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Convexity and uniqueness in a free boundary problem arising in combustion theory

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Abstract. We consider solutions to a free boundary problem for the heat equation, describing the propagation of flames. Suppose there is a bounded domain $\Omega \subset Q_T = \mathbb{R}^n \times (0,T)$ for some $T > 0$ and a function $u > 0$ in Ω such that

$$u_t = \Delta u \,, \qquad \text{in } \Omega \,,$$

$$u = 0 \text{ and } |\nabla u| = 1 \,, \qquad \text{on } \Gamma := \partial\Omega \cap Q_T \,,$$

$$u(\cdot, 0) = u_0 \,, \qquad \text{on } \overline{\Omega}_0 \,,$$

where Ω_0 is a given domain in $\mathbb{R}^n$ and u_0 is a positive and continuous function in Ω_0, vanishing on $\partial\Omega_0$. If Ω_0 is convex and u_0 is concave in Ω_0, then we show that (u, Ω) is unique and the time sections Ω_t are convex for every $t \in (0,T)$, provided the free boundary Γ is locally the graph of a Lipschitz function and the fixed gradient condition is understood in the classical sense.

1. Introduction and main result.

In this paper we consider solutions to a free boundary problem for the heat equation. Suppose there is a domain $\Omega \subset Q_T := \mathbb{R}^n \times (0,T)$

for some $T > 0$ and a positive smooth function u in Ω such that

$$u_t = \Delta u\,, \qquad \text{in } \Omega\,, \tag{1}$$

$$u = 0 \text{ and } |\nabla u| = 1\,, \qquad \text{on } \Gamma\,, \tag{2}$$

$$u(\cdot, 0) = u_0\,, \qquad \text{on } \overline{\Omega}_0\,, \tag{3}$$

where $\Gamma := \partial\Omega \cap Q_T$ is the (free) lateral boundary of Ω, $\Omega_0 \subset \mathbb{R}^n$ is the initial domain and u_0 is a prescribed positive continuous function in Ω_0, that vanishes continuously on $\Gamma_0 := \partial\Omega_0$. Then we say the pair (u, Ω) or, when there is no ambiguity, Ω to be a *solution to problem* (**P**). This problem, in mathematical framework, was introduced by L. A. Caffarelli and J. L. Vázquez [CV]. It describes propagation of so-called premixed equi-diffusional flames in the limit of high activation energy. In this problem the time sections

$$\Omega_t = \{x \in \mathbb{R}^n :\ (x, t) \in \Omega\} \tag{4}$$

represent the *unburnt* (*fresh*) *zone* in time t, $\Gamma_t := \partial\Omega_t$ corresponds to the *flame front*, and $u = c\,(T_c - T)$ is the *normalized temperature*. For further details in combustion theory we refer to paper [V] of J. L. Vázquez.

The existence of weak solutions to problem (**P**) as well as their regularity under suitable conditions on the data were established in [CV]. However, we should not expect any uniqueness result unless we impose some special geometrical restrictions. In this paper we study the case when the initial domain Ω_0 is bounded and *convex*, and the initial function u_0 is *concave*. Throughout the paper we make the following assumptions concerning solutions (u, Ω) to problem (**P**). First, the boundary of Ω consists of three parts

$$\partial\Omega = \Omega_0 \cup \overline{\Gamma} \cup \Omega_T\,, \tag{5}$$

where Ω_T is a nonvoid open set in the plane $t = T$. The presence of nonempty Ω_T excludes the *extinction phenomenon* in time $t \in [0, T]$. This assumption is rather of technical character, that can be avoided with the following simple procedure. Consider the *extinction time*

$$T_\Omega = \sup\{t : \Omega_t \neq \varnothing\}\,. \tag{6}$$

Then every domain $\Omega^{(\tau)} = \Omega \cap \{0 < t < \tau\}$, $\tau \in (0, T_\Omega)$, has nonempty "upper bound" Ω_τ. Therefore we can consider first $\Omega^{(\tau)}$ instead of Ω and then let $\tau \longrightarrow T_\Omega$.

