Rev. Mat. Iberoam, **17** (2001), 433–469

Parabolic equations involving 0^{th} and 1^{st} order terms with L^1 data

Thierry Goudon and Mazen Saad

Abstract. This paper is devoted to general parabolic equations involving 0^{th} and 1^{st} order terms, in linear and nonlinear expressions, while the data only belong to L^1. Existence and entropic-uniqueness of solutions are proved.

1. Introduction.

In this paper, we are concerned with the following general parabolic equation

$$(1.1) \qquad \begin{cases} \partial_t u - \nabla \cdot (A(t,x)\nabla u) \\ +B(t,x,u,\nabla u) = f\,, & \text{in } (0,T)\times\Omega\,, \\ u_{|t=0} = u_0\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } (0,T)\times\partial\Omega\,, \end{cases}$$

where Ω is a regular open bounded set in $\mathbb{R}^N$ and B involves the unknown u and its first derivatives. Precisely, B splits into terms which are linear with respect to u and ∇u and a nonlinear term as follows

$$(1.2) \qquad B(t,x,u,\nabla u) = b(t,x)\cdot\nabla u + d(t,x)\,u + g(t,x,u,\nabla u)\,.$$

Here, A, b and d are given functions defined on $Q = (0,T) \times \Omega$ with values in $\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N$ and $\mathbb{R}$, respectively. Our basic requirement on A, b, d is

$$A \in (L^\infty(Q))^{N\times N}\,, \qquad d \in L^\infty(Q)\,, \tag{1.3}$$

$$b \in (L^\infty(Q))^N\,, \qquad \nabla \cdot b \in L^\infty(Q)\,. \tag{1.4}$$

As usual, we also assume that there exists $\underline{a} > 0$ such that the matrix A satisfies

$$A(t,x)\,\xi \cdot \xi \geq \underline{a}\,|\xi|^2\,, \tag{1.5}$$

for almost every $(t,x) \in Q$ and for all $\xi \in \mathbb{R}^N$. The function $g : Q \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is measurable on Q for all $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^N$, continuous with respect to $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^N$, almost everywhere in Q. Furthermore, g is required to satisfy both a sign condition and a growth condition with respect to the gradient variable since we suppose that

$$\lambda\, g(t,x,\lambda,\xi) \geq 0\,, \tag{1.6}$$

there exists $0 \leq \sigma < 2$ such that

$$|g(t,x,\lambda,\xi)| \leq h(|\lambda|)\,(\gamma(t,x) + |\xi|^\sigma) \tag{1.7}$$

holds for all $\lambda \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, and almost everywhere in Q, with $\gamma \in L^1(Q)$; h being a non decreasing function on $\mathbb{R}^+$. Main difficulties in this work arise from the fact that we consider data which only belong to L^1, namely

$$u_0 \in L^1(\Omega)\,, \qquad f \in L^1(Q)\,. \tag{1.8}$$

Many physicals models lead to elliptic and parabolic problems with L^1-data. For instance, in [10] the authors study the modelling of an electronical device. The derived elliptic system coupled the temperature (denoted u) and the electronical potential (denoted Φ). The temperature equation is considered as an elliptic equation where the second member $f = |\nabla\Phi|^2$ belongs to $L^1(\Omega)$. In [11], a Fokker-Planck equation arising in populations dynamics is studied. The initial density of individuals, *i.e.* u_0, is considered to be positive and belongs to $L^1(\Omega)$.

Models of turbulent flows in oceanography and climatology also lead to such kind of problems (see [14] and the references therein).

Consider an incompressible flow described by a velocity field $u(t,x) = \overline{u} + u'$ where $\overline{u}$ is the mean field and u' is related to some fluctuations. Let $k = |\overline{u'}|^2$. For small Reynolds number, the following academic model can be used as a simplification of more general (k, ε) models

$$\partial_t k + \overline{u} \cdot \nabla_x k - \operatorname{div}_x((\nu + \nu_t)\nabla_x k) + k^{3/2} = \nu_t \, |\nabla_x \overline{u} +^t \nabla_x \overline{u}|^2 \,,$$

where ν_t can depend on k. It is quite natural to expect that the right hand side lies in $L^1(Q)$ and, for given ν, ν_t and $\overline{u}$, the above equation can be considered as a simplified version of (1.1). In [14] more complicated and coupled models are dealt with.

