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Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality

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Abstract. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by a Schrödinger operator $-A = \Delta - V$, where V is a nonnegative potential that belongs to a certain reverse Hölder class. We define a Hardy space H^1_A be means of a maximal function associated with the semigroup $\{T_t\}_{t>0}$. Atomic and Riesz transforms characterizations of H^1_A are shown.

1. Introduction and main results.

Let $A = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where $V \not\equiv 0$ is a nonnegative potential. We will assume that V belongs to reverse Hölder class $\mathcal{H}_q$ for some $q \geq d/2$, that is, V is locally integrable and

$$(1.0) \qquad \Big(\frac{1}{|B|}\int_B V^q\,dx\Big)^{1/q} \leq C\Big(\frac{1}{|B|}\int_B V\,dx\Big), \qquad \text{for every ball } B\,.$$

Trivially, $\mathcal{H}_q \subset \mathcal{H}_p$ provided $1 < p \leq q < \infty$. It is well known, *cf.* [Ge], that if $V \in \mathcal{H}_q$, then there is $\varepsilon > 0$ such that $V \in \mathcal{H}_{q+\varepsilon}$. Moreover, the measure $V(x)\,dx$ satisfies the doubling condition

$$\int_{B(y,2r)} V(x)\,dx \le C \int_{B(y,r)} V(x)\,dx\,.$$

We note that if V is a polynomial then $V \in \mathcal{H}_q$ for every $1 < q < \infty$.

Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by $-A$ and $T_t(x,y)$ be their kernels. Since V is nonnegative the Feynman-Kac formula implies that

$$0 \le T_t(x,y) \le \tilde{T}_t(x,y) = (4\pi\, t)^{-d/2} \exp\Big(-\frac{|x-y|^2}{4\,t}\Big)\,. \tag{1.1}$$

Obviously, by (1.1) the maximal operator

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| \tag{1.2}$$

is of weak-type (1,1).

We say that a function f is in the Hardy space H^1_A if

$$\|f\|_{H^1_A} = \|\mathcal{M}f\|_{L^1} < \infty\,.$$

The aim of this article is to present an atomic characterization of H^1_A.

For $n \in \mathbb{Z}$ we define the sets $\mathcal{B}_n$ by

$$\mathcal{B}_n = \{x:\ 2^{n/2} \le m(x,V) < 2^{(n+1)/2}\}\,, \tag{1.3}$$

where

$$m(x,V) = \Big(\sup\Big\{r>0:\ \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\,dy \le 1\Big\}\Big)^{-1}\,. \tag{1.4}$$

For more details concerning the function $m(x,V)$ and its applications in studying the Schrödinger operator A we refer the reader to [Fe] and [Sh].

Since $0 < m(x,V) < \infty$, we have $\mathbb{R}^d = \bigcup \mathcal{B}_n$.

A function a is an atom for the Hardy space H^1_A associated to a ball $B(x_0,r)$ if

(i) $$\operatorname{supp} a \subset B(x_0,r),$$

(ii) $$\|a\|_{L^\infty} \le \frac{1}{|B(x_0,r)|}\,,$$

(iii) $$\text{if}\ \ x_0 \in \mathcal{B}_n\ \text{then}\ r \le 2^{1-n/2}\,,$$

(iv) $$\text{if}\ x_0 \in \mathcal{B}_n,\ \text{and}\ r \le 2^{-1-n/2}\ \text{then}\ \int a(x)\,dx = 0\,.$$

The atomic norm in H^1_A is defined by

$$\|f\|_{A-\text{atom}} = \inf\left\{\sum |c_j|\right\},$$

where the infimum is taken over all decompositions $f = \sum c_j\, a_j$, where a_j are H^1_A atoms.

The main result of this article is the following

Theorem 1.5. *Assume that $V \not\equiv 0$ is a nonnegative potential such that $V \in \mathcal{H}_{d/2}$, then the norms $\|f\|_{H^1_A}$ and $\|f\|_{A-\text{atom}}$ are equivalent, that is, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_A} \le \|f\|_{A-\text{atom}} \le C\|f\|_{H^1_A}\,.$$

For $j = 1, 2, \ldots, d$, let us define the Riesz transforms R_j setting

$$R_j f = \frac{\partial}{\partial x_j} A^{-1/2}\,. \tag{1.6}$$

It was proved in [Sh] that if $V \in \mathcal{H}_{d/2}$ then the operators R_j are are bounded on L^p for $1 < p \le d$. It turns out that these operators characterize our Hardy space H^1_A, that is the following theorem holds.