Next, we assume that for every $(x_0, t_0) \in \overline{\Gamma}$ there exists a neighborhood V in $\mathbb{R}^n \times \mathbb{R}$ such that (after a suitable rotation of x-axes)

$$\Omega \cap V = \{(x,t) = (x', x_n, t) : \ x_n > f(x', t)\} \cap V \cap Q_T \, , \tag{7}$$

where f is a *Lipschitz* function, defined in

$$V' = \{(x', t) : \text{ there exists } x_n \text{ with } (x', x_n, t) \in V\} \, .$$

Further, for u we assume that it is continuous up to the boundary $\partial\Omega$ and can be extended smoothly through Ω_T. The gradient condition in (2) is understood in the classical sense

$$\lim_{\Omega_t \ni y \to x} |\nabla u(y,t)| = 1 \, , \tag{8}$$

for every $x \in \partial\Omega_t$, $0 < t \le T$.

The main result of this paper is as follows.

Theorem 1. *In problem* (**P**) *let* Ω_0 *be a bounded convex domain and* u_0 *be a concave function in* Ω_0. *Suppose that* (u, Ω) *is a solution to this problem in the sense described above. Then* (u, Ω) *is a unique solution. Moreover, the time sections* Ω_t *of* Ω *are convex for every* $t \in (0, T)$.

The plan of the paper is as follows. In Section 2 we prove a theorem on the convexity of level sets of solutions to a related Dirichlet problem. In Section 3 we recall some properties of caloric functions in Lipschitz domains. And finally in Section 4 we prove Theorem 1.

2. Convexity of level sets.

In this section we establish some auxiliary results, which are, however, of independent interest.

Let u_0 and Ω_0 be as in problem (**P**) and a domain $\Omega \subset Q_T$ meets conditions (5) and (7). Then by the Petrowski criterion [P] Ω is a regular domain for the Dirichlet problem for the heat equation (in the Perron sense), and its parabolic boundary is given by

$$\partial_p \Omega = \overline{\Omega}_0 \cup \Gamma \, . \tag{9}$$

We fix one such domain Ω and denote by u the solution to the Dirichlet problem

$$\begin{cases} u_t = \Delta u\,, & \\ u = u_0\,, & \text{on } \overline{\Omega}_0\,, \\ u = 0\,, & \text{on } \Gamma\,. \end{cases} \tag{10}$$

Theorem 2. *Let the time sections Ω_t of the domain Ω be convex for $t \in [0,T]$. Let also u_0 be a concave function on $\overline{\Omega}_0$, positive in Ω_0 and vanishing on $\partial\Omega_0$. Then the level sets*

$$\mathcal{L}_s(u(\cdot,t)) = \{x \in \Omega_t :\ u(x,t) > s\} \tag{11}$$

are convex for every fixed $s > 0$ and $t \in (0,T)$, where u is the solution to the Dirichlet problem (10).

The proof is based on the Concavity maximum principle originally due to N. Korevaar [K1] and [K2]. For a function v on Ω set

$$\mathcal{C}(x,y,t) = \frac{v(x,t)+v(y,t)}{2} - v\Big(\frac{x+y}{2}, t\Big)\,. \tag{12}$$

The function $\mathcal{C}$ is defined on an open subset D of the fiber product

$$\widetilde{\Omega} = \{(x,y,t) :\ (x,t), (y,t) \in \Omega\}.$$

Note that $D = \widetilde{\Omega}$ if the time sections of Ω are convex. Note also that if v is extended to the "upper bound" Ω_T of Ω, then $\mathcal{C}$ is extended to the "upper bound" D_T of D. We denote $\partial_p D = \overline{D} \setminus (D \cup D_T)$.

Lemma (Concavity maximum principle)**.** *Let v in $C^{2,1}_{x,t}(\Omega) \cap C(\Omega \cup \Omega_T)$ satisfy to a parabolic equation*

$$v_t = a^{ij}(t,\nabla v)\, v_{ij} + b(t,x,v,\nabla v)\,, \qquad \text{in } \Omega \tag{13}$$

with smooth coefficients and such that b is nonincreasing in v and jointly concave in (x,v). Then either $\mathcal{C} \le 0$ in $D \cup D_T$ or

$$0 < \sup_{(x,y,t)\in D\cup D_T} \mathcal{C}(x,y,t) = \limsup_{(x,y,t)\to\partial_p D} \mathcal{C}(x,y,t)\,.$$

PROOF. See [K2], the proof of Theorem 1.6. Though the result is proved there for cylindrical domains, the proof is valid also in our case.

