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On proximity relations for valuations dominating a two-dimensional regular local ring

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Abstract. The purpose of this paper is to define a new numerical invariant of valuations centered in a regular two-dimensional regular local ring. For this, we define a sequence of non-negative rational numbers $\delta_\nu = \{\delta_\nu(j)\}_{j\geq 0}$ which is determined by the proximity relations of the successive quadratic transformations at the points determined by a valuation ν. This sequence is characterized by seven combinatorial properties, so that any sequence of non-negative rational numbers having the above properties is the sequence associated to a valuation.

0. Introduction.

Valuations centered in a two-dimensional regular local ring have been studied and classified by Zariski, Abhyankar and Lipman (see for example [1]). More recently, there has been a revival of interest in this subject (see [15], [13], [7], [9], ...).

The main purpose of this paper is to define a new numerical invariant of valuations centered in a regular two-dimensional local ring. One advantage of our invariant over those of [15] is that it works for a general regular local ring of dimension two; in particular, we do not assume that the residue field is algebraically closed.

The idea of proximity to classify singularities of analytically irreducible plane curves was developed by Enriques (see [6]) and can be adapted to the situation above (see [7], [9], [13], ...).

Several invariants can be associated to proximity relations (the refined proximity matrix, the multiplicity sequence, the semigroup-length sequence, ..., see [13]). Here we will introduce a new one which is a sequence of non-negative rational numbers $\delta_\nu = \{\delta_\nu(j)\}_{j\geq 0}$ (later called proximity sequence), where the proximity relations are codified.

In what follows, all rings considered will be commutative and with a unit element. For a local ring R, we will denote by $M(R)$ its maximal ideal.

Throughout this paper, R will be a two-dimensional regular noetherian local ring and we will consider a fixed sequence

$$(*) \qquad R = R_0 \subset R_1 \subset \cdots \subset R_n \subset \cdots ,$$

where R_{i+1} is a quadratic transform of R_i (*i.e.* R_{i+1} is a localization at a maximal ideal of a ring $R[x^{-1}M(R_i)]$ with $x \in M(R_i)$ and $x \notin (M(R_i))^2$.

For $i > 0$, we will denote by

$$e_{i-1} = \Big[\frac{R_i}{M(R_i)} : \frac{R_{i-1}}{M(R_{i-1})}\Big] .$$

It should be remembered that $S = \cup_{i\geq 0} R_i$ is a valuation ring. (See [1]). If ν is the valuation of S then ν is the only valuation of the quotient field of R centered at the maximal ideal of R_i for all $i \geq 0$.

The main goal of the paper is the characterization of the properties of the proximity sequence in the following sense: the properties that a sequence of non-negative rational numbers $\{\delta(j)\}_{j\geq 0}$ must satisfy in order to be the sequence associated to a valuation ν (or equivalently to a sequence $(*)$). Therefore these properties characterize the class of all valuations with the same associated sequence δ_ν. This gives rise to a notion of equisingularity of valuations.

For this, we see that all such sequence can be realized taking $R = \mathbb{Q}(t_1, \ldots, t_n, \ldots)[[X, Y]]$, $\mathbb{Q}$ being the field of rational numbers. In general, this is not possible for any R. If, in addition, the sequence satisfies that $\delta(j)$ is an integer for all $j \geq 0$ (or equivalently all rings of $(*)$ have the same residue field) then there is a valuation ν such that its associated proximity sequence is the given one.

We are also interested in other properties of the proximity sequence. In particular, if R is a complete ring then there is a non-zero principal prime ideal J of R such that J "goes through" R_n for all $n \geq 0$ (*i.e.* $J_n \neq R_n$, where J_n is the strict quadratic transform of J in R_n) if and only if there is N_0, such that $\delta_\nu(n) = 0$ for all $n \geq N_0$. In this situation, δ_ν characterizes the equisingularity classes of analytically irreducible plane curves. So we also have an explicit description of the different equisingularity classes.

The paper is organized as follows:

In Section 1 we outline some definitions and properties of proximity relations.