Theorem 1.7. *If $V \in \mathcal{H}_{d/2}$ is a nonnegative potential, $V \not\equiv 0$, then there is a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_A} \le \|f\|_{L^1} + \sum_{j=1}^{d}\|R_j f\|_{L^1} \le C\|f\|_{H^1_A}\,. \tag{1.8}$$

2. Auxiliary lemmas.

Lemma 2.0. *There is a constant C such that for every $R > 2$ and every n if $x \in \mathcal{B}_n$, then*

$$\{k:\ B(x, 2^{-n/2}R) \cap \mathcal{B}_k \ne \varnothing\} \subset [n - C\log_2 R, n + C\log_2 R]\,.$$

PROOF. [Sh, Lemma 1.4] asserts that there exist constants $C > 0$, $c > 0$, and $k_0 > 0$ such that for every $x, y \in \mathbb{R}^d$ we have

$$m(y,V) \leq C\,(1 + |x-y|\,m(x,V))^{k_0}\,m(x,V) \tag{2.1}$$

and

$$m(y,V) \geq \frac{c\,m(x,V)}{(1+|x-y|\,m(x,V))^{k_0/(k_0+1)}}\,. \tag{2.2}$$

If $x \in \mathcal{B}_n$ and $y \in B(x, 2^{-n/2}R)$ then $|x-y|\,m(x,V) \leq 2\,R$ and by (2.1)

$$m(y,V) \leq C\,(1+2\,R)^{k_0}\,2^{n/2} \leq C\,2^{(n+C\log_2 R)/2}\,.$$

On the other hand applying (2.2), we obtain

$$m(y,V) \geq \frac{c\,2^{n/2}}{(1+2\,R)^{k_0/(k_0+1)}} \geq c\,2^{(n-C\log_2 R)/2}\,.$$

This completes the proof of the lemma.

Lemma 2.3. *There is a constant C and a collection of balls $B_{(n,k)} = B(x_{(n,k)}, 2^{2-n/2})$, $n \in \mathbb{Z}$, $k = 1, 2, \ldots,$ such that $x_{(n,k)} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B(x_{(n,k)}, 2^{-n/2})$, and*

$$\#\{(n',k') :\ B(x_{(n,k)}, R\,2^{-n/2}) \cap B(x_{(n',k')}, R\,2^{-n'/2}) \neq \varnothing\} \leq R^C\,,$$

for every (n,k) and $R \geq 2$.

From Lemma 2.3, we deduce

Corollary 2.4. *There exist constants $C > 0$ and $l_0 > 0$ such that for $l \geq l_0$ and every $x_{(n',k')}$ we have*

$$\sum_{(n,k)} (1+2^{n/2}|x_{(n',k')} - x_{(n,k)}|)^{-l} + \sum_{(n,k)} (1+2^{n'/2}|x_{(n',k')} - x_{(n,k)}|)^{-l} \leq C\,.$$

Another consequence of Lemma 2.3 is

Lemma 2.5. *There are nonnegative functions* $\psi_{(n,k)}$ *such that*

$$\psi_{(n,k)} \in C_c^\infty(B(x_{(n,k)}, 2^{1-n/2}))\,, \tag{2.6}$$

$$\sum_{(n,k)} \psi_{(n,k)}(x) = 1\,, \tag{2.7}$$

$$\|\nabla\psi_{(n,k)}\|_{L^\infty} \le C\,2^{n/2}\,. \tag{2.8}$$

3. Local maximal functions.

Lemma 3.0. *For ever* $l > 0$ *there is a constant* C_l *such that*

$$T_t(x,y) \le C_l\,(1+m(x,V)\,|x-y|)^{-l}\,|x-y|^{-d}\,, \qquad \textit{for } x,y \in \mathbb{R}^d\,. \tag{3.1}$$

Moreover, there is an $\varepsilon > 0$ *such that for every* $C' > 0$ *there exists* C *such that*

$$|T_t(x,y) - \tilde{T}_t(x,y)| \le C\,\frac{(|x-y|\,m(x,V))^\varepsilon}{|x-y|^d}\,, \tag{3.2}$$

for $|x-y| \le C'\,m(x,V)^{-1}$.

PROOF. Let $\Gamma(x,y,\tau)$, $\tilde{\Gamma}(x,y,\tau)$ be the kernels of the operators $(A + i\,\tau)^{-1}$ and $(-\Delta + i\,\tau)^{-1}$, $\tau \in \mathbb{R}$. It is proved in [Sh] (see [Sh, Theorem 2.7]) that for every $l > 0$ there is a constant C_l such that

$$\begin{aligned} |\Gamma(x,y,\tau)| \le\ & \frac{C_l}{(1+|\tau|^{1/2}\,|x-y|)^l\,(1+m(x,V)\,|x-y|)^l} \\ & \cdot \frac{1}{|x-y|^{d-2}}\,. \end{aligned} \tag{3.3}$$

By the functional calculus, $T_t(x,y) = c\int_{\mathbb{R}} e^{it\tau}\,\Gamma(x,y,\tau)\,d\tau$. Thus (3.1) is easily deduced from (3.3).

