Revista Matemática Iberoamericana
Vol. 16, N.º 3, 2000

An elliptic semilinear equation with source term involving boundary measures: the subcritical case

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Abstract. We study the boundary behaviour of the nonnegative solutions of the semilinear elliptic equation in a bounded regular domain Ω of $\mathbb{R}^N$ $(N \geq 2)$,

$$\begin{cases} \Delta u + u^q = 0\,, & \text{in } \Omega\,, \\ u = \mu\,, & \text{on } \partial\Omega\,, \end{cases}$$

where $1 < q < (N+1)/(N-1)$ and μ is a Radon measure on $\partial\Omega$. We give a priori estimates and existence results. They lie on the study of the superharmonic functions in some weighted Marcinkiewicz spaces.

1. Introduction.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a C^2 boundary $\partial\Omega$. Here we study the behaviour near the boundary of the nonnegative solutions of the semilinear elliptic equation

$$-\Delta u = u^q\,, \qquad \text{in } \Omega\,, \tag{1.1}$$

where $1 < q < (N+1)/(N-1)$. By solution of (1.1) we mean any function u such that $u^q \in L^1_{\mathrm{loc}}(\Omega)$ and satisfying the equation in $\mathcal{D}'(\Omega)$.

We denote by $\rho(x)$ the distance from any point $x \in \Omega$ to $\partial\Omega$, and by $B(x,r)$ the open ball of center x and radius $r > 0$. Let $\mathcal{G}$ be the Green function of the Laplacian in Ω, defined on the set $\{(x,y) \in \overline{\Omega} \times \overline{\Omega} : \; x \neq y\}$. Let $\mathcal{P}$ be the Poisson kernel defined on $\Omega \times \partial\Omega$ by $\mathcal{P}(x,z) = -\partial\mathcal{G}(x,z)/\partial n$. We call $\mathcal{M}(\Omega)$ and $\mathcal{M}(\partial\Omega)$ the spaces of Radon measures on Ω and $\partial\Omega$, and $\mathcal{M}^+(\Omega)$ and $\mathcal{M}^+(\partial\Omega)$ the cones of nonnegative ones.

Observe that any nonnegative and superharmonic function U in Ω satisfies $U \in L^1_{\mathrm{loc}}(\Omega)$. From the Herglotz theorem, there exist some unique $\varphi \in \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}^+(\partial\Omega)$ such that U admits the integral representation

$$U = G(\varphi) + P(\mu)\,, \tag{1.2}$$

where, for almost any $x \in \Omega$,

$$G(\varphi)(x) = \int_\Omega \mathcal{G}(x,y)\,d\varphi(y)\,, \qquad P(\mu)(x) = \int_{\partial\Omega} \mathcal{P}(x,z)\,d\mu(z)\,, \tag{1.3}$$

moreover $\int_\Omega \rho\, d\varphi < +\infty$. Reciprocally, for any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho\, d|\varphi| < +\infty$ and $\mu \in \mathcal{M}(\partial\Omega)$, the function U defined by (1.2) lies in $L^1_{\mathrm{loc}}(\Omega)$, and satisfies

$$-\Delta U = \varphi\,, \qquad \text{in } \mathcal{D}'(\Omega)\,,$$

see for example [12]. We shall say that U is the *integral solution* of problem

$$\begin{cases} -\Delta U = \varphi\,, & \text{in } \Omega\,, \\ U = \mu\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

Hence any solution u of (1.1) in Ω satisfies $\int_\Omega \rho\, u^q\, dx < +\infty$, and there exists a measure $\mu \in \mathcal{M}^+(\partial\Omega)$ such that

$$\begin{cases} -\Delta u = u^q\,, & \text{in } \Omega\,, \\ u = \mu\,, & \text{on } \partial\Omega\,, \end{cases} \tag{1.5}$$

in the integral sense. Our aim is to give *a priori estimates* for any solution of equation (1.1) near the boundary, and also to obtain *existence results* for a given measure μ on $\partial\Omega$.

The problem with the other sign

$$\begin{cases} \Delta u = u^q \,, & \text{in } \Omega \,, \\ u = \mu \,, & \text{on } \partial\Omega \,, \end{cases} \tag{1.6}$$

has been studied in [20] and [24] in the subcritical case $1 < q < (N+1)/(N-1)$, and in the supercritical case in [25]. Another approach coming from the probabilistic point of view is done in [14], [15], [22], which gives results in agreement with the previous ones in the case $1 < q \le 2$. It seems that the probabilistic techniques do not apply to our case. Our approach has to be compared to the methods of P. L. Lions used in [23] for the problem of an interior isolated singularity. Our proofs lie essentially on the study of the superharmonic functions in some weighted Marcinkiewicz spaces.

Let us recall some classical results for the interior problem for a better understanding. Let $x_0 \in \Omega$ and consider any nonnegative solution $w \in C^2(\Omega \backslash \{x_0\})$ of the equation

$$-\Delta w = w^q \,, \qquad \text{in } \Omega \backslash \{x_0\} \,. \tag{1.7}$$

When $1 < q < N/(N-2)$, one can give upper and lower bounds by using Serrin's methods of [28], see for example [4, Lemma 6.4]. The precise behaviour of w was obtained in [23]. First $w^q \in L^1_{\mathrm{loc}}(\Omega)$, and there exists some $\gamma \ge 0$ such that

$$-\Delta w = w^q + \gamma\, \delta_0 \,, \qquad \text{in } \mathcal{D}'(\Omega) \,, \tag{1.8}$$

from the Brézis-Lions Lemma, see [9]. Then the following estimates hold near x_0

$$\gamma E(x_0, x) \le w(x) \le \gamma\, E(x_0, x)\,(1 + o(1)) \,, \tag{1.9}$$

when $\gamma > 0$, where E is the fundamental solution of the Laplace equation. And the remaining term can be precised according to the values of N, q, see [23]. The function w can be extended as a function $w \in C^2(\Omega)$ if $\gamma = 0$. Concerning the existence of solutions of (1.8) for a given γ, there exists some finite positive γ^* such that the equation (1.8) admits a solution $w \ge 0$, with $w = 0$ on $\partial\Omega$, if and only if $\gamma \in [0, \gamma^*]$. If $q \ge N/(N-2)$, then $\gamma = 0$, see again [23]. If moreover $q \le (N+2)/(N-2)$, we have the estimate near x_0

$$w(x) \le C\, |x - x_0|^{-2/(q-1)} \tag{1.10}$$

with $C = C(N, q)$, see [17] and [11].

Now let us come back to the boundary problem. As in [20] we can define another concept of solution. Let $C_0^{1,1}(\overline{\Omega})$ be the space of C^1 functions vanishing on $\partial\Omega$ with Lipschitz continuous gradient. For any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho \, d|\varphi| < +\infty$ and any $\mu \in \mathcal{M}(\partial\Omega)$, we shall say that a function U is *weak solution* of problem (1.4) if $U \in L^1(\Omega)$ and

$$\int_\Omega U(-\Delta\xi)\, dx = \int_\Omega \xi \, d\varphi - \int_{\partial\Omega} \frac{\partial\xi}{\partial n}\, d\mu\,, \tag{1.11}$$

for any $\xi \in C_0^{1,1}(\overline{\Omega})$. In Section 2, we first verify that the integral solution coincides with the weak one, and hence is in $L^1(\Omega)$. Then we give regularity results of the general weak solution U of (1.4) in some Marcinkiewicz spaces with a weight of the form ρ^β ($\beta \in \mathbb{R}$). They lie on precise estimates of the Green and Poisson kernel. Up to our knowledge, most of them are new, more especially as the measure φ may be unbounded, and can present an interest in themselves. They are fundamental to obtain a priori estimates and existence results for the problem (1.5), above all in the most delicate case $N/(N-1) \leq q < (N+1)/(N-1)$.

In Section 3, we give an a priori estimate for the function $G(P^q(\mu))$, for any $\mu \in \mathcal{M}^+(\partial\Omega)$:

Theorem 1.1. *Assume that* $1 < q < (N+1)/(N-1)$. *Then* $P^q(\mu) \in L^1(\Omega, \rho\, dx)$ *for any* $\mu \in \mathcal{M}^+(\partial\Omega)$, *and there exists a constant* $K = K(N, \Omega, q)$ *such that,*

$$G(P^q(\mu)) \leq K\, \mu(\partial\Omega)^{q-1}\, P(\mu)\,, \qquad \textit{in } \Omega\,. \tag{1.12}$$

This result is interesting from two points of view. Above all it allows to construct supersolutions, hence to get existence results. Concerning the a priori estimates for (1.5), setting $u = P(\mu) + v$, the function v satisfies

$$v = G(u^q) \leq 2^{q-1}\left(G(P^q(\mu)) + G(v^q)\right), \qquad \text{almost everywhere in } \Omega\,,$$

hence any estimate on $G(P^q(\mu))$ gives informations on v.

In Section 4 we prove our main result, which is an a priori estimate of any solution of (1.5) in terms of the solution $P(\mu)$ of the associated linear problem. It lies on the results of Section 2. It also uses the estimate (1.12), which in fact can be shown almost as a necessary condition of existence of solutions, by using recent techniques of [8].

Theorem 1.2. *Assume that* $1 < q < (N+1)/(N-1)$. *Let* $\mu \in \mathcal{M}^+(\partial\Omega)$, *and* u *be any nonnegative solution of* (1.5). *Then there exists a constant* $C = C(N, q, \Omega, \mu(\partial\Omega))$, *such that*

$$P(\mu) \leq u \leq C\left(P(\mu) + \rho\right), \qquad \text{in } \Omega \tag{1.13}$$

(*and* $u \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ *for any* $\alpha \in (0,1)$ *if* $\mu = 0$).

More precisely, if $\mu = \sigma\,\delta_a$ *for some* $a \in \partial\Omega$ *and* $\sigma > 0$, *then near the point* a,

$$\sigma\, P(\delta_a)(x) \leq u(x) \leq \sigma\, P(\delta_a)(x)\left(1 + O\left(|x-a|^{N+1-(N-1)q}\right)\right). \tag{1.14}$$

This result applies in particular to any solution u of (1.1), such that $u \in C(\overline{\Omega}\backslash\{a\})$ and $u = 0$ on $\partial\Omega\backslash\{a\}$, since its trace is necessarily of the form $\mu = \sigma\,\delta_a$ for some $\sigma \geq 0$. Notice also that in case $q < (N+1)/(N-1)$, Theorem 1.2 extends some a priori estimates of [13], [18] to the case of unbounded boundary data.

In Section 5, we use for proving our second main theorem, which gives existence results.

Theorem 1.3. *Assume that* $1 < q < (N+1)/(N-1)$. *Let* $\mu \in \mathcal{M}^+(\partial\Omega)$ *with* $\mu(\partial\Omega) = 1$, *and* $\sigma \geq 0$. *Then there exists some finite positive* σ^* *such that the problem*

$$\mathcal{S}_\sigma \equiv \begin{cases} -\Delta u = u^q\,, & \text{in } \Omega\,, \\ u = \sigma\,\mu\,, & \text{on } \partial\Omega\,, \end{cases} \tag{1.15}$$

admits a solution if and only if $\sigma \in [0, \sigma^*]$.

The existence of solutions for small σ is a direct consequence of Theorem 1.1. The existence of an interval $[0, \sigma^*]$ is an adaptation of some results of [8].

