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Topological sectors for Ginzburg-Landau energies

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1. Introduction.

1.1. Ginzburg-Landau functionals.

Let Ω be the annulus $\{x \in \mathbb{R}^2 : 1/4 < |x| < 1\} \subset \mathbb{R}^2$. For maps $u \in H^1(\Omega, \mathbb{R}^2) = W^{1,2}(\Omega, \mathbb{R}^2)$ we consider the Ginzburg-Landau functional

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\,\varepsilon^2} \int_\Omega (1 - |u|^2)^2 \,, \tag{1.1}$$

where ε is a small parameter. For $\Lambda \in \mathbb{R}^+$ we define the energy level set E_ε^Λ as

$$E_\varepsilon^\Lambda := \{u \in H^1(\Omega, \mathbb{R}^2) : \; E_\varepsilon(u) < \Lambda\} \,. \tag{1.2}$$

One of the main purposes of this paper is to show that given $\Lambda > 0$, for ε small enough, E_ε^Λ may be multiply connected. Moreover, the connected components of E_ε^Λ may be classified by the degree of u (since u is not S^1-valued, we have to be careful in order to define its degree – this is the main technical problem of our work).

Functionals like E_ε play an important role in many low temperature physics phenomena like superfluidity. We can also find closely related functionals in the theory of superconductivity and in two-dimensional Higgs models. In our work we will consider one of these superconductivity models: the gauge-covariant Ginzburg-Landau model, where the energy functional may be written as

$$F_\varepsilon(u,A) = \frac{1}{2}\int_{\mathbb{R}^2} |dA|^2 + \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{4\,\varepsilon^2}\int_\Omega (1-|u|^2)^2\,,$$

where $u \in H^1(\Omega,\mathbb{R}^2)$, as before, and $A \in H^1(\mathbb{R}^2,\mathbb{R}^2)$ is the gauge potential one-form,

$$A = A_1\,dx^1 + A_2\,dx^2 \cong \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = (A_1, A_2)\,.$$

Here, as we will often do in this paper, we used the natural identification (given by the $\mathbb{R}^2$ scalar product) between the one-form A and the vector with the same components which we also denote by A. In equation (1.3) the expression $\nabla_A u$ denotes the covariant derivative of u, *i.e.* $\nabla_A u = \nabla u - \imath A u$.

This model was introduced by Ginzburg and Landau in the 50's for the study of phase transitions in superconducting materials (see the remarks on physics below).

The main feature of the functional F_ε is its invariance under gauge transformations. For a function $\phi \in W^{2,2}(\mathbb{R}^2,\mathbb{R})$, the gauge transformation associated to ϕ is the map $(u,A) \longmapsto (u_\phi, A_\phi)$ given by

$$\begin{cases} u_\phi = \exp(\imath\,\phi)\,u\,, & \text{in } \Omega\,, \\ A_\phi = A + d\phi\,, & \text{in } \mathbb{R}^2\,. \end{cases} \tag{1.4}$$

In this case we say that (u,A) is gauge-equivalent to (u_ϕ, A_ϕ) and we denote this by $(u,A) \sim (u_\phi, A_\phi)$. Saying that F_ε is gauge-invariant means that

$$F_\varepsilon(u_\phi, A_\phi) = F_\varepsilon(u,A)\,, \qquad \text{for all } \phi \in W^{2,2}(\mathbb{R}^2,\mathbb{R})\,. \tag{1.5}$$

This gauge-invariance follows easily from the facts that

$$(u_\phi, A_\phi) \in H^1(\Omega,\mathbb{R}^2) \times H^1(\mathbb{R}^2,\mathbb{R}^2)\,, \qquad |u_\phi| = |u|\,,$$

$$dA_\phi = dA + d\,d\phi = dA\,, \tag{1.6}$$

$$\nabla_{A_\phi} u_\phi = \exp(\imath\,\phi)\,\nabla_A u\,, \text{ and thus } |\nabla_{A_\phi} u_\phi| = |\nabla_A u|\,. \tag{1.7}$$

The only quantities which are significant from the physics point of view are those, like $|u|$, $\nabla_A u$ and the magnetic field $h = \star dA$, which are invariant under gauge transformations. Other important gauge-invariant quantities are the current $J = (\imath\, u, \nabla_A u)$ and, the one which we are more concerned about in this paper, the degree of u along a smooth closed curve γ, diffeomorphic to S^1, such that $|u| \neq 0$ on γ. In integral form, this degree is given by

$$\deg(u,\gamma) = \frac{1}{2\pi}\int_\gamma \frac{u}{|u|} \times \partial_\tau\Big(\frac{u}{|u|}\Big)\, d\tau\,, \tag{1.8}$$

where τ denotes the unit tangent to γ.

It is easy to see that gauge-equivalence defines an equivalence relation in $H^1(\Omega,\mathbb{R}^2) \times H^1(\mathbb{R}^2,\mathbb{R}^2)$. A physical state of our system is associated not with an individual configuration (u, A), but with a whole equivalence class $[u, A] := \{(v, B) \in H^1(\Omega,\mathbb{R}^2) \times H^1(\mathbb{R}^2,\mathbb{R}^2) : (v, B) \sim (u, A)\}$. We denote the physical space by $H_{gi} = [H^1(\Omega,\mathbb{R}^2) \times H^1(\mathbb{R}^2,\mathbb{R}^2)]/\sim$, and also consider F_ε as a functional defined on H_{gi}.

As in the case of E_ε, we define the energy level sets of F_ε by

$$F_\varepsilon^\Lambda := \{[v, B] \in H_{gi} :\ F_\varepsilon([v, B]) < \Lambda\}\,.$$

Since the functional E_ε does not involve the connection, it is a little easier to deal with than the functional F_ε. Nevertheless, as we will see in our work, most of the mathematical difficulties are already encountered in the study of E_ε. In fact, after some additional technical arguments, we deduce the classification result for the components of the level sets of F_ε, from the corresponding result for E_ε. Therefore, we start by considering the functional E_ε given by (1.1).

1.2. Degree of a map and definition of topological sectors.

We consider a fixed number $\Lambda > 0$, and focus our attention on the level set E_ε^Λ defined by (1.2). First, we remark that since the notion of degree we define is continuous in $W^{1,2}(\Omega) \cap E_\varepsilon^\Lambda$ and that smooth maps are dense in $W^{1,2}(\Omega) = H^1(\Omega)$, it suffices to consider the case where $u \in W^{1,2}(\Omega) \cap C^\infty$. Hence, without loss of generality, we will always assume that u is smooth in this paper.

Based on the work of B. White [28] (see also the work of F. Bethuel [6]), for maps $u \in W^{1,2}(\Omega, S^1)$, *i.e.* for the case when $|u| \equiv 1$, we can define the degree of u in Ω, $\deg(u,\Omega)$, as the degree of the restriction of u to a one-dimensional skeleton of Ω – for instance, in case u is continuous, this can be any circle $S_r = \{x : |x| = r\}$, for $1/4 < r < 1$

(if u is not continuous we might need to move the circle slightly in order to have a "nice" restriction). The degree can then be written, in integral form, as

$$\deg(u,\Omega) = \deg(u,S_r) = \frac{1}{2\pi}\int_{S_r} \frac{u}{|u|} \times \partial_\tau\Big(\frac{u}{|u|}\Big)\,d\tau\,. \tag{1.9}$$

This definition of the degree will always give us an integer, and it classifies the homotopy classes of $W^{1,2}(\Omega,S^1)$. Our purpose is to extend this notion to all $u \in E_\varepsilon^\Lambda$ for ε sufficiently small. In this context, our first result is given by the following Theorem.

Theorem 1. *Given $\Lambda \in \mathbb{R}^+$, there exists $\varepsilon_0 > 0$, depending only on Λ, such that for $\varepsilon < \varepsilon_0$, we can define a continuous map*

$$\begin{aligned} \chi : E_\varepsilon^\Lambda &\longrightarrow \mathbb{Z}\,,\\ u &\longmapsto \deg(u,\Omega)\,, \end{aligned} \tag{1.10}$$

such that this map coincides with the classical notion of degree mentioned above when u has values in S^1 (i.e. when $u \in W^{1,2}(\Omega,S^1)\cap E_\varepsilon^\Lambda$).

Usually we call the map χ the global degree in Ω and, as above, we denote $\chi(u) = \deg(u,\Omega)$. For each $n \in \mathbb{Z}$, $\chi^{-1}(n) = \{u \in E_\varepsilon^\Lambda : \deg(u,\Omega) = n\}$, is an open and closed subset of E_ε^Λ which we call the n^{th} topological sector of E_ε^Λ, and we also denote it by $\mathrm{top}_n(E_\varepsilon^\Lambda)$.

REMARK. In fact, what we prove in Theorem 1 is that the degree of u is constant inside each connected component of E_ε^Λ – we do not show that different connected components correspond to different values of the degree, which would give us a complete classification of the components by the degree of its members. We will come back to this question later on.

The asymptotic behavior, when $\varepsilon \longrightarrow 0$ of critical points of the functionals E_ε and F_ε was extensively studied by many authors. Among them we would like to single out the work of F. Bethuel, H. Brezis and F. Hélein [8] regarding the functional E_ε, and those of F. Bethuel and T. Rivière [9] and [10] which concern the functional F_ε.

We will give a rough description of the proof of Theorem 1 at the end of the Introduction. This proof is rather technical and will be done in sections 2 to 8. The Euler-Lagrange equations for the functional E_ε are called the Ginzburg-Landau equations. They can be written as

$$-\Delta u = \frac{1}{\varepsilon^2}\,u\,(1-|u|^2)\,, \qquad \text{in } \Omega\,. \tag{1.11}$$

In the context of the gauge invariant model, we can also extend the definition of degree to any configuration $[v, B] \in F_\varepsilon^\Lambda$ provided ε is small enough. In fact, we prove

Theorem 2. *Given $\Lambda \in \mathbb{R}^+$, there exists $\varepsilon_0 > 0$, depending only on Λ, such that for $\varepsilon < \varepsilon_0$, we can define a continuous map*

$$\begin{aligned} \hat{\chi} : F_\varepsilon^\Lambda &\longrightarrow \mathbb{Z}\,, \\ [u, A] &\longmapsto \deg\left([u, A], \Omega\right), \end{aligned} \tag{1.12}$$

such that this map coincides with the classical notion of degree mentioned above when u has values in S^1 (i.e. when $u \in W^{1,2}(\Omega, S^1) \cap F_\varepsilon^\Lambda$). Usually we call the map $\hat{\chi}$ the global degree in Ω and, as above, we denote $\hat{\chi}(u, A) = \deg\left([u, A], \Omega\right)$.

Minimizing E_ε inside each component of E_ε^Λ (or F_ε inside each component of F_ε^Λ), we will obtain solutions of (1.11) which are locally minimizing, *i.e.* critical points of E_ε (respectively, F_ε) which are local minima. These are the solutions that should be associated with permanent currents.

Moreover, we will show in the next subsection, that as a corollary of Theorems 1 and 2, we can also prove the existence of mountain-pass points for E_ε (which correspond to mountain-pass type solutions of (1.11)). An analogous reasoning gives the existence of mountain-pass points for F_ε. This result is stated in Theorem 4. Unlike the solutions obtained minimizing the energy inside each topological sector, the solutions of (1.11) we obtain in Theorem 4 will not necessarily be local minimizers of E_ε, and are probably unstable.

1.3. Mountain-pass solutions and threshold energies.

We start by the crucial, although elementary, remark that when $\Lambda = \infty$, we have that $E_\varepsilon^\infty = H^1(\Omega)$, *i.e.* the whole affine space $H^1(\Omega, \mathbb{R}^2)$. This space has obviously an unique component and furthermore, given any two elements $u_0, u_1 \in H^1(\Omega, \mathbb{R}^2)$ there is a natural path between them: the straight line segment $\gamma : [0, 1] \longrightarrow H^1(\Omega, \mathbb{R}^2)$, defined by

$$\gamma(s) := (1 - s)\, u_0 + s\, u_1\,, \qquad \text{for } s \in [0, 1]\,. \tag{1.13}$$

Likewise, $F_\varepsilon^\infty = H_{gi}$, which is the projection (continuous image) of $H^1 \times H^1$, and thus is connected. Given two states $[u_0], [u_1] \in H_{gi}$ we may consider the straight line between two of their representatives, $u_0, u_1 \in H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ and consider the projection in H_{gi} of the straight line in $H^1 \times H^1$ between u_0 and u_1.

An important example of a map of degree $n \in \mathbb{Z}$, in $H^1(\Omega, S^1) \subset H^1(\Omega, \mathbb{R}^2)$ (and for which we can thus use the classical definition of the degree), is the map

$$w_n(r,\theta) := \exp(\imath\, n\, \theta) = \frac{z^n}{|z|^n}\,. \tag{1.14}$$

Using (1.9) it is easy to check that $\deg(w_n, \Omega) = n$ and moreover, we can see that the energy, $E_\varepsilon(w_n)$, of the maps w_n, $n \in \mathbb{Z}$, is independent of ε and is given by

$$E_\varepsilon(w_n) = \frac{1}{2}\int_\Omega |\nabla w_n|^2 = \frac{1}{2}\int_{1/4}^1 r \int_0^{2\pi} \frac{n^2}{r^2}\, d\theta\, dr = \pi\, n^2\, \log 4\,. \tag{1.15}$$

Hence, given $\Lambda \in \mathbb{R}^+$, let

$$n_0 := \left[\sqrt{\frac{\Lambda}{\pi \log 4}}\,\right],$$

be the largest integer less than or equal to $\sqrt{\Lambda/(6\pi \log 4)}$. From equation (1.15) it follows that, at least for $n \in [-n_0, \dots, n_0]$, the topological sector $\mathrm{top}_n(E_\varepsilon^\Lambda)$ will be non-empty, and this independently of the value of $\varepsilon > 0$.

Likewise, for F_ε we could take $w_n(r,\theta) := [(\exp(\imath\, n\, \theta), 0)]$. All the rest of the discussion also easily extends to the case of F_ε.

Let $\Lambda \in \mathbb{R}^+$ be given, and let $\varepsilon < \varepsilon_0$ (where ε_0 is as in Theorem 1). Suppose that for some $n \in \mathbb{Z}$ both $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$ are non-empty, and consider two maps

$$u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda)\,, \qquad u_1 \in \mathrm{top}_{n+1}(E_\varepsilon^\Lambda)\,.$$

Let $\gamma : [0,1] \longrightarrow H^1(\Omega)$ be a path between u_0 and u_1 (*i.e.* $\gamma(0) = u_0$ and $\gamma(1) = u_1$). Recall that, as we mentioned above, such a path always exists because $H^1(\Omega, \mathbb{R}^2)$ is an affine space. Then, γ cannot be entirely contained in E_ε^Λ – if this were so, u_0 and u_1 would be in the same path component of E_ε^Λ, and hence also in the same component of E_ε^Λ

which contradicts our assumption (since, by Theorem 1, the topological sectors $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$ are disjoint open and closed subsets of the energy level set E_ε^Λ). Hence, there exists some $s \in (0,1)$ such that $\gamma(s) \notin E_\varepsilon^\Lambda$, which is equivalent to saying that $E_\varepsilon(\gamma(s)) \geq \Lambda$. A standard Min-Max argument will then yield the existence of generalized critical values of E_ε of the form

$$c_n := \inf_{\gamma \in \mathcal{V}} \max_{s \in [0,1]} E_\varepsilon(\gamma(s)) \,. \tag{1.16}$$

where $\mathcal{V} := \{\gamma \in C^0([0,1], H^1(\Omega, \mathbb{R}^2)) : \gamma(0) = u_0, \text{ and } \gamma(1) = u_1\}$, is the space of continuous paths in $H^1(\Omega)$ between u_0 and u_1. The value c_n will be a generalized critical value of E_ε. To make sure it is actually a critical value we use the following

Theorem 3. *The functionals E_ε and F_ε satisfy the Palais-Smale condition (in $H^1(\Omega, \mathbb{R}^2)$ and H_{gi}, respectively).*

This implies that c_n is a critical value of E_ε and hence, there exists a map $u \in H^1(\Omega)$ such that u is a critical point of E_ε and $E_\varepsilon(u) = c_n$. This u is probably not a local minimum of E_ε. All this discussion extends to the case of F_ε. Thus, we have proved

Theorem 4. *Suppose that for some $\Lambda \in \mathbb{R}^+$, we have that for some $\varepsilon < \varepsilon_0$ (where ε_0 is given Theorem 1) there exists $n \in \mathbb{Z}$ such that the topological sectors $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$ are both non-empty. Then, there are mountain-pass type critical points of E_ε or, equivalently, there exist mountain-pass type solutions of the Ginzburg-Landau equations (1.11).*

More precisely, consider two maps

$$u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda) \qquad \text{and} \qquad u_1 \in \mathrm{top}_{n+1}(E_\varepsilon^\Lambda)\,,$$

and let c_n be defined as in (1.16). Then, there exists a map $u \in H^1(\Omega, \mathbb{R}^2)$ such that u is a critical point of E_ε and $E_\varepsilon(u) = c_n$.

