \delta$. It follows that

$$\begin{aligned}(2\pi\, n)^{(N+1)/2}\, &\mathbb{E}[\phi(X_1^n, Y_1^n)] \\ &\leq n^{(N+1)/2} \int_{|u|+\|v\|<\delta} |\hat{\phi}(u,v)|\, |\hat{p}(u,v)|^n\, \lambda_1(du)\, \lambda_N(dv) \\ &\quad + n^{(N+1)/2} \int_{|u|+\|v\|\geq\delta} |\hat{\phi}(u,v)|\, |\hat{p}(u,v)|^n\, \lambda_1(du)\, \lambda_N(dv)\end{aligned}$$

$$\leq n^{(N+1)/2} \int_{|u|+\|v\|<\delta n^{(N+1)/2}} \left|\hat{\phi}\Big(\frac{u}{\sqrt{n}}, \frac{v}{\sqrt{n}}\Big)\right| e^{-(n/4)C(u/\sqrt{n}, v/\sqrt{n})} \cdot \lambda_1(du)\, \lambda_N(dv)$$
$$+ n^{(N+1)/2}\, \rho^n\, \|\hat{\phi}\|_1$$
$$\leq \|\hat{\phi}\|_\infty \int_{\mathbb{R}\times\mathbb{R}^N} e^{-C(u,v)/4} \lambda_1(du)\, \lambda_N(dv) + n^{(N+1)/2}\, \rho^n\, \|\hat{\phi}\|_1\ .$$

Now set $\phi_{x,y}(x', y') = \phi(x+x', y+y')$ for any $(x, y) \in \mathbb{R}\times\mathbb{R}^N$ and note that $\hat{\phi}_{x,y}(u, v) = e^{iux+i\langle v,y\rangle}\hat{\phi}(u, v)$; the functions $\hat{\phi}_{x,y}$ and $\hat{\phi}$ thus have the same compact support and satisfies the equalities $\|\hat{\phi}_{x,y}\|_1 = \|\hat{\phi}\|_1$ and $\|\hat{\phi}_{x,y}\|_\infty = \|\hat{\phi}\|_\infty$. For any $(x, y) \in \mathbb{R} \times \mathbb{R}^N$ one thus has

$$|(2\pi\, n)^{(N+1)/2}\, \mathbb{E}[\phi_{x,y}(X_1^n, Y_1^n)]|$$
$$\leq \|\hat{\phi}\|_\infty \int_{\mathbb{R}\times\mathbb{R}^N} e^{-C(u,v)/4}\, \lambda_1(du)\, \lambda_N(dv) + n^{(N+1)/2}\, \rho^n\, \|\hat{\phi}\|_1\ .$$

The assertion ii) thus holds for any function ϕ whose Fourier transform has a compact support. To achieve the proof of ii) it suffices to show that for any function φ in $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ with $\delta > 4$ there exists a function ϕ whose Fourier transform has a compact support and $|\varphi| \leq \phi$. It is an immediate consequence of the following result; we thank here J. P. Conze for helpfull discussions about this fact.

Lemma 2.3. *Set*

$$h_\varepsilon(x) = \frac{1}{1 + |x|^{4+\varepsilon}}\ ,$$

for any $x \in \mathbb{R}$. If $\varepsilon > 0$ there exists a function $\overline{h_\varepsilon}$ greater than h_ε and whose Fourier transform has a compact support in $\mathbb{R}$.

Proof. Set

$$\overline{h_\varepsilon}(x) = C\Big(\frac{\sin^2 x}{x^2} + \frac{\sin^2 \alpha\, x}{x^2}\Big)$$

for some α and C in $\mathbb{R}^{*+}$ which will depend on ε. Assume $\alpha \notin \mathbb{Q}$, the function $\overline{h_\varepsilon}$ is strictly positive on $\mathbb{R}$; it thus suffices to show that there exists $\alpha \notin \mathbb{Q}$ such that

$$\lim_{x\to+\infty} x^{2+\varepsilon}(\sin^2 x + \sin^2(\alpha\, x)) = +\infty\, .$$

If such a real did not exist, then for any $\alpha \notin \mathbb{Q}$ there should exist a sequence $\{x_n\}_{n\geq 1}$ which tends to $+\infty$ and a constant $C_\varepsilon > 0$ such that for all $n \geq 1$,

$$\sin^2 x_n + \sin^2(\alpha\, x_n) \leq \frac{C}{x_n^{2+\varepsilon}}\ .$$

So there should exist two strictly increasing sequences of integers $\{k_n\}_{n\geq 1}$ and $\{l_n\}_{n\geq 1}$ such that

$$|x_n - k_n\,\pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}}\ , \qquad |\alpha\, x_n - l_n\,\pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}}$$

which implies

$$\left|\alpha - \frac{l_n}{k_n}\right| \leq \frac{C''}{k_n^{2+\varepsilon/2}}$$

for some positive constants C' and C''. This leads to a contradiction because for almost all $\alpha \in \mathbb{R}$ (with respect with the Lebesgue measure), this last inequality has at most a finite number of solutions in $\mathbb{N}^2$ [2]. The lemma is proved.