In ([16, p. 110]), the author studies the Navier-Stokes equations completed by an equation for the temperature ($u = T$). In this case, if we denote by v the velocity of the fluid, then the temperature equation reduces to (1.1) with $b = v$, $d = \operatorname{div}(v) = 0$, $g = 0$ and $f = (\partial_i v_j + \partial_j v_i)^2 \in L^1(Q)$. Note that for compressible flows the divergence of the velocity does not vanish, and the temperature equation can be considered with linear terms having the form $b \cdot \nabla u + du$. These linear terms introduce new difficulties in the sense that the compactness results developped in [3], [4], [16] do not apply directly to (1.1) which needs further technical investigations.

Assuming $B = 0$, existence results for such parabolic problems with non regular data are established in [4] (see also [3], [10]) while uniqueness questions, in the sense of entropic or renormalized formulations, are considered in [17], [1]. Existence-uniqueness of renormalized solution for a linear parabolic equation involving a first order term with a free divergence coefficient is discussed in [16]. Taking into account the g term, the corresponding elliptic problem, with an integrable source term, is treated in [9] when $\sigma < 2$ and the critical case $\sigma = 2$ is dealt with in [5]. In [6], the g term appears in (1.1), still neglecting the linear terms involving b and d, with the restriction that g does not vanish for large value of u, which induces some regularizing effects in the equation. Note that in view of the quoted papers, our results extend to more general Leray-Lions operators ; however, to avoid technical complications and to emphasize the influence of the term B we restrict our attention on a simple operator satisfying (1.2). Let us now introduce some definitions and give the statement of our main results.

For the sake of clarity, we dropped the dependence on t, x of A, b, d and g. When no confusion can arise, we will follow this convention in the sequel.

Definition 1. *By weak solution of* (1.1) *we shall mean any function* $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap C^0(0,T;L^1(\Omega))$ *such that* $g(u,\nabla u)$ *belongs to* $L^1(Q)$ *and satisfying*

$$\begin{aligned}
&\int_\Omega u\,\phi(T,x)\,dx - \int_\Omega u_0\,\phi(0,x)\,dx - \int_Q u\,\partial_t\phi(t,x)\,dx\,dt \\
&+ \int_Q A\nabla u\cdot\nabla\phi\,dx\,dt + \int_Q (g(u,\nabla u) + b\cdot\nabla u + du)\,\phi\,dx\,dt \qquad (1.9)\\
&\quad = \int_Q f\,\phi\,dx\,dt\,,
\end{aligned}$$

for all $T > 0$, $\phi \in C^0(0,T;W_0^{1,q'}(\Omega)) \cap C^1(0,T;L^{q'}(\Omega))$ *and for all* q *such that* $1 \le q < (N+2)/(N+1)$ *and* $1/q + 1/q' = 1$.

All terms in (1.9) are clearly defined (by duality $L^q, L^{q'}$), except those involving $g(u,\nabla u)$. However, since $1 \le q < (N+2)/(N+1)$, we have $q' = q/(q-1) > N$ and by Sobolev's embedding the test function ϕ actually lies in $L^\infty(Q)$ so that the integral of $g(u,\nabla u)\phi$ makes sense.

Theorem 1. *Assume that* (1.3)-(1.8) *hold. Then, there exists a weak solution of* (1.1), *in the sense of Definition* 1.

Let us recall the definition of the truncated function T_k. Let $k \in \mathbb{R}^+$. We set

$$T_k(z) = \begin{cases} z\,, & \text{if } |z| \le k\,,\\ k\,, & \text{if } z > k\,,\\ -k\,, & \text{if } z < -k\,, \end{cases} \qquad (1.10)$$

and we denote $S_k(z) = \int_0^z T_k(\tau)\,d\tau$.

Definition 2. *Let* $g = 0$. *We say that* u *is a entropic solution of* (1.1) *if* $u \in C^0(0,T;L^1(\Omega))$ *satisfies* $T_k(u) \in L^2(0,T;H_0^1(\Omega))$ *for all* $k > 0$, $\nabla u \in L^1(Q)$ *and*

$$\begin{aligned}
&\int_\Omega S_k(u-\psi)(T)\,dx - \int_\Omega S_k(u_0 - \psi(0,\cdot))\,dx \\
&+ \int_0^T \langle \partial_t\psi, T_k(u-\psi)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt \\
&+ \int_Q A\nabla u\cdot\nabla(T_k(u-\psi))dxdt \qquad (1.11)
\end{aligned}$$

$$+\int_Q (du + b\,\nabla u)\,T_k(u-\psi)\,dx\,dt$$
$$\leq \int_Q f\,T_k(u-\psi)\,dx\,dt$$

for all $k>0$ *and* $\psi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q) \cap C^0(0,T;L^1(\Omega))$ *with* $\partial_t\psi \in L^2(0,T;H^{-1}(\Omega))$.