It follows from [Sh], see [Sh, Lemma 4.5 and its proof], that for every $l, C' > 0$ there exists a constant $C > 0$ such that

$$|\Gamma(x,y,\tau) - \tilde{\Gamma}(x,y,\tau)| \le \frac{C_l}{(1+|\tau|^{1/2}\,|x-y|)^l}\,\frac{(|x-y|\,m(x,V))^\varepsilon}{|x-y|^{d-2}}\,, \tag{3.4}$$

for $|x-y| \le C' \, m(x,V)^{-1}$. Now the estimate (3.2) is a consequence of (3.4) and the formula $T_t - \tilde{T}_t = c \int_{\mathbb{R}} e^{it\tau} \, (\Gamma - \tilde{\Gamma}) \, d\tau$.

Since $T_t(x,y)$ is a symmetric function, we also have

$$(3.5)\quad T_t(x,y) \le C_l \, (1+m(y,V) \, |x-y|)^{-l} \, |x-y|^{-d} , \qquad \text{for } x,y \in \mathbb{R}^d .$$

We define the local maximal operators M_n $\widetilde{\mathcal{M}}_n$, and $\mathcal{M}_n$ putting

$$(3.6) \qquad M_n f(x) = \sup_{0<t\le 2^{-n}} |\tilde{T}_t f(x) - T_t f(x)| ,$$

$$(3.7) \qquad \widetilde{\mathcal{M}}_n f(x) = \sup_{0<t\le 2^{-n}} |\tilde{T}_t f(x)| ,$$

$$(3.8) \qquad \mathcal{M}_n f(x) = \sup_{0<t\le 2^{-n}} |T_t f(x)| .$$

Lemma 3.9. *There exists a constant $C > 0$ such that for every (n,k)*

$$\|M_n(\psi_{(n,k)} f)\|_{L^1} \le C \, \|f \, \psi_{(n,k)}\|_{L^1} \, .$$

PROOF. Set $B^*_{(n,k)} = B(x_{(n,k)}, 2^{(8-n)/2})$. Then by (3.2)

$$\|M_n(\psi_{(n,k)} \, f)\|_{L^1(B^*_{(n,k)})} \le C_{(n,k)} \, \|\psi_{(n,k)} \, f\|_{L^1} \, ,$$

where

$$C_{(n,k)} \le \sup_{y\in B_{(n,k)}} \int_{B^*_{(n,k)}} \frac{(|x-y| \, m(x,V))^{\varepsilon}}{|x-y|^d} \, dx \, .$$

It is easy to check that $C_{(n,k)} \le C$.

The task is now to estimate $\|M_n(\psi_{(n,k)} f)\|_{L^1((B^*_{(n,k)})^c)}$. According to (1.1), we obtain

$$\|M_n(\psi_{(n,k)} \, f)\|_{L^1((B^*_{(n,k)})^c)} \le C'_{(n,k)} \, \|f \, \psi_{(n,k)}\|_{L^1} \, ,$$

where

$$C'_{(n,k)} = 2 \sup_{y\in B_{(n,k)}} \int_{(B^*_{(n,k)})^c} \big(\sup_{0<t\le 2^{-n}} \tilde{T}_t(x,y) \big) \, dx \le C' \, .$$

This finishes the proof of the lemma.

Let

$$\mathcal{M}_{(n,k)}f(x) = \sup_{0<t\le 2^{-n}} |T_t(\psi_{(n,k)}\, f)(x) - \psi_{(n,k)}(x)\, T_t f(x)|\,. \tag{3.10}$$

Lemma 3.11. *There is a constant C such that*

$$\sum_{(n,k)} \|\mathcal{M}_{(n,k)}f\|_{L^1} \le C\,\|f\|_{L^1}\,. \tag{3.12}$$

PROOF.

$$[\psi_{(n,k)}, T_t]f(x) = \sum_{(n',k')} T_{t,(n,k),(n',k')}f(x)\,,$$

where

$$T_{t,(n,k),(n',k')}f(x) = \int f(y)\, T_t(x,y)(\psi_{(n,k)}(x) - \psi_{(n,k)}(y))\, \psi_{(n',k')}(y)\, dy\,.$$

Let

$$\mathcal{M}_{(n,k),(n',k')}f(x) = \sup_{0<t\le 2^{-n}} |T_{t,(n,k),(n',k')}f(x)|\,.$$

Set $J_{(n,k)} = \{(n',k') : |x_{(n',k')} - x_{(n,k)}| \le C'\, 2^{-n/2}\}$, and $I_{(n,k)} = \{(n',k') : |x_{(n',k')} - x_{(n,k)}| > C'\, 2^{-n/2}\}$. Note that the number of elements in $J_{(n,k)}$ is bounded by a constant independent of (n,k). Moreover, taking C' is sufficiently large we see that if $(n',k') \in I_{(n,k)}$ then $B^{**}_{(n,k)} \cap B^{**}_{(n',k')} = \varnothing$, where $B^{**}_{(n,k)} = B(x_{(n,k)}, 2^{8-n/2})$. Furthermore, $|x-y| \sim |x_{(n,k)} - x_{(n',k')}|$ for $x \in B_{(n,k)}$, $y \in B_{(n',k')}$, provided $(n',k') \in I_{(n,k)}$.