Likewise, if we consider two states $\Phi_0 \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and $\Phi_1 \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$, and let c_n be defined by

$$c_n := \inf_{\gamma \in \mathcal{V}} \max_{s \in [0,1]} F_\varepsilon(\gamma(s))\,, \tag{1.17}$$

where now $\mathcal{V} := \{\gamma \in C^0([0,1], H_{gi}) : \gamma(0) = \Phi_0, \text{ and } \gamma(1) = \Phi_1\}$, is the space of continuous paths in H_{gi} between Φ_0 and Φ_1. Then, there *exists a state $\Phi = [(u, A)] \in H_{gi}$ such that Φ is a critical point of F_ε and $F_\varepsilon(\Phi) = c_n$.*

REMARK 1. The number c_n defined by (1.16) is called the *threshold energy* for the transition from the state u_0 to the state u_1. It will be the infimum of the energies for which such a transition is possible. This concept will play a crucial role in the physical behavior of our system. We will come back to this point in the remarks on physics (see below).

REMARK 2. In Theorem 4, for simplicity, we just considered transitions from a state $u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda)$ to a state u_1 belonging to the adjacent state $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$. However, both the concept of threshold energy and the result stated in Theorem 4 are immediately generalizable to the case where $u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $u_1 \in \mathrm{top}_k(E_\varepsilon^\Lambda)$, for any two distinct integers $n, k \in \mathbb{Z}$. As usual, this remark and the previous one extend to the setting of the gauge-covariant functional F_ε.

REMARK 3. All these results extend to the setting of more general domains considered in Theorem 6, stated below.

1.4. Remarks on physics.

1.4.1. Ginzburg-Landau theory.

In the Ginzburg-Landau theory of superconductivity, the conducting electrons are described as a fluid existing in two phases, the superconducting one and the normal one. In the superconducting state the material has an infinite electrical conductivity and magnetic fields are repelled from the interior of the sample (this is the so called Meissner effect).

On a microscopic scale, the superconducting state is described by the theory of Bardeen, Cooper and Schrieffer (BCS). In this theory, the existence of superconductivity is due to a pairing of the conducting electrons forming the so called Cooper pairs. For small applied forces, these pairs behave as a single particle (a boson) of twice the charge of the electron. At a macroscopic scale the behavior of the Cooper pairs is described by a complex-valued function u, called the condensate wave function (or order parameter). The density $|u(x)|^2$ is proportional to the density of pairs of superconducting electrons.

The Ginzburg-Landau model is a phenomenological model which extends Landau's theory of second order phase transitions. It was proposed well before the microscopic theory (BCS) existed, but it can be obtained as an approximation to the macroscopic consequences of this theory. This model gives us a system of equations which describe the interaction between the condensate wave function, u, and the electromagnetic vector potential, A. In this model the parameter $\kappa = \varepsilon^{-1}$ (which depends on the material we consider and on the temperature) plays a crucial role in determining the behavior of our system.

If $\kappa < 1/\sqrt{2}$, the material is called a type I superconductor. If one applies an exterior magnetic field to the sample, then there is a critical value, H_c, such that when the applied magnetic field H increases beyond H_c, the sample passes suddenly from the superconducting phase to the normal phase. On the other hand, if $\kappa \geq 1/\sqrt{2}$, the behavior is quite different and the transition between the superconducting and the normal phase is done gradually. These materials are called type II superconductors and they are characterized by two critical values of the applied magnetic field: the first, H_{c1}, corresponds to the critical field above which the two phases coexist, and the second, H_{c2}, corresponds to the critical field above which all the sample will be in the normal phase. Between these two critical values the normal and superconducting phase will coexist: the normal state will be confined in vortices or filaments whose number will increase as the applied field increases. The flux lines of the magnetic field inside the material will be concentrated inside these vortices (since they are repelled by the part of the sample that is in the superconducting phase). For a detailed description of the physics involved in the phenomena of superconductivity and superfluidity see, for instance, the works of D. Saint-James, G. Sarma and E. J. Thomas [26], and of D. Tilley and T. Tilley [27]. For a more mathematical approach see the work of A. Jaffe and C. Taubes [20].

1.4.2. Permanent currents.

A very interesting phenomenon in superconductivity, that motivates our work, is the existence of permanent currents in a superconducting ring. The experiment is the following: a ring of superconducting material in the normal state is submitted to a fixed external magnetic field (subcritical), and then the temperature of the system is decreased until temperatures below the critical temperature corresponding to the applied field are attained. The applied field is then turned off and there is a current that persists inside the superconducting ring. Furthermore it was observed that such a current does not dissipate with time – there were experiments where the current persisted for several years without any dissipation, thus the name permanent current.

This behavior of the system indicates that we should be in presence of an energy functional having multiple wells (local minima) separated by very high barriers. The main purpose of our work is to show that even in the simple models considered in this paper, the energy functionals E_ε and F_ε have this type of structure.

The big height of the barriers would be associated to the "permanent" character of these currents. In fact, considering the possibility of the system tunneling through the barrier, thus moving from one energy well into another (and eventually to the ground state), the associated probability should be proportional to $\exp(-h)$, where h is the height of the barrier relative to the initial state of the system. Thus, having very high barriers will yield transition probabilities close to zero and therefore justify the "permanent" character of our currents.

1.4.3. Transitions between states and threshold energies.

The natural question is then to describe the transitions between two different sectors – thus, the notion of threshold energy for such transitions (defined in equation (1.16)) is a crucial one for the physical behavior of our system. We remark that in the setting of the gauge-invariant model, as we mentioned before, physical states of the system are represented by gauge-equivalence classes (defined by (1.4)) of configurations of our system – thus the configuration (u, A) is just a particular representative of the state $[u, A]$. Therefore, we shouldn't consider paths between configurations in the space $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$, but paths between states in the quotient space of $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ by the gauge-equivalence relation, which we denote by H_{gi} (this is the physical space).

The threshold energy c_n for a transition between a state $[u_0, A_0] \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and a state $[u_1, A_1] \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$ will be of the order of $|\log \varepsilon|$. It is easy to see that it is at most of this order. Indeed, we can prove the following upper bound for the transition energy.

Theorem 5. *Let c_n be the threshold energy for the transition between the state $[u_0, A_0] \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and the state $[u_1, A_1] \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$, defined as in* (1.16)*. Then,*

$$c_n \leq M_n |\log \varepsilon| + L_n \ , \tag{1.18}$$

where M_n and L_n are constants that depend only on n and our domain Ω.

We will give an intuitive proof of Theorem 5. Let $\Lambda > \pi \log(4)\,(n+1)^2$ and suppose that we want to describe a path from the configuration $(u_n, A_n) = (\exp(\imath\, n\, \theta), 0) \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ to the configuration $(u_{n+1}, A_{n+1}) = (\exp(\imath\,(n+1)\,\theta), 0) \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$. We remark that once we construct a path in the space $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ between (u_n, A_n) and (u_{n+1}, A_{n+1}), we can obtain a path between the corresponding physical states $[u_n, A_n]$ and $[u_{n+1}, A_{n+1}]$ in the quotient space H_{gi} by projecting the original path. The general case of a transition between $(v_0, B_0) \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and $(v_1, B_1) \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$ can be proved in a similar way.

Physically, the path we construct corresponds to bringing a positive unit charge of size ε from a point P arbitrarily close to infinity, to the origin. By a positive unit charge of size ε at a point $z_s \in \mathbb{C}$, we mean the map

$$f_{z_s}(z) = \frac{z - z_s}{|z - z_s|}\, \varphi_\varepsilon(z - z_s)\,, \tag{1.19}$$

where $\varphi_\varepsilon(\cdot) = \varphi(\cdot/\varepsilon)$, and $\varphi \in C_0^\infty(\mathbb{R}^2)$ is such that

$$\begin{cases} \varphi(x) = 0\,, & \text{if } |x| < 1\,, \\ \varphi(x) = 1\,, & \text{if } |x| > 2\ , \\ 0 \leq \varphi(x) \leq 1\,, & \text{for all } x\,, \\ |\nabla \varphi(x)| \leq 2\,, & \text{for all } x\,. \end{cases} \tag{1.20}$$

Hence f_{z_s} is a unit vortex at z_s which is “smoothened out” in a ball of radius $2\,\varepsilon$ around z_s. Then,

$$F_\varepsilon(f_{z_s}, 0) \leq C_1\,|\log \varepsilon| + C_2 \ , \tag{1.21}$$

where C_1 and C_2 are constants.

Let $M \in \mathbb{R}^+$ be an arbitrarily big number, and let $z_s = (1 - s)\,(-M) \in \mathbb{C}$, for $s \in [0,1]$. This will be a path from the point $(-M)$ in the negative real axis, to the origin. Using z_s we construct the path in $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ defined by

$$(v_s, B_s) := (f_{z_s} u_n, 0)\,, \qquad \text{for } s \in [0,1]\,.$$

We can check that (v_0, B_0) is arbitrarily close in $H^1(\Omega)$ norm to (u_n, A_n) – in fact, we would obtain the configuration (u_n, A_n) if we chose $M = +\infty$. Hence, in particular, for big values of M, we certainly have $(v_0, B_0) \in \mathrm{top}_n(F_\varepsilon^\Lambda)$. Furthermore, $(v_1, B_1) = (u_{n+1}, A_{n+1})$ and we can obtain estimate (1.18) as a consequence of the bound (1.21).

Hence we see that the path corresponding to passing a positive unit charge "of size ε" from the outside of our annulus, to the hole inside the annulus, corresponds to increasing by one the degree of our map and requires that we go to an energy level of order $|\log \varepsilon|$. To prove that any transition between $\mathrm{top}_n(F_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$ also requires passing through energy levels of order $|\log \varepsilon|$, thus proving that c_n is of order $|\log \varepsilon|$, is a very delicate problem. We will show a way to solve this problem and obtain very precise estimates for the threshold energies in a forthcoming work ([1]).

1.5. The case of more general domains.

In Theorem 1 we considered a very particular domain – the annulus $\Omega = \{x \in \mathbb{R}^2 : 1/4 < |x| < 1\}$. However, once we have the result for the annulus, it is not hard to extend it to the case of a general open subset $D \subset \mathbb{R}^2$, or even the case of a domain in a Riemannian manifold $\mathcal{M}$. We define the energy functional just as in (1.1) but replacing Ω by our new domain D,

$$E_\varepsilon(u, D) = \frac{1}{2} \int_D |\nabla u|^2 + \frac{1}{4\,\varepsilon^2} \int_D (1 - |u|^2)^2\,, \tag{1.22}$$

and we define the corresponding level sets

$$E_\varepsilon^\Lambda(D) := \{u \in H^1(D, \mathbb{R}^2) : \; E_\varepsilon(u, D) < \Lambda\}\,.$$

We start by fixing a set of representatives of generators of $\pi_1(D)$ (the first homotopy group of D), $\{\gamma_j, j \in J\}$, such that each $\gamma_j : S^1 \longrightarrow D$, is an injective closed smooth curve inside our open set D. Hence, γ_j will have a tubular neighborhood $\Gamma_j \subset \Omega$. We may suppose that for each j there is a positive number, $\delta_j > 0$, such that for each j

i) $\Gamma_j = \{x \in D : \ \mathrm{dist}\,(x, \gamma_j) < \delta_j\}$.

ii) There is a diffeomorphism

$$\Phi_j : \Gamma_j \longrightarrow S^1 \times (0,1)\,,$$

such that $\gamma_j(\theta) = \Phi_j^{-1}(\theta, 1/2)$, and the Jacobian of Φ_j is uniformly bounded from above and away from zero, *i.e.* there is a constant $C_j > 0$ such that

$$\frac{1}{C_j} < |\nabla \Phi_j(x)| < C_j\,, \qquad \text{for all } x \in \Gamma_j\,. \tag{1.24}$$

Let $\hat{\Omega} := S^1 \times (1/4, 3/4)$. This set is topologically an annulus just like our standard set Ω considered before. Let $Y_j := \Phi_j^{-1}(\hat{\Omega})$. Given a map $u \in E_\varepsilon^\Lambda(D)$ we consider the map $w_j = u \circ \Phi_j^{-1} : \hat{\Omega} \longrightarrow \mathbb{R}^2$. The map w_j belongs to $E_\varepsilon^\varrho(\hat{\Omega})$, where ϱ is a constant that depends only on Λ and the constant C_j in (1.24). Thus, we can apply Theorem 1 replacing Ω and Λ by $\hat{\Omega}$ and ϱ, respectively. Hence for ε sufficiently small $\deg\,(w_j, \hat{\Omega})$ is well defined. We set, for each $j \in J$,

$$\deg\,(u, Y_j) := \deg\,(w_j, \hat{\Omega})\,. \tag{1.25}$$

Suppose that the index set J is finite ($J = \{1, \dots, m\}$), *i.e.* suppose that we fix a finite number of (representatives of) generators of $\pi_1(D)$. We define the topological type of $u \in E_\varepsilon^\Lambda(D)$ as the m-tuple of integers

$$\chi(u) := (\deg\,(u, Y_1), \dots, \deg\,(u, Y_m))\,. \tag{1.26}$$

By the previous argument, this $\chi(u) \in \mathbb{Z}^m$ is well defined for sufficiently small ε. The continuity of χ in $W^{1,2}(D, \mathbb{R}^2)$ topology inside $E_\varepsilon^\Lambda(D)$ (which is an immediate consequence of the continuity of $\deg\,(u, \Omega)$ proved in section 7) will then allow us to assert that, since $\mathbb{Z}^m$ is discrete, for each $P \in \mathbb{Z}^m$, its inverse image by χ, *i.e.* $\chi^{-1}(P) = \{u \in E_\varepsilon^\Lambda(D) : \chi(u) = P\}$, will be an open and closed subset of $E_\varepsilon^\Lambda(D)$. For each $P \in \mathbb{Z}^m$, we call $\chi^{-1}(P)$ the P-topological sector of $E_\varepsilon^\Lambda(D)$. We have thus proved the following Theorem which extends the classification given by Theorem 1 to this more general setting.

Theorem 6. *Let D be an open subset of $\mathbb{R}^2$ or a domain in a Riemann manifold $\mathcal{M}$. Let $\gamma_1, \dots, \gamma_m$ be simple, closed and smooth curves which are a set of representatives of generators of $\pi_1(D)$. Given $\Lambda > 0$ there exists $\varepsilon_0 > 0$, depending on Λ, such that for $\varepsilon < \varepsilon_0$ we can define a continuous map*

$$\begin{aligned} \chi : E_\varepsilon^\Lambda(D) &\longrightarrow \mathbb{Z}^m\,, \\ u &\longmapsto (\deg(u, Y_1), \dots, \deg(u, Y_m))\,, \end{aligned} \tag{1.27}$$

such that for the special case where $u \in E_\varepsilon^\Lambda(D) \cap W^{1,2}(D, S^1)$, we recover the classical notion of degree of a S^1 valued map. Therefore, given $P = (P_1, \dots, P_m) \in \mathbb{Z}^m$, the subset $\chi^{-1}(P) \subset E_\varepsilon^\Lambda(D)$ will be an open and closed subset of $E_\varepsilon^\Lambda(D)$.

The same argument in the context of the superconductivity model will give a similar extension of Theorem 2.

1.6. Idea of the proof of Theorem 1.

The maps $u \in E_\varepsilon^\Lambda$ may take values close to zero, which creates big technical problems for defining their degree. However, this can only happen in a set of small measure. We will start by studying, in sections 2, 3 and 4 the set $G(\zeta)$ where $|u|$ is smaller than an appropriately chosen $\zeta \in (1/2, 3/4)$. For technical reasons (to avoid problems that may appear near the boundary $\partial\Omega$) we will concentrate on the components of $G(\zeta)$ that intersect an interior annulus

$$Y := \left\{ x \in \mathbb{R}^2 : \; \frac{1}{2} < |x| < \frac{3}{4} \right\}.$$

Using Sard's Lemma we will see that for sufficiently small ε, these components of G may be included in a finite number of simply-connected sets, which we denote by $W_k, k = 1, \dots, \tilde{N}$. Their boundaries will be closed smooth curves, $V_k = \partial W_k$, and $|u| = \zeta$ on each of the V_k's.

In Section 2 we see, using the coarea formula, that the sum of the lengths of the V_k's will tend to zero when $\varepsilon \longrightarrow 0$. Furthermore, the coarea formula also gives us a bound on the L^1 norm of ∇u on $V = \bigcup V_k$. Since $|u| = \zeta > 1/2$ on V_k, it makes sense to talk about $\deg(u, V_k)$.

In Section 3 we the obtain an uniform bound on $\sum |\deg(u, V_k)|$ using the estimate for $\|\nabla u\|_{L^1(V)}$ (and consequently we will also have uniform bounds on $|\deg(u, V_k)|$ for each k). Thus, we see that for all $u \in E_\varepsilon^\Lambda$ the number of V_k's such that $\deg(u, V_k) \neq 0$ (which we call the "charged" V_k's) is uniformly bounded by a constant depending only on Λ. Suppose that the charged V_k's are $V_1, \dots, V_{N_2}$.

In Section 4 we will focus our attention on the "uncharged" V_k's (*i.e.* those for which $\deg(u, V_k) = 0$). We will see, again using the estimate for $\|\nabla u\|_{L^1(V)}$ obtained in Section 2, that the number of "uncharged" V_k's such that the oscillation of u is bigger than or equal to $\pi/3$, is also uniformly bounded. Suppose they are $V_{N_2+1}, \dots, V_N$. Moreover, for the remaining V_k's, *i.e.* the "uncharged" ones such that the oscillation of u is smaller than $\pi/3$ (which will be $V_{N+1}, \dots, V_{\tilde{N}}$), we are able to prove that the energy minimizing extension to W_k of $u_{|V_k}$ will have absolute value which is uniformly bounded away from zero – hence we will show that these sets are rather "harmless".