Obviously, $T_k(u-\psi)$ lies in $L^\infty(Q)$ and S_k is $k-$Lipschitzian; hence with the requirements $\nabla u \in L^1(Q)$ and $T_k(u-\psi) \in L^2(0,T;H_0^1(\Omega))$, both term in (1.11) clearly makes sense except the product $A\nabla u \cdot \nabla(T_k(u-\psi))$. Remark now that $\nabla(T_k(u-\psi)) = \chi_{|u-\psi|\leq k}\nabla(u-\psi)$ can be estimated by $\chi_{|u|\leq k+\|\psi\|_{L^\infty}}|\nabla u| + |\nabla\psi| = |\nabla T_{k+\|\psi\|_{L^\infty}}(u)| + |\nabla\psi|$ which belongs to L^2 since one chooses the test function ψ in $L^\infty(Q)$. Therefore $A\nabla u \cdot \nabla(T_k(u-\psi))$ is integrable.

Theorem 2. *Let* $g=0$. *Assume that* (1.3)-(1.8) *hold. Then, there exists a unique entropic solution of* (1.1).

The strategy we adopt is rather close to those introduced in [4]. However, new difficulties arise essentially related to the influence of the linear 0^{th} and 1^{st} order terms. Then, this paper is organized as follows. First, Section 2 is devoted to an independent preliminary result which will be used to derive a bound in L^q on the gradient of the solutions, in despite of the perturbation induced by the additional terms of lower order. In Section 3, we deal with sequences u_ε of approximate solutions. We establish some a priori estimates on these solutions and we translate the obtained bounds in terms of compactness properties. Then, we explain how we can pass to the limit as $\varepsilon \longrightarrow 0$ in the weak formulation satisfied by u_ε. In Section 4, we are concerned with the uniqueness of entropic solution. Finally, in Section 5, we slightly weaken the regularity assumption concerning the coefficient b.

2. A preliminary result.

The main idea in the proof of Theorem 1 consists in deriving a $L^q(0,T;W_0^{1,q}(\Omega))$ estimate on the solutions depending only on the L^1 norm of the data f and u_0. Such an estimate will appear as a consequence of the following lemma.

Lemma 1. *Let $u \in L^2(0,T;H^1_0(\Omega))$ satisfy*

$$\sup_{t\in(0,T)} \int_\Omega |u|\,dx \le \beta\,, \tag{2.1}$$

and

$$\int_{B_n} |\nabla u|^2\,dx\,dt \le C_0 + C_1 \int_{E_n} |\nabla u|\,dx\,dt\,, \qquad \text{for all } n \in \mathbb{N}, \tag{2.2}$$

where

$$B_n = \{(t,x) \in Q : \ n \le |u(t,x)| \le n+1\}\,,$$

and

$$E_n = \{(t,x) \in Q : \ |u(t,x)| > n+1\}\,.$$

Then, for all $1 \le q < (N+2)/(N+1)$, there exists $C > 0$, depending on β, C_0, C_1, $|\Omega|$, T, and q such that

$$\|u\|_{L^q(0,T;W^{1,q}_0(\Omega))} \le C\,. \tag{2.3}$$

PROOF. In [4], [10] inequality (2.2) appears with $C_1 = 0$ and is used to derive (2.3). Here, the additional term is related to the influence of the first order term $b\cdot\nabla u$ in the equation as we shall see in next section (see Proposition 1). However, exploiting carefully the fact that the integral in the right hand side is only taken over the large values of the unknow, we can obtain (2.3) as a consequence of (2.2).