In conclusion, in the subcritical case we have completely extended the results of an interior punctual singularity to any boundary measure singularity. The next step, that is the study of the case $q \geq (N+1)/(N-1)$, is still open.

Note added in proof. In the moment this article was in printing, we received a preprint of H. Amann and P. Quittner [1], where they consider more general problems with interior and boundary bounded measures, and use duality methods. In case of problem (1.5), they get a regularity result in $W^{1-\varepsilon,1}(\Omega)$ for any $\varepsilon \in (0,1)$, and prove the existence of at least two solutions, under the condition $q < N/(N-1)$.

2. Regularity of the weak solutions.

2.1. About the Green and Poisson kernels.

Here we recall and complete some classical estimates for the Green function and the Poisson kernel. For almost any $y \in \Omega$ and $z \in \partial\Omega$, the functions $\mathcal{G}(\cdot, y)$ and $\mathcal{P}(\cdot, z)$ are the integral solutions of

$$\begin{cases} -\Delta\mathcal{G}(\cdot, y) = \delta_y\,, & \text{in } \Omega\,, \\ \mathcal{G}(\cdot, y) = 0\,, & \text{on } \partial\Omega\,, \end{cases} \qquad \begin{cases} -\Delta\mathcal{P}(\cdot, z) = 0\,, & \text{in } \Omega\,, \\ \mathcal{P}(\cdot, z) = \delta_z\,, & \text{on } \partial\Omega\,, \end{cases}$$

where δ_y, δ_z are the Dirac masses at points $y \in \Omega$, and $z \in \partial\Omega$.

Proposition 2.1. *There exists a constant $c_N = c_N(N, \Omega)$ such that:*

i) *For any $(x, y) \in \Omega \times \Omega$ with $x \neq y$,*

$$\mathcal{G}(x, y) \leq \begin{cases} c_N\,|x-y|^{2-N}\,, & \textit{if } N \geq 3\,, \\ c_2\,(1 + |\ln|x-y||)\,, & \textit{if } N = 2\,, \end{cases} \tag{2.1}$$

$$\mathcal{G}(x, y) \leq c_N\,\rho(x)\,|x-y|^{1-N}\,, \tag{2.2}$$

$$\mathcal{G}(x, y) \leq c_N\,\rho(x)\,\rho(y)\,|x-y|^{-N}\,, \tag{2.3}$$

$$\mathcal{G}(x, y) \leq \begin{cases} c_N\,\dfrac{\rho(x)}{\rho(y)}\,|x-y|^{2-N}\,, & \textit{if } N \geq 3\,, \\ c_2\,\dfrac{\rho(x)}{\rho(y)}\,(1 + |\ln|x-y||)\,, & \textit{if } N = 2\,, \end{cases} \tag{2.4}$$

and

$$|\nabla_x \mathcal{G}(x, y)| \leq c_N\,|x-y|^{1-N}\,, \tag{2.5}$$

$$|\nabla_x \mathcal{G}(x,y)| \le c_N\, \rho(y)\, |x-y|^{-N}, \tag{2.6}$$

$$|\nabla_x \mathcal{G}(x,y)| \le c_N\, \frac{\rho(y)}{\rho(x)}\, |x-y|^{1-N}. \tag{2.7}$$

ii) *For any* $(x,z) \in \Omega \times \partial\Omega$,

$$\begin{aligned} c_N^{-1}\, \rho(x)\, |x-z|^{-N} &\le \mathcal{P}(x,z) \\ &\le c_N\, \rho(x)\, |x-z|^{-N} \\ &\le c_N\, |x-z|^{1-N}, \end{aligned} \tag{2.8}$$

$$|\nabla_x \mathcal{P}(x,z)| \le c_N\, |x-z|^{-N}. \tag{2.9}$$

Proof. But for (2.4) and (2.7), all these estimates are well known. They are deduced from the explicit expression of $\mathcal{G}$ in an half-space, and extended to any C^2 bounded open set. For the lower estimate of (2.8), see [21]. Let us prove (2.4): it is a consequence of (2.1) and (2.2). Indeed that is true in the set $\{\rho(y) \le 2\,\rho(x)\}$. Now suppose $\rho(y) > 2\,\rho(x)$. Let $x^* \in \partial\Omega$ such that $|x - x^*| = \rho(x)$. Then

$$|x-y| \ge |x^*-y| - |x^*-x| \ge \rho(y) - \rho(x) \ge \frac{\rho(y)}{2},$$

hence (2.2) implies if $N \ge 3$

$$\mathcal{G}(x,y) \le c_N\, \frac{\rho(x)}{|x-y|}\, |x-y|^{2-N} \le 2\, c_N\, \frac{\rho(x)}{\rho(y)}\, |x-y|^{2-N},$$

thus (2.4) holds with a new constant c_N; likewise if $N = 2$. Similarly (2.5) and (2.6) imply (2.7).

Remark 2.1. Notice that (2.2) can be deduced from (2.1) and (2.5) when $N \ge 3$. Indeed (2.1) implies (2.2) in the set $\{|x-y| \le 2\,\rho(x)\}$. Now suppose that $|x-y| > 2\,\rho(x)$. Defining x^* as above, we have $[x, x^*) \subset \Omega$, and from (2.5),

$$\begin{aligned} \mathcal{G}(x,y) &= |\mathcal{G}(x,y) - \mathcal{G}(x^*,y)| \\ &\le |x-x^*| \sup_{t\in[x,x^*)} |\nabla_x \mathcal{G}(t,y)| \\ &\le c_N\, \rho(x) \sup_{t\in[x,x^*)} |y-t|^{1-N} \le 2^{N-1}\, c_N\, \rho(x)\, |x-y|^{1-N}, \end{aligned}$$

since $|y-t| \geq |y-x| - |t-x| \geq |y-x| - \rho(x) \geq |x-y|/2$. Hence (2.2) holds. Similarly (2.2) and (2.6) imply (2.3) for any $N \geq 2$. And (2.6) also implies the upper estimate (2.8), since $\mathcal{P}(x,z) = -\partial\mathcal{G}(x,y)/\partial n$ and $\mathcal{G}$ is of class C^1 in the set $\{(x,y) \in \overline{\Omega} \times \overline{\Omega} : \ x \neq y\}$. The estimates (2.2), (2.5) and (2.6) are proved in [33] in the more general framework of a Lyapounov open set. And (2.9) is proved in a $C^{1,\alpha}$ open set in [26].

As a consequence we can compare the integral and weak solutions of (1.4):

Corollary 2.2. *For any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho\, d|\varphi| < +\infty$ and any $\mu \in \mathcal{M}(\partial\Omega)$, a function U is weak solution of problem* (1.4) *if and only if it is given by the representation* (1.2). *Consequently,* (1.4) *has a unique weak solution U in $L^1(\Omega)$, and*

$$\|U\|_{L^1(\Omega)} \leq C\Big(\int_\Omega \rho\, d|\varphi| + \int_{\partial\Omega} d|\mu|\Big)\,, \tag{2.10}$$

for some constant $C = C(N,\Omega)$.

Proof. By considering the positive and negative parts of φ and μ, we can assume that the two measures are nonnegative. Let us prove that the integral solution U is a weak solution. The main point is to prove that $U \in L^1(\Omega)$. From (2.2) and (2.8),

$$\begin{aligned}\int_{\Omega\times\Omega} \mathcal{G}(x,y)\, dx\, d\varphi(y) &\leq c_N \int_\Omega \Big(\int_\Omega |x-y|^{1-N}\, dx\Big)\rho(y)\, d\varphi(y)\\ &\leq C\int_\Omega \rho\, d\varphi\,,\end{aligned} \tag{2.11}$$

with another constant $C = C(N,\Omega)$, and

$$\begin{aligned}\int_{\Omega\times\partial\Omega} \mathcal{P}(x,z)\, dx\, d\mu(z) &\leq c_N \int_\Omega \Big(\int_{\partial\Omega} |x-z|^{1-N}\, dx\Big) d\mu(z)\\ &\leq C\int_{\partial\Omega} d\mu\,,\end{aligned} \tag{2.12}$$

hence $U \in L^1(\Omega)$ and

$$\int_\Omega U(x)\, dx \leq C\Big(\int_\Omega \rho\, d\varphi + \int_{\partial\Omega} d\mu\Big)\,.$$

Now for any $\xi \in C_0^{1,1}(\overline{\Omega})$, we have

$$\mathcal{G}(x,y)\,\Delta\xi(x) \in L^1(\Omega \times \Omega, dx\, d\varphi(y))\,,$$

$$\mathcal{P}(x,z)\,\Delta\xi(x) \in L^1(\Omega \times \partial\Omega, dx\, d\mu(z))\,,$$

from (2.11) and (2.12). Then U is a weak solution from the Fubini theorem, and (2.10) follows. Reciprocally, if U is a weak solution of problem (1.4), then $-\Delta U = \varphi$ in $\mathcal{D}'(\Omega)$, and there exists a unique measure $\widetilde{\mu} \in \mathcal{M}^+(\partial\Omega)$ such that $U = G(\varphi) + P(\widetilde{\mu})$. Then U is a weak solution for the problem with data φ and $\widetilde{\mu}$. Hence for any $\xi \in C_0^{1,1}(\overline{\Omega})$,

$$\int_{\partial\Omega} \frac{\partial\xi}{\partial n}\, d\mu = \int_{\partial\Omega} \frac{\partial\xi}{\partial n}\, d\widetilde{\mu}\,,$$

which implies $\widetilde{\mu} = \mu$. Then U is the integral solution of (1.4), and (2.10) follows again.

REMARK 2.2. Thus for any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho\, d|\varphi| < +\infty$ and any $\mu \in \mathcal{M}(\partial\Omega)$, the problem (1.4) is well posed in $L^1(\Omega)$. We find again in a very short way the result of [7] where φ is a measurable function with $\rho\,\varphi \in L^1(\Omega)$ and $\mu \in L^1(\partial\Omega)$.

The lower estimate of the Poisson kernel (2.8) also shows that the value $q = (N+1)/(N-1)$ is a natural barrier for the problem (1.5):

Corollary 2.3. *Assume that $q \geq (N+1)/(N-1)$. Then problem* (1.5) *has no solution for a positive measure concentrated at some point $a \in \partial\Omega$.*

PROOF. For any $\mu \in \mathcal{M}^+(\partial\Omega)$, if (1.5) has a solution u, then $u \geq P(\mu)$, but $\rho\, u^q \in L^1(\Omega)$, hence also $\rho\, P^q(\mu)$. Suppose that $\mu > 0$ with $\operatorname{supp}\mu = \{a\}$, that means $\mu = \sigma\,\delta_a$ for some $\sigma > 0$. From (2.8), we have

$$\begin{aligned}\int_\Omega P^q(\delta_a)\,\rho\,dx &\geq c_N^{-q} \int_\Omega |x-a|^{-Nq}\,\rho^{q+1}\,dx \\ &\geq 2^{-(q+1)} c_N^{-q} \int_{\{x\in\Omega | \rho(x)\geq |x-a|/2\}} |x-a|^{q+1-Nq}\,dx\,.\end{aligned}$$

But the set $\{x \in \Omega :\ \rho(x) \geq |x-a|/2\}$ contains the intersection of a cone of vertex a and angle $\pi/3$ with a small ball of center a. Hence the integral is divergent, since $q \geq (N+1)/(N-1)$. Then we arrive to a contradiction.