Obviously,

$$\|\mathcal{M}_{(n,k),(n',k')}f\|_{L^1} \le C_{(n,k),(n',k')}\, \|f\|_{L^1(B_{(n',k')})}\,,$$

where

$$\begin{aligned} C_{(n,k),(n',k')} &\\ \le \sup_{y\in B_{(n',k')}} \int \Big(&\sup_{0<t\le 2^{-n}} |T_t(x,y)\, (\psi_{(n,k)}(x) \\ &\qquad - \psi_{(n,k)}(y))\, \psi_{(n',k')}(y)|\Big)\, dx\,. \end{aligned}$$

If $(n', k') \in J_{(n,k)}$ then, by (1.1) and (2.8), we have

$$C_{(n,k),(n',k')} \leq C \sup_{y \in B_{(n',k')}} \int \Big(\sup_{0<t\leq 2^{-n}} 2^{n/2} \, |x-y| \, \tilde{T}_t(x,y) \Big) \, dx \leq C \, .$$

If $(n', k') \in I_{(n,k)}$ then using (3.1), we get

$$\begin{aligned} & C_{(n,k),(n',k')} \\ & \leq \sup_{y \in B_{(n',k')}} \int \frac{C_{2l} \, \psi_{(n,k)}(x) \, \psi_{(n',k')}(y) \, dx}{|x-y|^d \, (1 + m(x,V) \, |x-y|)^{2l}} \\ & \leq \sup_{y \in B_{(n',k')}} \int \frac{C_{2l} \, \psi_{(n,k)}(x) \, \psi_{(n',k')}(y) \, dx}{|x-y|^d \, (1 + 2^{n/2} \, |x-y|)^l \, (1 + 2^{n/2} \, |x_{(n,k)} - x_{(n',k')}|)^l} \\ & \leq \frac{C}{(1 + 2^{n/2} \, |x_{(n,k)} - x_{(n'k')}|)^l} \, . \end{aligned}$$

Applying the above estimates, we obtain

$$\begin{aligned} & \sum_{(n,k)} \|\mathcal{M}_{(n,k)} f\|_{L^1} \\ & \leq \sum_{(n,k)} \sum_{(n',k')} \|\mathcal{M}_{(n,k),(n',k')} f\|_{L^1} \\ & \leq C \sum_{(n,k)} \sum_{(n',k') \in J_{(n,k)}} \|f\|_{L^1(B(x_{(n,k)}, C 2^{-n/2}))} \\ & \quad + C \sum_{(n,k)} \sum_{(n',k') \in I_{(n,k)}} (1 + 2^{n/2} \, |x_{(n,k)} - x_{(n',k')}|)^{-l} \, \|f\|_{L^1(B_{(n',k')})} \, . \end{aligned}$$

Finally, by Corollary 2.4, we get (3.12).

4. Proof of Theorem 1.5.

In this section we prove our main theorem. First we recall some results from the theory of local Hardy spaces, *cf.* [Go].

We say that a function f is in the local Hardy space $\mathbf{h}^1_n$ if

$$\|f\|_{\mathbf{h}^1_n} = \|\widetilde{\mathcal{M}}_n f\|_{L^1} < \infty \, . \tag{4.0}$$

A function $\tilde{a}$ is an atom for the local Hardy space $\mathbf{h}^1_n$ if there is a ball $B(x_0, r)$, $r \le 2^{1-n/2}$ such that

$$\operatorname{supp} \tilde{a} \subset B(x_0, r), \tag{4.1}$$

$$\|\tilde{a}\|_{L^\infty} \le |B(x_0, r)|^{-1}, \tag{4.2}$$

$$\text{if } r \le 2^{-1-n/2}, \text{ then } \int \tilde{a}(x)\, dx = 0 . \tag{4.3}$$

The atomic norm in $\mathbf{h}^1_n$ is defined by

$$\|f\|_{\mathbf{h}^1_{\tilde{a},n}} = \inf \Big(\sum_j |c_j| \Big), \tag{4.4}$$

where the infimum is taken over all decompositions $f = \sum c_j\, \tilde{a}_j$, where $\tilde{a}_j$ are $\mathbf{h}^1_n$ atoms.