2.2. A local limit theorem for a killed random walk on a half space.

In [7], we proved the following

Theorem 2.4. *Let the hypotheses of Theorem 2.2 hold. Then for any continuous function with compact support φ on $\mathbb{R}^-$ we have*

$$\lim_{n\to+\infty} n^{3/2}\,\mathbb{E}\,[[T_+ > n]; \varphi(X_1^n)] = \frac{1}{\sigma(X_1)\,\sqrt{2\pi}} \int_{-\infty}^{0} \varphi(x)\,\lambda_1^- * U^{*-}(dx)\,,$$

where λ_1^- denotes the restriction of the Lebesgue measure on $\mathbb{R}^-$ and U^{*-} is the σ-finite measure on $\mathbb{R}^-$ defined by

$$U^{*-}(\mathcal{B}) = \sum_{k=1}^{+\infty} \mathbb{E}\,[\mathbf{1}_{\mathcal{B}}(X_1^{T_{*-}^{(k)}})]$$

for any Borel set $\mathcal{B}$. In the same way, one has

$$\lim_{n\to+\infty} n^{3/2}\,\mathbb{E}\,[[T_{*+} > n]; \varphi(X_1^n)] = \frac{1}{\sigma(X_1)\,\sqrt{2\pi}} \int_{-\infty}^{0} \varphi(x)\,\lambda_1^- * U^{-}(dx)\,,$$

where U^- is the σ-finite measure on $\mathbb{R}^-$ defined by

$$U^-(\mathcal{B}) = \sum_{k=1}^{+\infty} \mathbb{E}\,[\mathbf{1}_{\mathcal{B}}(X_1^{T_-^{(k)}})]$$

for any Borel set $\mathcal{B}$.

Recall that the random walks $\{X_1^{T_-^{(k)}}\}_{k\geq 1}$ and $\{X_1^{T_{*-}^{(k)}}\}_{k\geq 1}$ are transient on $\mathbb{R}^-$; it follows that the series $\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_+ > k]; \varphi(x+X_1^k)]$ and $\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_{*+} > k]; \varphi(x + X_1^k)]$ do converge. Furthermore one has

$$\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_+ > k]; \varphi(x + X_1^k)] = \int_{-\infty}^{0} \varphi(x)\, U^{*-}(dx)$$

and

$$\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_{*+} > k]; \varphi(x + X_1^k)] = \int_{-\infty}^{0} \varphi(x)\, U^{-}(dx)\,.$$

Let us now state the following

Theorem 2.5. *Let the hypotheses of Theorem* 2.2 *hold. Then*:

i) *For any continuous function φ with compact support on $\mathbb{R}^- \times \mathbb{R}^N$ one has*

$$\lim_{n\to+\infty} n^{(N+3)/2}\, \mathbb{E}\,[[T_+ > n]; \varphi(X_1^n, Y_1^n))] = \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \int_{\mathbb{R}^-\times\mathbb{R}^N} \varphi(x,y)\, \lambda_1^- * U^{*-}(dx)\, \lambda_N(dy)\,.$$

ii) *For any continuous function f with compact support on $\mathbb{R}$ and any g in $\mathcal{H}_\delta(\mathbb{R}^N)$ with $\delta > 4$, the sequence*

$$\{n^{(N+3)/2}\, \mathbb{E}\,[[T_+ > n]; f(X_1^n)\, g(y + Y_1^n)]\}_{n\geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

In the same way, one has

$$\lim_{n\to+\infty} n^{(N+3)/2}\, \mathbb{E}\,[[T_{*+} > n]; \varphi(X_1^n, Y_1^n)] = \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \int_{\mathbb{R}^-\times\mathbb{R}^N} \varphi(x,y)\, \lambda_1^- * U^{-}(dx)\, \lambda_N(dy)$$

and the sequence

$$\{n^{(N+3)/2}\,\mathbb{E}\,[[T_+ > n]; f(X_1^n)\, g(y + Y_1^n)]\}_{n\geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

PROOF. We prove this theorem by induction over N. Theorem 2.2 deals with the case $N = 0$; we will suppose that this result hold for some $N \geq 0$ and we consider a sequence $(X_n, Y_n, Z_n)_{n\geq 1}$ of independent identically distributed random variables on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. By a classical argument in probability theory, it suffices to show the above convergence hold for $\varphi(x, y, z) = e^{ax}\,\mathbf{1}_{\mathbb{R}^-}(x)\,\phi(y)\,\psi(z)$ where $a \in \mathbb{R}^{*+}$ and ϕ, ψ are $\mathbb{C}$-valued functions whose Fourier transform are continuous with compact supports. By the inverse Fourier transform one has

$$\begin{aligned} I_n &= \mathbb{E}\,[[T_+ > n]; e^{aX_1^n}\,\phi(Y_1^n)\,\psi(Z_1^n)] \\ &= \frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^N\times\mathbb{R}} \hat{\phi}(v)\,\hat{\psi}(w)\,\alpha_n(a, v, w)\,\lambda_N(dv)\,\lambda_1(dw) \end{aligned}$$

with $\alpha_n(a, v, w) = \mathbb{E}\,[[T_+ > n]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}]$.

The Spitzer's factorisation for random walks on $\mathbb{R}$ gives for all $a > 0$, for all $s \in [0, 1[$

$$\sum_{n=0}^{+\infty} s^n\,\mathbb{E}\,[[T_+ > n]; e^{aX_1^n}] = \exp\Big(\sum_{n=1}^{+\infty} \frac{s^n}{n}\,\mathbb{E}\,[[X_1^n < 0]; e^{aX_1^n}]\Big).$$

Using the fact that $\mathbb{R}^+ \times \mathbb{R}^{N+1}$ and $\mathbb{R}^{*-} \times \mathbb{R}^{N+1}$ are semi-groups in $\mathbb{R}^{N+2}$, Ch. Sunyach extended this factorisation to the multidimensionnal case ([9, Corollary 3, p. 553 and Theorem 5, p. 556]); for any $a > 0$, $v \in \mathbb{R}^N$, $w \in \mathbb{R}$ and $s \in [0, 1[$ one thus has

$$\begin{aligned} &\sum_{n=0}^{+\infty} s^n\,\mathbb{E}\,[[T_+ > n]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}] \\ &\qquad\qquad = \exp\Big(\sum_{n=1}^{+\infty} \frac{s^n}{n}\,\mathbb{E}\,[[X_1^n < 0]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}]\Big) \end{aligned}$$

that is

$$(n + 1)\,\alpha_{n+1}(a, v, w) = \sum_{k=0}^{n} \beta_{n+1-k}(a, v, w)\,\alpha_k(a, v, w)$$

with $\beta_n(a,v,w) = \mathbb{E}[[X_1^n < 0]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}]$. Finally

$$I_n = \frac{1}{n+1}\sum_{k=0}^{n} I_{n,k}$$

with

$$I_{n,k} = \frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^N\times\mathbb{R}} \beta_{n+1-k}(a,v,w)\,\alpha_k(a,v,w) \cdot \hat{\phi}(v)\,\hat{\psi}(w)\,\lambda_N(dv)\,\lambda_1(dw)\,.$$