Let $1 \le q < 2$. From (2.2), we first notice that

$$\begin{aligned}\int_{B_n} |\nabla u|^2\,dx\,dt &\le C_0 + C_1\Big(\int_{E_n} |\nabla u|^q\,dx\Big)^{1/q} |E_n|^{(q-1)/q}\\ &\le C_0 + C_1\,\|\nabla u\|_{L^q(Q)} |E_n|^{(q-1)/q}\end{aligned} \tag{2.4}$$

holds by using Holder's inequality. Thus, applying again Holder's inequality, we obtain

$$\begin{aligned}\int_{B_n} |\nabla u|^q\,dx\,dt &\le |B_n|^{(2-q)/2}\Big(\int_{B_n} |\nabla u|^2\,dx\,dt\Big)^{q/2}\\ &\le |B_n|^{(2-q)/2}\Big(C_0^{q/2} + C_1^{q/2}\,\|\nabla u\|^{q/2}_{L^q(Q)} |E_n|^{(q-1)/2}\Big)\end{aligned} \tag{2.5}$$

by (2.4) and the elementary inequality $(a+b)^{q/2} \le a^{q/2} + b^{q/2}$. Let $r \ge 0$ to be chosen later. Clearly, one has

$$\text{(2.6)} \qquad \begin{cases} |B_n| \le \dfrac{1}{n^r} \displaystyle\int_{B_n} |u|^r \, dx \, dt \,, \\[2ex] |E_n| \le \dfrac{1}{n^r} \displaystyle\int_{E_n} |u|^r \, dx \, dt \le \dfrac{1}{n^r} \, \|u\|^r_{L^r(Q)} \,. \end{cases}$$

Hence, (2.5) becomes

$$\text{(2.7)} \qquad \begin{aligned} &\int_{B_n} |\nabla u|^q \, dx \, dt \\ &\qquad \le C_0^{q/2} \Big(\frac{1}{n}\Big)^{r(2-q)/2} \Big(\int_{B_n} |u|^r \, dx \, dt \Big)^{(2-q)/2} \\ &\qquad\quad + C_1^{q/2} \, \|\nabla u\|^{q/2}_{L^q(Q)} \, \|u\|^{r(q-1)/2}_{L^r(Q)} \Big(\frac{1}{n}\Big)^{r/2} \\ &\qquad\qquad \cdot \Big(\int_{B_n} |u|^r \, dx \, dt \Big)^{(2-q)/2} \,. \end{aligned}$$

Let $K \in \mathbb{N}$ to be determined. We split $\|\nabla u\|^q_{L^q(Q)}$ as follows

$$\text{(2.8)} \qquad \int_Q |\nabla u|^q \, dx \, dt = \sum_{n=0}^{K} \int_{B_n} |\nabla u|^q \, dx \, dt + \sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u|^q \, dx \, dt \,.$$

Since $|B_n| \le T\,|\Omega|$ and $|E_n| \le T\,|\Omega|$, we simply evaluate the first term in the right hand side of (2.8) as follows

$$\text{(2.9)} \qquad \sum_{n=0}^{K} \int_{B_n} |\nabla u|^q \, dx \, dt \le K C_2 \Big(1 + \|\nabla u\|^{q/2}_{L^q(Q)} \Big) \,,$$

by (2.5), where $C_2 = \max\{C_0^{q/2}(T\,|\Omega|)^{(2-q)/2}, C_1^{q/2}(T\,|\Omega|)^{1/2}\}$. Thus, by using Young's inequality in (2.8)-(2.9), we get

$$\text{(2.10)} \qquad \|\nabla u\|^q_{L^q(Q)} \le C(K) + \sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u|^q \, dx \, dt \,,$$

where $C(K)$ tends to infinity as K becomes large. It remains to proceed to the study of the series which appears in the right hand side.

Applying Holder's inequality on the series with exponents $2/(2-q)$ and $2/q$ and using (2.7), we have

$$\begin{aligned}
&\sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u|^q \, dx \, dt \\
&\leq C_0^{q/2} \Big(\sum_{n=K+1}^{\infty} \frac{1}{n^{r(2-q)/q}} \Big)^{q/2} \Big(\sum_{n=K+1}^{\infty} \int_{B_n} |u|^r \, dx \, dt \Big)^{(2-q)/2} \\
&\quad + C_1^{q/2} \, \|\nabla u\|_{L^q(Q)}^{q/2} \|u\|_{L^r(Q)}^{r(q-1)/2} \Big(\sum_{n=K+1}^{\infty} \frac{1}{n^{r/q}} \Big)^{q/2} \\
&\qquad \cdot \Big(\sum_{n=K+1}^{\infty} \int_{B_n} |u|^r \, dx \, dt \Big)^{(2-q)/2} \\
&\leq C_0^{q/2} \Big(\sum_{n=K+1}^{\infty} \frac{1}{n^{r(2-q)/q}} \Big)^{q/2} \|u\|_{L^r(Q)}^{r(2-q)/2} \\
&\quad + C_1^{q/2} \, \|\nabla u\|_{L^q(Q)}^{q/2} \, \|u\|_{L^r(Q)}^{r/2} \, \Big(\sum_{n=K+1}^{\infty} \frac{1}{n^{r/q}} \Big)^{q/2} .
\end{aligned} \tag{2.11}$$