2.2. Regularity of $G(\varphi)$ and $P(\mu)$.

Now we are going to complete the estimate (2.10) by much more precise estimates of the functions $G(\varphi)$ and $P(\mu)$ in Marcinkiewicz weighted spaces, with a power of the distance ρ as a weight function. Let us recall their definition. For any $k \in \mathbb{R}$ with $k \geq 1$, and any positive weight function $\eta \in C(\Omega)$, we denote by $L^k(\Omega, \eta\, dx)$ the space of measurable functions v on Ω such that

$$\|v\|_{L^k(\Omega,\eta\, dx)} = \Big(\int_\Omega |v|^k\, \eta\, dx\Big)^{1/k} < +\infty\,,$$

and the Marcinkiewicz space $M^k(\Omega, \eta\, dx)$ is the space of measurable functions v on Ω such that

$$\sup_{\lambda>0} \lambda \Big(\int_{\{x\in\Omega:\, |v(x)|\geq\lambda\}} \eta\, dx\Big)^{1/k} < +\infty\,.$$

And for any $k > 1$, $M^k(\Omega, \eta\, dx)$ is also the normed space of the v such that

$$\|v\|_{M^k(\Omega,\eta\, dx)} = \sup \frac{\displaystyle\int_\omega |v|\, \eta\, dx}{\Big(\displaystyle\int_\omega \eta\, dx\Big)^{1-1/k}} < +\infty\,,$$

where the supremum is taken over the measurable subsets ω of Ω such that $\int_\omega \eta\, dx$ is finite. We have $L^k(\Omega, \eta\, dx) \subset M^k(\Omega, \eta\, dx)$. If $\eta \in L^1(\Omega)$ (in particular $\eta = \rho^\beta$ with $\beta > -1$), then

$$M^k(\Omega, \eta\, dx) \subset L^m(\Omega, \eta\, dx)\,, \qquad \text{for any } m \in [1, k)\,.$$

If $\eta \equiv 1$, $L^k(\Omega, \eta\, dx) = L^k(\Omega)$, and $M^k(\Omega, \eta\, dx) = M^k(\Omega)$.

Recall that the solution w of the interior problem (1.7) lies in $M^{N/(N-2)}_{\mathrm{loc}}(\Omega)$ if $N \geq 3$, and in $L^p_{\mathrm{loc}}(\Omega)$ for any $p \geq 1$ if $N = 2$, see [9]. On the other part, from [20] and Corollary 2.2, for any nonnegative $\mu \in L^1(\partial\Omega)$, the function $P(\mu)$ lies in $M^{N/(N-1)}(\Omega) \cap M^{(N+1)/N-1)}(\Omega, \rho\, dx)$. The following Lemma extends the techniques used in [2] and [20]:

Lemma 2.4. *Let ν be a nonnegative bounded Radon measure on $D = \Omega$ or $\partial\Omega$, and $\eta \in C(\Omega)$ be a positive weight function. Let $\mathcal{H}$ be a continuous nonnegative function on $\{(x,t) \in \Omega \times D :\ x \neq t\}$. For any $\lambda > 0$, we set*

$$A_\lambda(t) = \{x \in \Omega\backslash\{t\} :\ \mathcal{H}(x,t) > \lambda\}\,, \tag{2.13}$$

$$m_\lambda(t) = \int_{A_\lambda(t)} \eta\, dx\,. \tag{2.14}$$

Suppose that for some $C \geq 0$ and $k > 1$

$$m_\lambda(t) \leq C\,\lambda^{-k}\,, \qquad \textit{for all } \lambda > 0\,. \tag{2.15}$$

Then the function

$$x \in \Omega \longmapsto H(x) = \int_D \mathcal{H}(x,t)\, d\nu(t) \tag{2.16}$$

is in $M^k(\Omega, \eta\, dx)$ and

$$\|H\|_{M^k(\Omega,\eta\, dx)} \leq \Big(1 + \frac{k}{k-1}\, C\Big)\nu(D)\,. \tag{2.17}$$

PROOF. Let ω be any measurable subset of Ω such that $\int_\omega \eta\, dx$ is finite. Then for any $\lambda > 0$, and any $t \in D$,

$$\int_\omega \mathcal{H}(x,t)\,\eta(x)\, dx \leq \int_{A_\lambda(t)} \mathcal{H}(x,t)\,\eta(x)\, dx + \lambda \int_\omega \eta(x)\, dx\,,$$

with

$$\int_{A_\lambda(t)} \mathcal{H}(x,t)\,\eta(x)\, dx = -\int_\lambda^{+\infty} \theta\, dm_\theta(t) = \lambda\, m_\lambda(t) + \int_\lambda^{+\infty} m_\theta(t)\, d\theta$$
$$\leq \frac{k}{k-1}\, C\,\lambda^{1-k}\,.$$

Choosing $\lambda = (\int_\omega \eta\, dx)^{-1/k}$, we get

$$\int_\omega \mathcal{H}(x,t)\,\eta(x)\, dx \leq \Big(1 + \frac{k}{k-1}\, C\Big)\Big(\int_\omega \eta\, dx\Big)^{1-1/k}\,,$$

and by integration over D with respect to the measure ν,

$$\int_\omega H(x)\,\eta(x)\,dx = \int_\omega \int_D \mathcal{H}(x,t)\,\eta(x)\,dx\,d\nu(t)$$
$$\le \Big(1+\frac{k}{k-1}\,C\Big)\nu(D)\Big(\int_\omega \eta\,dx\Big)^{1-1/k},$$

hence the conclusion.

Let us first complete the estimates of [20] for the function $P(\mu)$:

Theorem 2.5. *For any $\mu \in \mathcal{M}(\partial\Omega)$, let $\Psi = P(\mu)$ be the solution of the problem*

$$\begin{cases} -\Delta\Psi = 0\,, & \textit{in } \Omega\,,\\ \Psi = \mu\,, & \textit{on } \partial\Omega\,. \end{cases} \tag{2.18}$$

Then

$$\Psi \in M^{(N+\beta)/(N-1)}(\Omega, \rho^\beta\,dx)\,, \tag{2.19}$$

for any $\beta > -1$, and

$$|\nabla\Psi| \in M^{(N+\gamma)/N}(\Omega, \rho^\gamma\,dx)\,, \tag{2.20}$$

for any $\gamma > 0$. Moreover there exists constants $C = C(\Omega, N, \beta) > 0$ and $C' = C'(\Omega, N, \gamma) > 0$ such that

$$\|\Psi\|_{M^{(N+\beta)/(N-1)}(\Omega,\rho^\beta\,dx)} \le C\,|\mu|\,(\partial\Omega)\,, \tag{2.21}$$

$$\|\,|\nabla\Psi|\,\|_{M^{(N+\gamma)/N}(\Omega,\rho^\gamma\,dx)} \le C'\,|\mu|\,(\partial\Omega)\,. \tag{2.22}$$

Proof. First step: estimate of the function. We can suppose that μ is nonnegative. Let β be a real parameter. We shall apply Lemma 2.4 with

$$D = \partial\Omega\,, \quad \eta = \rho^\beta\,, \quad \nu = \mu\,, \quad \text{and} \quad \mathcal{H}(x,t) = \mathcal{P}(x,t)\,. \tag{2.23}$$

From (2.8), for any $t \in \partial\Omega$, and any $\lambda > 0$, and any $x \in A_\lambda(t)$,

$$\lambda \le c_N\,\rho(x)\;|x-t|^{-N} \le c_N\,|x-t|^{1-N}\,.$$

Hence if $\beta \geq 0$,

$$\begin{aligned}
m_\lambda(t) &\leq \int_{B(t,(c_N/\lambda)^{1/(N-1)})} \rho^\beta \, dx \\
&\leq \int_{B(t,(c_N/\lambda)^{1/(N-1)})} |x-t|^\beta \, dx \\
&\leq C\,\lambda^{-(N+\beta)/(N-1)} .
\end{aligned}$$

If $\beta < 0$, then

$$\begin{aligned}
m_\lambda(t) &\leq \int_{B(t,(c_N/\lambda)^{1/(N-1)})} \rho^\beta \, dx \leq \int_{B(t,(c_N/\lambda)^{1/(N-1)})} \Big(\lambda \, \frac{|x-t|^N}{c_N}\Big)^\beta \, dx \\
&\leq C\,\lambda^\beta \int_0^{(c_N/\lambda)^{1/(N-1)}} r^{N-1+N\beta} \, dr \\
&\leq C\,\lambda^{-(N+\beta)/(N-1)} ,
\end{aligned}$$

under the condition $\beta > -1$. Then Lemma 2.4 gives (2.19) and (2.21).

SECOND STEP: ESTIMATE OF THE GRADIENT. Let $i \in \{1, \ldots, N\}$. Here we use Lemma 2.4 with

$$D = \partial\Omega\,, \quad \eta = \rho^\gamma\,, \quad \nu = \mu\,, \quad \text{and} \quad \mathcal{H}(x,t) = \frac{\partial \mathcal{P}(x,t)}{\partial x_i}\,. \tag{2.24}$$

From (2.9), for any $t \in \partial\Omega$, and any $\lambda > 0$, and any $x \in A_\lambda(t)$,

$$\lambda \leq c_N \, |x-t|^{-N} .$$

Then if $\gamma \geq 0$,

$$m_\lambda(t) \leq \int_{B(t,(c_N/\lambda)^{1/N})} \rho^\gamma \, dx \leq \int_{B(t,(c_N/\lambda)^{1/N})} |x-t|^\gamma \, dx \leq C\,\lambda^{-(N+\gamma)/N} .$$

Hence if $\gamma > 0$, the function

$$Q_i(x) = \int_{\partial\Omega} \frac{\partial \mathcal{P}(x,t)}{\partial x_i} \, d\mu(t)$$

lies in $M^{(N+\gamma)/N}(\Omega, \rho^\gamma \, dx)$. But $Q_i = \partial P(\mu)/\partial x_i$ from the derivation theorem, so that (2.20) and (2.22) hold.

Let us now give precise estimates of $G(\varphi)$. They are one of the keys of Theorem 1.2.