Section 2 is devoted to an introduction of the invariant and to study its properties. In particular we see that it is equivalent to the refined proximity matrix.

In the last section we characterize δ_ν by its properties and when δ_ν is an invariant for the equisingularity of plane curves.

1. Preliminaries.

First we will outline some concepts about the proximity relations of $(*)$.

For $j > i$ we say that R_j is *proximate* to R_i if the valuation ring $V(R_i)$ of Ord_{R_i} contains R_j, where Ord_{R_i} is the usual valuation order of R_i (*i.e.* $\mathrm{Ord}_{R_i}(x)$ is the greatest non-negative integer d such that $x \in (M(R_i))^d$, x being a non-zero element of R_i). In this case, $V(R_i) = (R_j)_{\mathfrak{p}}$, where $\mathfrak{p}$ is a height one prime ideal of R_j containing $M(R_i)R_j$ and R_k is proximate to R_i for $i < k \leq j$.

Moreover, for $j > i$ it is easy to verify that $M(R_i)R_j = t_{ij}^{a_{ij}} u_{ij}^{b_{ij}}$, where $t_{ij}R_j = M(R_{j-1})R_j$, $(t_{ij}, u_{ij})R_j = M(R_j)$, $a_{ij} > 0$ and $b_{ij} \geq 0$. (a_{ij} and b_{ij} being integers). So R_j is proximate to R_{j-1} and at most to one other ring in $(*)$. In fact if $j > i+1$ and R_j is proximate to R_i we can write

$$R_k = \left(R_{k-1}\left[\frac{u_{i,k-1}}{t_{i,k-1}}\right]\right)_{(t_{i,k}, u_{i,k})},$$

with $t_{i,k} = t_{i,k-1}$ and $u_{i,k} = u_{i,k-1}/t_{i,k-1}$, $i+2 \geq k \geq j$. So $b_{i,j} = 1$ and this is also a sufficient condition for R_j to be proximate to R_i.

One also has $a_{i,j} = j + i - 1$ and $e_{k-1} = 1$, $i+2 \leq k \leq j$.

In general, for $j > i+1$ we say that R_j is a *satellite* of R_i if $b_{ij} \neq 0$, where $M(R_i)R_j = t_{ij}^{a_{ij}} u_{ij}^{b_{ij}}$ as above. If $b_{ij} = 0$ we say that R_j is *free* with respect to R_i.

This is simply Zariski's definition of satellite and free points. (See [17]). It should be noted that R_j is a satellite of R_i if and only if $\mathrm{Ord}_{R_j}\left(\sqrt{M(R_i)R_j}\,\right) = 2$. It is also easy to verify that R_j is a satellite of R_i if and only if there is a non-negative integer q with $j-1 > q \geq i$ such that R_j is proximate to R_q.

2. The invariant.

In this section we will use the above notations.

We define the function $\gamma : \mathbb{Z}_0 \longrightarrow \mathbb{Z}_0$ as follows: $\gamma(0) = 0$ and for $j \geq 1$, $\gamma(j) = 1 + \min\{k : R_j$ is proximate to $R_k\}$, where $\mathbb{Z}_0$ denotes the set of non-negative integers.

Thinking geometrically, this map computes the oldest exceptional divisor that "goes through" R_j.

On the other hand, note that $\gamma(j) < j$ if and only if R_j is a satellite of R_i for some $i < j-1$. So $\gamma(j) = j$ if and only if R_j is free with respect to R_i for all $i < j-1$.

The most interesting properties of γ are given in the following results.

Proposition 2.1. *We have the following statements*:

a) $\gamma(j) \leq j$.

b) *If* $\gamma(j) < i < j$ *then* $\gamma(i) = \gamma(j)$.

c) *For all* $j \geq 0$ *there is a non-negative integer* n *such that* $\gamma^n(j) = \gamma^{n+1}(j)$, *where* $\gamma^0 = \mathbf{1}_{\mathbb{Z}_0}$ *and* $\gamma^{k+1} = \gamma \circ \gamma^k$.

d) *If* $\gamma(j) < j$ *then* $\gamma(j) = j-1$ *or* $\gamma(j) = \gamma(j-1)$.