In Section 5, thanks to the uniform bound on N (the number of "charged" V_k's plus that of "uncharged" V_k's such that the oscillation of u is bigger than or equal to $\pi/3$), we can cover $V_1, \dots, V_N$ by a finite (uniformly bounded) number of balls, $B_1, \dots, B_m$, of radius of order at most ε^α for some $\alpha > 1/2$, and which are far away from each other (in the sense that suitable dilations of the B_i's are pairwise disjoint). Furthermore, we will see that $\deg(u, \partial B_i) = 0$, for all i. This means that though we may have individual singularities that are charged, at a scale of order $\varepsilon^{1/2}$ they cluster to form neutral structures.

In Section 6 we will finally give the good definition of the global degree of u in Ω, $\deg(u, \Omega)$. Let

$$T := \left\{ r \in \left(\frac{1}{2}, \frac{3}{4}\right) \text{ such that } S_r \cap G(\zeta) \neq \varnothing \right\},$$

and let

$$A := \left(\frac{1}{2}, \frac{3}{4}\right) \setminus T\,.$$

We show that $|T| \longrightarrow 0$, when $\varepsilon \longrightarrow 0$, and hence $|A| \longrightarrow 1/4$, when $\varepsilon \longrightarrow 0$. For $r \in A$ we define

$$f(r) := \deg(u, S_r) = \deg\Big(\frac{u}{|u|}, S_r\Big) \in \mathbb{Z}\,. \tag{1.28}$$

This function is well defined since for $r \in A, |u(r,\theta)| \geq \zeta$. As we mentioned before, for $u \in W^{1,2}(\Omega, S^1)$ this function is constant. In our case this might not be true, but by the results of Section 5, it cannot change too much: as a matter of fact, for ε sufficiently small, the value of f can only change when S_r intersects one of the balls B_i, and even when this occurs, the absolute value of f remains bounded by a constant that depends only on Λ. Outside these balls (*i.e.* when $S_r \cap B = \varnothing$, where $B := \bigcup B_i$) $f(r)$ will always have the same value (since $\deg(u, \partial B_i) = 0$). This is the value we use to define $\deg(u, \Omega)$, which will thus automatically be an integer. To recover this integer we can also integrate $f(r)$ over A and divide by the measure of A, thus defining

$$\widetilde{\text{adeg}}\,(u, \Omega) := \frac{1}{|A|} \int_A f(r)\, dr\,. \tag{1.29}$$

This quantity, $\widetilde{\text{adeg}}\,(u, \Omega)$, is called the approximate degree of u in Ω. In general, it is not an integer, but it will tend to the integer $\deg(u, \Omega)$ as $\varepsilon \longrightarrow 0$. In fact, let $Q = A \cap B = \bigcup (A \cap B_i)$. The measure of Q tends to zero when $\varepsilon \longrightarrow 0$ (it is bounded by $|B|$ which, in turn, is at most, of order $\varepsilon^\alpha < \varepsilon^{1/2}$). Furthermore, f remains uniformly bounded even inside Q, and hence, we can see that

$$|\widetilde{\text{adeg}}\,(u, \Omega) - \deg(u, \Omega)| < \frac{1}{4}\,, \tag{1.30}$$

for sufficiently small ε. Thus we can recover the integer $\deg(u, \Omega)$ as the closest integer to $\widetilde{\text{adeg}}\,(u, \Omega)$ for ε small.

In Section 7 we will prove, for sufficiently small ε, the continuity of $\widetilde{\text{adeg}}\,(u, \Omega)$ (and thus also of $\deg(u, \Omega)$) in $W^{1,2}(\Omega)$ norm, inside the level set E_ε^Λ we fixed. Using this continuity we will then conclude the proof of Theorem 1 in Section 5.

Finally, in the Appendix (Section 12) we prove a general covering Lemma of which we used a special case to obtain the balls B_i in Section 5.

1.7. Open questions and related results.

As we saw, many questions about this subject remain open, in particular in the borderline between the mathematics and the physical behavior of these systems, a considerable amount of work remains to be done. In this subsection we will discuss some of these problems shortly and mention some results of related interest. We start by mentioning a few problems we are working on at the moment.

In [1] we are able to carry out a more detailed study of the properties of the threshold energies we introduced above. In particular, using some techniques introduced by F. Bethuel and the author in [4], we can prove a more accurate version of the upper bound for the threshold energy c_n stated in Theorem 5. More precisely, we show that there exists a constant α_n, not depending on ε, such that $c_n \leq \pi\,|\log\varepsilon| + \alpha_n$.

This estimate is crucial to succeeding in obtaining (see [1]) a lower bound for c_n which is of the same order of the above, i.e. to showing that $c_n \geq \pi\,|\log\varepsilon| - \alpha_n$. Such a bound, as we mentioned, implies that the energy barriers have a height of at least $\pi\,|\log\varepsilon| - \alpha_n$, and therefore, since ε is supposed to be small, we will have very high barriers separating the wells. This agrees with what we expected considering the physical behavior of our system, as we described above.

Regarding the extension of our results to the 3-dimensional case, there is a substantial part we are able to do, but there are still some technical difficulties (which stem from the higher degree of liberty of the equivalent of the V_k's, which, in this setting, will be two-dimensional surfaces). Once we succeed in defining the degree, we can obtain mountain-pass solutions just as for the dimension 2, but proving that the threshold energy, c_n, is of order $|\log\varepsilon|$ should be considerably harder (for results on the structure of the singularities of the Abelian Higgs model in $\mathbb{R}^3$, see the works of T. Rivière [23] and [24]).

Our work was also motivated by the paper of S. Jimbo and Y. Morita [16]. In [16] the authors establish the existence of stable non-trivial solutions for the Ginzburg-Landau equations in the case the domain $\Omega \subset \mathbb{R}^3$ is a solid of revolution obtained by rotating a convex cross-section around the z-axis in $\mathbb{R}^3$. Thanks to this special geometry, they can find solutions using a separation of variables method. They show that the solutions constructed are stable for variations in a linear space that is transversal to the gauge-invariance of the problem.

Very recently, while this work was being finished, the author received a series of preprints of S. Jimbo, Y. Morita and J. Zhai [17], [18], [19] where they improve the techniques developed in [16] and introduce some new ideas to obtain very interesting results about stationary solutions of the Ginzburg-Landau equations in topologically non-trivial domains. The author also received recently a preprint J. Rubinstein and P. Sternberg [25], where the ideas of B. White and F. Bethuel concerning the homotopy classes for Sobolev functions are used, together with variational techniques, in a very ingenious way, to obtain a homotopy classification for the minimizers of the Ginzburg-Landau energy in the case the domain is topologically a torus in $\mathbb{R}^3$. One fundamental difference between these works and ours is that, since their authors are looking at critical points, they rely strongly on the Ginzburg-Landau equation to prove nice properties for these critical points, and then succeed in defining the degree of the stationary solutions using these properties. In our case, since we look at the whole level set of the energy, we cannot rely on the equation to help us define the degree. This, as we saw, poses many technical problems, but gives us a considerable amount of new information. Such information should enable us to have a better understanding about the formation of permanent currents and the transition processes between physical states.

Another important question is that of the evolution equation for Ginzburg-Landau. Recently there was some work of F. H. Lin [21], [22], and of S. Demoulini and D. Stuart [12] on the heat flow for Ginzburg-Landau. The author, F. Bethuel and Y. Guo have also obtained some results regarding the dynamical stability of symmetric vortices in the Maxwell-Higgs model (see [15] and [5]).

REMARKS ON NOTATION.

- Ω is the annulus $\{x \in \mathbb{R}^2 : 1/4 < |x| < 1\} \subset \mathbb{R}^2$. Its boundary, $\partial\Omega$, has two connected components: $\partial\Omega_1 = S_{1/4}$, the inner circle, and $\partial\Omega_2 = S_1$, the exterior circle. On $\partial\Omega$, $\nu(x)$ stands for the exterior unit normal to $\partial\Omega$ at x. Hence $\nu(x) = -x/|x|$ on $\partial\Omega_1$, and $\nu(x) = x/|x|$ on $\partial\Omega_2$. For $x \in \partial\Omega$, $\tau(x)$ stands for the unit tangent vector to $\partial\Omega$ at x, pointing in the sense of increasing θ.

- $\wedge$ denotes the wedge product of differential forms, and $\times$ represents the exterior product of two vectors in $\mathbb{R}^2$ (it is considered as a real number).

- We often use the natural identification between an one-form and the associated vector (given by the scalar product in $\mathbb{R}^2$).

- Although we would normally prefer to write vectors as columns, we will often write them as rows because it makes it easier to insert them in the text.

- We identify the vector $(v^1, v^2) \in \mathbb{R}^2$ with the complex number $v^1 + \imath v^2$. The scalar product in $\mathbb{C}$ is denoted by $(\ ,\ )$. So $(u, v) = (u\,\overline{v} + v\,\overline{u})/2$. With this notation we have that $u \times u_\tau = (\imath\, u, u_\tau)$.

Although this permanent switch between the vector and the complex number notation may be slightly confusing at the beginning, later on the reader will appreciate the convenience that stems from having both notations available.

• d denotes the exterior derivative and $\star$ denotes the Hodge star operator, which in $\mathbb{R}^2$ is the linear operator on $\mathbb{R}$-valued forms defined by

$$\star 1 = dx^1 \wedge dx^2\,,\ \star dx^1 = dx^2\,,\ \star dx^2 = dx^1\,,\ \text{and}\ \star dx^1 \wedge dx^2 = 1\,.$$

We have that for k-forms on $\mathbb{R}^2$, $\star\star = I^{(k(2-k))}$, where I denotes the identity. Hence $\star\star\alpha = \alpha$, if α is a zero-form or a two-form, and $\star\star\alpha = -\alpha$, if α is a one-form.

• $d^\star$ denotes the operator $\star^{-1}d\star$, where $\star^{-1}$ stands for the inverse operator of $\star$.

• In many of the estimates we obtain during the proof of Theorem 1, there are constants which depend on the domain considered. However, since we will have fixed as domain the annulus Ω, we will usually not mention such dependence explicitly in the text.

2. Coarea formula and control of the bad set.

As we mentioned before, the bad set consists of the places where $|u|$ is close to zero. Nevertheless, the presence of the potential term in E_ε (in particular, the presence of the ε^{-2} factor), assures us that for $u \in E_\varepsilon^\Lambda$, the measure of the set $\{x : |u| < 1/2\}$ will be very small when $\varepsilon \longrightarrow 0$. In fact, as we will see in this section, a more careful analysis using the coarea formula will allow us to prove much more about this set.

Suppose Λ and ε given and fix an element $u \in E_\varepsilon^\Lambda \cap C^\infty(\Omega)$. For each $\zeta \in [1/2, 3/4]$, let

$$V(\zeta) = \{x \in \Omega :\ |u(x)| = \zeta\}\,.$$

By Sard's Lemma we know that for almost every ζ, $V(\zeta)$ is a one-dimensional submanifold of Ω, hence we will suppose that the ζ we choose is in these conditions. We will now define as our bad set, the set G where $|u|$ is smaller than ζ. Let

$$G(\zeta) := \{x \in \Omega :\ |u(x)| < \zeta\}\,, \qquad \zeta \in \left[\frac{1}{2}, \frac{3}{4}\right].$$

It is easy to see that for small ε, the measure of $G(\zeta)$ will be very small. In fact,

$$
\begin{aligned}
\int_{G(\zeta)} (1-|u|^2)^2 &\geq \int_{G(\zeta)} (1-\zeta^2)^2 \\
&\geq |G(\zeta)|\,(1-\zeta^2)^2 \\
&\geq \Big(\frac{7}{16}\Big)^2\,|G(\zeta)|\,,
\end{aligned}
\tag{2.1}
$$

and,

$$
\int_{G(\zeta)} (1-|u|^2)^2 \leq 4\,\varepsilon^2\,\frac{1}{4\,\varepsilon^2}\int_\Omega (1-|u|^2)^2 \leq 4\,\varepsilon^2\,\Lambda\,. \tag{2.2}
$$

Combining (2.1) and (2.2) we obtain the desired bound on $|G(\zeta)|$,

$$
|G(\zeta)| \leq \Big(\frac{16}{7}\Big)^2\,4\,\varepsilon^2\,\Lambda = C\,\varepsilon^2 \underset{\varepsilon\to 0}{\longrightarrow} 0\,, \tag{2.3}
$$

where C is a constant depending only on the energy bound Λ.

2.1. The coarea formula.

Using the coarea formula of Federer and Flemming, we can obtain a considerable amount of information about the V_k's and the behavior of $u_{|V_k}$, for ζ conveniently chosen.

Here we will apply a special case of this formula which can be stated as follows (for a proof and more general forms of this result see, for instance, L. Evans and R. Gariepy [13]).

Theorem 7 (coarea formula (change of variables)). *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be Lipschitz. Then, for every Lebesgue summable function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, we have that*

i) *The restriction $g_{|f^{-1}\{y\}}$ is Hausdorff $\mathcal{H}^1$-measurable for almost every y.*

ii) *For every measurable set $X \subset \mathbb{R}^2$,*

$$
\int_X g|\nabla f|\,dx = \int_{\mathbb{R}} \Big(\int_{f^{-1}\{y\}\cap X} g\,d\mathcal{H}^1\Big)\,dy\,.
$$

REMARK. By Rademacher's Theorem, since f is Lipschitz, it is differentiable almost everywhere, and hence ∇f is defined almost everywhere $x \in X$.

2.2. Upper-bound for the length of the V_k's.

We start by proving that the length (Hausdorff one-dimensional measure) of the V_k's is small for small ε. As a matter of fact, if we denote $\Xi := \{x : 1/2 \leq |u| \leq 3/4\}$, it follows from the co-area formula that

$$\begin{aligned} \int_{1/2}^{3/4} \mathcal{H}^1(V(\zeta))\, d\zeta &= \int_\Xi |\nabla |u|\,| \\ &\leq \int_\Xi |\nabla u| \\ &\leq \Big(\int_\Xi |\nabla u|^2\Big)^{1/2} |\Xi|^{1/2}, \end{aligned} \tag{2.4}$$

where we used Cauchy-Schwarz for the last inequality. Moreover,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \leq E_\varepsilon(u) \leq \Lambda\,,$$

hence,

$$\Big(\int_\Omega |\nabla u|^2\Big)^{1/2} \leq \sqrt{2\,\Lambda}\,. \tag{2.5}$$

On the other hand, the measure of Ξ can also be estimated using the energy bound (just like we did for $G(\zeta)$, in fact $\Xi = G(1/2)$). We obtain

$$|\Xi| \leq \Big(\frac{16}{7}\Big)^2 \int_\Xi (1-|u|^2)^2 \leq \Big(\frac{32}{7}\Big)^2 \varepsilon^2 \Lambda\,. \tag{2.6}$$

From (2.4), (2.5) and (2.6), it follows that

$$\int_{1/2}^{3/4} \mathcal{H}^1(V(\zeta))\, d\zeta \leq \frac{32\sqrt{2}}{7}\,\varepsilon\,\Lambda\,.$$

Hence, except for ζ in a set $Z_1 \subset [1/2, 3/4]$ of measure at most $\sqrt{2}/70 \leq 1/40$,

$$\mathcal{H}^1(V(\zeta)) \leq \frac{70}{\sqrt{2}}\,\frac{32\sqrt{2}}{7}\,\Lambda\,\varepsilon = 320\,\Lambda\,\varepsilon\,. \tag{2.7}$$

2.3. Upper-bound for the $L^1(V(\zeta))$ norm of ∇u.

A different application of the coarea formula yields

$$\int_{0<\zeta<1}\int_{V(\zeta)} |\nabla u| = \int_\Omega |\nabla|u||\,|\nabla u| \leq \int_\Omega |\nabla u|^2\,. \tag{2.8}$$

Since we assume that $u \in E_\varepsilon^\Lambda$, from (2.8) it follows that

$$\int_{0<\zeta<1}\int_{V(\zeta)} |\nabla u| \leq 2\,E_\varepsilon^\Lambda(u) \leq 2\,\Lambda\,. \tag{2.9}$$

Using Fubini's Theorem, we will then have that except for ζ in a set $Z_2 \subset [1/2, 3/4]$ of measure at most $1/40$,

$$\int_{V(\zeta)} |\nabla u| \leq 80\,\Lambda\,. \tag{2.10}$$

Thus, except when ζ belongs to the set $Z_1 \cup Z_2$, whose measure is at most $1/20$, estimates (2.7) and (2.10) will be valid. For the rest of this paper we will choose a $\zeta \in (1/2,\, 3/4)$ such that estimates (2.7) and (2.10) are valid, and that $V(\zeta)$ is a one-dimensional submanifold of Ω. Hence, $V(\zeta)$ consists of a finite number of simple curves in Ω. Let $V_1, \ldots, V_{\breve{N}}$, denote the connected components of $V(\zeta)$. Equation (2.7) gives us an upper-bound on the length of each V_k,

$$\sum_{k=1}^{\breve{N}} \mathcal{H}^1(V_k) \leq \mathcal{H}^1(V(\zeta)) \leq 320\,\Lambda\,\varepsilon\,. \tag{2.11}$$

In particular,

$$\mathcal{H}^1(V_k) \leq 320\,\Lambda\,\varepsilon\,, \qquad \text{for all } k = 1, \ldots, \breve{N}\,. \tag{2.12}$$

Hence, for small ε, the length of each V_k will be small (the same being true for the sum of their lengths).