Theorem 4.5 ([Go])**.** *The norms $\| \cdot \|_{\mathbf{h}^1_n}$ and $\| \cdot \|_{\mathbf{h}^1_{\tilde{a},n}}$ are equivalent with constants independent of $n \in \mathbb{Z}$.*

Moreover, if $f \in \mathbf{h}^1_n$, $\operatorname{supp} f \subset B(x, 2^{1-n/2})$, then there are $\mathbf{h}^1_n$ atoms $\tilde{a}_j$ such that $\operatorname{supp} \tilde{a}_j \in B(x, 2^{2-n/2})$ and

$$f = \sum_j c_j\, \tilde{a}_j\ , \qquad \sum_j |c_j| \le C\, \|f\|_{\mathbf{h}^1_n} \tag{4.6}$$

with a constant C independent of n.

PROOF OF THEOREM 1.5. We first assume that $f \in H^1_A$. Lemma 3.9 implies

$$\begin{aligned} &\|\widetilde{\mathcal{M}}_n(\psi_{(n,k)} f)\|_{L^1} \\ &\qquad \le C\, (\|\mathcal{M}_n(\psi_{(n,k)} f)\|_{L^1} + \|\psi_{(n,k)} f\|_{L^1}) \\ &\qquad \le C\, (\|\psi_{(n,k)}(\mathcal{M}_n f)\|_{L^1} + \|\mathcal{M}_{(n,k)} f\|_{L^1} + \|\psi_{(n,k)} f\|_{L^1}) . \end{aligned}$$

From Lemma 3.11 we conclude that

$$\sum_{(n,k)} \|\widetilde{\mathcal{M}}_n(\psi_{(n,k)} f)\|_{L^1} \le C\, (\|\mathcal{M} f\|_{L^1} + \|f\|_{L^1}) . \tag{4.7}$$

Application of Theorem 4.5 gives

$$\psi_{(n,k)}f = \sum_j c_j^{(n,k)}\, a_j^{(n,k)}\,, \qquad \text{where } a_j^{(n,k)} \text{ are } H^1_A \text{ atoms}, \tag{4.8}$$

and

$$\sum_j |c_j^{(n,k)}| \le C\, \|\widetilde{\mathcal{M}}_n(\psi_{(n,k)}f)\|_{L^1}\,. \tag{4.9}$$

Finally, using (4.7) and (4.8), we obtain the required H^1_A atomic decomposition

$$f = \sum_{(n,k)} \sum_j c_j^{(n,k)}\, a_j^{(n,k)} \text{ and } \sum_{(n,k)} \sum_j |c_j^{(n,k)}| \le C\, \|\mathcal{M}f\|_{L^1}\,, \tag{4.10}$$

and the inequality $\|f\|_{A-\text{atom}} \le C\, \|f\|_{H^1_A}$ is proved.

In order to prove the converse inequality we need only to show that that there exists a constant $C > 0$ such that for every H^1_A atom a

$$\|\mathcal{M}a\|_{L^1} \le C\,. \tag{4.11}$$

Let a be an H^1_A atom associated to a ball $B(x_0, r)$. Assume that $x_0 \in \mathcal{B}_n$. Then, by definition, $r \le 2^{1-n/2}$. Theorem 4.5 combined with Lemma 3.9 implies that $\|\mathcal{M}_n a\|_{L^1} \le C$. Therefore what is left is to show that

$$\Big\| \sup_{t>2^{-n}} |T_t a(x)| \Big\|_{L^1(dx)} \le C\,.$$

If $x \in B(x_0, 2^{(8-n)/2})$ then

$$\sup_{t>2^{-n}} |T_t a(x)| \le \sup_{t>2^{-n}} \int \tilde{T}_t(x,y)\, |a(y)|\, dy \le C\, 2^{nd/2}\,,$$

and, consequently,

$$\Big\| \sup_{t>2^{-n}} |T_t a(x)| \Big\|_{L^1(B(x_0, 2^{(8-n)/2}))} \le C\,.$$

If $x \notin B(x_0, 2^{(8-n)/2})$ and $y \in B(x_0, 2^{1-n/2})$ then $|x-y| \ge 2^{(2-n)/2}$. Moreover, $m(y,V) \sim 2^{n/2}$. Applying (3.5) we get

$$\begin{aligned}
&\int_{B(x_0,2^{(8-n)/2})^c} \sup_{t>2^{-n}} |T_t a(x)|\, dx \\
&\le \int_{B(x_0,2^{(8-n)/2})^c} \int |a(y)|\, C_l\, (1 + m(y,V)\, |x-y|)^{-l}\, \frac{1}{|x-y|^d}\, dy\, dx \\
&\le C_l \int 2^{dn/2} (1 + 2^{n/2}\, |x|)^{-l}\, dx \le C\,.
\end{aligned}$$

5. Characterization of H^1_A by the Riesz transforms.

In this section we prove Theorem 1.7. Our proof of it is very much in the spirit of the proof of Theorem 1.5.