Proof. a) Follows from the definition of γ.

b) If $m+1 = \gamma(j) < j$ then R_j is proximate to R_m and also R_i is proximate to R_m for $m+1 \leq i \leq j$. So $\gamma(i) = m+1 = \gamma(j)$.

c) By a) we have $0 \leq \cdots \leq \gamma^k(j) \leq \cdots \leq \gamma(j) \leq j$. So there is an n such that $\gamma^n(j) = \gamma^{n+1}(j)$.

d) As $\gamma(j) < j$, if $\gamma(j) \neq j-1$ then $\gamma(j) < j-1 < j$ by a). And by b) $\gamma(j) = \gamma(j-1)$.

In what follows we will denote by

$$n(j) = \min\{n \in \mathbb{Z}_0 : \ \gamma^n(j) = \gamma^{n+1}(j)\}.$$

Proposition 2.2. *With the above notations, let us assume that $k < j$, then we have*:

1) *If $\gamma(j) = k$ then $n(j) = n(k) + 1$.*

2) *If $\gamma(j) = \gamma(k)$ then $n(j) = n(k)$.*

Proof. Note that $\gamma^{n(k)+1}(j) = \gamma^{n(k)}(k) = \gamma^{n(k)+1}(k) = \gamma^{n(k)+2}(j)$, so $n(j) \leq n(k) + 1$.

On the other hand, $\gamma^{n(j)-1}(k) = \gamma^{n(j)}(j) = \gamma^{n(j)+1}(j) = \gamma^{n(j)}(k)$, so $n(j) \geq n(k) + 1$ and we have 1).

The proof of 2) is similar.

Now we have the conditions to define the invariant, which we will call *proximity sequence.*

We define $\delta_\nu = \{\delta_\nu(j)\}_{j\geq 0}$ as follows: $\delta_\nu(0) = 0$ and for $j \geq 1$

$$\delta_\nu(j) = n(j) + 1 - \frac{1}{e_{j-1}}.$$

First of all, we will see that the sequence δ_ν characterizes the proximity relations of ν (or equivalently of $(*)$).

Proposition 2.3. *With the above notations, the following statements are equivalent*:

a) *R_j is free with respect to R_i for all $i < j - 1$.*

b) *$\delta_\nu(j) = 1 - (1/e_{j-1})$.*

c) *$\delta_\nu(j) < 1$.*

Proof. R_j is free with respect to R_i for all $i < j - 1$ if and only if $\gamma(j) = j$, so if and only if $\delta_\nu(j) = 1-(1/e_{j-1})$ or equivalently $\delta_\nu(j) < 1$.

Proposition 2.4. *With the above notations, if $i < j - 1$ the following statements are equivalent*:

a) *R_j is proximate to R_i.*

b) $n(i+1)+1 = \delta_\nu(i+1)+1/e_i = \delta_\nu(k) = \delta(j)$, $i+2 \leq k \leq j$.

Proof. In order to see that a) implies b), we note that R_k is proximate to R_i, $i+1 \leq k \leq j$. So $\gamma(k) = i+1$ for $i+2 \leq k \leq j$. Then, by definition of δ_ν we have $\delta_\nu(i+1)+1/e_i = 1+n(i+1) = n(k) = \delta_\nu(k)$, $i+2 \leq k \leq j$.

On the other hand, by Proposition 2.3 R_j is proximate to R_h, $h < j-1$.

If $h < i$ then by a) implies b) we have that $\delta_\nu(k) = \delta_\nu(j)$ for $h+2 \leq k \leq j$. In particular, $\delta_\nu(i+1) = \delta_\nu(j)$. Yet $\delta_\nu(i+1) < n(i+1)+1 = \delta_\nu(j)$, which is a contradiction.

If $i < h$ then also by a) implies b) $\delta_\nu(h+1) < \delta_\nu(j)$, which is also a contradiction.

