Spectral multipliers
on metabelian groups

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0. Introduction.

Let G be a Lie group, X_j right invariant vector fields on G, which generate (as a Lie algebra) the Lie algebra of G,

$$L = -\sum X_j^2 \ .$$

Then L is called sublaplacian, and it well-known that L is positive definite and essentially selfadjoint on $C_c^\infty(G) \subset L^2(G)$, where $L^2(G)$ is taken with respect to a left-invariant Haar measure dg. By the spectral theorem, for any bounded Borel measurable function $F : [0, \infty) \longmapsto \mathbb{C}$ the operator $F(L)f = \int_0^\infty F(\lambda) \, dE(\lambda) \, f$ is bounded on $L^2(G)$. We are interested in the behavior of $F(L)$ on L^p.

This question has a long history. Classical results for polynomial growth case are [15], [16], [5], [18], [1], [7], [22] for exponential growth [8], [23], [2], [3]. Newer results show that connection with growth is more complicated [11], [19], [12], [10], [9], [4], [6], [13], [20], [14], [21], [17].

In this paper we consider $L^1(G)$ boundedness of $F(L)$ for (some) metabelian G and a distinguished L on G. Of the main interest is that the group is of exponential growth, and possibly higher rank. Previously positive results about higher rank groups where only about Iwasawa type groups. Also, our groups may be unimodular, so it is the second positive result (after [13]) about unimodular groups, and the first giving a family of examples.

1. Results.

Let $G = \mathbb{R}^n \ltimes \mathbb{R}^m$, adjoint action is semisimple, $L = L_0 + L_1$, L_0 lives on $\mathbb{R}^n$, L_1 lives on $\mathbb{R}^m$ and is a sum of (squares of) eigenvectors for adjoint action. More precisely, assume that λ_j, $j = 1, \ldots, m$ are linear forms on $\mathbb{R}^n$, e_j, $j = 1, \ldots, m$ is the canonical basis of $\mathbb{R}^m$, linear operator $A(x) : \mathbb{R}^m \longmapsto \mathbb{R}^m$ is given by the formula $A(x) e_j = \lambda_j(x) e_j$ and

$$(x_1, y_1)(x_2, y_2) = (x_1 + x_2, \exp(A(-x_2)) y_1 + y_2).$$

The right-invariant vector fields are:

$$X_j = \partial_{x_j}$$

and

$$Y_j = \exp(-\lambda_j(x)) \partial_{y_j}.$$

We assume that

$$L = -\sum X_j^2 - \sum Y_j^2 = L_0 + L_1.$$

We can transform general L_0 to our form, but for L_1 the assumption is somewhat restrictive.

In this paper we identify convolution operators with functions:

$$\exp(-tL) f = \exp(-tL) * f.$$

Theorem 1.1. *If G and L are as above, then there exists C such that*

$$\|\exp(-(1 + i\,s)\,L)\|_{L^1} \leq C(1 + |s|^{3m+n}).$$

Thorem 1.2. *For every compactly supported $F \in C^{3m+n+1}$ the operator $F(L)$ is bounded on $L^1(G)$.*

Theorem 1.2 is a straightforward consequence of Thorem 1.1.

Before the proof of Thorem 1.1 we need a lemma about "symbols". We consider it as well-known, but the form given below is adjusted to our needs.

Lemma 1.3. *There is C such that if E is a normed space, $f : \mathbb{R} \longmapsto E$, $|f|$ is integrable, $b \geq 1$,*

$$\sup |\hat{f}| \leq a \,,$$

$$\sup |\omega \, \partial_\omega \, \widehat{f}(\omega)| \leq a \, b \,,$$

$$\sup |\omega^2 \, \partial_\omega^2 \, \widehat{f}(\omega)| \leq a \, b^2 \,,$$

then

$$|f(x)| \leq \frac{C \, a \, b}{|x|} \,.$$

REMARK. The lemma remains valid as long as $\hat{f}$ is reasonably defined (like $f \in S(\mathbb{R}, E^*)^*$, where $S(\mathbb{R}, E^*)$ consists of E^* valued Schwartz class functions).