First we recall the characterization of the local Hardy spaces $\mathbf{h}^1_n$ by means of local Riesz transforms. Let ζ be a C^∞ function on $\mathbb{R}^d$ such that $\zeta(x) = 0$ for $|x| \geq 1$ and $\zeta(x) = 1$ for $|x| < 1/2$. We define the local Riesz transforms $\mathcal{R}_j^{[n]}$ by

$$\mathcal{R}_j^{[n]} f = f * \mathcal{R}_j^{[n]} \,, \tag{5.0}$$

where

$$\mathcal{R}_j^{[n]}(x) = c_d\, \zeta(2^{n/2}\, x)\, \frac{x_j}{|x|^{d+1}} \,.$$

We have

Theorem 5.1 *There is a constant $C > 0$ such that for every integer n*

$$C^{-1}\, \|f\|_{\mathbf{h}^1_n} \leq \|f\|_{L^1} + \sum_{j=1}^{d} \|f * \mathcal{R}_j^{[n]}\|_{L^1} \leq C\, \|f\|_{\mathbf{h}^1_n} \,. \tag{5.2}$$

Throughout this section we shall assume that $V \in \mathcal{H}_{d/2}$ is a non-negative potential, $V \not\equiv 0$.

Let us denote by $R_j(x,y)$ the integral kernel of the operator

$$\frac{\partial}{\partial x_j}\, A^{-1/2} \,.$$

Lemma 5.3 *There exists a constant $C > 0$ such that for every (n,k)*

$$C_{(n,k)} = \sup_{y \in B_{(n,k)}} \int_{B(x_{(n,k)},\, 2^{8-n/2})^c} |R_j(x,y)|\, dx \leq C \,.$$

PROOF. By [Sh, p. 538] we have that for every $l > 0$ there is a constant C_l such that

$$|R_j(x,y)| \leq \frac{C_l}{(1 + m(y,V)\, |x-y|)^l} \cdot \Big(\frac{1}{|x-y|^{d-1}} \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\, dz + \frac{1}{|x-y|^d} \Big) \,. \tag{5.4}$$

Let us note that if $y \in B_{(n,k)}$ and $x \notin B^{**}_{(n,k)} = B(x_{(n,k)}, 2^{8-n/2})$, then $|x-y| \sim |x - x_{(n,k)}|$. Thus

$$\begin{aligned}
C_{(n,k)} \leq C_l \sup_{y \in B_{(n,k)}} \int_{(B^{**}_{(n,k)})^c} & \frac{1}{(1+2^{n/2}\,|x-x_{(n,k)}|)^l\,|x-x_{(n,k)}|^{d-1}} \\
& \cdot \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz\,dx \\
+ C_l \int_{(B^{**}_{(n,k)})^c} & \frac{1}{(1+2^{n/2}\,|x-x_{(n,k)}|)^l\,|x-x_{(n,k)}|^{d}}\,dx \\
= C'_{(n,k)} + C''_{(n,k)}\,. &
\end{aligned}$$

Obviously $C''_{(n,k)} \leq C$. We now turn to estimate $C'_{(n,k)}$.

$$\begin{aligned}
C'_{(n,k)} &\leq C_l \int_{(B^{*}_{(n,k)})^c} \Big(\frac{V(z)}{(1+2^{n/2}\,|z-x_{(n,k)}|)^l\,|z-x_{(n,k)}|^{d-1}} \\
&\qquad \cdot \int_{B(z,|x_{(n,k)}-z|/2)} \frac{1}{|z-x|^{d-1}}\,dx \Big)\,dz \\
&\leq C_l \int_{(B^{*}_{(n,k)})^c} \frac{V(z)}{(1+2^{n/2}\,|z-x_{(n,k)}|)^l\,|z-x_{(n,k)}|^{d-2}}\,dz\,.
\end{aligned}$$

[Sh, Lemma 1.8] asserts that if $\rho\, m(x,V) \geq 1$ then

$$\frac{1}{\rho^{d-2}} \int_{B(x,\rho)} V(z)\,dz \leq C\,(\rho\, m(x,V))^{k_0}\,, \tag{5.5}$$

for some $k_0 > 0$. Therefore

$$\begin{aligned}
C'_{(n,k)} &\leq C_l \sum_{i=0}^{\infty} \int_{B(x_{(n,k)},2^{i+1-n/2})} \frac{V(z)}{(1+2^i)^l\,(2^{i-n/2})^{d-2}}\,dz \\
&\leq C_l \sum_{i=0}^{\infty} \frac{1}{(1+2^i)^l}\,(2^{i+1-n/2}\,2^{n/2})^{k_0} \\
&\leq C\,.
\end{aligned}$$

Corollary 5.6. *There is a constant $C > 0$ such that for every (n,k) we have*

$$\|R_j(\psi_{(n,k)}f)\|_{L^1((B^{**}_{(n,k)})^c)} \leq C\,\|\psi_{(n,k)}f\|_{L^1}\,. \tag{5.7}$$