So $h = i$ and we have that b) implies a).

Proposition 2.5. *With the above notations, the proximity sequence δ_ν has the following properties*:

1) $\delta_\nu(j) \geq 0$.

2) $\delta_\nu(0) = 0$.

3) $\delta_\nu(1) < 1$.

4) *If $\delta_\nu(j) \geq 1$ then $\delta_\nu(i)$ is an integer.*

5) *If $\delta_\nu(j) < 1$ then $1/(1-\delta_\nu(j))$ is an integer.*

6) *If $\delta_\nu(j+1) < \delta_\nu(j)$ then $\delta_\nu(j+1) < 1$.*

7) $\delta_\nu(j+1) \leq 1 + \delta_\nu(j)$.

Proof. 1) and 2) follow from the definition of δ_ν.

3) As $\gamma(1) = 1$ we have $n(1) = 0$ and $\delta_\nu(1) = 1 - 1/e_1 < 1$.

4) If $\delta_\nu(j) \geq 1$ then $\gamma(j) \neq j$, so R_j is proximate to R_q, with $q < j-1$. So $e_{j-1} = 1$ and $\delta_\nu(j) = n(j)$ is an integer.

5) If $\delta_\nu(j) < 1$ then $\gamma(j) = j$ and $\delta_\nu(j) = 1 - 1/e_{j-1}$, so $e_{j-1} = 1/(1-\delta_\nu(j))$ is an integer.

6) If $\delta_\nu(j+1) < \delta_\nu(j)$ and $\delta_\nu(j+1) \geq 1$, then $\delta_\nu(j) \geq 1$. So R_{j+1} is proximate to R_q, $q < j$ and R_j is proximate to R_h, $h < j-1$. Therefore $\gamma(j+1) = q+1$, $\gamma(j) = h+1$, $e_{j-1} = e_j = 1$, $\delta_\nu(j+1) = n(j+1) = n(q+1)+1$ and $\delta_\nu(j) = n(j) = n(h+1)+1$.

If $q < j-1$ then $q = h$ and $\delta_\nu(j+1) = \delta_\nu(j)$, which is a contradiction.

So $q = j-1$ and $\gamma(j+1) = j$. Then by 2.2 we have $n(j+1) = n(j)+1$ and

$$\delta_\nu(j+1) = n(j+1) + 1 - \frac{1}{e_j} = n(j) + 1 + 1 - \frac{1}{e_{j-1}} = \delta_\nu(j) + 1\,,$$

which is also a contradiction. So $\delta_\nu(j+1) < 1$.

7) We have three possibilities:

• $\gamma(j+1) = j+1$, in this case $\delta_\nu(j+1) < 1$ and always $\delta_\nu(j+1) \leq \delta_\nu(j) + 1$.

• $\gamma(j+1) = j$, in this case we have $n(j+1) = n(j)+1$, see 2.2. So

$$\delta_\nu(j+1) = n(j+1) + 1 - \frac{1}{e_j} \leq \delta_\nu(j) + 1\,.$$

• $\gamma(j+1) = \gamma(j)$, in this case we have $n(j+1) = n(j)$, see 2.2. So

$$\delta_\nu(j+1) = n(j+1) + 1 - \frac{1}{e_j}$$

and

$$\delta_\nu(j) = n(j) + 1 - \frac{1}{e_{j-1}}\,,$$

then

$$\delta_\nu(j+1) = \delta_\nu(j) + \frac{1}{e_{j-1}} - \frac{1}{e_j} < \delta_\nu(j) + 1\,.$$

To finish this section we will compare the proximity sequence with other invariants. Namely, we will see that it defines equivalent data to the refined proximity matrix.

It should be remembered (see [13]) that the *refined proximity matrix* $P_\nu = (p_{ij})_{i,j\geq 0}$ is given by $p_{ii} = 1$,

$$p_{ij} = -\Big[\frac{R_j}{M(R_j)} : \frac{R_i}{M(R_i)}\Big]$$

if R_j is proximate to R_i and $p_{ij} = 0$ for the rest. Note that P_ν is an upper triangular matrix.