Set

$$I = \frac{1}{(2\pi)^{(N+2)/2}\sqrt{|C|}} \int_{\mathbb{R}^N\times\mathbb{R}} \sum_{k=0}^{+\infty} \mathbb{E}\Big[[T_+ > k]; \frac{e^{aX_1^k}}{a}\Big] \cdot \phi(y)\,\psi(z)\lambda_N(dy)\,\lambda_1(dz)\,,$$

since

$$I = \lambda_1^- * U^{*-}(e^{a\cdot})\,\lambda_N(\phi)\,\lambda_1(\psi)\,,$$

it suffices to show that $\{n^{(N+4)/2} I_n\}_{n\geq 1}$ converges to I, that is

1) for all $k > 0$, $\lim_{n\to+\infty} n^{(N+2)/2} I_{n,k} = I_{*k}$,

2) $\sum_{k=0}^{+\infty} |I_{*k}| < +\infty$ and $\sum_{k=0}^{+\infty} I_{*k} = I$,

3) $\limsup_{l\to+\infty}\limsup_{n\to+\infty} n^{(N+2)/2} \sum_{k=l}^{n} |I_{n,k}| = 0$.

To prove the assertion 1), note that

$$I_{n,k} = \mathbb{E}[[T_+ > k]\cap[X_{k+1}^{n+1} > 0]; e^{aX_1^{n+1}}\,\phi(Y_1^{n+1})\,\psi(Z_1^{n+1})]\,,$$

by the local limit theorem on $\mathbb{R}^{N+2}$ the assertion 1) follows with

$$I_{*k} = \frac{1}{2\pi^{(N+2)/2}\sqrt{|C|}}\,\frac{\mathbb{E}[[T_+ > k]; e^{aX_1^k}]}{a} \cdot \int_{\mathbb{R}^N} \phi(y)\,\lambda_N(dy) \int_{\mathbb{R}} \psi(z)\,\lambda_1(dz)\,.$$

The fact that the series $\sum_{k=0}^{+\infty} |I_{*k}|$ converges is a direct consequence of Theorem 2.4. To prove the assertion 3), note that

$$\begin{aligned}
|I_{n,k}| &\le \mathbb{E}\,[[T_+ > k] \cap [X_{k+1}^{n+1} < 0]; e^{aX_1^{n+1}}\, |\phi(Y_1^{n+1})|\, |\psi(Z_1^{n+1})|] \\
&\le \mathbb{E}\Big[[T_+ > k]; e^{aX_1^k} \int_{\mathbb{R}^- \times \mathbb{R}^N \times \mathbb{R}} e^{ax}\, |\phi(y + Y_1^k)| \\
&\qquad\qquad \cdot |\psi(z + Z_1^k)|\, p^{*(n+1-k)}(dx\, dy\, dz)\Big] \\
&\le \frac{C(a,\phi,\psi)}{(n+1-k)^{(N+2)/2}}\, \mathbb{E}\,[[T_+ > k]; e^{aX_1^k}] \qquad \text{by Theorem 2.2.ii)} \\
&\le \frac{C_1}{(n+1-k)^{(N+2)/2}\, k^{3/2}} \qquad \text{by Theorem 2.4.}
\end{aligned}$$

On the other hand

$$\begin{aligned}
|I_{n,k}| &\le \|\psi\|_\infty \int_{\mathbb{R}^- \times \mathbb{R}^N \times \mathbb{R}} \mathbb{E}[[T_+ > k]; e^{aX_1^k}\, |\psi(y + Y_1^k)| \\
&\qquad\qquad \cdot e^{ax}\, p^{*(n+1-k)}(dx\, dy\, dz)] \\
&\le \frac{\|\psi\|_\infty\, C(a,\phi)}{k^{(N+3)/2}} \\
&\qquad \cdot \mathbb{E}\,[[X_{k+1}^{n+1} < 0]; e^{aX_{k+1}^{n+1}}] \qquad \text{by hypothesis of induction} \\
&\le \frac{C_2}{k^{(N+3)/2}\, \sqrt{n+1-k}}\,.
\end{aligned}$$

The assertion 3) follows since for any $\varepsilon > 0$ one has

$$\begin{aligned}
n^{(N+2)/2} \sum_{k=l}^{n} |I_{n,k}| &\le C_1 \sum_{k=l}^{[n(1-\varepsilon)]} \frac{n^{(N+2)/2}}{k^{3/2}\, (n+1-k)^{(N+2)/2}} \\
&\quad + C_2 \sum_{[n(1-\varepsilon)+1]}^{n} \frac{n^{(N+2)/2}}{k^{(N+3)/2}\, \sqrt{n+1-k}} \\
&\le \frac{C_1}{\varepsilon^{(N+2)/2}} \sum_{k=l}^{[n(1-\varepsilon)]} \frac{1}{k^{3/2}} \\
&\quad + \frac{C_2}{\sqrt{n}\, (1-\varepsilon)^{(N+3)/2}} \sum_{[n(1-\varepsilon)+1]}^{n} \frac{1}{\sqrt{n+1-k}}
\end{aligned}$$

$$\leq C\Big(\frac{1}{\varepsilon^{(N+2)/2}\sqrt{l}} + \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^{(N+3)/2}}\Big).$$

Since ε is arbitrarily small, the assertion 3) follows.