Lemma 5.8. *There exists a constant C such that*

$$\sum_{(n,k)} \|R_j(\psi_{(n,k)}f) - \psi_{(n,k)}R_jf\|_{L^1} \leq C\,\|f(x)\|_{L^1(dx)}\,. \tag{5.9}$$

PROOF. For fixed (n,k) we have

$$\|[\psi_{(n,k)}, R_j]f\|_{L^1} \leq \sum_{(n',k')} C_{(n',k')}\,\|f\|_{L^1(B_{(n',k')})}\,, \tag{5.10}$$

where

$$\begin{aligned} C_{(n',k')} \leq \sup_{y\in B_{(n',k')}} \int |R_j(x,y)(\psi_{(n,k)}(x) \\ - \psi_{(n,k)}(y))\,\psi_{(n',k')}(y)|\,dx\,. \end{aligned} \tag{5.11}$$

Let $J_{(n,k)}$ and $I_{(n,k)}$ be as in the proof of Lemma 3.11.

If $(n',k') \in J_{(n,k)}$ then, by Lemma 2.5,

$$\begin{aligned} C_{(n',k')} &\leq \sup_{y\in B_{(n,k)}} \int_{(B^{**}_{(n,k)})^c} |R_j(x,y)|\,dx \\ &\quad + \sup_{y\in B_{(n',k')}} \int_{B^{**}_{(n,k)}} C\,|R_j(x,y)|\,2^{n/2}\,|x-y|\,dx \\ &= S_1 + S_2\,. \end{aligned}$$

Lemma 5.3 clearly forces $S_1 \leq C$.

Applying (5.4) and the theorem on fractional integrals we obtain

$$\begin{aligned} S_2 \le \sup_{y\in B_{(n',k')}} \int_{B^{**}_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l} \frac{2^{n/2}\,|x-y|}{|x-y|^{d-1}} \\ \cdot \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz\,dx \\ + \sup_{y\in B_{(n',k')}} \int_{B^{**}_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l} \frac{2^{n/2}\,|x-y|}{|x-y|^{d}}\,dx \\ \le \sup_{y\in B_{(n',k')}} C \int_{B^{**}_{(n,k)}} \Big(\frac{1}{|x-y|^{d-1}} \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz \\ + \frac{2^{n/2}}{|x-y|^{d-1}} \Big)\,dx \\ \le \sup_{y\in B_{(n',k')}} C \int_{B(x_{(n,k)},C2^{-n/2})} \frac{V(z)}{|z-y|^{d-2}}\,dz \\ + \sup_{y\in B_{(n',k')}} C \int_{B^{**}_{(n,k)}} \frac{2^{n/2}}{|x-y|^{d-1}}\,dx\,. \end{aligned}$$

Let us note that the Hölder inequality and the fact that $V \in \mathcal{H}_{d/2+\varepsilon}$ for some $\varepsilon > 0$ imply that

$$\int_{B(x,\rho)} \frac{V(z)}{|z-x|^{d-2}}\,dz \le \frac{C}{\rho^{d-2}} \int_{B(x,\rho)} V(z)\,dz\,. \tag{5.12}$$

Therefore $S_2 \le C$.

If $(n',k') \in I_{(n,k)}$, then

$$C_{(n',k')} \le \sup_{y\in B_{(n',k')}} \int_{B_{(n,k)}} |R_j(x,y)|\,dx\,.$$

Using (5.4) we get

$$\begin{aligned} C_{(n',k')} \le \sup_{y\in B_{(n',k')}} \int_{B_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l\,|x-y|^{d-1}} \\ \cdot \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz\,dx \\ + \sup_{y\in B_{(n',k')}} \int_{B_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l\,|x-y|^{d}}\,dx\,. \end{aligned}$$

Since $|x-y| \sim |x_{(n,k)} - x_{(n',k')}|$ for $x \in B_{(n,k)}$, $y \in B_{(n',k')}$, we have

$$C_{(n',k')} \leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \cdot \sup_{y\in B_{(n',k')}} \int_{B(x_{(n,k)},|x_{(n,k)}-y|/2)} \frac{V(z)}{|y-z|^{d-2}}\,dz + \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l}\,.$$

It is not difficult to check that $B(x_{(n,k)}, |x_{(n,k)}-y|/2) \subset B(y, C|x_{(n,k)} - x_{(n',k')}|)$ for $y \in B_{(n',k')}$, with C independent of (n,k) and (n',k'). Thus

$$C_{(n',k')} \leq \frac{C_l}{(1+2^{n'/2}|x_{(n,k)} - x_{(n',k')}|)^l} \cdot \Big(1 + \sup_{y\in B_{(n',k')}} \int_{B(y,C|x_{(n,k)}-x_{(n',k')}|)} \frac{V(z)}{|y-z|^{d-2}}\,dz\Big)\,.$$