REMARK. There are several formulations of this principle in the elliptic case. The strongest version states that it is sufficient to require harmonic concavity of b in (x, v) instead of concavity; see B. Kawohl [Ka], and A. Greco and G. Porru [GP]. In the parabolic case, in order to use such an extension, it seems necessary to assume also the nonnegativeness of v_t; see A. Kennington [Ke].

PROOF OF THEOREM 2. Assume first, that the functions f in the local representations (7) of Ω are smooth in (x', t) and strictly convex in x' and that u_0 is smooth. These assumptions imply the smoothness up to $\partial\Omega$ of the solution u to (10). Also, the positivity of u_0 implies the positivity of u. Define now $v = \log(u)$. We claim then that $v(\cdot, t)$ are concave functions in Ω_t for every $t \in (0, T]$. Clearly, this will imply the statement of the theorem. For this purpose, we consider the concavity function $\mathcal{C}$, defined above, and show that $\mathcal{C} \le 0$ on $D \cup D_T$. Suppose the contrary. Then take a maximizing sequence $(x_k, y_k, t_k) \in D \cup D_T$ such that

$$\lim \mathcal{C}(x_k, y_k, t_k) = \sup_{D \cup D_T} \mathcal{C} > 0 . \tag{14}$$

Without loss of generality we may assume that there exists limit

$$(x_0, y_0, t_0) = \lim (x_k, y_k, t_k) .$$

Direct calculation shows, that v satisfies

$$v_t = \Delta v + |\nabla v|^2 , \tag{15}$$

in Ω and hence the Concavity maximum principle is applicable. Hence we may assume $(x_0, y_0, t_0) \notin D \cup D_T$. We want to exclude also the other possibilities. First, the case $t_0 = 0$ and $x_0, y_0 \in \Omega_0$ is impossible, since $v(\cdot, 0) = \log(u_0)$ is concave in Ω_0. Next, $x_0 \in \Gamma_{t_0}$ but $y_0 \ne x_0$ is also excluded by the strict convexity of Ω_t's, since then $\mathcal{C}(x_k, y_k, t_k) \longrightarrow -\infty$. So, it remains to consider the last case $x_0 = y_0 \in \Gamma_{t_0}$. We observe now that by the boundary point lemma, the outward spatial normal derivatives $u_\nu < 0$ on $\Gamma \cup \Gamma_T$. Besides, $u_\nu < 0$ also on Γ_0 since u_0 is concave and positive in Ω_0 and vanishes on Γ_0. By the smoothness assumption we have therefore $u_\nu \le -\varepsilon_0 < 0$ on $\overline{\Gamma}$. Hence we can carry out the same reasonings as in [CS, Proof of Lemma 3.1] (see also [GP, Lemma 3.2]) to obtain that $\liminf \mathcal{C}(x_k, y_k, t_k) < 0$, which contradicts (14). Therefore $\mathcal{C} \le 0$ in $D \cup D_T$ and $v(\cdot, t)$ is concave in Ω_t for every $t \in (0, T]$. This proves the theorem in the considering case.

To prove the theorem in the general case, we use approximation of Ω by domains with smooth lateral boundary and with strictly convex time sections, and relevant smooth concave approximations of u_0.

3. On caloric functions.

In this section we recall some properties of caloric functions in Lipschitz domains. They will be used in the next section, where we prove Theorem 1. The main reference here is the paper [ACS] by I. Athanasopoulos, L. Caffarelli and S. Salsa.

As in the previous section we consider a domain Ω, satisfying conditions (5) and (7). Let also u be the solution to (10). Consider a neighborhood V of a point $(x_0, t_0) \in \Gamma$, where (7) holds. The function u vanishes on $\Gamma \cap V$, is positive and satisfies the heat equation in $\Omega \cap V$. In other words, u is *caloric*.

We start with the following lemma from [ACS], which states that a caloric function u is "almost harmonic" in time sections near the lateral boundary Γ.