The proof of ii) is also made by induction over N. If $g \in \mathcal{H}_\delta(\mathbb{R}^{N+1})$ there exist $\phi \in \mathcal{H}_\delta(\mathbb{R}^N)$ and $\psi \in \mathcal{H}_\delta(\mathbb{R}^1)$ such that $|g| \leq \phi \otimes \psi$. We set

$$I_n(y,z) = \mathbb{E}\,[[T_+ > n]; e^{aX_1^n}\,\phi(y+Y_1^n)\,\psi(z+Z_1^n)]$$

and we have

$$I_n(y,z) = \frac{1}{n+1}\sum_{k=0}^{n} I_{n,k}(y,z)$$

with

$$I_{n,k}(y,z) = \mathbb{E}\,[[T_+ > k]\cap[X_{k+1}^{n+1} < 0]; e^{aX_1^{n+1}}\,\phi(y+Y_1^{n+1})\,\psi(z+Z_1^{n+1})]\,.$$

As above, one has

$$|I_{n,k}(y,z)| \leq \inf\Big\{\frac{C_1}{(n+1-k)^{(N+2)/2}\,k^{3/2}}, \frac{C_2}{k^{(N+3)/2}\sqrt{n+1-k}}\Big\}$$

which proves that the sequence

$$\Big\{n^{(N+2)/2}\sum_{k=0}^{n}|I_{n,k}(y,z)|\Big\}_{n\geq 1}$$

is uniformly bounded in y, z. This achieves the proof of ii).

The convergence of the sequence

$$\{n^{(N+3)/2}\mathbb{E}[[T_{*+} > n]; \varphi(X_1^n, Y_1^n)]\}_{n\geq 1}$$

is obtained with similar arguments.

2.3. Behaviour of the process $((X_1^n, \max\{0, X_1^1, \ldots, X_1^n\}, Y_1^n))_{n\geq 0}$.

For any $n \geq 0$ set $\mathcal{X}_1^n = \max\{0, X_1^1, \ldots, X_1^n\}$ and let T_n be the random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by $T_n = \inf\{0 \leq k \leq n : \mathcal{X}_1^n =$

$X_1^k\}$; for any continuous function φ with compact support on $\mathbb{R}^{N+1}$ we have

$$\begin{aligned}
&\mathbb{E}[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)] \\
&= \sum_{k=0}^{n} \mathbb{E}[[T_n = k]; \varphi(X_1^k, -X_{k+1}^n, Y_1^n)] \\
&= \sum_{k=0}^{n} \mathbb{E}[[0 < X_1^k, X_1^1 < X_1^k, \dots, X_1^{k-1} < X_1^k, \\
&\qquad\qquad X_1^{k+1} \leq X_1^k, \dots, X_1^n \leq X_1^k]; \varphi(X_1^k, -X_{k+1}^n, Y_1^n)] \\
&= \sum_{k=0}^{n} \mathbb{E}[[X_1^1 > 0, \dots, X_1^k > 0] \cap [X_{k+1}^{k+1} \leq 0, \dots, X_{k+1}^n \leq 0]; \\
&\qquad\quad \varphi(X_1^k, -X_{k+1}^n, Y_1^n)] \, .
\end{aligned}$$

One obtains the following factorisation

$$\mathbb{E}[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)] = \sum_{k=0}^{n} J_{n,k}(\varphi)$$

with

$$\begin{aligned}
&J_{n,k}(\varphi) \\
&\quad = \int_{\mathbb{R}^{N+1}} \varphi(x, -x', y + y') \, P_-^k((0,0), dx\, dy) \, P_{*+}^{n-k}((0,0), dx'\, dy') \, .
\end{aligned}$$

The behaviour of the process $(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)$ is thus closely related to the one of the iterates of the transition kernels P_- and P_{*+}. Using this factorisation one proves the

Theorem 2.6. *Suppose that the hypotheses of Theorem* 2.2 *hold.*

Then, for any continuous function with compact support on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N$ *the sequence*

$$\{n^{(N+3)/2} \, \mathbb{E}[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)]\}_{n \geq 1}$$

converges to

$$\frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \int_{\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^N} \varphi(s,-t,y)\, U^{*+}(ds)\, \lambda_1^- * U^-(dt)\, \lambda_N(dy)$$
$$+ \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \cdot \int_{\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^N} \varphi(s,-t,y)\, \lambda_+ * U^{*+}(ds)\, U^-(dt)\, \lambda_N(dy)\,.$$

Furthermore, for any continuous function f with compact support on $\mathbb{R}^+\times\mathbb{R}^+$ and any g in $\mathcal{H}_\delta(\mathbb{R}^N)$, the sequence

$$\{n^{(N+3)/2}\, \mathbb{E}[f(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n)\, g(y+Y_1^n)]\}_{n\geq 1}$$

is bounded, uniformly in $y\in\mathbb{R}^N$.

PROOF. We only proof the first assertion; the second one may obtained with obvious modifications as in Theorem 2.5. Set $\varphi(x,t,y) = \varphi_1(x)\,\varphi_2(t)\,\varphi_3(y)$ where φ_1,φ_2 and φ_3 are continuous with compact support. Fix $k\geq 0$; by Theorem 2.5, the sequence

$$\left\{n^{(N+3)/2}\int_{\mathbb{R}^-\times\mathbb{R}^N} \varphi_2(x')\,\varphi_3(y+y')\, P_{*+}^{n-k}((0,0), dx'\,dy')\right\}_{n\geq 1}$$

is bounded uniformly in $y\in\mathbb{R}^N$ and converges to

$$\frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}}\int_{-\infty}^0 \varphi_2(-t)\,\lambda_1^- * U^-(dt)\,\lambda_N(\varphi_3)\,.$$