Theorem 2.6. *For any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho^\alpha \, d|\varphi| < +\infty$, with $\alpha \in [0,1]$, let $\Phi = G(\varphi)$ be the solution of problem*

$$\begin{cases} -\Delta\Phi = \varphi\,, & \text{in } \Omega\,, \\ \Phi = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{2.25}$$

i) *Then if $N \geq 3$,*

$$\Phi \in M^{(N+\beta)/(N-2+\alpha)}(\Omega, \rho^\beta \, dx)\,, \tag{2.26}$$

for any $\beta \in (-N/(N+\alpha-1), \alpha N/(N-2))$ if $\alpha \neq 0$, for any $\beta \in (-N/(N-1), 0]$ if $\alpha = 0$. In any case, there exists some $C = C(\Omega, N, \alpha, \beta) > 0$ such that

$$\|\Phi\|_{M^{(N+\beta)/(N-2+\alpha)}(\Omega,\rho^\beta dx)} \leq C \int_\Omega \rho^\alpha \, d|\varphi|\,. \tag{2.27}$$

If $N = 2$, and $\alpha \in (0,1]$,

$$\Phi \in M^{(2+\beta-\varepsilon)/\alpha}(\Omega, \rho^\beta \, dx)\,, \tag{2.28}$$

for any $\beta \in (-2/(1+\alpha), +\infty)$ and $\varepsilon > 0$ small enough; if $\alpha = 0$, then

$$\Phi \in M^p(\Omega, \rho^\beta \, dx)\,, \tag{2.29}$$

for any $\beta \in (-2, 0]$ and $p \in (\max\{1, -\beta\}, +\infty)$; with similar continuity properties in those spaces.

ii) *For any $N \geq 2$,*

$$\begin{aligned} &|\nabla\Phi| \in M^{(N+\gamma)/(N-1+\alpha)}(\Omega, \rho^\gamma \, dx)\,, \\ &\frac{\Phi}{\rho} \in M^{(N+\gamma)/(N-1+\alpha)}(\Omega, \rho^\gamma \, dx)\,, \end{aligned} \tag{2.30}$$

for any $\gamma \in [0, \alpha N/(N-1))$ if $\alpha \in (0,1)$, any $\gamma \in (0, N/(N-1))$ if $\alpha = 1$, and $\gamma = 0$ if $\alpha = 0$, and there exists some $C' = C(\Omega, N, \alpha, \gamma) > 0$ such that

$$\Big\| |\nabla\Phi| + \frac{\Phi}{\rho} \Big\|_{M^{(N+\gamma)/(N-1+\alpha)}(\Omega,\rho^\gamma dx)} \leq C' \int_\Omega \rho^\alpha \, d|\varphi|\,. \tag{2.31}$$

PROOF. FIRST STEP: ESTIMATE OF THE FUNCTION. Here also we can assume that φ is nonnegative. Let $\alpha \in [0,1]$ be fixed, and β be a real parameter. We have

$$G(\varphi)(x) = \int_\Omega \frac{\mathcal{G}(x,y)}{\rho^\alpha(y)} \, \rho^\alpha(y) \, d\varphi(y) \, .$$

We shall apply Lemma 2.4 with

$$D = \Omega \, , \quad \eta = \rho^\beta \, , \quad \nu = \rho^\alpha \, \varphi \, , \quad \text{and} \quad \mathcal{H}(x,t) = \frac{\mathcal{G}(x,t)}{\rho^\alpha(t)} \, . \tag{2.32}$$

i) First assume $N \geq 3$. From (2.1) and (2.2), for any $x, t \in \Omega$ with $x \neq t$,

$$\begin{aligned} \mathcal{G}(x,t) &\leq c_N \, |x-t|^{(2-N)(1-\alpha)} \, (\rho(t) \, |x-t|^{1-N})^\alpha \\ &\leq c_N \, \rho^\alpha(t) \, |x-t|^{2-N-\alpha} \, . \end{aligned} \tag{2.33}$$

Moreover, from (2.1) and (2.4),

$$\begin{aligned} \mathcal{G}(x,t) &\leq c_N \, |x-t|^{(2-N)(1-\alpha)} \Big(\frac{\rho(t)}{\rho(x)} \, |x-t|^{2-N} \Big)^\alpha \\ &\leq c_N \, \frac{\rho^\alpha(t)}{\rho^\alpha(x)} \, |x-t|^{2-N} \, , \end{aligned} \tag{2.34}$$

and from (2.2) and (2.3),

$$\begin{aligned} \mathcal{G}(x,t) &\leq c_N \, (\rho(x) \, |x-t|^{1-N})^{(1-\alpha)} \, (\rho(x) \, \rho(t) \, |x-t|^{-N})^\alpha \\ &\leq c_N \, \rho(x) \, \rho^\alpha(t) \, |x-t|^{1-N-\alpha} \, . \end{aligned} \tag{2.35}$$

Then for any $\lambda > 0$, and any $x \in A_\lambda(t)$, from (2.33)

$$\lambda \leq c_N \, |x-t|^{2-N-\alpha} \, , \tag{2.36}$$

and from (2.34) and (2.35),

$$\rho^\alpha(x) \leq \frac{c_N}{\lambda} \, |x-t|^{2-N} \qquad \text{and} \qquad \rho(x) \geq \frac{\lambda}{c_N} \, |x-t|^{N-1+\alpha} \, . \tag{2.37}$$

First suppose that $\alpha > 0$ and $\beta > 0$. Then

$$\begin{aligned} m_\lambda(t) &\le \int_{B(t,(c_N/\lambda)^{1/(N-2+\alpha)})} \rho^\beta\,dx \\ &\le \int_{B(t,(c_N/\lambda)^{1/(N-2+\alpha)})} \Big(\Big(\frac{c_N}{\lambda}\Big)|x-t|^{2-N}\Big)^{\beta/\alpha}\,dx \\ &\le C\,\lambda^{-\beta/\alpha}\int_0^{(c_N/\lambda)^{1/(N-2+\alpha)}} r^{N-1-(N-2)\beta/\alpha}\,dr \\ &\le C\,\lambda^{-(N+\beta)/(N-2+\alpha)}\,, \end{aligned}$$

under the condition $\beta < \alpha N/(N-2)$. Now suppose that $\beta \le 0$. Then

$$\begin{aligned} m_\lambda(t) &\le \int_{B(t,(c_N/\lambda)^{1/(N-2+\alpha)})} \rho^\beta\,dx \\ &\le \int_{B(t,(c_N/\lambda)^{1/(N-2+\alpha)})} \Big(\Big(\frac{\lambda}{c_N}\Big)|x-t|^{N-1+\alpha}\Big)^{\beta}\,dx \\ &\le C\,\lambda^{\beta}\int_0^{(c_N/\lambda)^{1/(N-2+\alpha)}} r^{N-1+(N-1+\alpha)\beta}\,dr \\ &\le C\,\lambda^{-(N+\beta)/(N-2+\alpha)}\,, \end{aligned}$$

under the condition $\beta > -N/(N-1+\alpha)$. Hence Lemma 2.4 applies and gives the estimates (2.26) and (2.27) for Φ.

ii) Now assume $N = 2$. Then (2.33) and (2.34) are replaced by

$$\mathcal{G}(x,t) \le c_2\,(1+|\ln|x-t||)^{(1-\alpha)}\,(\rho(t)\,|x-t|^{-1})^\alpha\,, \tag{2.38}$$

$$\mathcal{G}(x,t) \le c_2\,\frac{\rho^\alpha(t)}{\rho^\alpha(x)}\,(1+|\ln|x-t||)\,, \tag{2.39}$$

and (2.35) is still valid. Then (2.36) and (2.37) become

$$\lambda \le c_2\,|x-t|^{-\alpha}(1+|\ln|x-t||)^{(1-\alpha)}\,, \tag{2.40}$$

$$\rho^\alpha(x) \le \frac{c_2}{\lambda}(1+|\ln|x-t||) \quad \text{and} \quad \rho(x) \ge \frac{\lambda}{c_2}\,|x-t|^{1+\alpha}\,. \tag{2.41}$$

First suppose $\alpha \in (0,1]$. Notice that (2.40) and (2.41) imply for any $\varepsilon > 0$,

$$\lambda \le C_\varepsilon\,|x-t|^{-\alpha-\varepsilon}\,, \qquad \rho^\alpha(x) \le \frac{C_\varepsilon}{\lambda}\,|x-t|^{-\varepsilon}\,,$$

with $C_\varepsilon = C_\varepsilon(N, \Omega, \varepsilon)$, since Ω is bounded. If $\beta > 0$, then

$$\begin{aligned} m_\lambda(t) &\leq \int_{B(t,(C_\varepsilon/\lambda)^{1/(\alpha+\varepsilon)})} \rho^\beta \, dx \\ &\leq \int_{B(t,(C_\varepsilon/\lambda)^{1/(\alpha+\varepsilon)})} \Big(\Big(\frac{C_\varepsilon}{\lambda}\Big)|x-t|^{-\varepsilon}\Big)^{\beta/\alpha} dx \end{aligned}$$

so that for any small $\varepsilon > 0$,

$$m_\lambda(t) \leq C'_\varepsilon \, \lambda^{-(2+\beta-\varepsilon)/\alpha} ,$$

with $C'_\varepsilon = C'_\varepsilon(\Omega, \alpha, \beta, \varepsilon)$, hence $\Phi \in M^{(2+\beta-\varepsilon)/\alpha}(\Omega, \rho^\beta \, dx)$. In case $\beta \leq 0$, we find

$$\begin{aligned} m_\lambda(t) &\leq \int_{B(t,(C_\varepsilon/\lambda)^{1/(\alpha+\varepsilon)})} \rho^\beta \, dx \\ &\leq C_\varepsilon \, \lambda^\beta \int_0^{(C_\varepsilon/\lambda)^{1/(\alpha+\varepsilon)}} r^{1+(1+\alpha)\beta} \, dr \\ &\leq C''_\varepsilon \, \lambda^{\beta-(2+(1+\alpha)\beta)/(\alpha+\varepsilon)} , \end{aligned}$$

with $C''_\varepsilon = C''_\varepsilon(\Omega, \alpha, \beta, \varepsilon)$, under the condition $\beta > -2/(1+\alpha)$, hence the same conclusion holds. Now suppose $\alpha = 0$ and $-2 < \beta \leq 0$. Observe that the condition (2.40) implies $|x - t| \leq C_\Omega \, e^{-\lambda/c_2}$, with for example $C_\Omega = e\,(1 + (\operatorname{diam}\Omega)^2)$. Then from (2.41),

$$\begin{aligned} m_\lambda(t) &\leq \int_{B(t,C_\Omega e^{-\lambda/c_2})} \rho^\beta \, dx \\ &\leq C \, \lambda^\beta \int_0^{C_\Omega e^{-\lambda/c_2}} r^{1+\beta} \, dr \\ &\leq C \, \lambda^\beta \, e^{-(\beta+2)\lambda/c_2} \\ &\leq C_p \, \lambda^{-p} , \end{aligned}$$

for any $p > -\beta$, hence $\Phi \in M^p(\Omega, \, \rho^\beta \, dx)$ for any $p \in (\max\{1, -\beta\}, +\infty)$.