3. Properties of the V_k's which are far from $\partial\Omega$.

We consider the interior subdomain $Y := \{(r,\theta) : \; 1/2 < r < 3/4\} \subset \Omega$, *i.e.*, the interior annulus consisting of the points whose distance to the origin lies between $1/2$ and $3/4$. For technical reasons, we will also have to consider a slightly enlarged subdomain, $\hat{Y} := \{(r,\theta) : 3/8 < r < 7/8\}$. Hence, $Y \Subset \hat{Y} \Subset \Omega$.

We start by proving that for ε sufficiently small, the V_k's that intersect $\hat{Y}$ are closed curves that stay away from the boundary of Ω.

Lemma 1. *If ε is sufficiently small, then $V_k \cap \hat{Y} \neq \varnothing$, implies that V_k is a closed curve and* $\operatorname{dist}(V_k, \partial\Omega) > 1/16$*.*

PROOF. Suppose that $V_k \cap \hat{Y} \neq \varnothing$. Then, since $\operatorname{dist}(\hat{Y}, \partial\Omega) = 1/8$, for $\operatorname{dist}(V_k, \partial\Omega)$ to be smaller than $1/16$, it is necessary that $\operatorname{diam}(V_k) \geq 1/16$. However, from (2.12) it follows that

$$\operatorname{diam}(V_k) \leq \mathcal{H}^1(V_k) \leq 320\,\Lambda\,\varepsilon\,.$$

Hence, for $\varepsilon < \Lambda/5120$ we must have that $\operatorname{diam}(V_k) < 1/16$, and thus, $\operatorname{dist}(V_k, \partial\Omega) > 1/16$.

The fact that V_k is then a closed curve, follows from it being a one-dimensional submanifold of Ω which does not touch $\partial\Omega$.

Henceforth, we will always suppose that ε is chosen sufficiently small for the result in Lemma 1 to be true. Suppose that the V_k's that intersect $\hat{Y}$ are $V_1, \ldots, V_{\overline{N}}$. They will be closed curves and thus, by Jordan's Curve Theorem, we can define the domain W_k enclosed by V_k (W_k is the bounded component of $\mathbb{R}^2 \setminus V_k$, and in particular, $V_k = \partial W_k$).

Among $V_1, \ldots, V_{\overline{N}}$ we will only consider those which are maximal in the following sense: for $i, j \leq \overline{N}$, if $V_i \subset W_j$ then we disregard V_i and just keep V_j in our list (so we always keep only the exterior curves). Suppose that $V_1, \ldots, V_{\tilde{N}}$, for some $\tilde{N} \leq \overline{N}$, are the maximal curves we obtain. These are the V_k's that will interest us for the rest of this paper (unless stated otherwise, henceforth we will always assume $k \leq \tilde{N}$).

3.1. Estimates for $\deg(u, V_k)$.

By the definition of $V(\zeta)$, the restriction of u to V_k will have values in the circle of radius ζ, *i.e.* $u_{|V_k} : V_k \longrightarrow S_\zeta$, where we denote $S_\zeta = \{z \in \mathbb{R}^2 : |z| = \zeta\}$. Therefore, we can define the degree of u: as usual we consider the map

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} := \frac{u}{|u|} : V_k \longrightarrow S^1 ,$$

and we define

$$\deg(u, V_k) := \deg(v, V_k) := \frac{1}{2\pi} \int_{V_k} v \times \frac{\partial v}{\partial \tau}\, d\tau , \tag{3.1}$$

where τ denotes, as usual, the arc-length parameter on V_k.

Since $u = |u|\, v$, we have that

$$\nabla u = \nabla(|u|\, v) = \begin{pmatrix} \dfrac{\partial |u|}{\partial x^1}\, v^1 + |u|\, \dfrac{\partial v^1}{\partial x^1} & \dfrac{\partial |u|}{\partial x^2}\, v^1 + |u|\, \dfrac{\partial v^1}{\partial x^2} \\ \dfrac{\partial |u|}{\partial x^1}\, v^2 + |u|\, \dfrac{\partial v^2}{\partial x^1} & \dfrac{\partial |u|}{\partial x^2}\, v^2 + |u|\, \dfrac{\partial v^2}{\partial x^2} \end{pmatrix} . \tag{3.2}$$

Thus,

$$\begin{aligned} |\nabla u|^2 = |u|^2 \Big(\Big(\frac{\partial v^1}{\partial x^1}\Big)^2 + \Big(\frac{\partial v^1}{\partial x^2}\Big)^2 + \Big(\frac{\partial v^2}{\partial x^1}\Big)^2 + \Big(\frac{\partial v^2}{\partial x^2}\Big)^2 \Big) \\ + ((v^1)^2 + (v^2)^2) \\ \cdot \Big(\Big(\frac{\partial |u|}{\partial x^1}\Big)^2 + \Big(\frac{\partial |u|}{\partial x^2}\Big)^2 \Big) + |u|\, \frac{\partial |u|}{\partial x^1} \Big(v\, \frac{\partial v}{\partial x^1} \Big) + |u|\, \frac{\partial |u|}{\partial x^2} \Big(v\, \frac{\partial v}{\partial x^2} \Big) . \end{aligned} \tag{3.3}$$

But since $|v| = C^{te} = 1$, it follows that

$$(v^1)^2 + (v^2)^2 = |v|^2 = 1 ,$$

and,

$$v\, \frac{\partial v}{\partial x^i} = \frac{1}{2}\, \frac{\partial}{\partial x^i}\, (v\, v) = 0 .$$

Thus, (3.3) yields

$$|\nabla u|^2 = |u|^2\, |\nabla v|^2 + |\nabla |u|\, |^2 . \tag{3.4}$$

Hence, in particular,

$$|\nabla u|^2 \geq |u|^2 \, |\nabla v|^2 \, . \tag{3.5}$$

For $x \in V_k$, since $|u(x)| = \zeta \geq 1/2$, this yields

$$|\nabla u|^2 \geq \zeta^2 \, |\nabla v|^2 \geq \frac{1}{4} \, |\nabla v|^2 \, , \tag{3.6}$$

which, in turn, implies that on V_k,

$$|\nabla u| \geq \frac{1}{2} \, |\nabla v| \, . \tag{3.7}$$

From equations (3.1) and (3.7) it follows that

$$|\deg(u, V_k)| = |\deg(v, V_k)| \leq \int_{V_k} \left| v \times \frac{\partial v}{\partial \tau} \right| d\tau \leq \int_{V_k} |\nabla v| \leq 2 \int_{V_k} |\nabla u| \, .$$

Therefore, using equation (2.10), we obtain a bound on the absolute value of the degree of u in each of the V_k, for all $k = 1, \ldots, \tilde{N}$ (we remark that this bound is also valid for $\tilde{N} < k \leq \hat{N}$ as long as V_k is a closed curve – so that we have no problem defining $\deg(u, V_k)$),

$$|\deg(u, V_k)| \leq 2 \int_{V_k} |\nabla u| \leq 2 \int_{V(\zeta)} |\nabla u| \leq 160 \, \Lambda \, . \tag{3.8}$$

Moreover, we even have a bound on the sum of the absolute values of these degrees,

$$\sum_{k=1}^{\tilde{N}} |\deg(u, V_k)| \leq 2 \sum_{k=1}^{\tilde{N}} \int_{V_k} |\nabla u| \leq 2 \int_{V(\zeta)} |\nabla u| \leq 160 \, \Lambda \, , \tag{3.9}$$

which gives a bound on the number $N_2 := \#\{k \, : \, V_k \cap \hat{Y} \neq \varnothing, \text{ and } \deg(u, V_k) \neq 0\}$, *i.e.*, the number of "charged" V_k's that intersect $\hat{Y}$. In fact, we obtain

$$N_2 \leq \sum_{k=1}^{\tilde{N}} |\deg(u, V_k)| \leq 160 \, \Lambda \, . \tag{3.10}$$

REMARK. We will often refer to a V_k such that $\deg(u, V_k) \neq 0$ as a "charged" (or topologically charged) singularity of u, and to one such that $\deg(u, V_k) = 0$ as a "uncharged" (or neutral or topologically uncharged) singularity of u. This terminology is unprecise but helps convey the essential difference between the behavior of u on these two types of sets.

Using this terminology, the charged V_k's that intersect $\hat{Y}$ are $V_1, \ldots, V_{N_2}$, and the neutral ones are $V_{N_2+1}, \ldots, V_{\tilde{N}}$.

4. The "uncharged" V_k's.

Although the charged V_k's are the only ones that may change the value of $f(r) = \deg(u, S_r)$, defined in (6.1), in order to prove that these cannot be isolated, we will need some control of u on the uncharged V_k's (*i.e.*, $V_{N_2+1}, \ldots, V_{\tilde{N}}$), and on the energy minimizing extensions of u to the W_k's that lie inside them. Thus, in this section we will always suppose $k \in \{N_2+1, \ldots, \tilde{N}\}$.

The restriction of u to $V_k = \partial W_k, g_k : V_k \longrightarrow S_\zeta$, has degree zero (since we are considering only the "uncharged" V_k's) and W_k is simply connected, hence g_k can be written as

$$g_k = \zeta \exp(\imath\, \theta_k)\,, \tag{4.1}$$

where $\theta_k : V_k \longrightarrow \mathbb{R}$, is a smooth lifting of $u_{|V_k}$. For $x \in V_k$ we have that

$$|\nabla \theta_k|^2 = |\nabla(\exp(\imath\, \theta_k))|^2 = \left|\nabla\left(\frac{u}{|u|}\right)\right|^2 .$$

Therefore, by (3.4),

$$|\nabla u|^2 = \zeta^2\, |\nabla \theta_k|^2 + |\nabla |u|\,|^2\,, \tag{4.2}$$

and, in particular,

$$|\nabla \theta_k| \leq \frac{|\nabla u|}{\zeta}\,. \tag{4.3}$$

As usual, we define the oscillation of θ_k as

$$\operatorname{osc}(\theta_k) := \sup_{x \in V_k} (\theta_k(x)) - \inf_{x \in V_k} (\theta_k(x))\,. \tag{4.4}$$

We will prove that the number of V_k's for which θ_k can oscillate considerably, is uniformly bounded (by a constant depending only on the energy bound Λ).

Lemma 2. *Given $\Lambda \in \mathbb{R}^+$, there is a constant $M \in \mathbb{R}^+$ such that, for all $\varepsilon > 0$, for all $u \in E_\varepsilon^\Lambda$, if*

$$I := \Big\{ k \in \{N_2 + 1, \dots, \tilde{N}\},\ \textit{such that}\ \operatorname{osc}(\theta_k) > \frac{\pi}{3} \Big\},$$

then,

$$\#I \leq M = \frac{480\,\Lambda}{\pi}\,. \tag{4.5}$$

PROOF. By the fundamental Theorem of Calculus,

$$\operatorname{osc}(\theta_k) = \sup_{x,y \in V_k} (\theta_k(x) - \theta_k(y)) \leq \int_{V_k} \Big|\frac{\partial \theta_k}{\partial \tau}\Big| \leq \int_{V_k} |\nabla \theta_k|\,.$$

Then, using equations (2.10) and (4.3) we obtain

$$\begin{aligned}
\frac{\pi}{3}\,\#I &\leq \sum_{k\in I} \operatorname{osc}(\theta_k) \\
&\leq \sum_{k\in I} \int_{V_k} |\nabla \theta_k| \\
&\leq \sum_{k\in I} \frac{1}{\zeta} \int_{V_k} |\nabla u| \\
&\leq 2 \int_{V(\zeta)} |\nabla u| \\
&\leq 160\,\Lambda\,.
\end{aligned}$$

Hence,

$$\#I \leq \frac{3}{\pi}\,160\,\Lambda = \frac{480}{\pi}\,\Lambda\,. \tag{4.6}$$

Thus, we have proven Lemma 2 with $M = 480\,\Lambda/\pi$.

If $\deg(u, V_k) = 0$, we know that there exist smooth extensions of $g = u_{|V_k} : V_k \longrightarrow S_\zeta$ to $\overline{W}_k$, where $S_\zeta = \{x \in \mathbb{R}^2 : |x| = \zeta\} \simeq S^1$, and W_k is the domain inside V_k (in the sense of Jordan's curve Theorem). Let $H^1_g := \{u \in H^1(W_k, \mathbb{C}) : u = g \text{ on } V_k\}$. Then, as in the work of F. Bethuel, H. Brezis and F. Hélein [7], we know that

$$\mu_g := \min_{u \in H^1_g} E_\varepsilon(u)\,, \tag{4.7}$$

is achieved by some map u_ε, and furthermore, u_ε satisfies the Euler equation

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\, u_\varepsilon\,(1 - |u_\varepsilon|^2)\,, & \text{in } W_k\,, \\ u_\varepsilon = g = u\,, & \text{on } V_k\,. \end{cases} \tag{4.8}$$

This elliptic system will allow us to prove some sort of maximum principle for u_ε which will give us upper and lower bounds for $|u_\varepsilon|$ in terms of the oscillation of $g = u_{|V_k}$ or, more precisely, in terms of $\operatorname{osc}(\theta_k)$. In particular, we will be able to prove that if the oscillation of θ_k is small enough, then $|u_\varepsilon|$ stays bounded away from zero in W_k. Together with Lemma 2 this will imply that the number of W_k's for which $|u_\varepsilon|$ can be close to zero, is uniformly bounded.

We start by proving an upper bound for $|u_\varepsilon|$. The following Lemma is just an adaptation of [7, Proposition 2] to our situation.

Lemma 3. *Let u_ε be a solution of* (4.8)*. Then, $|u_\varepsilon| \le 1$, in W_k.*

PROOF. We start by observing that

$$\Delta(|u_\varepsilon|^2) = 2\,u_\varepsilon\,\Delta u_\varepsilon + 2\,|\nabla u_\varepsilon|^2\,.$$

Hence, by (4.8),

$$\Delta(|u_\varepsilon|^2) = \frac{2}{\varepsilon^2}\,|u_\varepsilon|^2\,(|u_\varepsilon|^2 - 1) + 2\,|\nabla u_\varepsilon|^2 \ge \frac{2}{\varepsilon^2}\,|u_\varepsilon|^2\,(|u_\varepsilon|^2 - 1)\,. \tag{4.9}$$

Therefore, $v_\varepsilon := |u_\varepsilon|^2 - 1$, will satisfy

$$\begin{cases} \Delta v_\varepsilon - \dfrac{2}{\varepsilon^2}\,|u_\varepsilon|^2\,v_\varepsilon \ge 0\,, & \text{in } W_k\,, \\ v_\varepsilon = \zeta^2 - 1\,, & \text{on } V_k = \partial W_k\,. \end{cases}$$

Since $-(2/\varepsilon^2)\,|u_\varepsilon|^2 \le 0$, the maximum principle implies that (see, for instance, [14, Corollary 3.2])

$$\sup_{W_k} v_\varepsilon \le \sup_{V_k} v^+\,, \tag{4.10}$$

where $v^+(x) := \max\{v_\varepsilon(x), 0\}$. Hence, since $v^+(x) := \max\{\zeta^2-1, 0\} = 0$, on V_k, it follows that

$$\sup_{W_k} |u_\varepsilon|^2 - 1 = \sup_{W_k} v_\varepsilon \le 0\,.$$

Thus,

$$\sup_{W_k} |u_\varepsilon| \le 1\,. \tag{4.11}$$

This concludes the proof of Lemma 3.

Using this Lemma and equation (4.8), we are now able to obtain

Proposition 1. *Suppose that* $\operatorname{osc}(\theta_k) \le \pi/3$. *Let* u_ε *be the minimizer of* (4.7). *Then,*

$$|u_\varepsilon(x)| \ge \frac{1}{2}\,\zeta \ge \frac{1}{4}\,, \qquad \textit{for all } x \in W_k\,. \tag{4.12}$$

PROOF. If $\operatorname{osc}(\theta_k) \le \pi/3$, then $u(V_k)$ is contained in an arch $\hat\beta$ of S_ζ, of amplitude at most $\pi/3$. Let a and b be the endpoints of $\hat\beta$, and let B be the domain bounded by the straight line $\hat r$ passing through a and b, and the unit circle S^1. We claim that the maximum principle implies that

$$u_\varepsilon(W_k) \subset B\,. \tag{4.13}$$

By Lemma 3 we already know that $|u_\varepsilon| \le 1$, so it suffices to prove that $u_\varepsilon(W_k)$ and the origin lie on opposite sides of the straight line $\hat r$ defined above.

Choose coordinates y^1, y^2 in the image space such that the y^2 axis is parallel to $\hat r$ (*i.e.*, it is the straight line through the origin parallel to the segment $\overline{a\,b}$), and the y^1 axis cuts the segment $\overline{a\,b}$ perpendicularly at its midpoint. In these coordinates we may write

$$u_\varepsilon(x) = \begin{pmatrix} u_\varepsilon^1(x) \\ u_\varepsilon^2(x) \end{pmatrix} = \zeta \exp(\imath\,\theta_k)\,,$$

where, we are taking the positive y^1 axis as the origin for the angle θ_k.