By the dominated convergence theorem, one thus obtains, for any fixed $i\geq 1$

$$\lim_{n\to+\infty} n^{(N+3)/2}\sum_{k=0}^{i} J_{n,k}(\varphi)$$
$$= \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}}\sum_{k=0}^{i}\mathbb{E}[[T_- > k]; \varphi_1(X_1^k)] \cdot \int_{-\infty}^0 \varphi_2(-t)\,\lambda_1^- * U^-(dt)\,\lambda_N(\varphi_3)\,.$$

In the same way one has

$$\lim_{n\to+\infty} n^{(N+3)/2} \sum_{k=n-i+1}^{n} J_{n,k}(\varphi)$$
$$= \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \sum_{k=0}^{i} \mathbb{E}\,[[T_{*+} > k]; \varphi_2(-X_1^k)]$$
$$\cdot\, \lambda_1^+ * U^{*+}(\varphi_1)\, \lambda_N(\varphi_3)\,.$$

Note that the sums $\sum_{k=0}^{i} \mathbb{E}\,[[T_- > k]; \varphi_1(X_1^k)]$ and $\sum_{k=0}^{i} \mathbb{E}\,[[T_{*+} > k]; \varphi_2(X_1^k)]$ converges respectively to $U^{*+}(\varphi_1)$ and $\int_{-\infty}^{0} \varphi_2(-t)\, U^-(dt)$. To obtain the theorem it suffices to check that

$$\limsup_{i\to+\infty} \limsup_{n\to+\infty} \Big| n^{(N+3)/2} \sum_{k=i+1}^{n-i} J_{n,k}(\varphi)\Big| = 0\,,$$

one has

$$\Big| n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} J_{n,k}(\varphi)\Big|$$
$$\leq n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} \mathbb{E}\,[[T_- > k]; |\varphi_1(X_1^k)|]$$
$$\cdot \int_{\mathbb{R}^-\times\mathbb{R}^N} |\varphi_2(x')|\, |\varphi_3(y+y')|\, P_{*+}^{n-k}((0,0), dx'\, dy')$$
$$\leq C(\varphi_2,\varphi_3) \sum_{k=i+1}^{[n/2]} \mathbb{E}\,[[T_- > k]; |\varphi_1(X_1^k)|]$$
$$\cdot \frac{n^{(N+3)/2}}{(n-k)^{(N+3)/2}} \qquad \text{by Theorem 2.5.ii)}$$
$$\leq C(\varphi) \sum_{k=i+1}^{+\infty} \frac{1}{k^{3/2}}\,.$$

The same upperbound holds for the term

$$n^{(N+3)/2} \sum_{k=[n/2]+1}^{n-i} J_{n,k}(\varphi)\,.$$

This achieves the proof.

3. A local limit theorem for a particular class of solvable groups.

Recall that $G = G_{d,N} = \mathbb{R}^{*+} \times \mathbb{R}^d \times \mathbb{R}^N$ with the composition law

$$g \cdot g' = (a\,a', a\,u' + u, b + b'),$$

for all $g = (a, u, b)$, for all $g' = (a', u', b') \in G_{d,N}$.

The proof of Theorem 1.1 is closed to the one of the local limit theorem for the affine group of the real line given in [7]; we just give here the main steps of the demonstration.

Let us first introduce some helpfull notations. Let $g_n = (a_n, u_n, b_n)$, $n = 1, 2, \ldots$ be independent and identically distributed random variables with distribution μ. Denote by $\mathcal{F}_n$ the σ-algebra generated by the variables $g_1, g_2, \ldots, g_n$, $n \geq 1$. For any $n \geq 1$, set $G_1^n = g_1 \cdots g_n = (A_1^n, U_1^n, B_1^n)$; we have $A_1^n = a_1 \cdots a_n$, $U_1^n = \sum_{k=1}^n a_1 \cdots a_{k-1}\, u_k$ and $B_1^n = b_1 + \cdots + b_n$. More generally, if $1 \leq m \leq n$, set $A_m^n = a_m \cdots a_n$, $U_m^n = \sum_{k=m}^n a_m \cdots a_{k-1}\, u_k$, $B_m^n = b_m + \cdots + b_n$ and set $A_m^n = 1$, $U_m^n = 0$, $B_m^n = 0$ otherwise.

Let $\tilde{\mu}$ be the image of μ by the map

$$g = (a, u, b) \longmapsto \tilde{g} = \Big(\frac{1}{a}, \frac{u}{a}, b\Big),$$

if $\tilde{g}_n = (\tilde{a}_n, \tilde{u}_n, \tilde{b}_n)$, $n = 1, 2, \ldots$ are independent and identically distributed random variables with distribution $\tilde{\mu}$ on G, set $\tilde{G}_m^n = \tilde{g}_m \cdots \tilde{g}_n = (\tilde{A}_m^n, \tilde{U}_m^n, \tilde{B}_m^n)$.

In order to obtain the asymptotic behaviour of the power of convolution μ^{*n} we use the fact that the sequence $\{U_1^n\}_{n\geq 1}$ behaves like the maximum of the variables $A_1^1, \ldots, A_1^n$. These idea was already used in [7]. Set $\mathcal{A} = \{g = (a, u, b) \in G : \ a > 1\}$ and consider the transition kernel $P_{\mathcal{A}}$ associated with $(\mu, \mathcal{A})$ and defined by

$$P_{\mathcal{A}}(g, \mathcal{B}) = \int_G \mathbf{1}_{\mathcal{A}^c \cap \mathcal{B}}(g\,h)\, \mu(dh)$$

for any Borel set $\mathcal{B} \subset G$ and any $g \in G$. The probabilistic interpretation of $P_{\mathcal{A}}$ is the following one: if $T_{\mathcal{A}} = \inf\{n \geq 1 : \ G_1^n \in \mathcal{A}\}$ is the first entrance time in $\mathcal{A}$ of the random walk $\{G_1^n\}_{n\geq 0}$ then

$$P_{\mathcal{A}}^n(e, \mathcal{B}) = \mathbb{P}\,[[T_{\mathcal{A}} > n] \cap [G_1^n \in \mathcal{B}]], \qquad \text{for all } n \geq 1.$$