Second step: estimate of the gradient and of Φ/ρ. In the same way, we take

$$\begin{aligned} &D = \Omega \,, \qquad \eta = \rho^\gamma \,, \qquad \nu = \rho^\alpha \varphi \,, \qquad \text{and} \\ &\mathcal{H}(x,t) = \frac{\partial \mathcal{G}(x,t)}{\partial x_i} \, \frac{1}{\rho^\alpha(t)} \qquad \text{or} \qquad \mathcal{H}(x,t) = \frac{\mathcal{G}(x,t)}{\rho(x)} \, \frac{1}{\rho^\alpha(t)} \,. \end{aligned} \tag{2.42}$$

As above, for any $N \geq 2$, from (2.5) and (2.6), for any $x, t \in \Omega$ with $x \neq t$,

$$\Big|\frac{\partial \mathcal{G}(x,t)}{\partial x_i}\Big| \leq c_N\, \rho^\alpha(t)\, |x-t|^{1-N-\alpha}\,, \tag{2.43}$$

and similarly from (2.2) and (2.3),

$$\frac{\mathcal{G}(x,t)}{\rho(x)} \leq c_N\, \rho^\alpha(t)\, |x-t|^{1-N-\alpha}\,. \tag{2.44}$$

And from (2.5) and (2.7),

$$\Big|\frac{\partial \mathcal{G}(x,t)}{\partial x_i}\Big| \leq c_N\, \frac{\rho^\alpha(t)}{\rho^\alpha(x)}\, |x-t|^{1-N}\,, \tag{2.45}$$

and similarly from (2.2), which is symmetrical in x and y,

$$\frac{\mathcal{G}(x,t)}{\rho(x)} \leq \frac{\rho^\alpha(t)}{\rho^\alpha(x)}\, |x-t|^{1-N}\,. \tag{2.46}$$

Then for any $\lambda > 0$, and any $x \in A_\lambda(t)$, from (2.43) and (2.45), or from (2.44) and (2.46),

$$\lambda \leq c_N\, |x-t|^{1-N-\alpha}\,, \qquad \text{and} \qquad \rho^\alpha(x) \leq \frac{c_N}{\lambda}\, |x-t|^{1-N}\,.$$

First assume that $\alpha > 0$ and $\gamma \geq 0$. Then

$$\begin{aligned} m_\lambda(t) &\leq \int_{B(t,(c_N/\lambda)^{1/(N-1+\alpha)})} \rho^\gamma\, dx \\ &\leq \int_{B(t,(c_N/\lambda)^{1/(N-1+\alpha)})} \Big(\Big(\frac{c_N}{\lambda}\Big)|x-t|^{1-N}\Big)^{\gamma/\alpha} dx \\ &\leq C\,\lambda^{-\gamma/\alpha} \int_0^{(c_N/\lambda)^{1/(N-1+\alpha)}} r^{N-1-(N-1)\gamma/\alpha}\, dr \\ &\leq C\,\lambda^{-(N+\gamma)/(N-1+\alpha)}\,, \end{aligned}$$

under the condition $\gamma < \alpha N/(N-1)$. Then Lemma 2.4 applies if $\gamma > \alpha - 1$, that is $\gamma > 0$ in case $\alpha = 1$. Now assume $\alpha = \gamma = 0$. Then we get directly

$$m_\lambda(t) \leq \int_{B(t,(c_N/\lambda)^{1/(N-1)})} dx \leq C\,\lambda^{-N/(N-1)}\,.$$

Hence the functions Φ/ρ and

$$R_i(x) = \int_{\partial\Omega} \frac{\partial \mathcal{G}(x,y)}{\partial x_i}\, d\varphi(y)$$

lie in $M^{(N+\gamma)/(N-1+\alpha)}(\Omega, \rho^\gamma\, dx)$, and satisfy

$$\left\| \frac{\Phi}{\rho} + |R_1| + \cdots + |R_N| \right\|_{M^{(N+\gamma)/(N-1+\alpha)}(\Omega,\, \rho^\gamma\, dx)} \le C' \int_\Omega \rho^\alpha\, d|\varphi|\,.$$

In order to obtain (2.30) and (2.31), it remains to prove that $\partial G(\varphi)/\partial x_i = R_i$ in $\mathcal{D}'(\Omega)$. The result is true when $\varphi \in L^\infty(\Omega)$: in that case, following the proof of [19, Lemma 4.1], we have $G(\varphi) \in C^1(\Omega)$ and $\partial G(\varphi)/\partial x_i = R_i$ in Ω. In the general case where $\varphi \in \mathcal{M}^+(\Omega)$ with

$$\int_\Omega \rho^\alpha\, d\varphi < +\infty\,,$$

we consider a sequence of nonnegative functions $f_n \in L^\infty(\Omega)$, bounded in $L^1(\Omega, \rho^\alpha\, dx)$, converging weakly to φ, Then the sequence $\{G(f_n)\}$ converges in $L^1(\Omega)$ to $G(\varphi)$ from (2.10). And $\{\partial G(f_n)/\partial x_i\}$ converges in $L^1(\Omega, \rho\, dx)$ to R_i. Hence $\partial G(\varphi)/\partial x_i = R_i$ in $\mathcal{D}'(\Omega)$, and in fact in $L^1_{\mathrm{loc}}(\Omega)$.

Remark 2.3. As a consequence, we get estimates of Ψ and Φ in weighted Sobolev spaces. Recall that for any $k > 1$, and any real γ,

$$W^{1,k}(\Omega, \rho^\gamma\, dx) = \{v \in L^k(\Omega, \rho^\gamma\, dx) :\ |\nabla v| \in L^k(\Omega, \rho^\gamma\, dx)\}\,,$$

endowed with the norm

$$\|v\|_{W^{1,k}(\Omega,\rho^\gamma dx)} = \|v\|_{L^k(\Omega,\, \rho^\gamma dx)} + \|\, |\nabla v|\, \|_{L^k(\Omega,\rho^\gamma dx)}\ ,$$

and $W^{1,k}_0(\Omega,\, \rho^\gamma\, dx)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,k}(\Omega,\, \rho^\gamma\, dx)$. From [16], it is also given by

$$W^{1,k}_0(\Omega, \rho^\gamma\, dx) = \{u \in L^k(\Omega, \rho^{\gamma-k}\, dx) :\ |\nabla u| \in L^k(\Omega, \rho^\gamma\, dx)\}\,,$$

if $k \neq \gamma + 1$, and

$$W^{1,k}_0(\Omega, \rho^\gamma\, dx) = \Big\{u \in L^k\Big(\Omega, \rho^{\gamma-k}\Big(\ln\Big(\frac{R}{\rho}\Big)\Big)^{-k} dx\Big) :$$
$$|\nabla u| \in L^k(\Omega, \rho^\gamma\, dx)\Big\}\,,$$

if $k = \gamma + 1$, where $R > \max\{e^2, \operatorname{diam}\Omega\}$. And

$$W_0^{1,k}(\Omega, \rho^\gamma\, dx) = W^{1,k}(\Omega, \rho^\gamma\, dx)\,, \qquad \text{if } \gamma + 1 \leq 0 \text{ or } \gamma + 1 > k\,.$$

Then one verifies that, for any $\mu \in \mathcal{M}(\partial\Omega)$,

$$\Psi = P(\mu) \in W^{1,s}(\Omega, \rho^\gamma\, dx)\,, \tag{2.47}$$

for any $\gamma > 0$ and $s \in [1, (N+\gamma)/N)$. And for any $\alpha \in [0,1]$ and any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho^\alpha\, d|\varphi| < +\infty$,

$$\Phi = G(\varphi) \in W_0^{1,s}(\Omega, \rho^\gamma\, dx)\,, \tag{2.48}$$

for any $\gamma \in [0, N\alpha/(N-1))$ if $\alpha \in (0,1)$, any $\gamma \in (0, N/(N-1))$ if $\alpha = 1$, and $\gamma = 0$ if $\alpha = 0$, and for any $s \in [1, (N+\gamma)/(N-1+\alpha))$. And P and G map bounded subsets into bounded sets in those spaces. For any measure $\varphi \in \mathcal{M}(\Omega)$, one finds again the well-known result $\Phi = G(\varphi) \in W_0^{1,s}(\Omega)$ for any $s \in [1, N/(N-1))$.

If $\alpha \in (0,1)$, we can improve the estimates (2.26) and (2.30) by using *interpolation in weighted spaces*. These results will not be used in the sequel, but they deserve to be mentionned.

Theorem 2.7. *Assume that $\alpha \in (0,1)$. Then for any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho^\alpha\, d|\varphi| < +\infty$, and any $N \geq 3$,*

$$G(\varphi) \in L^{(N+\beta)/(N-2+\alpha)}(\Omega, \rho^\beta\, dx)\,, \tag{2.49}$$

for any $\beta \in (-N/(N-1+\alpha), \alpha N/(N-2))$, and

$$\|G(\varphi)\|_{L^{(N+\beta)/(N-2+\alpha)}(\Omega, \rho^\beta\, dx)} \leq C \int_\Omega \rho^\alpha\, d|\varphi|\,. \tag{2.50}$$

And for any $N \geq 2$,

$$|\nabla G(\varphi)| + \frac{G(\varphi)}{\rho} \in L^{(N+\gamma)/(N-1+\alpha)}(\Omega, \rho^\gamma\, dx)\,, \tag{2.51}$$

for any $\gamma \in [0, \alpha N/(N-1))$, and

$$\left\| |\nabla G(\varphi)| + \frac{G(\varphi)}{\rho} \right\|_{L^{(N+\gamma)/(N-1+\alpha)}(\Omega, \rho^\gamma\, dx)} \leq C' \int_\Omega \rho^\alpha\, d|\varphi|\,, \tag{2.52}$$

hence

$$G(\varphi) \in W_0^{1,(N+\gamma)/(N-1+\alpha)}(\Omega, \rho^\gamma\, dx)\,, \tag{2.53}$$

for any $\gamma \in [0, \alpha N/(N-1))$, *and*

$$\|G(\varphi)\|_{W_0^{1,(N+\gamma)/(N-1+\alpha)}(\Omega,\rho^\gamma dx)} \le C \int_\Omega \rho^\alpha\, d|\varphi|\,. \tag{2.54}$$

PROOF. For a given $k > 0$, and any $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_{i,k} > k+\alpha_i - N$, $\gamma_i > \alpha_i - 1$, for $i = 1, 2$, and any $\theta \in (0,1)$ we can verify that the spaces of interpolation are given by

$$[L^1(\Omega, \rho^{\alpha_1}\, dx), L^1(\Omega, \rho^{\alpha_2}\, dx)]_\theta = L^1(\Omega, \rho^\alpha\, dx)\,,$$

$$\begin{aligned}[][M^{(N+\beta_1)/(k+\alpha_1)}(\Omega, \rho^{\beta_1}\, dx), M^{(N+\beta_2)/(k+\alpha_2)}(\Omega, \rho^{\beta_2}\, dx)]_\theta \\ = M^{(N+\beta)/(k+\alpha)}(\Omega, \rho^\beta\, dx)\,,\end{aligned}$$

where α, β are given by the relations

$$\alpha = (1-\theta)\,\alpha_1 + \theta\,\alpha_2\,, \qquad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}\,, \tag{2.55}$$

$$p_i = \frac{N+\beta_i}{k+\alpha_i} \qquad \text{and} \qquad p = \frac{N+\beta}{k+\alpha}\,. \tag{2.56}$$

From the Marcinkiewicz theorem, if a transformation maps continuously $L^1(\Omega, \rho^{\alpha_i}\, dx)$ into $M^{(N+\beta_i)/(k+\alpha_i)}(\Omega, \rho^{\beta_i}\, dx)$ for $i = 1, 2$, it also maps continuously $L^1(\Omega, \rho^\alpha\, dx)$ into $L^{(N+\beta)/(k+\alpha)}(\Omega, \rho^\beta\, dx)$, see [29]. Let us show that the estimates (2.49) and (2.51) can be obtained by interpolation of the estimates (2.26), (2.27) and (2.30), (2.31) for $\alpha_1 = 0$ and $\alpha_2 = 1$, *with the exception of the case* $\beta = 0$ *for* (2.51). First take $k = N-2$, and observe that

$$\begin{gathered}\beta_i \in \Big(-\frac{N}{N-1+\alpha_i}, \frac{\alpha_i N}{N-2}\Big) \\ \text{if and only if} \\ \frac{1}{p_i} \in \Big(\frac{N-2}{N}, \frac{N-1+\alpha_i}{N}\Big),\end{gathered} \tag{2.57}$$

so that from (2.55) and (2.56), if $\beta_i \in (-N/(N-1+\alpha_i), \alpha_i N/(N-2))$, then

$$\beta \in \Big(-\frac{N}{N-1+\alpha}, \frac{\alpha N}{N-2}\Big). \tag{2.58}$$

Reciprocally, for any $\alpha \in (0,1)$ and β satisfying (2.58), taking $\alpha_1 = 0$ and $\alpha_2 = 1$ and defining p by (2.56) with $k = N-2$, we set

$$p_1 = p_2 = p\,, \qquad \text{if } \frac{1}{p} \leq \frac{N-2+\alpha}{N}\,,$$

$$\frac{1}{p_1} = \frac{1}{p} - \frac{\alpha}{N}\,, \qquad \frac{1}{p_2} = \frac{1}{p} + \frac{1-\alpha}{N}\,, \qquad \text{if } \frac{1}{p} > \frac{N-2+\alpha}{N}\,.$$