Since the amplitude $\beta := \operatorname{osc}(\theta_k) \le \pi/3$, the y^1 coordinate of the endpoints a and b satisfies

$$\begin{aligned}
\ell &:= y^1(a) \\
&= y^1(b) \\
&= \min_{x\in V_k} y^1(u(x)) \\
&= \zeta \cos\left(\frac{\beta}{2}\right) \\
&\ge \zeta \cos\left(\frac{\pi}{6}\right) \\
&= \frac{\zeta}{2} \\
&\ge \frac{1}{4}\,.
\end{aligned} \tag{4.14}$$

On the other hand, since u_ε is a minimizer of E_ε, hence a critical point, it is a solution of equation (4.8). In particular u_ε^1 will satisfy

$$\begin{cases} -\Delta u_\varepsilon^1 = \dfrac{1}{\varepsilon^2}\, u_\varepsilon^1 \,(1 - |u_\varepsilon|^2)\,, & \text{in } W_k\ , \\ u_\varepsilon^1 \ge \ell\,, & \text{on } V_k = \partial W_k\ . \end{cases} \tag{4.15}$$

Doing a reflection of u across the y^2 axis in order to make the image lie in the right half-plane, we obtain the map

$$\tilde u_\varepsilon(x) = \begin{pmatrix} \tilde u_\varepsilon^1(x) \\ \tilde u_\varepsilon^2(x) \end{pmatrix} := \begin{pmatrix} |u_\varepsilon^1(x)| \\ u_\varepsilon^2(x) \end{pmatrix}\,,$$

which satisfies

$$E_\varepsilon(\tilde u_\varepsilon) = E_\varepsilon(u_\varepsilon) = \min_{v\in H_g^1(W_k,\mathbb{C})} E_\varepsilon(u)\,.$$

Hence, $\tilde u_\varepsilon$ is also a minimizer, and thus critical point, of E_ε, and therefore, $\tilde u_\varepsilon^1 = |u_\varepsilon^1|$, satisfies

$$\begin{cases} -\Delta \tilde u_\varepsilon^1 = \dfrac{1}{\varepsilon^2}\, \tilde u_\varepsilon^1 \,(1 - |u_\varepsilon|^2)\,, & \text{in } W_k\ , \\ \tilde u_\varepsilon^1 \ge \ell\,, & \text{on } V_k = \partial W_k\ . \end{cases} \tag{4.16}$$

Using Lemma 3 we see that the right-hand side of (4.16) is always non-negative. Hence, $-\Delta\tilde{u}_\varepsilon \geq 0$, and thus the maximum principle assures us that

$$\min_{\overline{W}_k} \tilde{u}_\varepsilon^1 = \min_{V_k} \tilde{u}_\varepsilon^1 \geq \ell\,.$$

Consequently, using (4.14) we obtain

$$\min_{\overline{W}_k} |u_\varepsilon^1| \geq \ell \geq \frac{\zeta}{2} \geq \frac{1}{4}\,. \tag{4.17}$$

Since u_ε^1 is continuous and W_k is connected, $u_\varepsilon^1(W_k)$ has to be connected. Thus, using (4.17) and the fact that $u_\varepsilon^1(x) \geq \ell$ on V_k, we know that we must have

$$u_\varepsilon^1(x) \geq \ell\,, \qquad \text{for all } x \in W_k\,. \tag{4.18}$$

This, together with equation (4.11), proves claim (4.13). In particular, from (4.18) it follows that

$$|u_\varepsilon| = \sqrt{(u_\varepsilon^1)^2 + (u_\varepsilon^2)^2} \geq |u_\varepsilon^1| \geq \ell \geq \frac{\zeta}{2} \geq \frac{1}{4}\,, \quad \text{for all } x \in W_k\,, \tag{4.19}$$

which is equation (4.12).

REMARK. The same method we used to prove claim (4.13) will give us the slightly more precise result

$$u_\varepsilon(W_k) \subset A \subset B\,, \tag{4.20}$$

where A is the closed set bounded by the half-lines $\dot{0}\,a$ and $\dot{0}\,b$, the segment $\overline{a\,b}$ and the circle S^1. In fact, all we have to do to prove this result is to, instead of using a reflection relative to an axis parallel to the segment $\overline{a\,b}$, as before, we have to consider reflections with respect to axii which approach $\overline{0\,a}$ (and others which approach $\overline{0\,b}$) on the outside of the set A defined above.

5. Blow-up of the energy around an isolated "charged" singularity.

5.1. The covering argument.

For simplicity, we will do one more renumbering of the V_k's, $k = 1, \ldots, \tilde{N}$ such that

a) $\deg(u, V_k) \neq 0$ and $V_k \cap Y \neq \varnothing$ if and only if $k \in \{1, \ldots, N_1\}$.

b) $\deg(u, V_k) \neq 0$, $V_k \cap \hat{Y} \neq \varnothing$ and $V_k \cap Y = \varnothing$ if and only if $k \in \{N_1 + 1, \ldots, N_2\}$.

c) $\deg(u, V_k) = 0$, $V_k \cap \hat{Y} \neq \varnothing$ and $\operatorname{osc}(\theta_k) > \pi/3$ if and only if $k \in \{N_2 + 1, \ldots, N\}$.

d) $\deg(u, V_k) = 0$, $V_k \cap \hat{Y} \neq \varnothing$ and $\operatorname{osc}(\theta_k) \leq \pi/3$ if and only if $k \in \{N + 1, \ldots, \tilde{N}\}$.

From (3.10) it follows that

$$N_1 \leq N_2 \leq 160\,\Lambda\,. \tag{5.1}$$

On the other hand, Lemma 2 implies that

$$N = N_2 + \#I \leq 160\,\Lambda + \frac{480}{\pi}\,\Lambda \leq 320\,\Lambda\,. \tag{5.2}$$

We remark that (5.2) gives a bound for N which is valid for all $u \in E_\varepsilon^\Lambda$ and which, moreover, depends only on Λ and not on ε. We have no similar bound for $\tilde{N}$, the total number of V_k's that intersect $\hat{Y}$. However, as we will see in this section, a bound on N like (5.2) is enough since Proposition 1 will allow us to prove that the V_k's in condition d) (*i.e.*, those for which $\deg(u, V_k) = 0$ and $\operatorname{osc}(\theta_k) \leq \pi/3$ are "harmless" – in fact, Proposition 1 gives us a good enough control over the behavior of u inside these V_k's for our estimates of lower bounds on the energy of an isolated charged singularity to go through, regardless of the the presence of V_k's of type d) in its neighborhood. We will need the following two rather technical Lemmas to obtain these lower bounds.

The first one is a covering argument that will allow us to see that $W_1, \ldots, W_N$ can be subdivided into groups, each of which is contained in some ball of radius of order bigger than $\sqrt{\varepsilon}$, and that the different balls are, in some sense, far apart (this type of technique has recently been used by several authors like M. Strüwe or F. Bethuel, H. Brezis and F. Hélein or still F.H. Lin in [22] – our approach is closer to that of the latter).

The second Lemma will then serve to prove that if any of the balls B_j which intersect Y were charged, then we would have to pay a very high price (of order $|\log\varepsilon|$) in energy.

Lemma 4. *Fix $\Lambda \in \mathbb{R}^+$. Let $u \in E_\varepsilon^\Lambda$, and $W_1, \dots, W_N$ be defined as above. Then, for ε sufficiently small, there is an integer $m \le N$, a family of numbers $\alpha_1, \dots, \alpha_m \in (1/2, 1]$, and a family of balls B_j, $j = 1, \dots, m$, of centers x_j and radii r_j such that*

i) $r_j \le C\varepsilon^{\alpha_j}$.

ii) $\displaystyle\bigcup_{i=1}^N W_i \subset \bigcup_{j=1}^m B_j$.

iii) *The enlarged balls $\tilde{B}_j := B(x_j, \varepsilon^{-\alpha_j/(2^{N+1}+1)}\, r_j)$ are pairwise disjoint.*

PROOF. We have fixed $\Lambda \in \mathbb{R}^+$, and we are looking at maps $u \in E_\varepsilon^\Lambda$, for ε sufficiently small (to be chosen later). We define $W_1, \dots, W_N$ as above (thus they will be open, simply-connected subsets of $\Omega \subset \mathbb{R}^2$, such that $\partial W_k = V_k$). By equation (5.2) we know that there exists a uniform bound on N depending only on the energy level Λ we are considering, and not on ε – to be able to change ε while having an uniform bound on the number m of balls used in the covering is crucial for our argument to work.

On the other hand, by (2.12) we have that

$$\operatorname{diam}(W_k) \le \frac{1}{2}\,\mathcal{H}^1(V_k) \le 160\,\Lambda\,\varepsilon\,. \tag{5.13}$$

Hence our Lemma follows from the more general covering argument stated in Lemma 7 of the Appendix. In fact, it corresponds to the special case where $C = 160\,\Lambda$ and $\alpha = 1$.

5.2. Lower-bound for the energy around an isolated charged singularity.

Lemma 5. *Let R_1, $R_2 \in \mathbb{R}^+$ be such that $R_1 < R_2$. Let Ω be the annulus $\Omega = B(0, R_2) \setminus B(0, R_1)$, and $u \in H^1(\Omega, \mathbb{C})$ be such that exists*

$\sigma \in \mathbb{R}^+$ *such that* $|u(x)| \geq \sigma > 0$, *for all* $x \in \Omega$, *and* $\deg(u, S_{R_1}) = \deg(u, S_{R_2}) = d \neq 0$. *Then,*

$$E_\varepsilon(u) \geq \pi\, d^2\, \sigma^2 \log\left(\frac{R_2}{R_1}\right). \tag{5.4}$$

PROOF. We have that

$$E_\varepsilon(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,, \qquad \text{for all } u \in H^1(\Omega, \mathbb{C})\,. \tag{5.5}$$

Hence, we will concentrate on obtaining a lower bound for the Dirichlet energy of u (the right hand side of (5.5)). Since, by hypothesis, $|u| \geq \sigma > 0$, we may define

$$v := \frac{u}{|u|} \in H^1(\Omega, S^1)\,, \text{ and } \deg(v, S_{R_1}) = \deg(v, S_{R_2}) = d \neq 0\,.$$

By (3.5) we know that

$$|\nabla u|^2 \geq |u|^2\, |\nabla v|^2 \geq \sigma^2\, |\nabla v|^2\,. \tag{5.6}$$

We define

$$\mathcal{V}_d = \{v \in H^1(\Omega, S^1) :\ \deg(v, S_{R_1}) = \deg(v, S_{R_2}) = d\}\,.$$

From (5.5) and (5.6) it follows that

$$E_\varepsilon(u) \geq \frac{1}{2}\int |\nabla u|^2 \geq \sigma^2 \inf_{v \in \mathcal{V}_d}\left(\frac{1}{2}\int |\nabla v|^2\right). \tag{5.7}$$

The problem of determining

$$\inf_{v \in \mathcal{V}_d}\left(\frac{1}{2}\int |\nabla u|^2\right)$$

has already been extensively studied. In fact we can reduce it, using an associated linear problem (see, for instance, [8, Theorems I.1 and II.1, and their Corollaries]), to determining the Dirichlet energy of a harmonic map Φ such that

$$\begin{cases} \Delta\Phi = 0\,, & \text{in } \Omega\,, \\ \Phi = 0\,, & \text{on } S_{R_2}\,, \\ \Phi = C\,, & \text{on } S_{R_1}\,, \\ \displaystyle\int_{S_{R_i}} \frac{\partial \Phi}{\partial \nu} = 2\pi\, d\,, & \end{cases} \tag{5.8}$$

where C is some constant, and ν is the outward normal to B_{R_1} and also the outward normal to B_{R_2} (so ν will point inside Ω on S_{R_1} and outside on S_{R_2}).

We can easily check that $\Phi = d\log(r/R_2)$ is a solution of (5.8). Therefore, by the proof of [8, Theorem I.1] (see step 1 of that proof – it is essentially a consequence of Poincaré's Lemma) we know that for all $v \in H^1(\Omega, S^1)$: $\deg(v, S_{R_i}) = d$, $i = 1, 2$,

$$\begin{aligned}\int_\Omega |\nabla v|^2 &\geq \int_\Omega |\nabla \Phi|^2 \\ &= \int_\Omega \left|\frac{d}{r}\right|^2 \\ &= \int_0^{2\pi} d\theta \int_{R_1}^{R_2} r\,\frac{d^2}{r^2}\,dr \\ &= 2\pi\, d^2 \log\left(\frac{R_2}{R_1}\right).\end{aligned} \tag{5.9}$$

Combining equations (5.7) and (5.9) we obtain

$$E_\varepsilon(u) \geq \pi\,\sigma^2\, d^2 \log\left(\frac{R_2}{R_1}\right),$$

which is the desired result.

We are now ready to prove the main result of this section.

Theorem 8. *Let $\Lambda \in \mathbb{R}$ be fixed and $u \in E_\varepsilon^\Lambda$. Then, there exists $\varepsilon_0 > 0$ (depending only on Λ) such that if $\varepsilon < \varepsilon_0$, then $B_j \cap Y \neq \varnothing$ implies that* $\deg(u, \partial B_j) = 0$*, where the balls B_j are given by Lemma* 4.

PROOF. Suppose that for some ε, sufficiently small to apply Lemma 4, there exists $u \in E_\varepsilon^\Lambda$ such that in Lemma 4 we obtained a ball B_j such that $B_j \cap Y \neq \varnothing$ and $\deg(u, \partial B_j) \neq 0$. Since $B_j \cap Y \neq \varnothing$, if ε is sufficiently small (depending only on Λ) $\tilde{B}_j \subset \hat{Y}$ (because the radius of $\tilde{B}_j$ tends to zero when $\varepsilon \longrightarrow 0$). Thus, since in the covering argument we took care of all the V_k's such that $V_k \cap \hat{Y} \neq \varnothing$ and $\deg(u, V_j) \neq 0$ or $\operatorname{osc}(\theta_k) > \pi/3$, we know that the annulus $D_j := \tilde{B}_j \setminus B_j$ may only intersect uncharged V_k's such that $\operatorname{osc}(\theta_k) \leq \pi/3$ (what we called V_k's of type d) in the beginning of this section).

We may suppose, without loss of generality, that the V_k's that intersect D_j are $V_{N+1}, \dots, V_{\hat{N}}$, for some $\hat{N} \leq \tilde{N}$. We know that $\operatorname{osc}(\theta_k) \leq \pi/3$, $k = N+1, \dots, \hat{N}$. However, we cannot apply Lemma 5 directly to u on D_j since a priori we have no lower bound on $|u|$ inside $W_{N+1}, \dots, W_{\hat{N}}$. Nevertheless, if we replace u inside each of the W_k, $k = N+1, \dots, \hat{N}$, by the corresponding minimizer of (4.7), we will decrease the energy and, at the same time, by Proposition 1, we will have a lower bound on the absolute value of the map obtained. Let

$$\tilde{u} := \begin{cases} u\,, & \text{in } D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k\,, \\ u_\varepsilon\,, & \text{in } W_k\,,\ k = N+1, \dots, \hat{N}\,, \end{cases} \tag{5.10}$$

where u_ε is the minimizer of E_ε in W_k with boundary value u. In particular, u_ε satisfies equation (4.8). By construction, $|\tilde{u}| \geq \zeta \geq 1/2$ in $D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k$, and by Proposition 1, $|\tilde{u}| = |u_\varepsilon| \geq 1/4$ in W_k, $k = N+1, \dots, \hat{N}$. Therefore,

$$|\tilde{u}| \geq \frac{1}{4}\,, \qquad \text{in } D_j\,. \tag{5.11}$$

Hence, $\deg(\tilde{u}, \partial\tilde{B}_j) = \deg(\tilde{u}, \partial B_j) = d \neq 0$.Thus, we may apply Lemma 5 to $\tilde{u}$ in D_j. Denoting the energy of a map w in a domain G by

$$E_\varepsilon(w, G) := \frac{1}{2}\int_G |\nabla w|^2 + \frac{1}{4\,\varepsilon^2}\int_G (1 - |w|^2)^2\,,$$

this Lemma yields

$$E_\varepsilon(\tilde{u}, D_j) \geq \pi d^2 \Big(\frac{1}{4}\Big)^2 \log\big(\varepsilon^{-\alpha_j/(2^{N+1}+1)}\big)\,. \tag{5.12}$$

Since $\alpha_j \geq 1/2$ (by Lemma 4), we have that

$$E_\varepsilon(\tilde{u}, D_j) \geq \frac{\pi\, d^2}{16} \log\big(\varepsilon^{-1/(2(2^{N+1}+1))}\big) = -\frac{\pi\, d^2}{32\,(2^{N+1}+1)} \log\varepsilon\,. \tag{5.13}$$

We claim that, for ε sufficiently small

$$E_\varepsilon(u, \Omega) \geq E_\varepsilon(\tilde{u}, D_j)\,. \tag{5.14}$$

PROOF OF CLAIM (5.14). We have that

$$
\begin{aligned}
E_\varepsilon(\tilde{u}, D_j) &= E_\varepsilon\Big(\tilde{u}, D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k\Big) + \sum_{k=N+1}^{\hat{N}} E_\varepsilon(\tilde{u}, W_k \cap D_j) \\
&\le E_\varepsilon\Big(\tilde{u}, D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k\Big) + \sum_{k=N+1}^{\hat{N}} E_\varepsilon(\tilde{u}, W_k)\,.
\end{aligned}
\tag{5.15}
$$

By construction, $\tilde{u} = u$ on $D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k$, we have that

$$
E_\varepsilon\Big(\tilde{u}, D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k\Big) = E_\varepsilon\Big(u, D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k\Big)\,,
$$

and, on the other hand, by the definition of u_ε as the minimizer of (4.7), we also have that

$$
E_\varepsilon(\tilde{u}, W_k) \le E_\varepsilon(u, W_k)\,, \qquad \text{for } k = N+1, \dots, \hat{N}\,.
$$

Therefore, it follows from (5.15) that

$$
\begin{aligned}
E_\varepsilon(\tilde{u}, D_j) &\le E_\varepsilon\Big(u, D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k\Big) + \sum_{k=N+1}^{\hat{N}} E_\varepsilon(u, W_k) \\
&= E_\varepsilon(u, D_j \cup W_{N+1} \cup \cdots \cup W_{\hat{N}}) \\
&\le E_\varepsilon(u, \Omega)\,,
\end{aligned}
$$

since $W_k \subset \hat{Y} \subset \Omega$, $k = N+1, \dots, \hat{N}$, if ε is sufficiently small. This concludes the proof of claim (5.14).