In the same way, set $\mathcal{A}' = \{g \in G : \ a(g) \geq 1\}$, let $\tilde{P}_{\mathcal{A}'}$ be the operator associated with $(\tilde{\mu}, \mathcal{A}')$ and denote by $\tilde{T}_{\mathcal{A}'}$ the first entrance time in $\mathcal{A}'$ of the random walk $\{\tilde{G}_1^n\}_{n\geq 1}$; one has

$$\tilde{P}^n_{\mathcal{A}'}(e, \mathcal{B}) = \mathbb{P}\,[[\tilde{T}_{\mathcal{A}'} > n] \cap [\tilde{G}_1^n \in \mathcal{B}]]\,, \qquad \text{for all } n \geq 1\,.$$

As in Section 2.3, we introduce the first time at which the random walk $\{A_1^n\}_{n\geq 1}$ reaches its maximun on $\mathbb{R}^{*+}$; for any continuous function φ with compact support on G, we thus obtain

$$\mathbb{E}\,[\varphi(G_1^n)] = \sum_{k=0}^{n} I_{n,k}(\varphi)\,,$$

where

$$I_{n,k}(\varphi) = \int_{G\times G} \varphi\Big(\frac{a'}{a}, \frac{u+u'}{a}, b+b'\Big)\, \tilde{P}^k_{\mathcal{A}'}(e, da\, du\, db)\, P^{n-k}_{\mathcal{A}}(e, da'\, du'\, db')\,.$$

We now give the main steps of the proof of Theorem 1.1 under hypothesis G1, G2 and G3.

First step. *Control of the central terms of the sum* $\sum_{k=0}^{n} I_{n,k}(\varphi)$.

We show here that

$$\limsup_{i\to+\infty} \limsup_{n\to+\infty} \sum_{k=i}^{n-i} I_{n,k}(\varphi) = 0\,.$$

Without loss of generality, one may suppose that the support of φ is included in $\mathbb{R}^{*+} \times (\mathbb{R}^{*+})^d \times \mathbb{R}^N$; for any $\varepsilon > 0$ there exist a constant $C > 0$ and a positive function ϕ with compact support on $\mathbb{R}^N$ such that

$$\varphi(a, u, b) \leq C\, \frac{a^\varepsilon}{\|u\|^{2\varepsilon}}\, \phi(b)\,,$$

it follows that for any (α, β) in $\mathbb{R}^{*+} \times \mathbb{R}^N$

$$\begin{aligned}
&\mathbb{E}\Big[[T_{\mathcal{A}} > l]; \varphi\Big(\frac{A_1^l}{\alpha}, \frac{u + U_1^l}{\alpha}, \beta + B_1^l\Big)\Big] \\
&\leq C\, \alpha^\varepsilon\, \mathbb{E}\Big[[a_1 \leq 1] \cap \Big[\max\{A_2^2, \ldots, A_2^l\} \leq \frac{1}{a_1}\Big]; \frac{(A_1^l)^\varepsilon}{\|u + U_1^l\|^{2\varepsilon}}\, \phi(\beta + B_1^l)\Big] \\
&\leq C\, \alpha^\varepsilon \int_G \mathbb{E}\Big[\frac{(A_2^l)^\varepsilon}{\max\{A_2^2, \ldots, A_2^l\}^{2\varepsilon}}\, \phi(\beta + b + B_2^l)\Big]\, \frac{\mu(da\, dv\, db)}{a^\varepsilon\, \|v\|^{2\varepsilon}}\,,
\end{aligned}$$

the last inequality being a consequence of the fact that $\|u+U_1^l\| \geq \|u_1\|$ $\mathbb{P}$-almost surely and

$$\mathbf{1}_{\{\max\{A_2^2,\dots,A_2^l\}\leq 1/a_1\}} \leq \frac{1}{a_1^{2\varepsilon}\max\{A_2^2,\dots,A_2^l\}^{2\varepsilon}}\,.$$

By Theorem 2.6 one obtains

$$l^{(N+3)/2}\,\mathbb{E}\Big[[T_{\mathcal{A}} > l];\varphi\Big(\frac{A_1^l}{\alpha},\frac{u+U_1^l}{\alpha},\beta+B_1^l\Big)\Big] \leq C_1(\varphi)\,\alpha^{\varepsilon}\,.$$

The same upperbound holds under hypotheses G1, G2 and G'3 (see [7, Lemma 3.1]).

It readily follows that

$$\begin{aligned} n^{(N+3)/2}\sum_{k=i}^{[n/2]} I_{n,k}(\varphi) &\leq 2^{(N+3)/2}\sum_{k=i}^{[n/2]}(n-k)^{(N+3)/2} I_{n,k}(\varphi) \\ &\leq C_1(\varphi)\sum_{k=i}^{[n/2]}\mathbb{E}\,[[\tilde{T}_{\mathcal{A}'} > k];(\tilde{A}_1^k)^{\varepsilon}] \\ &\leq C_2(\varphi)\sum_{k=i}^{[n/2]}\frac{1}{k^{3/2}} \end{aligned}$$

and so

$$\limsup_{i\to+\infty}\limsup_{n\to+\infty} n^{(N+3)/2}\sum_{k=i}^{[n/2]} I_{n,k}(\varphi) = 0\,.$$

The control of the sum $\sum_{k=[n/2]}^{n-i} I_{n,k}(\varphi)$ goes along the same lines.

Second step. *Convergence of the sequence*

$$l^{(N+3)/2}\,\mathbb{E}\Big[[T_{\mathcal{A}} > l];\varphi\Big(\frac{A_1^l}{\alpha},\frac{u+U_1^l}{\alpha},\beta+B_1^l\Big)\Big]$$

for any $(\alpha,u,\beta)\in\,]0,1]\times(\mathbb{R}^{*+})^d\times\mathbb{R}^N$.