Then p_1 and p_2 satisfy (2.57), and we can interpolate between these values, with β_1, β_2 given by (2.56). Thus G maps continuously $L^1(\Omega, \rho^\alpha dx)$ into $L^{(N+\beta)/(N-2+\alpha)}(\Omega, \rho^\beta\, dx)$ and (2.50) follows on $L^1(\Omega, \rho^\alpha\, dx)$. Now take $k = N-1$, replace β_1, β_2, β by γ_1, γ_2, γ in (2.56): now

$$p_i = \frac{N+\gamma_i}{N-1+\alpha_i} \qquad \text{and} \qquad p = \frac{N+\gamma}{N-1+\alpha}\,. \tag{2.59}$$

Observe that

$$\gamma_1 = 0 \qquad \text{if and only if} \qquad \frac{1}{p_1} = \frac{N-1}{N}\,, \tag{2.60}$$

$$\gamma_2 \in \Big[0, \frac{N}{N-1}\Big) \qquad \text{if and only if} \qquad \frac{1}{p_2} \in \Big(\frac{N-1}{N}, 1\Big]\,, \tag{2.61}$$

so that from (2.55) and (2.56), if $\gamma_2 \in (0, N/(N-1))$,

$$\gamma \in \Big(0, \alpha\,\frac{N}{N-1}\Big). \tag{2.62}$$

Reciprocally, for any $\alpha \in (0,1)$ and γ satisfying (2.62), taking $\alpha_1 = 0$ and $\alpha_2 = 1$ and defining p by (2.59), we set

$$\frac{1}{p_1} = \frac{N-1}{N}\,, \qquad \frac{1}{p_2} = \frac{\dfrac{1}{p} - \dfrac{1-\alpha}{p_1}}{\alpha}\,.$$

Then p_1 and p_2 satisfy (2.60), (2.61), and we can interpolate between these values, with γ_1, γ_2 given by (2.59). Hence (2.51) and (2.52) follow on $L^1(\Omega, \rho^\alpha\,dx)$ when $\gamma \neq 0$. In case $\gamma = 0$, we interpolate between $\alpha_1 = \alpha - \varepsilon$ and $\alpha_2 = \alpha + \varepsilon$ for $\varepsilon > 0$ small enough, with $\gamma_1 = \gamma_2 = 0$, and get again (2.51) and (2.52).

Now consider any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho^\alpha\, d|\varphi| < +\infty$. Then there exists a bounded sequence of functions $f_n \in L^1(\Omega, \rho^\alpha\,dx)$ converging weakly to φ. The sequence $\{G(f_n)\}$ is bounded in $W_0^{1,s}(\Omega)$ for any $s \in [1, N/(N-1+\alpha))$, from (2.48). After an extraction it converges to some function Φ strongly in $L^s(\Omega)$ and almost everywhere in Ω. Then Φ is a weak solution of problem (2.25), hence $\Phi = G(\varphi)$. Moreover $\{G(f_n)\}$ is bounded in $L^{(N+\beta)/(N-2+\alpha)}(\Omega, \rho^\beta\,dx)$ for any $\beta \in (-N/(N-1+\alpha), \alpha N/(N-2))$. And $\{|\nabla G(f_n)|\}$ is bounded in $L^{(N+\gamma)/(N-1+\alpha)}(\Omega, \rho^\gamma\,dx)$ for any $\gamma \in [0, \alpha N/(N-1))$. Since those spaces are reflexive, we get (2.49) and (2.50), (2.51) and (2.52) by going to the weak limit after a new extraction. Then (2.53) and (2.54) follow.

Remark 2.4. Let us mention that the result $\Phi \in W_0^{1,s}(\Omega)$ with $s = N/(N-1+\alpha)$ can be proved by duality, see [12]. Notice that the value of s given in [12] is not correct, due to a small error in the parameters of the Sobolev injection.

Remark 2.5. Assume $N \geq 2$. From (2.48), we deduce that

$$G \text{ is compact from } L^1(\Omega, \rho^\alpha\,dx) \text{ into } L^p(\Omega, \rho^\beta\,dx)\,,$$

for any $\alpha \in (0,1]$, $\beta \in (-N/(N+\alpha-1), N\alpha/(N-2))$ or $\alpha = \beta = 0$, and $p \in [1, (N+\beta)/(N+\alpha-2))$ and $p > -\beta$. It comes from the compactness of the Sobolev injection

$$W_0^{1,s}(\Omega, \rho^\gamma\,dx) \subset L^p(\Omega, \rho^\beta\,dx)\,,$$

when $1 \leq s \leq p < +\infty$ and $N/p - N/s + 1 > 0$ and $(N+\beta)/p - (N+\gamma)/s + 1 > 0$, with $\gamma + 1 \neq s$, see [27]. In the case $\alpha = 1$ and $\beta = 0$, we find again a result cited in [10].

2.3. Application to the problems 1.1 and 1.4.

Combining theorems 2.5 and 2.6, we deduce regularity results for the problem (1.4). In particular, taking $\alpha = 1$, we get the following:

Corollary 2.8. *For any $\varphi \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho\, d|\varphi| < +\infty$ and any $\mu \in \mathcal{M}(\partial\Omega)$, the solution U of problem* (1.4) *satisfies*

$$\left\{\begin{array}{l} U \in M^{(N+\beta)/(N-1)}(\Omega, \rho^\beta\, dx)\,, \\ \quad \text{for any } \beta \in \Big(-1, \dfrac{N}{N-2}\Big)\,,\ \text{if } N \geq 3\,, \\ U \in L^k(\Omega, \rho^\beta\, dx)\,, \\ \quad \text{for any } \beta \in (-1, +\infty) \text{ and } k \in [1, 2+\beta)\,,\ \text{if } N = 2\,, \end{array}\right. \tag{2.63}$$

$$\left\{\begin{array}{l} |\nabla U| \in M^{(N+\gamma)/N}(\Omega, \rho^\gamma dx)\,, \\ \quad \text{for any } \gamma \in \Big(0, \dfrac{N}{N-1}\Big),\ \text{if } N \geq 2\,, \end{array}\right. \tag{2.64}$$

(*hence $U \in W_0^{1,s}(\Omega, \rho^\gamma\, dx)$ for any $\gamma \in (0, N/(N-1))$ and $s \in [1, (N+\gamma)/N)$). And in any case,*

$$\|U\|_{M^{(N+\beta)/(N-1)}(\Omega, \rho^\beta dx)} \leq C\Big(\int_\Omega \rho\, d|\varphi| + |\mu|(\partial\Omega)\Big)\,, \tag{2.65}$$

if $N \geq 3$,

$$\|\,|\nabla U|\,\|_{M^{(N+\gamma)/N}(\Omega, \rho^\gamma\, dx)} \leq C\Big(\int_\Omega \rho\, d|\varphi| + |\mu|(\partial\Omega)\Big)\,, \tag{2.66}$$

if $N \geq 2$.

This gives an interior regularity result for problem (1.5):

Corollary 2.9. *If $1 < q < (N+1)/(N-1)$, then any solution u of* (1.1) *is a classical solution in Ω.*

PROOF. Applying Corollary 2.8 to u we get in particular

$$u \in M^{(N+1)/(N-1)}(\Omega, \rho\, dx)\,, \qquad \text{if } N \geq 3$$

(and $u \in M^{3-\varepsilon}(\Omega, \rho\, dx)$ if $N = 2$). Then $u^q \in L^{k_0}_{\text{loc}}(\Omega)$ for some $k_0 > 1$, since $q < (N+1)/(N-1)$. If $N = 2$, then from Schauder estimates, $u \in C^\infty(\Omega)$. In case $N \geq 3$ and $k_0 < N/2$, we can make a usual bootstrapp: from the L^p regularity theory, $u \in W^{2,k_0}_{\text{loc}}(\Omega)$, hence from the Sobolev injection $u^q \in L^{k_1}_{\mathrm{loc}}(\Omega)$ for $k_1 = N\,k_0/q\,(N-2\,k_0) > k_0$, since $q < N/(N-2)$. By induction $u^q \in L^{k_n}_{\mathrm{loc}}(\Omega)$ for

$$k_n = N\,\frac{k_{n-1}}{q}\,\frac{1}{N-2\,k_{n-1}} > k_{n-1}\,, \tag{2.67}$$

till $k_n < N/2$. But if $k_n < N/2$ for any $n \in \mathbb{N}$, then $k_n \longrightarrow \ell = N\,(q-1)/2\,q < 1$, which is impossible. Then changing slightly k_0 if necessary, we find some $n_0 \in \mathbb{N}$ such that $k_{n_0} > N/2$, hence $u \in C^\infty(\Omega)$.

3. Estimate of $G(P^q(\mu))$.

Now we assume that $1 < q < (N+1)/(N-1)$, and we prove Theorem 1.1. First observe that for any $\mu \in \mathcal{M}^+(\partial\Omega)$, we have $P(\mu) \in M^{(N+1)/(N-1)}(\Omega, \rho\,dx)$ from Theorem 2.5. In particular,

$$P^q(\mu) \in L^1(\Omega, \rho\,dx)\,,$$

since $q < (N+1)/(N-1)$, hence $G(P^q(\mu))$ is well defined and lies in $L^1(\Omega)$ from Corollary 2.2. And $P(\mu) \in C^0(\Omega)$, since $\mathcal{P}$ is continuous, hence also $G(P^q(\mu)) \in C^0(\Omega)$.