Combining (5.13) and (5.14) we have that for ε sufficiently small,

$$
E_\varepsilon(u, \Omega) \ge -\frac{\pi\, d^2}{32\,(2^{N+1}+1)} \log\varepsilon \ge C\, d^2\, |\log\varepsilon|\,,
\tag{5.16}
$$

where C is a positive constant only depending on Λ (in fact, using equation (5.2) we may choose $C = \pi/(32\,(2^{320\Lambda+1}+1)) > 0$).

If, as we supposed, $d \neq 0$, then, since $u \in E_\varepsilon^\Lambda$, we would have that $C\, d^2 |\log\varepsilon| \le \Lambda$, for all ε sufficiently small. However, this is clearly not true for $\varepsilon \leq \exp(-\Lambda/(C\,d^2))$. Hence, d must be zero, which concludes the proof of Theorem 8.

REMARK. Theorem 8 proves rigorously our idea that as ε gets small the charged V_k's have to cluster, giving rise to "neutral" ($\deg = 0$) B_j's, or to "drift" towards the boundary $\partial\Omega$ (thus exiting the interior domain Y). Hence, in the interior of Ω, and for a distance scale of order $\varepsilon^{1/2}$, the charged singularities shouldn't be "perceptible".

6. Definition of the degree of u in Ω.

In this section we define the degree of u in Ω, which is an integer, and show that this integer is well defined.

Let

$$v := \frac{u}{|u|} : \hat{Y} \setminus \bigcup_{k=1}^{\tilde{N}} W_k \longrightarrow S^1 ,$$

and

$$A := \left\{ r \in \left(\frac{1}{2}, \frac{3}{4}\right) : \; S_r \cap V_k = \varnothing \,, \text{ for all } k = 1, \ldots, \tilde{N} \right\}.$$

As before, for $r \in A$, we define

$$f(r) := \frac{1}{2\pi} \int_{S_r} v \times \frac{\partial v}{\partial \tau} = \deg(u, S_r) \,, \tag{6.1}$$

and we define the approximate degree as

$$\operatorname{adeg}(u) := \frac{1}{2\pi|A|} \int_A \int_{S_r} v \times \frac{\partial v}{\partial \tau}\, d\tau\, dr = \frac{1}{|A|} \int_A f(r)\, dr \,. \tag{6.2}$$

The function f may only change value when we cross a charged V_k since if $r_1, r_2 \in A$, $r_2 > r_1$, then

$$f(r_2) - f(r_1) = \sum_{k \in I_{r_1,r_2}} \deg(u, V_k) \,, \tag{6.3}$$

$$I_{r_1,r_2} = \{k : \; V_k \subset B(0, r_2) \setminus B(0, r_1)\} \,.$$

By (5.3), (5.2), Lemma 4 and Theorem 8, inside Y we can cover all the charged V_k's by an uniformly bounded number of balls $B_1, \ldots, B_m$, with $m \leq 320\,\Lambda$, and such that $r_j = \mathrm{radius}\,(B_j) \leq 160\,\Lambda\,\varepsilon^{1/2}$, and $\deg\,(u, \partial B_j) = 0$. Hence the function f will always have the same value in $\breve{A} := A \setminus B$, where $B := \bigcup_{j=1}^{m} \{r : \ S_R \cap B_j \neq \varnothing\}$. This is the value we use to define $\deg\,(u, \Omega) \in \mathbb{Z}$.

When $\varepsilon \longrightarrow 0$ the approximate degree ($\mathrm{adeg}\,(u)$) approaches this value. In fact, from (5.2) and Lemma 4, it follows that

$$|B| \leq 2 \sum_{j=1}^{m} r_j \leq 2\, m\, 160\,\Lambda\,\varepsilon^{1/2} \leq (320\,\Lambda)^2\,\varepsilon^{1/2}\,. \tag{6.4}$$

Furthermore, even inside $A \cap B$ the value of $f(r) = \deg\,(u, S_r)$ is uniformly bounded – equations (3.9) and (6.3) imply that

$$|f - \deg\,(u, \Omega)| \leq \sum_{k=1}^{N_1} |\deg\,(u, V_k)| \leq 160\,\Lambda\,. \tag{6.5}$$

Thus, using (2.7), (6.4) and (6.5), we obtain

$$\begin{aligned} |\mathrm{adeg}\,(u) - \deg\,(u)| &= \Big| \frac{1}{|A|} \int_A f(r)\, dr - \frac{1}{|A|} \int_A \deg\,(u, \Omega)\, dr \Big| \\ &\leq \frac{1}{|A|} \int_A |f(r) - \deg\,(u, \Omega)| \\ &\leq \frac{1}{|A|}\, |B|\, 160\,\Lambda \\ &\leq \frac{(320\,\Lambda)^3}{2\Big(\dfrac{1}{4} - \mathcal{H}^1(V(\zeta))\Big)}\, \varepsilon^{1/2} \\ &\leq \frac{(320\,\Lambda)^3}{\dfrac{1}{2} - 320\,\Lambda\,\varepsilon}\, \varepsilon^{1/2}\,. \end{aligned} \tag{6.6}$$

Since this bound depends only on Λ and ε (and not on u), we will have that $\mathrm{adeg}\,(u)$ will converge to $\deg\,(u, \Omega) \in \mathbb{Z}$, uniformly in $u \in E_\varepsilon^\Lambda$. Hence, given Λ, we know that for ε sufficiently small

$$|\mathrm{adeg}\,(u) - \deg\,(u)| \leq \frac{1}{4}\,,$$

and therefore, the knowledge of $\mathrm{adeg}\,(u)$ will determine the integer $\deg\,(u)$ as desired.

REMARK. Of course we can also obtain $\deg(u,\Omega)$ by evaluating $f(r) = \deg(u, S_r)$ for any $r \in \breve{A} = A \setminus B$. The problem is that the process of obtaining the balls B_j that define B is very elaborate – hence our choice of also showing how to obtain $\deg(u,\Omega)$ using the approximate degree. We remark also that the B_j's obtained using Lemma 4, and thus also B, are not uniquely determined. However, using estimate (6.4), it is easy to check that (for sufficiently small ε, as usual) the value of $\deg(u,\Omega)$ obtained by evaluating $f(r)$ in $\breve{A}$, is independent of the particular B_j's used in the process.

7. Continuity of $\deg(u,\Omega)$.

This section is devoted to showing that the notion of $\deg(u,\Omega)$ we defined in the previous section (Section 6) is continuous in $H^1(\Omega)$ topology inside each level set of the Ginzburg-Landau energy (1.1). This result will be stated in Theorem 9 at the end of the section.

Let $\Lambda \in \mathbb{R}^+$ be given and $\varepsilon < \varepsilon_0$ (with ε_0 defined as in Theorem 8) and consider $u_1, u_2 \in E_\varepsilon^\Lambda$. Suppose $B_1^i, \ldots, B_{m_i}^i$, are the balls obtained when applying Lemma 4 to u_i, $i = 1, 2$, and V_k^i, $k = 1, \ldots, \tilde{N}_i$, $i = 1, 2$, denote the corresponding V_k's. We define, as before, $v_i := u_i/|u_i|$,

$$A_i := \Big\{ r \in \Big(\frac{1}{2}, \frac{3}{4}\Big) : \; S_r \cap V_k^i = \varnothing, \text{ for all } k = 1, \ldots, \tilde{N}, \\ \text{and } S_r \cap B_j^i = \varnothing, \text{ for all } j = 1, \ldots, m_i \Big\},$$

$$f_i(r) := \frac{1}{2\pi} \int_{S_r} v_i \times \frac{\partial v_i}{\partial \tau}\, d\tau\,, \qquad \text{for } r \in A_i\,.$$

Then, denoting $A := A_1 \cap A_2$,

$$\deg(u_i,\Omega) = \frac{1}{|A_i|} \int_{A_i} f_i(r)\, dr = \frac{1}{|A|} \int_A f_i(r)\, dr\,, \tag{7.1}$$

since $f_i(r) = C^{te} = \deg(u_i,\Omega)$ in A_i (hence also in $A \subset A_i$). Therefore, denoting $G := \{(r,\theta) : \; r \in A, \; \theta \in [0, 2\pi)\}$,

$$|\deg(u_1,\Omega) - \deg(u_2,\Omega)| \\ = \frac{1}{2\pi|A|} \Big| \int_A \int_{S_r} \Big(\frac{u_1}{|u_1|} \times \partial_\tau\Big(\frac{u_1}{|u_1|}\Big) - \frac{u_2}{|u_2|} \times \partial_\tau\Big(\frac{u_2}{|u_2|}\Big)\Big)\, d\tau\, dr \Big| \tag{7.2}$$

$$= \frac{1}{2\pi|A|}\Big|\int_A\int_{S_r}\Big(\frac{u_1}{|u_1|^2}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|^2}\times\frac{\partial u_2}{\partial\tau}\Big)\,d\tau\,dr\Big|,$$

since

$$\begin{aligned}\frac{u_i}{|u_i|}\times\partial_\tau\Big(\frac{u_i}{|u_i|}\Big) &= \frac{u_i}{|u_i|}\times\Big(\frac{1}{|u_i|}\frac{\partial u_i}{\partial\tau}\Big)+\frac{u_i}{|u_i|}\times\Big(u_i\,\partial_\tau\Big(\frac{1}{|u_i|}\Big)\Big)\\ &= \frac{u_i}{|u_i|}\times\Big(\frac{1}{|u_i|}\frac{\partial u_i}{\partial\tau}\Big),\end{aligned}$$

because $u_i\times u_i=0$.

Furthermore, from equation (2.11) and Lemma 4, it follows that $|A_1|$, $|A_2|$ and $|A|\longrightarrow 1/4$ uniformly when $\varepsilon\longrightarrow 0$, and thus, in particular, we have that for ε sufficiently small (independent of the particular choice of $u_1,u_2\in E_\varepsilon^\Lambda$), $|A|>1/(2\pi)$. Hence, equation (7.2) yields that for all ε as above,

$$\begin{aligned}&|\deg(u_1,\Omega)-\deg(u_2,\Omega)|\\ &\quad= \frac{1}{2\pi|A|}\Big|\int_A\int_{S_r}\Big(\frac{u_1}{|u_1|^2}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|^2}\times\frac{\partial u_2}{\partial\tau}\Big)\,d\tau\,dr\Big|\\ &\quad\le \frac{1}{2\pi|A|}\int_G\Big|\frac{u_1}{|u_1|^2}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|^2}\times\frac{\partial u_2}{\partial\tau}\Big|\\ &\quad\le \Big\|\frac{u_1}{|u_1|^2}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|^2}\times\frac{\partial u_2}{\partial\tau}\Big\|_{L^1(G)}.\end{aligned}\tag{7.3}$$

We can write the integrand in (7.3) as

$$\begin{aligned}&\frac{u_1}{|u_1|^2}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|^2}\times\frac{\partial u_2}{\partial\tau}\\ &\quad=\frac{1}{|u_1|}\,\frac{u_1}{|u_1|}\times\frac{\partial u_1}{\partial\tau}-\frac{1}{|u_2|}\,\frac{u_2}{|u_2|}\times\frac{\partial u_2}{\partial\tau}\\ &\quad=\Big(\frac{1}{|u_1|}-\frac{1}{|u_2|}\Big)\frac{u_1}{|u_1|}\times\frac{\partial u_1}{\partial\tau}-\frac{1}{|u_2|}\Big(\frac{u_1}{|u_1|}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|}\times\frac{\partial u_2}{\partial\tau}\Big).\end{aligned}\tag{7.4}$$

Moreover, one can write the last factor in (7.4) as

$$\frac{u_1}{|u_1|}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|}\times\frac{\partial u_2}{\partial\tau}$$

$$
\begin{aligned}
&= \frac{1}{|u_1|}\Big(u_1 \times \frac{\partial u_1}{\partial \tau} - u_2 \times \frac{\partial u_2}{\partial \tau}\Big) + \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big) u_2 \times \frac{\partial u_2}{\partial \tau} \\
&= \frac{1}{|u_1|}\Big((u_1 - u_2) \times \frac{\partial u_1}{\partial \tau} + u_2 \times \frac{\partial (u_1 - u_2)}{\partial \tau}\Big) \\
&\quad + \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big) u_2 \times \frac{\partial u_2}{\partial \tau} \,.
\end{aligned}
\tag{7.5}
$$

From (7.4) and (7.5) it follows that

$$
\begin{aligned}
&\frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial \tau} \\
&\quad = \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big) \frac{u_1}{|u_1|} \times \frac{\partial u_1}{\partial \tau} + \frac{1}{|u_1|\,|u_2|}\Big((u_1 - u_2) \times \frac{\partial u_1}{\partial \tau}\Big) \\
&\quad + \frac{1}{|u_1|}\Big(\frac{u_2}{|u_2|} \times \frac{\partial (u_1 - u_2)}{\partial \tau}\Big) + \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big)\Big(u_2 \times \frac{\partial u_2}{\partial \tau}\Big) \,.
\end{aligned}
\tag{7.6}
$$

On the other hand, since $|u_i| \geq 1/2$ in G, we have that

$$
\frac{1}{|u_i|} \leq 2\,,\ i = 1, 2\,,\ \text{and}\ \frac{1}{|u_1|\,|u_2|} \leq 4\,,\ \text{in } G\,. \tag{7.7}
$$

Furthermore, we have the following estimates for $v_i = u_i/|u_i|$,

$$
\Big\| \frac{u_i}{|u_i|} \Big\|_{L^\infty(\Omega)} = 1\,, \tag{7.8}
$$

$$
\Big\| \frac{u_i}{|u_i|} \Big\|_{L^2(G)} \leq \Big\| \frac{u_i}{|u_i|} \Big\|_{L^\infty(G)} |G|^{1/2} \leq |G|^{1/2} \leq |Y|^{1/2} = \frac{\sqrt{5\pi}}{4}\,. \tag{7.9}
$$

Regarding the tangential derivatives, we have that $|\partial u_i/\partial \tau| \leq |\nabla u_i|$, and thus,

$$
\Big\| \frac{\partial u_i}{\partial \tau} \Big\|_{L^2(G)} \leq \|\nabla u_i\|_{L^2(G)} \leq \|\nabla u_i\|_{L^2(\Omega)}\,, \tag{7.10}
$$

and also that

$$
\Big| \frac{\partial (u_1 - u_2)}{\partial \tau} \Big| \leq |\nabla (u_1 - u_2)|\,,
$$

which implies that

$$
\Big\| \frac{\partial (u_1 - u_2)}{\partial \tau} \Big\|_{L^2(G)} \leq \|\nabla (u_1 - u_2)\|_{L^2(G)} \leq \|\nabla (u_1 - u_2)\|_{L^2(\Omega)}\,. \tag{7.11}
$$

Finally, we can easily check that

$$\Big|\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big| = \frac{|\,|u_1| - |u_2|\,|}{|u_1|\,|u_2|} \le \frac{|u_1 - u_2|}{|u_1|\,|u_2|} \le 4\,|u_1 - u_2|\,,$$

which, in turn, yields

$$\Big\|\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big\|_{L^2(G)} \le 4\,\|u_1 - u_2\|_{L^2(G)} \le 4\,\|u_1 - u_2\|_{L^2(\Omega)}\,. \tag{7.12}$$

Moreover, since we supposed that $u_i \in E_\varepsilon^\Lambda$, we have, as in (2.5),

$$\|\nabla u_i\|_{L^2(G)} \le \|\nabla u_i\|_{L^2(\Omega)} \le \sqrt{2\,E_\varepsilon(u_i)} \le \sqrt{2\,\Lambda}\,. \tag{7.13}$$

Using the Cauchy-Schwarz inequality and equations (7.6), (7.7), (7.8), (7.9), (7.10), (7.11), (7.12) and (7.13), it follows from equation (7.3) that

$$\begin{aligned}
&|\deg(u_1,\Omega) - \deg(u_2,\Omega)| \\
&\quad \le \Big\|\frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial\tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial\tau}\Big\|_{L^1(G)} \\
&\quad \le \Big\|\frac{u_1}{|u_1|}\Big\|_{L^\infty(G)} \Big\|\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big\|_{L^2(G)} \|\nabla u_1\|_{L^2(G)} \\
&\qquad + 4\,\|u_1 - u_2\|_{L^2(G)}\,\|\nabla u_1\|_{L^2(G)} \\
&\qquad + 2\,\Big\|\frac{u_2}{|u_2|}\Big\|_{L^2(G)} \|\nabla(u_1 - u_2)\|_{L^2(G)} \\
&\qquad + \Big\|\frac{u_2}{|u_2|}\Big\|_{L^\infty(G)} \Big\|\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big\|_{L^2(G)} \|\nabla u_2\|_{L^2(G)} \\
&\quad \le 4\,\|\nabla u_1\|_{L^2(G)}\,\|u_1 - u_2\|_{L^2(G)} + 4\,\|\nabla u_1\|_{L^2(G)}\,\|u_1 - u_2\|_{L^2(G)} \\
&\qquad + 2\,|Y|^{1/2}\,\|\nabla(u_1 - u_2)\|_{L^2(G)} + 4\,\|\nabla u_2\|_{L^2(G)}\,\|u_1 - u_2\|_{L^2(G)} \\
&\quad \le \left(8\,\|\nabla u_1\|_{L^2(\Omega)} + 4\,\|\nabla u_2\|_{L^2(\Omega)}\right)\|u_1 - u_2\|_{L^2(\Omega)} \\
&\qquad + 2\,\frac{\sqrt{5\pi}}{4}\,\|\nabla(u_1 - u_2)\|_{L^2(\Omega)} \\
&\quad \le 12\,\sqrt{2\,\Lambda}\,\|u_1 - u_2\|_{L^2(\Omega)} + \frac{\sqrt{5\pi}}{2}\,\|\nabla(u_1 - u_2)\|_{L^2(\Omega)} \\
&\quad \le C\,\|u_1 - u_2\|_{H^1(\Omega)}\,,
\end{aligned}$$

where C is a constant that depends only on the energy bound Λ (we may take $C = 12\sqrt{2\Lambda} + \sqrt{5\pi}/2$). Therefore, we have proven the following Theorem which is the main result of this section.