Lemma 4 ([ACS, Lemma 5]). *There exist $\varepsilon > 0$ and a neighborhood Q of the point $(x_0, t_0) \in \Gamma$ such that the functions*

$$w_+ = u + u^{1+\varepsilon}, \qquad w_- = u - u^{1+\varepsilon}, \tag{16}$$

are respectively sub- and superharmonic in $Q \cap \Omega \cap \{t = t_0\}$.

We will need also the following lemma on asymptotic development of u near the boundary point (x_0, t_0).

Lemma 5 [ACS, Lemma 6]. *Suppose there exists an n-dimensional ball $B \subset \Omega^c \cap \{t = t_0\}$ such that $\overline{B} \cap \Gamma = \{(x_0, t_0)\}$. Then near x_0 in Ω_{t_0}*

$$u(x, t_0) = \alpha (x - x_0, \nu)^+ + o\left(|x - x_0|\right), \tag{17}$$

for some $\alpha \in [0, \infty)$ and where ν denotes the outward radial direction of B at (x_0, t_0).

In the next lemma we show that α in (17) is in fact the nontangential limit of $|\nabla u(y,t_0)|$ as $y \longrightarrow x_0$.

Lemma 6. *Under the conditions of Lemma* 5*, let also* $K \subset \Omega_{t_0}$ *be an* n*-dimensional truncated cone with the vertex at* (x_0,t_0) *such that* $|x-x_0| \le c_1 \operatorname{dist}(x,\Gamma_{t_0})$ *for every* $x \in K$ *and some constant* c_1*. Then*

$$\lim_{K \ni y \to x_0} \nabla u(y,t_0) = \alpha\,\nu\,, \tag{18}$$

where α *and* ν *are as in the asymptotic development* (17).

PROOF. By [ACS, Corollary 4], there exists a neighborhood V of the point (x_0,t_0) such that

$$|u_t(x,t)| \le c_2\,\frac{u(x,t)}{d_{x,t}}\,, \qquad d_{x,t} = \operatorname{dist}(x,\Gamma_t)\,, \tag{19}$$

for all $(x,t) \in V \cap \Omega$. Take an arbitrary sequence $y_k \longrightarrow x_0$, $y_k \in K$, and consider the functions

$$v_k(z) = \frac{u(y_k + r_k z, t_0)}{r_k}\,, \qquad r_k = |y_k - x_0|\,, \tag{20}$$

defined on the ball $B = B(0,\rho)$, $\rho = 1/(2\,c_1)$. Using (17) and (19), we obtain that for large k

$$|v_k(z)| < (\alpha+1)\,(1+\rho) \tag{21}$$

and

$$|\Delta v_k(z)| = r_k\,|\Delta u(y_k + r_k z, t_0)| = r_k\,|u_t(y_k + r_k z, t_0)| \le 2\,c_1\,c_2\,, \tag{22}$$

uniformly in B. Then $C^{1,\beta}$ norms of v_k are locally uniformly bounded in B for a $\beta \in (0,1)$; see *e.g.* [LU]. Therefore a subsequence of v_k converges locally in C^1 norm to a function v_0 in B. We may also assume that over this subsequence there exists $e_0 = \lim e_k$, where $e_k = (y_k - x_0)/|y_k - x_0|$. Then, using (17), we can compute that $v_0(z) = \alpha(z,\nu) + \alpha(e_0,\nu)$ in B, hence $\nabla v_0(0) = \alpha\nu$. Therefore, over a subsequence, $\lim \nabla u(y_k,t_0) = \lim \nabla v_k(0) = \nabla v_0(0) = \alpha\nu$. Since the sequence $y_k \longrightarrow x_0$, $y_k \in K$ was arbitrary, this proves the lemma.

4. Proof of the main theorem.

In this section Ω will be a solution to problem $(\mathbf{P})$, under conditions of Theorem 1. Denote by Ω^* the *spatial convex hull* of Ω, in the sense that the time sections Ω^*_t are the convex hulls of Ω_t for every $t \in (0,T)$. Since Ω is assumed to satisfy (5) and (7), Ω^* will also satisfy similar conditions. In particular, we may apply the results of two previous sections to Ω^*. The lateral boundary of Ω^* will be denoted by Γ^* and the solution to the Dirichlet problem, corresponding to (5), by u^*.