It is the more technical part of the proof and it uses and idea due to Afanasev [1]. Without loss of generality, one may suppose $\alpha = 1$, $u = 0$ and $\beta = 0$. For any $n \geq 1$, set

$$\mathbb{E}_n(\varphi) = n^{(N+3)/2}\,\mathbb{E}\,[[T_A > n];\varphi(A_1^n, U_1^n, B_1^n)]\,.$$

Fix $i \in \mathbb{N}$ such that $1 \le i \le n/2$ and consider

$$\mathbb{E}_n(\varphi, i) = n^{(N+3)/2}\, \mathbb{E}[[T_A > n]; \varphi(A_1^n, U_1^i + A_1^{n-i} U_{n-i+1}^n, B_1^n)]\,.$$

To obtain the claim, it suffices to prove that

a) $\limsup\limits_{i\to+\infty} \limsup\limits_{n\to+\infty} |\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| = 0,$

b) for any fixed $n \in \mathbb{N}$, the sequence $\{\mathbb{E}_n(\varphi, i)\}_{n\ge 1}$ converges to a finite limit.

Proof of convergence a). We use the equality

$$U_1^n = U_1^i + A_1^i\, U_{i+1}^{n-i} + A_1^{n-i}\, U_{n-i+1}^n\,,$$

without loss of generality one may suppose that φ is continuously differentiable, and so, for any $\varepsilon > 0$ there exists $C > 0$ and a positive function ϕ with compact support on $\mathbb{R}^N$ such that

$$|\varphi(a, u, b) - \varphi(a, v, b)| \le C\, a^{\varepsilon} \|u - v\|^{\varepsilon} \phi(b)\,,$$

consequently

$$\begin{aligned}
&|\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| \\
&\qquad \le C\, n^{(N+3)/2}\, \mathbb{E}[[T_A > n]; (A_1^n)^{\varepsilon}\, (A_1^i)^{\varepsilon}\, \|U_{i+1}^{n-i}\|^{\varepsilon}\, \phi(B_1^n)] \\
&\qquad \le C\, n^{(N+3)/2} \sum_{k=i+1}^{n-i} \mathbb{E}[[T_A > n]; (A_1^n)^{\varepsilon}\, (A_1^{k-1})^{\varepsilon}\, \|u_k\|^{\varepsilon}\, \phi(B_1^n)]\,.
\end{aligned}$$

Note that for $i \le k \le [n/2]$ one has

$$\begin{aligned}
&\mathbb{E}[[T_A > n]; (A_1^n)^{\varepsilon}\, (A_1^{k-1})^{\varepsilon}\, \|u_k\|^{\varepsilon}\, \phi(B_1^n)] \\
&\le \mathbb{E}\Big[[T_A > k-1] \cap \Big[\max\{A_{k+1}^{k+1}, \dots, A_{k+1}^n\} \le \frac{1}{A_1^k}\Big]; \\
&\qquad (A_1^n)^{\varepsilon}\, (A_1^{k-1})^{\varepsilon}\, \|u_k\|^{\varepsilon}\, \phi(B_1^n)\Big] \\
&\le \mathbb{E}[[T_A > k-1]; (A_1^{k-1})^{\varepsilon/2}\, a_k^{-\varepsilon/2}\, \|u_k\|^{\varepsilon} \\
&\qquad\qquad \cdot \max\{A_{k+1}^{k+1}, \dots, A_{k+1}^n\}^{-3\varepsilon/2}\, (A_{k+1}^n)^{\varepsilon}\, \phi(B_1^n)]\,.
\end{aligned}$$

By Theorem 2.6,

$$(n-k)^{(N+3)/2}\, \mathbb{E}[\max\{A_{k+1}^{k+1}, \dots, A_{k+1}^n\}^{-3\varepsilon/2}\, (A_{k+1}^n)^{\varepsilon}\, \phi(\beta + B_{k+1}^n)]$$

is bounded, uniformly in $\beta \in \mathbb{R}^N$ and so

$$(n-k)^{(N+3)/2}\, E\,[[T_A > n]; (A_1^n)^{\varepsilon}\,(A_1^{k-1})^{\varepsilon}\,\|u_k\|^{\varepsilon}\,\phi(B_1^n)] \le \frac{C_1}{k^{3/2}}\,.$$

When $[n/2] \le k \le n-i$ one obtains by a similar argument

$$k^{(N+3)/2}\,\mathbb{E}\,[[T_A > n]; (A_1^n)^{\varepsilon}\,(A_1^{k-1})^{\varepsilon}\,\|u_k\|^{\varepsilon}\,\phi(B_1^n)] \le \frac{C_2}{(n-k)^{3/2}}\,.$$

Finally one has

$$|\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| \le C_3\,\frac{1}{\sqrt{i}}\,,$$

convergence a) follows.

PROOF OF CONVERGENCE b). Fix an integer i; we have

$$\begin{aligned}&\mathbb{E}_n(\varphi, i)\\ &\quad = \int_G E_n(\varphi, g, h_1, h_2, \ldots, h_i)\, P^i_{\mathcal{A}}(e, dg)\,\mu(dh_1)\,\mu(dh_2)\cdots\mu(dh_i)\end{aligned}$$

with

$$\begin{aligned}&E_n(\varphi, g, h_1, h_2, \ldots, h_i)\\ &= \mathbb{E}\Big[\Big[\max\{A_{i+1}^{i+1}, \ldots, A_{i+1}^{n-i}\} \le \frac{1}{a(g)}\Big]\\ &\qquad \cap \Big[A_{i+1}^{n-i} \le \min\Big\{\frac{1}{a(g)}, \frac{1}{a(g)a(h_1)}, \ldots, \frac{1}{a(g)a(h_1)\cdots a(h_i)}\Big\}\Big];\\ &\qquad \varphi(a(g)\,A_{i+1}^{n-i}\,a(h_1)\cdots a(h_i), u(g) + a(g)\,A_{i+1}^{n-i}\,u(h_1\cdots h_i),\\ &\qquad B_1^{n-i} + b(h_1) + \cdots + b(h_i)]\,.\end{aligned}$$

Using Theorem 2.6, one may see that, for any $g, h_1, \ldots, h_i \in G$, the sequence

$$\{n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \cdots, h_i)\}_{n\ge 1}$$

converges to a finite limit. To obtain the convergence b), we have to use Lebesgue dominated convergence theorem and therefore, we have to obtain an appropriate upperbound for $n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \ldots, h_i)$.