Theorem 9. *Let $\Lambda > 0$ be given and ε be sufficiently small. Then, inside the level set E_ε^Λ the degree defined as above is continuous in $H^1(\Omega)$ topology, and there is a constant C, depending only on Λ, such that for all $u_1, u_2 \in E_\varepsilon^\Lambda$*

$$|\deg(u_1, \Omega) - \deg(u_2, \Omega)| \le C \, \|u_1 - u_2\|_{H^1(\Omega)} \,. \tag{7.14}$$

8. Proof of Theorem 1 and Theorem 6.

We start by proving Theorem 1, *i.e.* the case where Ω is of the special form we studied (the annulus $\Omega = \{x \in \mathbb{R}^2 : 1/4 < |x| < 1\}$). In this case we defined in Section 6 the map $\deg(u, \Omega)$ which has all the required properties of $\chi(u)$. Thus, we define $\chi(\cdot) := \deg(\cdot, \Omega) : E_\varepsilon^\Lambda \longrightarrow \mathbb{Z}$. Theorem 9 states that this map is continuous inside each level set of the Ginzburg-Landau energy. Since χ is a continuous map with values in the discrete set $\mathbb{Z}$, for each $k \in \mathbb{Z}$, $\chi^{-1}(k) = \{u \in E_\varepsilon^\Lambda : \chi(u) = k\}$, will be an open and closed subset of E_ε^Λ (in H^1 topology). We have thus succeeded in defining topological sectors inside E_ε^Λ. This concludes the proof of Theorem 1. Theorem 6 follows from Theorem 1 as described in the Introduction.

9. The Palais-Smale condition: proof of Theorem 3.

Suppose that u_n is a Palais-Smale sequence for E_ε, *i.e.* that there exists a constant M such that

$$E_\varepsilon(u_n) \le M \,, \qquad \text{for all } n \,, \tag{9.1}$$

$$d\,E_\varepsilon(u_n) \longrightarrow 0 \text{ in } (H^1)^* \text{ as } n \longrightarrow +\infty \,, \tag{9.2}$$

where $(H^1)^*$ is the dual of $H^1(\Omega, \mathbb{R}^2)$, and $d\,E_\varepsilon(u_n)$ denotes the differential of E_ε at u_n. We want to show that then u_n has a strongly convergent subsequence in H^1. This shall be achieved in two steps: first we prove that u_n is bounded in $H^1(\Omega, \mathbb{R}^2)$ and then we find a convergent subsequence.

9.1. Step 1: u_n is bounded in H^1.

Equation (9.1) can be written as

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{1}{4\,\varepsilon^2}\int_\Omega (1-|u_n|^2)^2 \le M\,, \qquad \text{for all } n\,, \tag{9.3}$$

and equation (9.2) means that there is a sequence $C_n \ge 0$, such that for all $v \in H^1(\Omega,\mathbb{R}^2)$,

$$\Big|\int_\Omega \nabla u_n \cdot \nabla v - \frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\, u_n \cdot v\Big| \le C_n\, \|v\|_{H^1(\Omega,\mathbb{R}^2)}\,, \tag{9.4}$$

which implies that there exists a sequence $b_n(v)$ such that $0 \le b_n(v) \le C_n$, for all n, v (and hence $b_n \longrightarrow 0$) and

$$\Big|\int_\Omega \nabla u_n \cdot \nabla v\Big| = b_n\, \|v\|_{H^1(\Omega,\mathbb{R}^2)} + \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\, u_n \cdot v\Big|\,. \tag{9.5}$$

Taking $v = u_n$ in (9.4) we obtain

$$\Big|\int_\Omega |\nabla u_n|^2 - \frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\, |u_n|^2\Big| \le C_n\, \|u_n\|_{H^1(\Omega,\mathbb{R}^2)}\,, \tag{9.6}$$

and thus

$$\Big|\int_\Omega |\nabla u_n|^2\Big| \le C_n\, \|u_n\|_{H^1(\Omega,\mathbb{R}^2)} + \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\, |u_n|^2\Big|\,. \tag{9.7}$$

First, using the Cauchy-Schwarz inequality and (9.3), we notice that,

$$\begin{aligned}
\Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\, |u_n|^2\Big| &= \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)^2 - \frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\Big| \\
&\le 4\,M + \frac{1}{\varepsilon^2}\Big(\int_\Omega (1-|u_n|^2)^2\Big)^{1/2} |\Omega|^{1/2} \\
&\le 4\,M + \frac{2}{\varepsilon}\, M^{1/2}\, |\Omega|^{1/2}\,.
\end{aligned} \tag{9.8}$$

Second, the same type of estimate yields

$$\begin{aligned}
\Big|\int_\Omega |u_n|^2\Big| &= \Big|\int_\Omega 1 - |u_n|^2 + 1\Big| \\
&\le \Big|\int_\Omega 1 - |u_n|^2\Big| + |\Omega| \\
&\le 2\,M^{1/2}\, |\Omega|^{1/2}\, \varepsilon + |\Omega| \\
&= |\Omega| + o\,(\varepsilon)\,.
\end{aligned} \tag{9.9}$$

From (9.7) and (9.8) it follows that

$$\int_\Omega |\nabla u_n|^2 \le C_n\,(\|u_n\|_{L^2} + \|\nabla u_n\|_{L^2}) + 4\,M + \frac{1}{\varepsilon}\,2\,M^{1/2}\,|\Omega|^{1/2}\,, \tag{9.10}$$

and, using (9.9), this yields

$$\begin{aligned}\|\nabla u_n\|_{L^2}^2 - C_n\,\|\nabla u_n\|_{L^2} &\le C_n\,(2\,M^{1/2}\,|\Omega|^{1/2}\,\varepsilon + |\Omega|)^{1/2}\\ &\qquad + 4\,M + \frac{1}{\varepsilon}\,2\,M^{1/2}\,|\Omega|^{1/2}\\ &= \hat{C}(M,\varepsilon)\,.\end{aligned} \tag{9.11}$$

Since $C_n \longrightarrow 0$ this implies that $\|\nabla u_n\|_{L^2(\Omega)}$ is bounded. Together with (9.9), which gives us a bound on $\|u_n\|_{L^2(\Omega)}$, this yields

$$\|u_n\|_{H^1(\Omega)} \le C(M,\varepsilon)\,. \tag{9.12}$$

which concludes the proof of the first step.

Step 2: u_n has a strongly convergent subsequence in H^1.

Since by (9.12) u_n is bounded in $H^1(\Omega,\mathbb{R}^2)$, it has a subsequence, which we will still denote by u_n which is weakly convergent in $H^1(\Omega,\mathbb{R}^2)$. Hence, using the fact that we have a compact embedding $H^1(\Omega,\mathbb{R}^2) \hookrightarrow L^2(\Omega)$, we know that, up to passing to a subsequence, there exists $u \in H^1(\Omega,\mathbb{R}^2)$ such that

$$u_n \longrightarrow u \text{ in } L^2(\Omega) \text{ and } \nabla u_n \rightharpoonup \nabla u \text{ in } L^2(\Omega)\,. \tag{9.13}$$

Therefore, we just need to prove strong convergence in $L^2(\Omega)$ of the gradients, $\nabla u_n \longrightarrow \nabla u$ in $L^2(\Omega)$. By (9.13) we already have weak convergence $\nabla u_n \rightharpoonup \nabla u$, thus we just need to prove the convergence of the $L^2(\Omega)$ norms in order to obtain strong convergence.

Since $H^1(\Omega) \hookrightarrow L^p(\Omega)$, for all $1 \le p < +\infty$, we have that

$$u_n \rightharpoonup u \text{ in } H^1 \text{ implies } u_n \longrightarrow u \text{ in } L^p\,, \quad \text{for all } 1 \le p < +\infty\,. \tag{9.14}$$

In particular

$$u_n \longrightarrow u \text{ in } L^4(\Omega) \text{ and } |u_n|^2 \longrightarrow |u|^2 \text{ in } L^4(\Omega)\,.$$

Thus, using Hölder's inequality,

$$(1-|u_n|^2)\,u_n \longrightarrow (1-|u|^2)\,u \text{ in } L^2(\Omega)\,,$$

$$(1-|u_n|^2)\,u_n\cdot u \longrightarrow (1-|u|^2)\,|u|^2 \text{ in } L^1(\Omega)\,, \tag{9.15}$$

and, since $u_n \longrightarrow u$ in $L^2(\Omega)$,

$$(1-|u_n|^2)\,u_n\cdot u_n \longrightarrow (1-|u|^2)\,|u|^2 \text{ in } L^1(\Omega)\,. \tag{9.16}$$

Taking $v = u \in H^1$ in equation (9.5) we obtain

$$\Big|\int_\Omega \nabla u_n\cdot\nabla u\Big| = b_n\,\|u\|_{H^1(\Omega,\mathbb{R}^2)} + \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\,u_n\cdot u\Big|\,. \tag{9.17}$$

Passing to the limit $n \longrightarrow +\infty$, using the fact that $\nabla u_n \rightharpoonup u$ weakly in $L^2(\Omega)$, $b_n \longrightarrow 0$ and (9.15), inequality (9.17) yields

$$\int_\Omega |\nabla u|^2 = \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)\,|u|^2\Big|\,. \tag{9.18}$$

On the other hand, passing to the limit in (9.7), using the fact that $C_n \longrightarrow 0$, (9.12), (9.16) and (9.18), we obtain

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2 \le \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)\,|u|^2\Big| = \int_\Omega |\nabla u|^2\,. \tag{9.19}$$

Since by the lower semi-continuity of the L^2 norm in weak topology we have that

$$\int_\Omega |\nabla u|^2 \le \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2\,,$$

equation (9.19) implies that

$$\int_\Omega |\nabla u|^2 = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2\,, \tag{9.20}$$

which concludes the proof of Theorem 3 for E_ε. For the case of the functional F_ε the same proof will work once we fix the Coulomb gauge. The reader interested in seeing how the gauge invariance affects Palais-Smale sequences in this problem may take a look at the appendix of [4].

10. Threshold energies and components of E_ε^Λ.

We can reformulate the statement of Theorem 4 and state the following Proposition.

Proposition 2. *Suppose that for some $\Lambda \in \mathbb{R}^+$, we have that for some $\varepsilon < \varepsilon_0$ (where ε_0 is given Theorem 1) there exist $n, k \in \mathbb{Z}$, $n \neq k$, such that the topological sectors $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_k(E_\varepsilon^\Lambda)$ are both non-empty. Then, there are mountain-pass type critical points of E_ε or, equivalently, there exist mountain-pass type solutions of the Ginzburg-Landau equations* (1.11).

More precisely, consider two non-empty components of E_ε^Λ, $\Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)$, and let $c_{n,k}(\Sigma_0, \Sigma_1)$ be defined as in (10.4). *Then, there exists a map $u \in H^1(\Omega, \mathbb{R}^2)$ which is a critical point of E_ε and such that $E_\varepsilon(u) = c_{n,k}(\Sigma_0, \Sigma_1)$.*

Since $H^1(\Omega)$ is locally pathwise connected and the level sets E_ε^Λ are open, their path components coincide with their components, so we can use the two concepts indistinguishably. Let $n, k \in \mathbb{Z}$ be two distinct integers, and let Σ_0 and Σ_1 be components of E_ε^Λ such that $\Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)$. Then, given $u_0, u_0' \in \Sigma_0$ and $u_1, u_1' \in \Sigma_1$, we know that there exist two paths γ_i, $i = 0, 1$, such that

$$\gamma_i : [0,1] \longrightarrow \Sigma_i\ ,\ \ \gamma_i(0) = u_i\ ,\ \ \gamma_i(1) = u_i'\ ,\ \ i = 0, 1\,.$$

In particular,

$$\gamma_i(s) < \Lambda\,, \qquad \text{for all } s \in [0,1]\,. \tag{10.1}$$

As usual, we define the composition operation for paths: let γ be a path from p to q, and σ be a path from q to r, then $\varrho = \gamma\,\sigma$ is the path from p to r defined by

$$\varrho(s) := \begin{cases} \gamma(2\,s)\,, & \text{for } 0 \le s \le \dfrac{1}{2}\,, \\[2ex] \sigma(2\,s - 1)\,, & \text{for } \dfrac{1}{2} \le s \le 1\,. \end{cases}$$

And we define the inverse path of γ, which we denote by γ^{-1}, as $\gamma^{-1}(s) := \gamma(1-s)$, for $s \in [0,1]$. Then, to any path $\gamma : [0,1] \longrightarrow H^1(\Omega)$ between u_0 and u_1, one can associate a path $\gamma' = \gamma_0^{-1}\,\gamma\,\gamma_1 : [0,1] \longrightarrow$

$H^1(\Omega)$ from u_0' to u_1'. And vice-versa, to any path $\gamma' : [0,1] \longrightarrow H^1(\Omega)$ between u_0' and u_1', one can associate a path $\gamma = \gamma_0\, \gamma'\, \gamma_1^{-1} : [0,1] \longrightarrow H^1(\Omega)$ from u_0 to u_1. With these definitions, from equation (10.1) it follows that

$$\max_{s\in[0,1]} E_\varepsilon(\gamma(s)) = \max_{s\in[0,1]} E_\varepsilon(\gamma'(s)) \geq \Lambda\,. \tag{10.2}$$

And hence,

$$\inf_{\gamma\in\mathcal{V}} \big(\max_{s\in[0,1]}(E_\varepsilon(\gamma(s)))\big) = \inf_{\gamma\in\mathcal{V}'} \big(\max_{s\in[0,1]}(E_\varepsilon(\gamma'(s)))\big) \geq \Lambda\,, \tag{10.3}$$

where,

$$\mathcal{V} := \{\gamma \in C^0([0,1], H^1(\Omega,\mathbb{R}^2)) :\ \gamma(0) = u_0,\ \text{and } \gamma(1) = u_1\}\,,$$

and

$$\mathcal{V}' := \{\gamma' \in C^0([0,1], H^1(\Omega,\mathbb{R}^2)) :\ \gamma'(0) = u_0',\ \text{and } \gamma'(1) = u_1'\}\,.$$

Thus, c_n, the threshold energy for a transition from u_0 to u_1 defined in (1.16), is well defined as a transition energy from a component Σ_0 of $\mathrm{top}_n(E_\varepsilon^\Lambda)$ to a component Σ_1 of $\mathrm{top}_k(E_\varepsilon^\Lambda)$. We can define,

$$c_{n,k}(\Sigma_0,\Sigma_1) := \inf_{\gamma\in\mathcal{V}_{n,k}(\Sigma_0,\Sigma_1)} \big(\max_{s\in[0,1]}(E_\varepsilon(\gamma(s)))\big)\,, \tag{10.4}$$

where,

$$\begin{aligned}
&\mathcal{V}_{n,k}(\Sigma_0,\Sigma_1)\\
&\qquad := \{\gamma \in C^0([0,1], H^1(\Omega,\mathbb{R}^2)) :\ \gamma(0) \in \Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \gamma(1) \in \Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)\}\,.
\end{aligned}$$

By the Mountain Pass Theorem we know that $c_{n,k}(\Sigma_0,\Sigma_1)$ is a generalized critical value of E_ε and, since by Theorem 3 the functional E_ε satisfies the Palais-Smale condition, this implies that $c_{n,k}(\Sigma_0,\Sigma_1)$ is also a critical value of E_ε, thus concluding the proof of Proposition 2 and Theorem 4.

Remark. For small ε and $n \neq k$, $c_{n,k}(\Sigma_0,\Sigma_1)$ shouldn't depend on the specific components $\Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)$, but only on n and k (*i.e.* only on the topological sectors themselves). This leads us back to the question of how many distinct components can there be inside a topological sector and how do they change when Λ changes. We expect that for certain values of Λ, $\mathrm{top}_n(E_\varepsilon^\Lambda)$ may not be connected, but that as we increase Λ the different components which existed at lower energies, should increase in size and eventually intersect thus becoming the same component. As a matter of fact, in [1] we will be able to prove that all the components in $\mathrm{top}_n(E_\varepsilon^\Lambda)$ can be connected by paths wich involve energies of, at most, something like $6\,\Lambda$, while to connect different topological sectors we will need energies like $\pi\,|\log\varepsilon|$, which for small enough ε is much bigger than $6\,\Lambda$. In this case $c_{n,k}(\Sigma_0, \Sigma_1)$ will depend only on n and k as we said.