Proposition 2.6. *The proximity sequence δ_ν determines the refined proximity matrix P_ν and vice-versa.*

Proof. First we note that $p_{00} = 1$, $p_{10} = 0$ and

$$p_{01} = -\Big[\frac{R_1}{M(R_1)} : \frac{R_0}{M(R_0)}\Big] = -e_0 = \frac{1}{\delta_\nu(1) - 1} .$$

So p_{01} and $\delta_\nu(1)$ are the same data.

Now let us assume that δ_ν determines p_{ij} for $0 \leq i,\ j \leq n$, $n \geq 1$. We have $p_{n+1,n+1} = 1$ and $p_{n+1,k} = 0$ for $0 \leq k \leq n$.

If R_{n+1} is free with respect to R_k for all $k < n$, then $p_{k,n+1} = 0$ for $k < n$ and

$$p_{n,n+1} = -\Big[\frac{R_{n+1}}{M(R_{n+1})} : \frac{R_n}{M(R_n)}\Big] = -e_n = \frac{1}{\delta_\nu(n+1) - 1} .$$

If R_{n+1} is proximate to R_k with $k < n$ then

$$n(k+1) + 1 = \delta_\nu(k+2) = \delta_\nu(k+3) = \cdots = \delta_\nu(n+1) = \delta_\nu(k) + \frac{1}{e_k} .$$

So

$$\frac{1}{\delta_\nu(n+1) - \delta_\nu(k+1)} = p_{k,n+1} .$$

Now

$$p_{n,n+1} = -\Big[\frac{R_{n+1}}{M(R_{n+1})} : \frac{R_n}{M(R_n)}\Big] = -e_n = -1$$

and $p_{j,n+1} = 0$ for $j < n$, and $j \neq k$.

So δ_ν determines P_ν.

Similar reasoning proves that P_ν determines δ_ν.

3. Valuations with a given δ.

Now we will prove the main result of this paper.

Theorem 3.1. *Let $\delta = \{\delta(j)\}_{j\geq 0}$ be a sequence of non-negative rational numbers having the seven properties of Proposition 2.5. Then there is a two dimensional regular noetherian local ring R and a valuation ν centered at $M(R)$ such that its proximity sequence is δ.*

Proof. We consider $R = \mathbb{Q}(t_1, \ldots, t_n, \ldots)\,[[X, Y]]$, where $\mathbb{Q}$ is the field of rational numbers, $\{t_1, \ldots, t_n, \ldots\}$ is a set of indeterminates over $\mathbb{Q}$ and X and Y are two indeterminates over $\mathbb{Q}(t_1, \ldots, t_n, \ldots)$.

We define $e_{j-1} = 1$ if $\delta(j) \geq 1$ and

$$e_{j-1} = \frac{1}{1-\delta(j)}\,, \qquad \text{if } \delta(j) < 1\,.$$

We put $R = R_0$ and

$$R_1 = \left(R\left[\frac{Y}{X}\right]\right)_{(X,(Y/X)^{e_0}-t_1)}\,.$$

Now let us assume that for $n \geq 1$ we have $R = R_0 \subset R_1 \subset \cdots \subset R_n$ such that for any valuation ν' centered at $M(R_n)$ we have that $\delta_{\nu'}(j) = \delta(j)$, for each $0 \leq j \leq n$, and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}\,[t_j^{1/e_{j-1}}]\,, \qquad \text{if } e_{j-1} > 1$$

and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}\,, \qquad \text{if } e_{j-1} = 1,\ 1 \leq j \leq n\,.$$

We have two possibilities:

1) $\delta(n+1) < 1$ (*i.e.* R_{n+1} must be free with respect to R_i for all $i < n$). In this case, let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define

$$R_{n+1} = \left(R_n\left[\frac{y_n}{x_n}\right]\right)_{(x_n,(y_n/x_n)^{e_n}-t_{n+1})}\,.$$

2) $\delta(n+1) \geq 1$ (*i.e.* R_{n+1} must be a satellite). In this case, we have $1 + \delta(n) \geq \delta(n+1) \geq \delta(n)$.