PROOF. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Fix $x_0 \neq 0$ and let $r = b/|x_0|$. Put $\widehat{f_1}(\omega) = \phi(\omega/r) \, \widehat{f}(\omega)$ and $\widehat{f_2}(\omega) = (1 - \phi(\omega/r)) \, \widehat{f}(\omega)$. We have

$$|f_1(x)| \leq \int |\widehat{f_1}| \, d\omega \leq \int_{-2r}^{2r} a \, d\omega = 4 \, a \, r \,,$$

and

$$|x^2 f_2(x)| \leq \int |\partial_\omega^2 \, \widehat{f_2}(\omega)| \, d\omega \,.$$

By the Leibnitz formula

$$\partial_\omega^2 \, \widehat{f_2}(\omega) = \left(1 - \phi\left(\frac{\omega}{r}\right)\right)\partial_\omega^2 \, \widehat{f}(\omega) - 2 \, r^{-1} \, \phi'\left(\frac{\omega}{r}\right)\partial_\omega \, \widehat{f}(\omega) + r^{-2} \, \phi''\left(\frac{\omega}{r}\right) \widehat{f}(\omega)$$

so

$$
\begin{aligned}
\int |\partial_\omega^2 \, \widehat{f_2}(\omega)| \, d\omega &\leq \int_{|\omega|>r} \frac{a \, b^2}{\omega^2} \, d\omega + \int_{2r>|\omega|>r} 2 \, C \, r^{-1} \, a \, b \, \omega^{-1} \, d\omega \\
&\quad + \int_{2r>|\omega|>r} C \, r^{-2} \, a \, d\omega \\
&\leq a \, b^2 \, r^{-1} + 4 \, C \, a \, b \, r^{-1} + 2 \, C \, a \, r^{-1} \\
&\leq C' \, a \, b^2 \, r^{-1} \,.
\end{aligned}
$$

Now

$$|f(x_0) \leq |f_1(x_0)| + |f_2(x_0)|$$
$$\leq 4\,a\,r + C'\,a\,b^2\,r^{-1}\,|x_0|^{-2}$$
$$= (4 + C')\,a\,b\,|x_0|^{-1}$$
$$= C''\,a\,b\,|x_0|^{-1}\ .$$

Proof of Theorem 1.1. We decompose the regular representation of G using Fourier transform in y variable. In coordinates

$$L = -\Delta_x - \sum \exp\left(-2\,\lambda_j(x)\right)\partial_{y_j}^2\ ,$$

where $\Delta_x = \sum \partial_{x_j}^2$.

If we denote by H_z the Fourier transform (in y variable) of L at z, then

$$H_z = -\Delta_x + \sum z_j^2 \exp\left(-2\,\lambda_j(x)\right).$$

$\operatorname{Re} H_z \geq 0$, provided that $\operatorname{Re} z_j > \operatorname{Im} z_j$, $j = 1, \ldots, m$, so $z \longmapsto \exp\left(-tH_z\right)$ is bounded holomorphic in the area given by the inequalities.

Considering $(t + i\,s)\,H_z$ we see that $\exp\left(-(t + i\,s)\,H_z\right)$ is bounded and holomorphic as long as $\operatorname{Re}(t + i\,s)\,z_j^2 \geq 0$, $j = 1, \ldots, m$. Moreover, we can estimate the integral kernels

$$\|\exp\left(-(2\,t + i\,s)\,H_z\right)\delta_0\|_{L^2} \leq \|\exp\left(-(t + i\,s)\,H_z\right)\|\,\|\exp\left(-tH_z\right)\delta_0\|_{L^2}\ .$$

By the Feynmann-Kac formula

$$\|\exp\left(-tH_z\right)\delta_0\|_{L^2} \leq \|q_t\|_{L^2} = c\,t^{-n/4}\ ,$$

where q_t is ordinary euclidean heat kernel.