We first consider the case where $\mu = \delta_a$ is a Dirac mass at a point a of $\partial\Omega$, hence $P(\mu)(\cdot) = P(\delta_a)(\cdot)\mathcal{P}(\cdot, a)$. Here we can give a more precise estimate near the point a:

Theorem 3.1. *Assume that $1 < q < (N+1)/(N-1)$. Let $a \in \partial\Omega$, and let $W = G(P^q(\delta_a))$ be the solution of*

$$\begin{cases} -\Delta W = \mathcal{P}^q(\cdot, a)\,, & \text{in } \Omega\,,\\ W = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.1}$$

Then there exists a constant $C = C(N, \Omega, q)$ such that

$$W(x) \le C\,\mathcal{P}(x,a)\,|x-a|^{N+1-(N-1)q}\,, \qquad \text{in } \Omega\,. \tag{3.2}$$

PROOF. From (2.8), we can majorize $\mathcal{P}^q(\cdot, a)$ by

$$\mathcal{P}^q(x,a) \le c_N^q\,\rho^q(x)\,|x-a|^{-Nq} \le c_N^q\,\rho(x)\,|x-a|^{-1-(N-1)q}\,.$$

Then for any $x \in \Omega$, from (1.3),

$$W(x) = \int_\Omega \mathcal{G}(x,y)\,\mathcal{P}^q(y,a)\,dy \le c_N^q \int_\Omega \mathcal{G}(x,y)\,\rho(y)\,|y-a|^{-1-(N-1)q}\,dy\,.$$

Now from (2.2) and (2.3),

$$W(x) \le c_N^{q+1}\,\rho(x) \int_\Omega f(x,y)\,dy\,,$$

where

$$f(x,y) = |y-a|^{-(N-1)q}\,|x-y|^{-N} \min\{|x-y|, |y-a|\}\,,$$

since $\rho(y) \le |y-a|$. Now we divide Ω in three parts

$$\Omega_1 = \Omega \cap B\Big(x, \frac{|x-a|}{2}\Big)\,,$$

$$\Omega_2 = \Omega \cap B\Big(a, \frac{|x-a|}{2}\Big)\,,$$

$$\Omega_3 = \Omega \backslash (\Omega_1 \cup \Omega_2)\,,$$

and integrate separately on each part. In the sequel C denotes constants which only depend on N, q and Ω. In Ω_1 we have $|x-a| \le 2\,|y-a|$, and

$$\begin{aligned}
\int_{\Omega_1} f(x,y)\,dy &\le \int_{\Omega_1} |y-a|^{-(N-1)q}\,|x-y|^{1-N}\,dy \\
&\le 2^{(N-1)q}\,|x-a|^{-(N-1)q} \int_{B(x,|x-a|/2)} |x-y|^{1-N}\,dy \\
&\le C\,|x-a|^{1-(N-1)q}\,.
\end{aligned}$$

In Ω_2 we have $|x-a| \le 2\,|x-y|$, and

$$\begin{aligned}
\int_{\Omega_2} f(x,y)\,dy &\le \int_{\Omega_2} |y-a|^{1-(N-1)q}\,|x-y|^{-N}\,dy \\
&\le 2^N\,|x-a|^{-N} \int_{B(a,|x-a|/2)} |y-a|^{1-(N-1)q}\,dy \\
&\le C\,|x-a|^{-N} \int_0^{|x-a|/2} r^{N-(N-1)q}\,dr \\
&\le C\,|x-a|^{1-(N-1)q}\,,
\end{aligned}$$

since $q < (N+1)/(N-1)$.

In Ω_3, we have $|x-a| \leq 2\min\{|y-a|, |y-x|\}$, hence $|y-a| \leq 3\,|y-x|$. Then we get

$$\begin{aligned} \int_{\Omega_3} f(x,y)\,dy &\leq \int_{\Omega_3} |y-a|^{1-(N-1)q}\,|x-y|^{-N}\,dy \\ &\leq 3^{-N}\int_{\Omega_3} |y-a|^{1-N-(N-1)q}\,dy \\ &\leq C\int_{|x-a|/2}^{+\infty} r^{-(N-1)q}\,dr \\ &\leq C\,|x-a|^{1-(N-1)q}\,. \end{aligned}$$

Then

$$W(x) \leq C\,\rho(x)\,|x-a|^{1-(N-1)q}\,,$$

and from the lower estimate of the Poisson kernel (2.8) we deduce (3.2).

Now we come to the general case.

Proof of Theorem 1.1. Let $\mu \in \mathcal{M}^+(\partial\Omega)$. We can reduce to the case $\mu(\partial\Omega) = 1$ by linearity of P and G. From (1.2), we have

$$P(\mu)(x) = \int_{\partial\Omega} \mathcal{P}(x,z)\,d\mu(z) = \int_{\partial\Omega} P(\delta_z)(x)\,d\mu(z)\,, \qquad \text{in } \Omega\,.$$

Then from the Jensen inequality,

$$P^q(\mu)(x) \leq \int_{\partial\Omega} P^q(\delta_z)(x)\,d\mu(z)\,, \qquad \text{in } \Omega\,.$$

And from the maximum principle,

$$G(P^q(\mu))(x) \leq G\Big(\int_{\partial\Omega} P^q(\delta_z)(x)\,d\mu(z)\Big) = \int_{\partial\Omega} G(P^q(\delta_z))(x)\,d\mu(z)\,.$$

Hence from (3.2),

$$\begin{aligned} G(P^q(\mu))(x) &\leq C\int_{\partial\Omega} \mathcal{P}(x,z)\,|x-z|^{N+1-(N-1)q}\,d\mu(z) \\ &\leq C\,P(\mu)(x)\,, \qquad \text{in } \Omega\,, \end{aligned}$$

since $N+1-(N-1)\,q>0$ and Ω is bounded.

4. A priori estimates.

Here we study the behaviour of the solutions of (1.5) for a given measure $\mu \in \mathcal{M}^+(\partial\Omega)$. First notice as in [13] that for any $q>1$, for any solution u of (1.5), $\|u^q\|_{L^1(\Omega,\rho\,dx)}$ is majorized independently of u: we have the estimate

$$\|u^q\|_{L^1(\Omega,\rho\,dx)} \le C\,(1+\mu(\partial\Omega))\,, \tag{4.1}$$

with $C=C(N,q,\Omega)$. Indeed consider a positive eigenvector Φ_1 for the first eigenvalue λ_1 of $(-\Delta)$ with Dirichlet conditions on $\partial\Omega$. Since u is a weak solution of (1.5), we have

$$\int_\Omega u\,(-\Delta\Phi_1)\,dx = \lambda_1 \int_\Omega u\,\Phi_1\,dx = \int_\Omega u^q\,\Phi_1\,dx + \int_{\partial\Omega} \frac{\partial\Phi_1}{\partial n}\,d\mu\,,$$

hence from Young inequality

$$\int_\Omega u^q\,\Phi_1\,dx \le \frac{1}{2}\int_\Omega u^q\,\Phi_1\,dx + (2\,\lambda_1^q)^{1/(q-1)} \int_\Omega \Phi_1\,dx + \int_{\partial\Omega} \Big|\frac{\partial\Phi_1}{\partial n}\Big|\,d\mu\,,$$

which implies (4.1), since $C^{-1}\rho \le \Phi_1 \le C^{-1}\rho$ in Ω, for some $C=C(N,\Omega)>0$.

Now we prove Theorem 1.2. We follow the technique of the interior problem, given in [23]. Once we have obtained the estimate (1.12), the proof goes quickly in case $q<N/(N-1)$. The main difficulty comes when $q\ge N/(N-1)$: in that case we really need the precise estimates of $G(\varphi)$ and $P(\mu)$ in Marcinkiewicz weighted spaces, proved in Section 2.2. We begin by the easiest case.

Proof of Theorem 1.2.

i) The simple case: $q<N/(N-1)$.

Let $\mu\in\mathcal{M}^+(\partial\Omega)$, and let u be any nonnegative solution of (1.5). Let us set

$$u = P(\mu) + v_1\,, \tag{4.2}$$

where $v_1 = G(u^q)$. Now

$$u \in M^{N/(N-1)}(\Omega)\,,\ \text{if } N\ge 3\,, \qquad u\in M^{2-\varepsilon}(\Omega)\,,\ \text{if } N=2\,, \tag{4.3}$$

from Corollary 2.8. Since $q < N/(N-1)$, $u^q \in L^{k_0}(\Omega)$ for some $k_0 > 1$. Now $u^q \leq 2^{q-1}\,(P^q(\mu) + v_1^q)$, hence from the maximum principle and from the estimate (1.12),

$$v_1 \leq 2^{q-1}\,(G(P^q(\mu)) + v_2) \leq C_1\,(P(\mu) + v_2)\,,$$

where $v_2 = G(v_1^q)$, hence

$$u \leq C_1'\,(P(\mu) + v_2)\,.$$

By induction for any $n \geq 2$, we can define the solution $v_n = G(v_{n-1}^q)$ of problem

$$\begin{cases} -\Delta v_n = v_{n-1}^q\,, & \text{in } \Omega\,, \\ v_n = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

such that

$$v_n \leq C_n\,(P(\mu) + v_{n+1})\,, \qquad u \leq C_n'\,(P(\mu) + v_{n+1})\,,$$

where C_n, C_n' only depend on N, q, Ω and $\mu(\partial\Omega)$. And $v_n \in L^{k_n}(\Omega)$ with k_n given by (2.67), hence there exists some $n_0 = n_0(N,q)$ such that $v_{n_0} \in C^0(\overline{\Omega})$. Then

$$u \leq C_{n_0}'\,(P(\mu) + v_{n_0+1})\,, \qquad \text{in } \Omega\,. \tag{4.4}$$

with $v_{n_0+1} = 0$ on $\partial\Omega$. Hence there exists a constant $C_0 > 0$ such that

$$v_{n_0+1}(x) \leq C_0\,\rho(x)\,, \qquad \text{in } \Omega\,, \tag{4.5}$$

and C_0 depends on $N, q, \Omega, \mu(\partial\Omega)$ and $\|u^q\|_{L^1(\Omega,\rho\,dx)}$, from the continuity properties given in Corollary 2.8, hence $C_0 = C_0(N, q, \Omega, \mu(\partial\Omega))$ from (4.1). Then (1.13) follows from (4.4) and (4.5). If $\mu = 0$, then $u \in C^\infty(\overline{\Omega})$ from Schauder estimates.

In case $\mu = \sigma\,\delta_a$ for some $a \in \partial\Omega$ and $\sigma > 0$, we get more precisely from Theorem 3.1

$$v_1 \leq 2^{q-1}\,(\sigma^q\,G(P^q(\delta_a)) + v_2) \leq C_1\,(P(\delta_a)\,|x-a|^{N+1-(N-1)q} + v_2)\,.$$

By induction we find

$$v_n \leq C_n\,(G(P^q(\delta_a)) + v_{n+1}) \leq C_n'\,(P(\delta_a)\,|x-a|^{N+1-(N-1)q} + v_{n+1})$$

and

$$u \leq P(\delta_a) + C''_n \left(P(\delta_a) \, |x-a|^{N+1-(N-1)q} + v_{n+1}\right).$$

Then we deduce (1.14).

ii) The case: $N/(N-1) \leq q < (N+1)/(N-1)$.
Let $p \geq 2$ be some fixed integer such that

$$\frac{1}{p} < N+1-(N-1)\,q\,.$$

Now for any $n \in [0,p]$, let $\beta_n = 1 - n/p \in [0,1]$. Now we start from the fact that

$$\begin{aligned} &u \in M^{(N+1)/(N-1)}(\Omega, \rho\, dx)\,, \qquad \text{if } N \geq 3\,,\\ &u \in M^{3-\varepsilon}(\Omega, \rho\, dx)\,, \qquad \text{if } N = 2\,, \end{aligned} \tag{4.6}$$

from Corollary 2.8. Let $v_0 = u$, hence

$$v_0^q \in L^{r_0}(\Omega, \rho^{\beta_0}\, dx)\,, \qquad \text{with } 1 < r_0 < \frac{N+\beta_0}{(N-1)\,q}\,.$$