• If $\delta(n+1) > \delta(n)$, then R_{n+1} must be proximate to R_{n-1}. (See 2.4). Let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define

$$R_{n+1} = \left(R_n\left[\frac{x_n}{y_n}\right]\right)_{(y_n,x_n/y_n)}\,.$$

• If $\delta(n+1) = \delta(n)$, then R_{n+1} must be proximate to R_k, with $k < n-1$. (See 2.4). In this case, we can take (x_n, y_n) a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$ and $M(R_k)R_n = x_n^a\, y_n R_n$.

We define

$$R_{n+1} = \left(R_n\left[\frac{y_n}{x_n}\right]\right)_{(x_n, y_n/x_n)} .$$

Now it is easy to see that $R = R_0 \subset R_1 \subset \cdots \subset R_n \subset R_{n+1}$ proves that for any valuation ν' centered at $M(R_{n+1})$ we have $\delta_{\nu'}(j) = \delta(j)$, for each $0 \leq j \leq n+1$, and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}[t_j^{1/e_{j-1}}], \qquad \text{if } e_{j-1} > 1$$

and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}, \qquad \text{if } e_{j-1} = 1,\ 1 \leq j \leq n+1 .$$

Now we will study the case in which $\delta_\nu(j)$ is an integer for all $j \geq 0$.

Theorem 3.2. *Let $\delta = \{\delta(j)\}_{j\geq 0}$ be a sequence of non-negative integers having the seven properties of Proposition* 2.5. *Let R be any regular noetherian local ring of dimension two. Then there is a valuation ν centered at $M(R)$ such that its proximity sequence is δ.*

PROOF. First we put $e_{j-1} = 1$ for all $j \geq 0$, $R = R_0$ and

$$R_1 = \left(R\left[\frac{y}{x}\right]\right)_{(x, y/x)},$$

(x, y) being any basis of $M(R)$.

Now let us assume that we have $R = R_0 \subset R_1 \subset \cdots \subset R_n$ such that for any valuation ν' centered at $M(R_n)$ we have $\delta_{\nu'}(j) = \delta(j)$, for each $0 \leq j \leq n$,

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}, \qquad 1 \leq j \leq n .$$

We have two possibilities:

1) $\delta(n+1) = 0$ (*i.e.* R_{n+1} must be free with respect to R_i, $i < n$). In this case, let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define

$$R_{n+1} = \left(R_n\left[\frac{y_n}{x_n}\right]\right)_{(x_n, (y_n/x_n))} .$$

2) $\delta(n+1) \geq 1$ (*i.e.* R_{n+1} must be a satellite). In this case, we have $1+\delta(n) \geq \delta(n+1) \geq \delta(n)$.

• If $\delta(n+1) > \delta(n)$, then R_{n+1} must be proximate to R_{n-1}. (See 2.4). Let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define
$$R_{n+1} = \left(R_n\left[\frac{x_n}{y_n}\right]\right)_{(y_n, x_n/y_n)} .$$

• If $\delta(n+1) = \delta(n)$, then R_{n+1} must be proximate to R_k, with $k < n-1$. (See 2.4). In this case, we can take (x_n, y_n) a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$ and $M(R_k)R_n = x_n^a\, y_n R_n$.

We define
$$R_{n+1} = \left(R_n\left[\frac{y_n}{x_n}\right]\right)_{(x_n, y_n/x_n)} .$$

Now it is easy to see that $R = R_0 \subset R_1 \subset \cdots \subset R_n \subset R_{n+1}$ proves that for any valuation ν' centered at $M(R_{n+1})$ we have $\delta_{\nu'}(j) = \delta(j)$, for each $0 \leq j \leq n+1$, and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}\,, \qquad 1 \leq j \leq n+1\,.$$

It should be noted that the above theorem is not true if δ is not a sequence of non-negative integers.