In the proof of Theorem 1 we use ideas of A. Henrot and H. Shahgholian [HS]. The key step is to prove the following lemma.

Lemma 7. *For every $x_0 \in \Gamma^*_{t_0}$, $0 < t_0 \le T$,*

$$\liminf_{\Omega^*_{t_0} \ni y \to x_0} |\nabla u^*(y,t_0)| \ge 1 \,. \tag{23}$$

Proof. From Lemma 4 it follows that there are ε and s_0 such that the function $w_+(y) = u^*(y,t_0) + u^{*(1+\varepsilon)}(y,t_0)$ is subharmonic in the ringshaped domain $\{u^*(\cdot,t_0) < s_0\}$. Let now $y \in \Omega^*_{t_0}$ and $u^*(y,t_0) = s < s_0$. Then $y \in \ell^*_s = \partial\mathcal{L}_s(u^*(\cdot,t_0))$. By Theorem 2, $\mathcal{L}^*_s = \mathcal{L}_s(u^*(\cdot,t_0))$ is convex and therefore there exists a supporting plane in $\mathbb{R}^n$ to $\mathcal{L}^*_s$ at the point y. After a suitable translation and rotation in spatial variable we may assume that $y = 0$, the supporting plane is $x_1 = 0$, and $\mathcal{L}^*_s \subset \{x_1 < 0\}$. Let $x^* \in \partial\Omega^*_{t_0}$ have the maximal positive x_1-coordinate. Since $\Omega^*_{t_0}$ is the convex hull of Ω_{t_0}, there must be $x^* \in \partial\Omega^*_{t_0} \cap \partial\Omega_{t_0}$. Take now $\beta \in (0,1)$ and consider a function $v(x) = w_+(x) + \beta\, x_1$. Since $\Omega^*_{t_0} \cap \{x_1 > 0\} \subset \{u^*(\cdot,t_0) < s_0\}$, v is subharmonic in $\Omega^*_{t_0} \cap \{x_1 > 0\}$ and therefore it must admit its maximum value on the boundary of this domain. Note that the maximum can be admitted either at x^* or at $y = 0$. We show that the former case cannot occur. Indeed, the plane $x_1 = x^*_1$ is supporting to the convex set $\Omega^*_{t_0}$ and therefore there exists a ball $B \subset {\Omega^*_{t_0}}^c \subset \Omega^c_{t_0}$, "touching" both boundaries $\partial\Omega^*_{t_0}$ and $\partial\Omega_{t_o}$ at x^* and with the outward radial direction $\nu = -e_1 = (-1,0,\dots,0)$. Therefore from Lemma 5 we will have the following asymptotic developments for u and u^* near x^* in Ω_{t_0} and $\Omega^*_{t_0}$ respectively

$$u(x,t_0) = \alpha(x^*_1 - x_1)^+ + o\left(|x - x^*|\right), \tag{24}$$

$$u^*(x,t_0) = \alpha^*(x^*_1 - x_1)^+ + o(|x - x^*|) \,. \tag{25}$$

Since (8) is satisfied at the point x^*, we conclude by Lemma 6 that $\alpha = 1$. Next, $u^* \geq u$ in Ω and hence $\alpha^* \geq \alpha = 1$. Observe now that w_+ admits the same representation as (25). Hence for the function $v(x)$ introduced above

$$v(x) = (\alpha^* - \beta)\,(x_1^* - x_1) + \beta\, x_1^* + o\,(|x - x^*|)\,. \tag{26}$$

Let now ν' be a spatial unit vector with $(\nu', e_1) < 0$ such that $x^* + h\,\nu' \in \Omega_{t_0}$ for small $h > 0$. The existence of such a ν' follows from the local representation of $\partial\Omega_{t_0}$ as the graph of a Lipschitz function. Then $v(x^* + h\,\nu') > v(x^*)$ by (26) and consequently v has no maximum at x^*. Therefore v admits its maximum at the origin $y = 0$. Hence

$$|\nabla w_+(0)| = \lim_{h\to 0+} \frac{w_+(0) - w_+(h\,e_1)}{h} \geq \lim_{h\to 0+} \frac{\beta\, h - 0}{h} = \beta\,. \tag{27}$$

Letting $\beta \longrightarrow 1$ we obtain that $|\nabla w_+(y)| \geq 1$, provided $u^*(y, t_0) < s_0$. Now observe that $\nabla w_+ = (1 + (1+\varepsilon)\,u^{*\varepsilon})\,\nabla u^*$. This proves the lemma.