Here again we define v_1 by (4.2), and $v_1 \leq u$. So that we can define $v_2 = G(v_1^q)$ in $L^1(\Omega)$. From Theorem 2.6, we have, for any $N \geq 2$ and $\varepsilon > 0$ small enough,

$$v_1 \in L^{(N+\beta)/(N-2+\beta_0)-\varepsilon}(\Omega, \rho^{\beta}\, dx)\,,$$

for any $\beta \in (-1, N/(N-2))$. Taking $\beta = \beta_1 = 1 - 1/p \in (0,1)$, we get

$$v_1^q \in L^{r_1}(\Omega, \rho^{\beta_1}\, dx)\,, \qquad \text{with } 1 < r_1 < \frac{N+\beta_1}{(N-2+\beta_0)\,q}\,, \tag{4.7}$$

since $N + \beta_1 - (N-2+\beta_0)\,q = N+1-(N-1)\,q - 1/p > 0$. For any $n \leq p$, assume by induction that $v_{n-1} = G(v_{n-2}^q)$ in $L^1(\Omega)$, and that

$$v_{n-1}^q \in L^{r_{n-1}}(\Omega, \rho^{\beta_{n-1}}\, dx)\,, \qquad \text{with } 1 < r_{n-1} < \frac{N+\beta_{n-1}}{(N-2+\beta_{n-2})\,q}\,,$$

then we can define $v_n = G(v_{n-1}^q)$ in $L^1(\Omega)$, and we get

$$v_n \in L^{(N+\beta)/(N-2+\beta_{n-1})-\varepsilon}(\Omega, \rho^{\beta}\, dx)\,,$$

for any $\beta \in (-1, \beta_{n-1}N/(N-2))$. Taking $\beta = \beta_n \geq 0$, we have $(N+\beta_n) - (N-2+\beta_{n-1})\,q > (n-1)\,(q-1)/p > 0$, hence

$$(4.8) \qquad v_n^q \in L^{r_n}(\Omega, \rho^{\beta_n}\, dx)\,, \qquad \text{with } 1 < r_n < \frac{N+\beta_n}{(N-2+\beta_{n-1})\,q}\,.$$

Now in case $n = p$, we have $\beta_p = 0$. This proves that $v_p^q \in L^{r_p}(\Omega)$, with $r_p > 1$ and we are reduced to the first case: there exists an integer $n_0 = n_0(N,q)$ such that $v_{n_0+p} \in C^0(\overline{\Omega})$. We deduce (1.13) and (1.14) as above.

In this proof we have used the estimate (1.12). In fact it is not really needed for getting a priori estimates, since we require that the problem admits a solution: the existence assumption in turn implies a condition of type (1.12). Adapting the arguments of [8] for the interior nonhomogeneous problem

$$\begin{cases} -\Delta u = u^q + f\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

with $f > 0$, we get the following:

Lemma 4.1. *Let $q > 1$ and $\mu \in \mathcal{M}^+(\partial\Omega)$. If the problem* (1.5) *admits a solution, then*

$$(4.9) \qquad G(P^q(\mu)) \leq \frac{1}{q-1}\,P(\mu)\,, \qquad \text{in } \Omega\,.$$

Proof. We can assume $\mu \neq 0$. For any $v, w \in C^2(\Omega)$ with v positive and harmonic, and any concave function F of class C^2 on the closure of the range of w/v, we have

$$(4.10) \qquad -\Delta\Big(v\,F\Big(\frac{w}{v}\Big)\Big) \geq F'\Big(\frac{w}{v}\Big)(-\Delta w)\,, \qquad \text{in } \Omega\,.$$

Suppose that problem (1.5) admits a solution u. Then we apply (4.10) with $v = P(\mu)$ and $w = u \geq v$, and

$$F(s) = \frac{1 - s^{1-q}}{q-1}\,, \qquad \text{on } [1, +\infty)\,.$$

It comes

$$P^q(\mu) = \Big(\frac{u}{P(\mu)}\Big)^{-q} u^q \leq -\Delta\Big(P(\mu)\, F\Big(\frac{u}{P(\mu)}\Big)\Big)\,,$$

and

$$G(P^q(\mu)) \leq P(\mu)\, F\Big(\frac{u}{P(\mu)}\Big) \leq \frac{1}{q-1}\, P(\mu)\,,$$

from the maximum principle.

Remark 4.1. In the supercritical case $q \geq (N+1)/(N-1)$, any solution u of (1.1), such that $u \in C(\overline{\Omega}\backslash \{a\})$ and $u = 0$ on $\partial\Omega\backslash\{a\}$, satisfies necessarily $u = 0$ on $\partial\Omega$, from Corollary 2.3. Considering the known behaviours for the problems (1.7) and (1.6), one can ask if an estimate of the type

$$u(x) \leq C\,\rho(x)\,|x-a|^{-(q+1)/(q-1)}$$

is true near the point a, at least if $q < (N+1)/(N-3)$. The question is entirely open.

5. Existence results.

Here we study the existence of solutions of problem (1.15). It is is based on the estimate of $G(P^q(\mu))$, which gives supersolutions.

Proof of Theorem 1.3.

First step: existence of solutions for small σ. Let $\mu \in \mathcal{M}^+(\partial\Omega)$ with $\mu(\partial\Omega) = 1$ and $\sigma > 0$. The function $\sigma\, P(\mu)$ is a subsolution of (1.15). We search a supersolution of (1.15) of the form

$$y = \sigma\, P(\mu) + a\, G(P^q(\sigma\,\mu))$$

with $a > 0$, and

$$a\, P^q(\sigma\,\mu) \geq y^q\,.$$

Since $q < (N+1)/(N-1)$, from Theorem 1.1, there exists a constant $K = K(N,\Omega,q)$ such that

$$y \leq \sigma(1 + a\,\sigma^{q-1}\,K)\,P(\mu)\,, \qquad \text{in } \Omega\,, \tag{5.1}$$

and y is a supersolution as soon as $a^{1/q} \geq 1 + a\,\sigma^{q-1} K$. As a consequence, taking the best value $a = (q/(q-1))^q$, if

$$\sigma \leq (q\,K)^{-1/(q-1)}\,\frac{q-1}{q}\,, \tag{5.2}$$

then $\mathcal{S}_\sigma$ has a solution.

Second step: interval of existence. Let

$$\Lambda = \{\sigma > 0 :\ \mathcal{S}_\sigma \text{ has a solution}\} \qquad \text{and} \qquad \sigma^* = \sup \Lambda\,.$$

Then from (4.9),

$$(\sigma^*)^{q-1}\,G(P^q(\mu)) \leq \frac{1}{q-1}\,P(\mu)\,, \qquad \text{in } \Omega\,, \tag{5.3}$$

hence σ^* is finite. For any $\sigma \in \Lambda$, $\mathcal{S}_\sigma$ has a solution u_σ. For any $\tau \in [0,\sigma)$, u_σ is a supersolution of (1.1) such that $u_\sigma \geq \tau\,P(\mu)$, hence $\mathcal{S}_\tau$ has a solution $u_\tau \leq u_\sigma$. Then Λ is an interval. At last, let us show that $\mathcal{S}_{\sigma^*}$ has a solution: let $\{\sigma_n\}$ be an increasing sequence with limit σ^*. Now u_{σ_n} is a weak solution of $\mathcal{S}_{\sigma_n}$; we use as a test function the unique solution $\xi > 0$ of problem $\xi = G(\xi^{1/q})$, introduced in [8], and get

$$\begin{aligned} \int_\Omega u_{\sigma_n}\,(-\Delta\,\xi)\,dx &= \int_\Omega u_{\sigma_n}\,\xi^{1/q}\,dx \\ &= \int_\Omega u_{\sigma_n}^q\,\xi\,dx - \sigma_n \int_{\partial\Omega} \frac{\partial \xi}{\partial n}\,d\mu\,. \end{aligned} \tag{5.4}$$

And $\partial\xi/\partial n < 0$, hence

$$\int_\Omega u_{\sigma_n}^q\,\xi\,dx \leq \frac{1}{q}\int_\Omega u_{\sigma_n}^q\,\xi\,dx + \frac{q-1}{q}\,|\Omega|\,,$$

so that $\{u_{\sigma_n}^q\}$ is bounded in $L^1(\Omega,\rho\,dx)$. And $u_{\sigma_n} \leq \sigma^* P(\mu)$ on $\partial\Omega$. Now $\{u_{\sigma_n}\}$ is bounded in $M^{N/(N-1)}(\Omega) \cap M^{(N+1)/(N-1)}(\Omega,\rho\,dx)$, from Corollary 2.8. Then we can go to the limit in the weak formulation of $\mathcal{S}_{\sigma_n}$, and construct a weak solution of $\mathcal{S}_{\sigma^*}$.

REMARK 5.1. Replacing K by $K_\mu = \|G(P^q(\mu))/P(\mu)\|_{L^\infty(\Omega)}$, we can estimate σ^* from (5.2) and (5.3)

$$\sigma^* \in \Big[(q\,K_\mu)^{-1/(q-1)}\,\frac{q-1}{q}, ((q-1)\,K_\mu)^{-1/(q-1)}\Big].$$

REMARK 5.2. Now assume $q \ge (N+1)/(N-1)$. We shall say that a measure $\mu \in \mathcal{M}^+(\partial\Omega)$ with $\mu(\partial\Omega) = 1$ is *admissible* if

$$P^q(\mu) \in L^1(\Omega, \rho\,dx) \quad \text{and } \mu \text{ satisfies the condition (1.12) for some } K > 0\,.$$

Then in the same way Theorem 1.3 applies to any admissible measure. Moreover, for such an admissible measure, following the techniques of [8], for any $\sigma \in [0, \sigma^*)$, we can construct a solution u_σ of $\mathcal{S}_\sigma$ satisfying the a priori estimate

$$\sigma\,P(\mu) \le u_\sigma \le C\,P(\mu)\,, \qquad \text{in } \Omega\,, \tag{5.5}$$

for some constant $C = C(\sigma)$. Indeed let $\tau \in (\sigma, \sigma^*)$ such that $\mathcal{S}_\tau$ admits a solution u_τ. Let us apply the inequality (4.10) with $v = \tau\,P(\mu)$ and $w = u_\tau \ge v$, and

$$F(s) = s\,(1+\varepsilon\,s^{q-1})^{-1/(q-1)} \qquad \text{and} \qquad \varepsilon = \Big(\frac{\tau}{\sigma}\Big)^{q-1} - 1\,, \text{ on } [1, +\infty)\,,$$

so that $F(1) = \sigma/\tau$ and $F'(s) = F^q(s)/s^q$. We obtain

$$-\Delta\Big(v\,F\Big(\frac{u_\tau}{v}\Big)\Big) \ge F'\Big(\frac{u_\tau}{v}\Big)(-\Delta u_\tau) = \Big(v\,F\Big(\frac{u_\tau}{v}\Big)\Big)^q, \qquad \text{in } \Omega\,.$$

Hence $z = v\,F(u_\tau/v)$ is a supersolution of (1.1). Then

$$z \ge \sigma\,P(\mu)\,, \text{ in } \Omega\,, \qquad \text{and } z = \sigma\,\mu\,, \text{ on } \partial\Omega\,,$$

and $\mathcal{S}_\sigma$ has a solution

$$u_\sigma \le z \le \varepsilon^{-1/(q-1)}\tau\,P(\mu)\,,$$

so that u_σ satisfies (5.5). Now for any $\tau \in (\sigma, \sigma^*)$, $\mathcal{S}_\tau$ admits a solution u_τ. Choosing $\tau = (\sigma + \sigma^*)/2$, we deduce that u_σ satisfies (5.5) with $C(\sigma) = \sigma^*\,((\sigma+\sigma^*)/2\,\sigma)^{q-1} - 1)^{-1/(q-1)}$. At last considering as above an increasing sequence $\{\sigma_n\}$ with limit σ^*, we prove that $\mathcal{S}_{\sigma^*}$ admits a solution u_{σ^*}.

An open question is to describe precisely those admissible measures.

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Recibido: 4 de febrero de 1.999

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