Using the fact that for any $\varepsilon > 0$ there exist $C > 0$ and a positive continuous function ϕ with compact support on $\mathbb{R}^N$ such that $|\varphi(a,u,b)| \leq C\, a^{\varepsilon}\phi(b)$, one thus obtains

$$n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \ldots, h_i) \leq C_1\, a(g)^{-3\varepsilon/2}\, a(h_1)^{\varepsilon} \cdots a(h_i)^{\varepsilon}$$

which allows us to use the Lebesgue dominated convergence theorem for ε small enough; convergence b) follows.

Consequently, $\{n^{(N+3)/2}\, I_{n,0}(\varphi)\}_{n\geq 1}$ converges to a finite limit; furthermore, for any $i \geq 1$ and any compact set $K \subset \mathbb{R}^{*+} \times \mathbb{R}^N$, the dominated convergence theorem ensures the existence of a finite limit as n goes to $+\infty$ for

$$\Big\{ n^{(N+3)/2} \sum_{k=0}^{i} I_{n,k}(\varphi, K) \Big\}_{n\geq 1},$$

where

$$\begin{aligned} I_{n,k}(\varphi, K) = \int_G & \mathbf{1}_K(g) \\ & \cdot \Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}, \frac{u(g)+u(h)}{a(g)}, b(g)+b(h)\Big) P_{\mathcal{A}}^{n-k}(e, dh) \Big) \\ & \cdot \tilde{P}^k_{\mathcal{A}'}(e, dg)\,. \end{aligned}$$

The following step shows that the indicator function $\mathbf{1}_K$ does not disturb too much the behaviour of theses integrals.

Third step. *Control of the residual terms.*

In the first step of the present proof, we have shown that, for any $\varepsilon > 0$ there exists $C_1 > 0$ such that

$$\begin{aligned} (n-k)^{(N+3)/2}\, \mathbb{E}\Big[[T_{\mathcal{A}} > n-k]; \varphi\Big(\frac{A_1^{n-k}}{\alpha}, \frac{u + U_1^{n-k}}{\alpha}, \beta + B_1^{n-k} \Big) \Big] \\ \leq C_1(\varphi)\, \alpha^{\varepsilon}\,. \end{aligned}$$

It follows that for any $0 < \delta < 1$

$$\sum_{k=1}^{i} \int_{\{g\in G : a(g) \leq \delta\}} \Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}, \frac{u(g)+u(h)}{a(g)}, b(g)+b(h)\Big) P_{\mathcal{A}}^{n-k}(e, dh) \Big)$$

$$
\begin{aligned}
&\cdot \tilde{P}^k_{\mathcal{A}'}(e, dg) \\
&\qquad \leq C_1 \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}} \, \mathbb{E}\,[[\tilde{T}_{\mathcal{A}'} > k]; (\tilde{A}^k_1)^{\varepsilon}] \\
&\qquad \leq C_1 \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}\, k^{3/2}} \, .
\end{aligned}
$$

On the other hand for any fixed $U > 0$, one has

$$
\begin{aligned}
&\sum_{k=1}^{i} \int_{\{g \in G : \|u(g)\| \geq U\}} \Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}, \frac{u(g)+u(h)}{a(g)}, b(h)+b(g)\Big) P^{n-k}_{\mathcal{A}}(e, dh)\Big) \\
&\qquad\qquad \cdot \tilde{P}^k_{\mathcal{A}'}(e, dg) \\
&\qquad \leq \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}} \, E\,[[\tilde{T}_{\mathcal{A}'} > k]; (\tilde{A}^k_1)^{\varepsilon} \, \|\tilde{U}_k\|^{\varepsilon/2}] \\
&\qquad \leq \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}\, k^{3/2}} \, .
\end{aligned}
$$

The last inequality being guaranteed by standart estimations. (see [7, Lemma 3.3] for more details).

References.

[1] Afanas'ev, V. I., On a maximum of a transient random walk in random environment. *Theory Probab. Appl.* **35** (1987), 205-215.

[2] Billingsley, P., *Ergodic theory and information.* Wiley Series in Probability and Mathematical statistics, 1964.

[3] Bougerol, Ph., Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré* **19** (1983), 369-391.

[4] Breiman, L., *Probability.* Addison-Wesley Publishing Company, 1964.

[5] Grincevicius, A. K., A central limit theorem for the group of linear transformation of the real axis. *Soviet Math. Doklady* **15** (1974), 1512-1515.

[6] Iglehart, D. L., Random walks with negative drift conditioned to stay positive. *J. Appl. Probab.* **11** (1974), 742-751.

[7] Le Page, E., Peigné, M., A local limit theorem on the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$. *Ann. Inst. H. Poincaré* **2** (1997), 223-252.

[8] Spitzer, F., *Principles of random walks.* D. Van Nostrand Company, 1964.

[9] Sunyach, Ch., Sur les fluctuations des marches aléatoires sur un groupe. *Lecture Notes in Math.* **1096** (1984), 549-558.

[10] Varopoulos, N. Th., Wiener-Hopf theory and nonunimodular groups. *J. Funct. Anal.* **120** (1994), 467-483.

[11] Varopoulos, N. Th., Analysis on Lie groups. *Revista Mat. Iberoamaricana* **12** (1996), 791-917.

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