Now using (5.12) we obtain

$$\begin{aligned} &C_{(n',k')} \\ &\leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \\ &\quad + \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \\ &\quad\cdot \sup_{y\in B_{(n',k')}} \frac{1}{|x_{(n,k)} - x_{(n',k')}|^{d-2}} \int_{B(y,C|x_{(n,k)}-x_{(n',k')}|)} V(z)\,dz\,. \end{aligned}$$

By virtue of (5.5) we get

$$\begin{aligned} C_{(n',k')} &\leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \cdot (1 + C\,(2^{n'/2}\,|x_{(n',k')} - x_{(n,k)}|)^{k_0}) \\ &\leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^{l-k_0}}\,. \end{aligned}$$

Now (5.9) follows easily from (5.10), Corollary 2.4, and Lemma 2.3.

Let $\tilde{R}_j f = (\partial/\partial x_j)\,\Delta^{-1/2} f$ denote the classical Riesz transforms and let $\tilde{R}_j(x,y)$ be their kernels.

Lemma 5.13. *There exists a constant $C>0$ such that*

$$\|R_j(\chi_{B_{(n,k)}} f) - \tilde{R}_j(\chi_{B_{(n,k)}} f)\|_{L^1(B^{**}_{(n,k)})} \le C\,\|\chi_{B_{(n,k)}} f\|_{L^1}\,. \tag{5.14}$$

Proof. The left-hand side of (5.14) is estimated by

$$C_{(n,k)}\,\|\chi_{B_{(n,k)}} f\|_{L^1}\,,$$

where

$$C_{(n,k)} \le \sup_{y\in B_{(n,k)}} \int_{B^{**}_{(n,k)}} |R_j(x,y) - \tilde{R}_j(x,y)|\,dx\,.$$

[Sh, Estimate (5.9)] says that for every $C'>0$ there is a constant $C>0$ such that

$$|R_j(x,y) - \tilde{R}_j(x,y)| \le \frac{C}{|x-y|^{d-1}}\Big(\int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz + \frac{1}{|x-y|}\,(|x-y|\,m(y,V))^{\varepsilon}\Big)\,,$$

for $|x-y| \le C'/m(y,V)$ and some $\varepsilon>0$. (In [Sh] this estimate is shown with $C'=1$. Actually the proof works for any C'). The theorem on fractional integrals leads to

$$C_{(n,k)} \le C \sup_{y\in B_{(n,k)}} \int_{B(y,C2^{-n/2})} \frac{V(z)}{|y-z|^{d-2}}\,dz + C \sup_{y\in B_{(n,k)}} \int_{B^{**}_{(n,k)}} \frac{(2^{n/2}\,|x-y|)^{\varepsilon}}{|x-y|^{d}}\,dx\,.$$

By virtue of (5.12) we have $C_{(n,k)} \le C$, and the proof is complete.

Proof of Theorem 1.7. Assume first that $\|f\|_{L^1} + \sum_{j=1}^{d} \|R_j f\|_{L^1} < \infty$. Lemmas 5.8 and 5.13 imply that

$$\sum_{(n,k)} \|\tilde{R}_j(\psi_{(n,k)} f)\|_{L^1(B^{**}_{(n,k)})} \le C\,(\|f\|_{L^1} + \|R_j f\|_{L^1})\,.$$

Now using Theorem 5.1, we obtain the required atomic decomposition

$$f = \sum_{(n,k)} \psi_{(n,k)} f = \sum_{(n,k)} \sum_i c_i^{(n,k)} \, a_i^{(n,k)} \,,$$

$$\sum_{(n,k)} \sum_i |c_i^{(n,k)}| \leq C \Big(\sum_{j=1}^{d} \|R_j f\|_{L^1} + \|f\|_{L^1} \Big) \,,$$

where $a_i^{(n,k)}$ are H_A^1 atoms.

To prove the converse inequality we only, by Theorem 1.5, need to show that

$$\|R_j a\|_{L^1} \leq C \,,$$

for every H_A^1 atom a with a constant C independent of a. Assume that a is an H_A^1 atom associated to a ball $B(x_0, r)$. If $x_0 \in \mathcal{B}_n$ then by definition $r \leq 2^{1-n/2}$ and there exists k such that $B(x_0, r) \subset B(x_{(n,k)}, 2^{2-n/2})$. By Lemma 5.3 we have

$$\|R_j a\|_{L^1((B^{**}_{(n,k)})^c)} \leq C \,.$$

On the other hand, since a is an atom for $\mathbf{h}_n^1$, Theorem 5.1 implies that $\|\tilde{R}_j a\|_{L^1(B^{**}_{(n,k)})} \leq C$. Applying Lemma 5.13, we get

$$\|R_j a\|_{L^1(B^{**}_{(n,k)})} \leq C \,,$$

which finishes the proof of the theorem.

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