REMARK. As usual, similar results are valid for F_ε.

11. A model for superconductivity.

In this section we will consider the gauge-invariant Ginzburg-Landau model (1.3), and prove that inside the level sets F_ε^Λ we can define topological sectors in a similar way to the one used for defining such sectors inside the level sets E_ε^Λ in theorems 1 and 6 which we proved in Section 8.

11.1. Gauge fixing.

Given a configuration $(v, B) \in F_\varepsilon^\Lambda$, we will show in this section how to choose a gauge equivalent configuration, $(u, A) \approx (v, B)$, such that we have the necessary control on A to allow us to bound the L^2 norm of ∇u by a constant depending only on the energy level Λ. In fact, to achieve this, all we need to do is to fix a Coulomb gauge over the unit disk $D = B(0,1) = \Omega \cup B(0, 1/4)$.

Proposition 3. *Given a configuration $(v, B) \in H^1$, there exists $(u, A) \approx (v, B)$ such that*

$$\left\{ \begin{array}{ll} d^\star A = 0\,, & \text{in } D\,, \\ A \cdot \nu = 0\,, & \text{on } \partial D = S^1\,. \end{array} \right. \tag{1.11}$$

The proof is just the same as that of [9, Propositions I.1 and I.2]. Now we remark that, since D is simply-connected, (11.1) implies that there exists $\zeta \in H^2(D,\mathbb{R})$ such that writing $\hat{\zeta} = \zeta\, dx^1 \wedge dx^2 = \star\zeta$,

$$\begin{cases} A = d^\star \hat{\zeta} = \star\, d\zeta\,, & \text{in } D\,,\\ \zeta = 0\,, & \text{on } \partial D\,. \end{cases} \tag{11.2}$$

It follows from (11.1) and (11.2) that ζ satisfies

$$\begin{cases} \Delta\zeta = d^\star\, d\zeta = \star\, dA\,, & \text{in } D\,,\\ \zeta = 0\,, & \text{on } \partial D\,. \end{cases} \tag{11.3}$$

This implies, using standard elliptic estimates, that

$$\|\zeta\|_{W^{2,2}(D)} \le \hat{C}\, \|dA\|_{L^2(D)}\ ,$$

which, together with (11.2) yields

$$\begin{aligned} \|A\|^2_{W^{1,2}(D)} &= \int_D |A|^2 + \int_D |\nabla A|^2 \\ &= \int_D |\nabla\zeta|^2 + \int_D |\nabla^2\zeta|^2 \\ &\le \|\zeta\|^2_{W^{2,2}(D)} \\ &\le \hat{C}\|dA\|^2_{L^2(D)} \\ &\le \hat{C}\, F_\varepsilon(u,A) \\ &< \hat{C}\,\Lambda\,, \end{aligned} \tag{11.4}$$

where $\hat{C}$ is a constant.

11.2. Global control of $|\nabla u|^2$.

The purpose of this subsection is to show how to obtain a bound on $\|\nabla u\|_{L^2(\Omega)}$ by a constant depending only on the energy level Λ.

Lemma 6. *Given $(v,B) \in F_\varepsilon^\Lambda$, let (u,A) be as in Proposition 3. Then,*

$$\int_\Omega |\nabla u|^2 \le C\,, \tag{11.5}$$

where C is a constant which only depends on Λ.

PROOF. Since, by construction, $F_\varepsilon(u,A) = F_\varepsilon(v,B) \leq \Lambda$, we have that, in particular,

$$\int_\Omega |\nabla_A u|^2 = \int_\Omega |\nabla u - \imath A u|^2 \leq 2 F_\varepsilon(u,A) \leq 2\Lambda \,. \tag{11.6}$$

Hence,

$$\begin{aligned}
\int_\Omega |\nabla u|^2 &= \int_\Omega |\nabla u - \imath A u + \imath A u|^2 \\
&\leq 2 \int_\Omega |\nabla u - \imath A u|^2 + 2 \int_\Omega |A u|^2 \\
&\leq 4 F_\varepsilon(u,A) + 2 \int_\Omega |A|^2 |u|^2 \qquad (11.7) \\
&\leq 4\Lambda + 2 \int_\Omega |A|^2 (|u|^2 - 1) + 2 \int_\Omega |A|^2 \\
&\leq 4\Lambda + 2 \int_\Omega |A|^2 |1 - |u|^2| + 2 \int_\Omega |A|^2 \,.
\end{aligned}$$

Using Hölder's inequality, and the fact that from the energy bound it follows that

$$\|1 - |u|^2\|^2_{L^2(\Omega)} \leq 4\varepsilon^2 F_\varepsilon(u,A) \leq 4\varepsilon^2 \Lambda \,,$$

we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^2 &\leq 4\Lambda + 2 \|A^2\|_{L^2(\Omega)} \|1 - |u|^2\|_{L^2(\Omega)} + 2 \|A\|^2_{L^2(\Omega)} \\
&\leq 4\Lambda + 4\varepsilon \Lambda^{1/2} \|A\|^2_{L^4(\Omega)} + 2 \|A\|^2_{L^2(\Omega)} \,.
\end{aligned} \tag{11.8}$$

Since we are in a two-dimensional domain it follows from the Sobolev Embedding Theorem that $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$, for all $q < +\infty$. hence, in particular, there exists a constant $\tilde{C}$ such that

$$\|A\|_{L^4(\Omega)} \leq \tilde{C} \|A\|_{W^{1,2}(\Omega)} \,. \tag{11.9}$$

Furthermore, from (11.4) we know that

$$\|A\|_{W^{1,2}(\Omega)} \leq \|A\|_{W^{1,2}(D)} \leq \sqrt{\hat{C}\Lambda} \,. \tag{11.10}$$

From equations (11.8), (11.9) and (11.10) it follows that for $\varepsilon < 1$ (as mentioned before, it is the case where ε is small that interests us),

$$\begin{aligned}\int_\Omega |\nabla u|^2 &\le 4\,\Lambda + 4\,\varepsilon\,\Lambda^{1/2}\,\tilde C^2\,\|A\|^2_{W^{1,2}(D)} + 2\,\|A\|^2_{W^{1,2}(D)} \\ &\le 4\,\Lambda + 4\,\Lambda^{1/2}\,\tilde C^2\,\hat C\,\Lambda + 2\,\Lambda\,\hat C = C\,,\end{aligned} \tag{11.11}$$

where C is a constant depending only on Λ.

11.3. Definition of $\deg([v,B],\Omega)$ and proof of Theorem 2.

Once we have the estimate (11.5), we can define $\deg(u,\Omega)$ as in the case of the initial model (1.1), since we will have all the estimates we used in the work that culminated with the definition of the degree in Section 6. Thus, for ε sufficiently small, $\deg(u,\Omega)$ is well defined, and hence we may define

$$\deg([v,B],\Omega) := \deg(u,\Omega)\,.$$

Once we have achieved this, Theorem 2 follows from the corresponding result for $\deg(u,\Omega)$ which, thanks to estimate (11.5), can be proven in a similar way to that we used for proving Theorem 1 (therefore, we omit this proof).

The generalization of Theorem 2 to the setting of Riemannian manifolds will then follow from Theorem 2 in an analogous way as Theorem 6 followed from Theorem 1.

12. Appendix: Covering Lemma.

This section is devoted to a general covering Lemma we used to prove Lemma 4.

Lemma 7. *Let $\varepsilon > 0$ and $W_1,\dots,W_n$ be connected open subsets of $\mathbb{R}^2$ such that there exist $C,\alpha > 0$ such that* $\operatorname{diam}(W_l) \le C\,\varepsilon^\alpha$*. Then, for ε sufficiently small, there is a family of numbers $\alpha_1,\dots,\alpha_m \ge \alpha/2$, and a family of balls $B_1,\dots,B_m$, with $m \le n$, such that, denoting by x_j the center of B_j, and by r_j its radius,*

i) $r_j \le C\varepsilon^{\alpha_j}$.

ii) $\displaystyle\bigcup_{l=1}^{n} W_l \subset \bigcup_{j=1}^{m} B_j$.

iii) *The enlarged balls* $\tilde{B}_j := B(x_j, \varepsilon^{-\alpha_j/(2^{n+1}+1)}\, r_j)$ *are pairwise disjoint.*

Proof. We start by defining

$$q_n := \frac{2^{n+1}}{2^{n+1}+1}\,,$$

$$p_k := \frac{1}{\displaystyle\sum_{j=0}^{k} 2^{-j}} = \frac{2^k}{2^{k+1}-1}\,,$$

for $k = 1, \dots, n$.

The proof of this Lemma is done by induction on the number k of components of $A = \bigcup_{l=1}^{n} W_l$. For $k = 1$, it suffices to consider a unique ball of radius $r_1 = C\,\varepsilon^{\alpha_1}$, with $\alpha_1 = 2\,\alpha/3 = \alpha\, p_1$, since, for ε sufficiently small,

$$\operatorname{diam}(A) \le \sum_{l=1}^{n} \operatorname{diam}(W_l) \le n\, C\, \varepsilon^{\alpha} \le C\, \varepsilon^{2\alpha/3}\,. \tag{12.1}$$

Hence, we can find a ball B_1, of radius $r_1 \le C\varepsilon^{2\alpha/3}$ containing $\bigcup_{l=1}^{n} W_l$.

Suppose that the result is always true if A has $\overline{n}$ components, for all $\overline{n} \le k-1 \le n-1$, and, furthermore, the number m of balls obtained in the covering process is at most $\overline{n}$ and each of the α_j's obtained satisfies

$$\alpha_j \ge \frac{\alpha}{\displaystyle\sum_{j=0}^{\overline{n}} 2^{-j}} = \alpha\, p_{\overline{n}} \ge \alpha\, p_{k-1}\,. \tag{12.2}$$

To complete the induction argument, we just have to show that then the result will still be true when A has k components, and that in this case $m \le k \le n$ and we can find α_j's such that

$$\alpha_j \ge \frac{\alpha}{\displaystyle\sum_{j=0}^{k} 2^{-j}} = \alpha\, p_k\,.$$

Let $A_1, \ldots, A_k$ be the connected components of A. Suppose that

$$\operatorname{diam}(A) \le 5\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} . \tag{12.3}$$

Then, for ε sufficiently small, we can include A in a ball B_1 of radius $r_1 \le \varepsilon^{\alpha p_k}$. In fact, it suffices that

$$3\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} \le C\, \varepsilon^{\alpha q_n p_k} .$$

This is always true, provided that ε is sufficiently small, since

$$\alpha\, q_n\, p_{k-1} > \alpha\, p_k \qquad \text{if and only if} \qquad \frac{p_{k-1}}{p_k} > \frac{1}{q_n} ,$$

and

$$\frac{p_{k-1}}{p_k} = 1 + \frac{2^{-k}}{p_{k-1}} > 1 + \frac{1}{2^{k+1}} = \frac{2^{k+1}}{2^{k+1}+1} \ge \frac{2^{n+1}}{2^{n+1}+1} = \frac{1}{q_n} .$$

Thus, if (12.3) is true, our proof will be completed. Hence, we may suppose that this is not so, *i.e.*, that

$$\operatorname{diam}(A) \ge 5\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} . \tag{12.4}$$

Let $y_1, y_2 \in \overline{A}$ be such that $|y_1 - y_2| = \operatorname{diam}(A)$, and consider the family of balls $B(y_1, r)$ for $r \in (0, \operatorname{diam}(A))$. Define $G_j := \{r : B(y_1, r) \cap A_j \neq \varnothing\}$, $j = 1, \ldots, k$. Each G_j will be an interval, and the sum of the lengths of the G_j's will be smaller than the sum of the diameters of the W_l's, which is at most $nC\varepsilon^\alpha$. Since $n\, C\, \varepsilon^\alpha \le n\, C\, \varepsilon^{\alpha q_n p_{k-1}}$, for all $\varepsilon \le 1$, it follows that the set

$$\hat{G} := (0, \operatorname{diam}(A)) \setminus \bigcup_{j=1}^{k} G_j ,$$

will have a measure of at least

$$5\, n\, C \varepsilon^{\alpha q_n p_{k-1}} - n\, C\, \varepsilon^{\alpha q_n p_{k-1}} = 4\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} .$$

Moreover, the set $\hat{G}$ is the union of, at most, $k-1$ subintervals of $(0, \operatorname{diam}(A))$ since it was obtained from the latter by removing the k open intervals G_j (among which one had endpoint 0 and another had endpoint diam (A)). Consequently, at least one of its components, which we will denote by $[a_0, b_0]$, will be such that

$$b_0 - a_0 \geq \frac{|\hat{G}|}{k-1} \geq \frac{4\,n}{k-1}\,\varepsilon^{\alpha q_n p_{k-1}} > 4\,\varepsilon^{\alpha q_n p_{k-1}}\,. \tag{12.5}$$

Let $\hat{A} = A \cap B(y_1, a_0)$, and $\tilde{A} = A \setminus B(y_1, b_0)$. Then, $A = \hat{A} \cup \tilde{A}$, and both $\hat{A}$ and $\tilde{A}$ include at least one of the A_j's. Hence, both $\hat{A}$ and $\tilde{A}$ have at most $k-1$ components and thus we can apply the induction step to each of them. It yields, since the sum of the number of components of $\hat{A}$ and $\tilde{A}$ is k, that there will be a total of $m \leq k$ balls $B_1, \ldots, B_m$, such that

a) $\hat{A} \subset B_1 \cup \cdots \cup B_{\overline{m}}$, $\tilde{A} \subset B_{\overline{m}+1} \cup \cdots \cup B_m$, for some $\overline{m} < m$.

b) Each B_j has center x_j and radius $r_j \leq C\,\varepsilon^{\alpha_j}$, where $\alpha_j \geq \alpha\,p_{k-1} \geq \alpha\,p_k$.

c) The enlarged balls $\tilde{B}_j := B(x_j, \varepsilon^{-\alpha_j/(2^{n+1}+1)}\,r_j)$ are pairwise disjoint for $j \in \{1, \ldots, \overline{m}\}$ and also for $j \in \{\overline{m}+1, \ldots, m\}$.

However, to obtain the disjointness of two $\tilde{B}_j$, one corresponding to $\hat{A}$ (*i.e.* $j \leq \overline{m}$) and the other to $\tilde{A}$ (*i.e.* $j > \overline{m}$), we need to use equation (12.5). In fact, if $j_1 \leq \overline{m}$ and $j_2 > \overline{m}$, then

$$|x_{j_1} - y_1| < a_0 + C\,\varepsilon^{\alpha_{j_1}} < a_0 + C\,\varepsilon^{\alpha q_n p_{k-1}}\,, \tag{12.6}$$

since $B_{j_1} \cap \hat{A} \neq \varnothing$, $\hat{A} \subset B(y_1, a_0)$ and by b), $\alpha_{j_1} \geq \alpha\,p_{k-1} > q_n\,\alpha\,p_{k-1}$. Similarly, we have that

$$|x_{j_2} - y_1| > b_0 - C\,\varepsilon^{\alpha_{j_2}} > b_0 - C\,\varepsilon^{\alpha q_n p_{k-1}}\,, \tag{12.7}$$

since $B_{j_2} \cap \tilde{A} \neq \varnothing$, $\tilde{A} \subset A \setminus B(y_1, b_0)$ and, by b), $\alpha_{j_2} \geq \alpha\,p_{k-1} > q_n\,\alpha\,p_{k-1}$.

Therefore, combining (12.6) and (12.7) we have

$$|x_{j_1} - x_{j_2}| > 2\,C\,\varepsilon^{\alpha q_n p_{k-1}}\,. \tag{12.8}$$

Since $\tilde{B}_{j_i}$ has radius

$$C\,\varepsilon^{q_n\,\alpha_{j_i}} < C\,\varepsilon^{\alpha q_n p_{k-1}}\,,$$

equation (12.8) implies that

$$\tilde{B}_{j_1} \cap \tilde{B}_{j_2} = \varnothing\,,$$

as desired. Consequently, the balls B_j obtained satisfy all the conditions required for the induction argument, and thus the proof of Lemma 7 in completed.

REMARK. Relative to the similar covering argument of Lin [22], our result has the advantage that we are able to keep the α_j always bigger than $\alpha/2$, which corresponds to keeping the balls B_j rather small – in Lin's result α_j may tend to zero when $n \longrightarrow \infty$. However, we also lose something, both because our proof is technically more complicated, but also because we obtain smaller (and more complex) expansion factors for the $\tilde{B}_j$'s. In fact, even Lin's expansion factors $(\varepsilon^{-\alpha_j/3})$ go to 1 when $n \longrightarrow \infty$, but ours $(\varepsilon^{-\alpha_j/(2^{n+1}+1)})$ will decrease to 1 considerably faster.

We prefered to privilege the scale of the balls because it enables us to assert that in our problem, at least at a scale $\varepsilon^{1/2}$, things appear neutral to an outside observer (and it also makes the energy explosion estimate (5.13) slightly neater). Using Lin's result, the scale would depend on n, and hence on Λ, which would be less satisfactory.

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