Consequently, by the Cauchy integral formula (for real z)

$$\|\partial_z^\alpha \exp\left(-(t + i\,s)\,H_z\right)\delta_0\|_{L^2} \leq C_\alpha\,|z_1|^{-\alpha_1} \cdots |z_m|^{-\alpha_m}\left(1 + \frac{|s|}{t}\right)^{|\alpha|}t^{-n/4}\ .$$

Applying Lemma 1.3 m times we get

$$\|\exp\left(-(t + i\,s)\,L\right)(\cdot, y)\|_{L^2} \leq C''\,(|y_1| \cdots |y_m|)^{-1}\left(1 + \frac{|s|}{t}\right)^m t^{-n/4}\ .$$

In [14] (as the first step in proof of Theorem 1.1) we proved that

$$(1.4) \qquad \int |\exp\left(-(1+i\,s)\,L\right)(g)|\, e^{d(g,0)}\, dg \le C\,\exp(C\,s^2)\,,$$

where $d(x,y)$ is the optimal control distance associated to L. One easily checks that

$$\{g:\; d(g,0) < r\} \subset \{(x,y):\; |x| < r,\; |y| < c_d \exp\left(c_d\,r\right)\}\,.$$

To estimate L^1 norm we put $r = C\,s^2$, $c = c_d\,C$, $A_j = \{(x,y):\; |x| < C\,s^2,\; |y_j| < \exp\left(-m\,c\,s^2\right),\; |y_l| < \exp\left(c\,s^2\right),\; l \ne j\}$. Note $|A_j| \le C\,s^{2n}$. We have

$$\|\exp(-(1+i\,s)\,L)\|_{L^1}$$

$$\le \int_{d(g,0)>r} |\exp\left(-(1+i\,s)\,L\right)(g)|\, dg$$

$$+ \int_{\substack{|x|<cs^2 \\ \exp(-mcs^2)\le|y_j|\le\exp(cs^2)}} |\exp\left(-(1+i\,s)\,L\right)((x,y))|\, dx\, dy$$

$$+ \sum_j \int_{A_j} |\exp\left(-(1+i\,s)\,L\right)|(g)\, dg$$

$$= I_\infty + I_0 + \sum I_j\,.$$

For I_∞ we use exponential estimate (1.4)

$$\int_{d(g,0)>r} |\exp\left(-(1+i\,s)\,L\right)(g)|\, dg$$

$$\le e^{-r} \int |\exp\left(-(1+i\,s)\,L\right)(g)|\exp\left(d(g,0)\right) dg$$

$$\le \exp\left(-C\,s^2\right) C \exp\left(C\,s^2\right) = C\,.$$

Next

$$I_j \le |A_j|^{1/2} \|\exp\left(-(1+i\,s)\,L\right)\|_{L^2} \le C\,|s|^n\,.$$

Finally

$$I_0 = \int_{\exp(-mcs^2)\leq|y_j|\leq\exp(cs^2)} \int_{|x|<Cs^2} |\exp(-(1+is)L)(x,y)|\,dx\,dy$$

$$\leq \int_{\exp(-mcs^2)\leq|y_j|\leq\exp(cs^2)} |\{x:\ |x|<c\,s^2\}|^{1/2}$$

$$\cdot \|\exp(-(1+is)L)(\cdot,y)\|_{L^2}\,dy$$

$$\leq \int_{\exp(-mcs^2)\leq|y_j|\leq\exp(cs^2)} c\,s^n\,C''(|y_1|\cdots|y_m|)^{-1}(1+|s|)^m\,dy$$

$$\leq C\,|s|^n(1+|s|)^m\left(2\int_{\exp(-mcs^2)}^{\exp(cs^2)}|y_1|^{-1}\,dy_1\right)^m$$

$$\leq C\,|s|^n\,(1+|s|)^m\,((m+1)\,c\,s^2)^m$$

$$\approx C'\,(1+|s|^{n+3m}).$$

2. Final remarks.

Our goal was to present the idea, so we used simple arguments even though we got weaker end result. If the estimates are done in a more involved way one may replace $n+3\,m$ in Theorem 1.1 by a smaller number (we checked that $(n+3\,m)/2$ is enough), however we expect that in Thorem 1.2 it is enough to have more than $n/2+m$ derivatives in L^2, and getting this requires new ideas. Also, constants in Theorem 1.2 grow exponentially with the diameter of support of F. We may get polynomial growth, but we would like to have a uniform bound on $\|F(tL)\|_{L^1}$.

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