Proof of Theorem 1. Prove first that the domain Ω coincides with its spatial convex hull Ω^*, studied above. For this purpose we apply the Lavrentiev principle. As a reference point we take $x_{\max} \in \Omega_0$, a maximum point for the initial function u_0. Without loss of generality we may assume that $x_{\max} = 0$. Since u_0 is concave,

$$u_0(\lambda\, x) \leq u_0(x)\,, \tag{28}$$

for every $\lambda \geq 1$ and $x \in \Omega_0(\lambda) = \lambda^{-1}\Omega_0$. For $\lambda \geq 1$ define

$$u_\lambda^*(x,t) = u^*(\lambda\, x, \lambda^2\, t)\,, \tag{29}$$

in $\Omega^*(\lambda) = \{(x,t) :\ (\lambda x, \lambda^2 t) \in \Omega^*\}$. Suppose now that $\Omega^* \not\subset \Omega$. Then

$$\lambda_0 = \inf\,\{\lambda :\ \Omega^*(\lambda) \subset \Omega\} > 1\,, \tag{30}$$

$\Omega^*(\lambda_0) \subset \Omega$, and there exists a common point $(x_0, t_0) \in \overline{\Gamma^*(\lambda_0)} \cap \Gamma$ with $t_0 \in (0, T)$. Show that this leads to a contradiction. Indeed, by construction, $u^*_{\lambda_0}$ satisfies the heat equation in $\Omega^*(\lambda_0)$. Comparing the values of $u^*_{\lambda_0}$ and u on the parabolic boundary $\partial_p\Omega^*(\lambda_0)$ (see (28)), we obtain that $u^*_{\lambda_0} \leq u$ in $\Omega^*(\lambda_0)$. Let now ν be the normal vector of a supporting plane in $\mathbb{R}^n$ to the convex domain $\Omega^*(\lambda_0)_{t_0}$ at the point x_0, pointing into $\Omega^*(\lambda_0)_{t_0}$. From lemmas 5, 6 and 7 and the definition of u^*_λ we conclude that $\nabla u^*_{\lambda_0}(x_0+h\,\nu,t_0)\longrightarrow\lambda_0\,\alpha^*\nu$ with $\alpha^*\geq 1$, as $h\longrightarrow 0+$. From elementary calculus there exists $\theta\in(0,1)$ such that

$$\frac{\dfrac{\partial}{\partial\nu}\,u(x_0+\theta\,h\,\nu,t_0)}{\dfrac{\partial}{\partial\nu}\,u^*_{\lambda_0}(x_0+\theta\,h\,\nu,t_0)}=\frac{u(x_0+h\,\nu,t_0)}{u^*_{\lambda_0}(x_0+h\,\nu,t_0)}\geq 1 \tag{31}$$

and hence

$$\limsup_{\Omega_{t_0}\ni y\to x_0}\frac{\partial}{\partial\nu}\,u(y,t_0)\geq\lim_{h\to 0+}\frac{\partial}{\partial\nu}\,u^*_{\lambda_0}(x_0+h\,\nu)=\lambda_0\,\alpha^*>1\,, \tag{32}$$

which violates condition (8) at the point (x_0,t_0). Therefore $\Omega^*=\Omega$, *i.e.* the time sections Ω_t are convex, for every $t\in(0,T)$.

It remains to prove the uniqueness of Ω. For this we make the following observation. Let Ω' be another solution. Then if everywhere in the proof of inclusion $\Omega^*\subset\Omega$ above we replace Ω^* by $(\Omega')^*$, but leave Ω unchanged, we will obtain that $(\Omega')^*\subset\Omega$. Since Ω and Ω' are interchangeable, also we will have $\Omega^*\subset\Omega'$. Therefore $\Omega'=\Omega$ and the proof of Theorem 1 is completed.

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