For example, let us consider $R = \mathbb{R}[[X,Y]]$, where $\mathbb{R}$ is the field of real numbers. Let $\delta = \{\delta(j)\}_{j\geq 0}$ be the sequence given by $\delta(0) = 0$, $\delta(1) = 2/3$ and $\delta(k) = 0$ for $k \geq 2$. If there is a valuation ν (or equivalently a sequence $(*)$) with δ as the proximity sequence, then $R/M(R)$ is isomorphic to $\mathbb{R}$ and

$$e_0 = \left[\frac{R_1}{M(R_1)} : \frac{R}{M(R)}\right] = 3\,,$$

which is a contradiction.

To finish the paper, we will clarify the relation between the proximity sequence and the classification of plane curve singularities.

For this, we need to assume that R is a complete ring.

Proposition 3.3. *Let us assume that there is a non-zero principal prime ideal J of $R = R_0$ such that $J_n \neq R_n$, J_n being the strict quadratic transform of J in R_n, $n \geq 0$. Then there is a non-negative integer N_0 such that $\delta_\nu(n) = 0$ for $n \geq N_0$.*

PROOF. By [2, Proposition 9.4 and Theorem 10.7], there is an N_0 such that JR_n has a normal crossing for $n \geq N_0$, that is $JR_n = x_n^{a_n} y_n^{b_n} R_n$, where (x_n, y_n) is a basis of R_n and a_n and b_n are non-negative integers.

On the other hand, by definition of strict quadratic transform of J we have

$$JR_n = \Big(\prod_{i=0}^{n-1} (M(R_i))^{d_i} \Big) J_n R_n \ ,$$

where $d_i = \mathrm{Ord}_{R_i}(J_i)$, $0 \leq i \leq n-1$.

We can thus assume that $J_n = y_n R_n$, with $b_n = 1$ and

$$\prod_{i=0}^{n-1} (M(R_i))^{d_i} R_n = x_n^{a_n} R_n \ .$$

Therefore R_n is free for $n \geq N_0$.

As $J_{n+1} \neq R_{n+1}$ we have

$$R_{n+1} = \Big(R_n \Big[\frac{y_n}{x_n} \Big] \Big)_{(x_n, y_n/x_n)} \ ,$$

so $e_n = 1$, for $n \geq N_0$.

Now, we have $\gamma(n) = n$ and $\delta(n) = 0$ for $n \geq N_0$.

Proposition 3.4. *With the above notations, let us assume that there is a non-negative integer N_0 such that $\delta_\nu(n) = 0$ for $n \geq N_0$. Then, there is a non-zero principal prime ideal J of $R = R_0$ such that $J_n \neq R_n$, J_n being the strict quadratic transform of J in R_n for all $n \geq 0$.*

PROOF. As $\delta_\nu(n) = 0$ for $n \geq N_0$ we have that R_n is free and $e_n = 1$, for $n \geq N_0$.

So we can write

$$R_{n+1} = \Big(R_n \Big[\frac{y_n}{x_n} \Big] \Big)_{(x_n, (y_n/x_n)+a)} \ ,$$

where (x_n, y_n) is a basis of $M(R_n)$ and $a_n \in R_{N_0}$, $n \geq N_0$.

Let us consider the ideal

$$J_{N_0} = (y_{N_0} + a_{N_0} \, x_{N_0} + a_{N_0+1} \, x_{N_0}^2 + \cdots) \, (R_{N_0})^* \ ,$$

where $(R_{N_0})^*$ is the complection of R_{N_0}.

It is now easy to see that $J = J_{N_0} \cap R$ is the required non-zero principal prime ideal of R.

It should be noted that Propositions 3.3 and 3.4 characterize the proximity sequences such that there is an analytically irreducible plane curve that "goes through" all the rings of $(*)$.

In addition, it is easy to verify that δ_ν is an invariant of the equisingularity class of such a curve. For a more specific treatment of proximity relations and plane curve singularities refer to [12].

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