Note that the conditions

$$r \, \frac{2-q}{q} > 1 \qquad \text{and} \qquad \frac{r}{q} > 1 \tag{2.12}$$

ensure the convergence of the series which appear in the right hand side of (2.11). Consequently, these terms become arbitrarily small when choosing K large enough as soon as (2.12) is fulfilled.

With the convention that $\delta(K)$ denotes quantities which tend to 0 as K goes to ∞, by combining (2.10) with (2.11), we get

$$\|\nabla u\|_{L^q(Q)}^q \leq C(K) + \delta(K) \Big(\|u\|_{L^r(Q)}^{r(2-q)/2} + \|\nabla u\|_{L^q(Q)}^{q/2} \|u\|_{L^r(Q)}^{r/2} \Big) . \tag{2.13}$$

Therefore, by using Young's inequality on the last term in the right side, it follows that

$$\|\nabla u\|_{L^q(Q)}^q \leq C(K) + \delta(K) \Big(\|u\|_{L^r(Q)}^{r(2-q)/2} + \|u\|_{L^r(Q)}^r \Big) , \tag{2.14}$$

where we keep the notation $C(K)$, $\delta(K)$ while the value of these terms may have changed, still with the meaning that $C(K) \longrightarrow \infty$, $\delta(K) \longrightarrow 0$ when K becomes large.

We denote by $q_\star = Nq/(N-q)$ the Sobolev conjugate of q. The Sobolev imbedding theorem implies that

$$\int_0^T \Big(\int_\Omega |u|^{q_\star}\,dx\Big)^{q/q_\star} dt \le C\int_Q |\nabla u|^q\,dx\,dt\,. \tag{2.15}$$

Assume now $1 < r < q_\star$ and set $1/r = \theta + (1-\theta)/q^\star$ with $0 < \theta < 1$. For almost everywere $t \in (0,T)$, one has

$$\|u(t,\cdot)\|^r_{L^r(\Omega)} \le \|u(t,\cdot)\|^{r\theta}_{L^1(\Omega)}\,\|u(t,\cdot)\|^{r(1-\theta)}_{L^{q_\star}(\Omega)}\,. \tag{2.16}$$

Integrating (2.16) with respect to time and recalling the bound (2.1) in $L^\infty(0,T,L^1(\Omega))$ yield

$$\|u\|^r_{L^r(Q)} \le \beta^{r\theta}\int_0^T \Big(\int_\Omega |u|^{q_\star} dx\Big)^{r(1-\theta)/q_\star} dt\,. \tag{2.17}$$

Choose now $r = q\,(N+1)/N$, noting that the convergence condition (2.12) is fulfilled as soon as $1 \le q < (N+2)/(N+1)$. Combining (2.14)-(2.17) with Young's inequality (since $(2-q)/2 < 1$) leads to

$$\begin{aligned}
&\int_0^T \Big(\int_\Omega |u|^{q_\star}\,dx\Big)^{q/q_\star} dt\\
&\le C(K) + \delta(K)\Big(\Big(\int_0^T \Big(\int_\Omega |u|^{q_\star}\,dx\Big)^{q/q_\star} dt\Big)^{(2-q)/2}\\
&\qquad + \int_0^T \Big(\int_\Omega |u|^{q_\star}\,dx\Big)^{q/q_\star} dt\Big)\\
&\le C(K) + \delta(K)\int_0^T \Big(\int_\Omega |u|^{q_\star}\,dx\Big)^{q/q_\star} dt\,.
\end{aligned} \tag{2.18}$$

We fix $K > 0$ so that, for instance, $1 - \delta(K) > 1/2$. Hence, we deduce from (2.18) that

$$\|u\|_{L^q(0,T,L^{q_\star}(\Omega))} \le C \tag{2.19}$$

holds, and the asserted estimate (2.3) follows easily from (2.17) and (2.14).

3. Proof of Theorem 1.

The proof falls naturally into several steps and we detail each of them separetely.

3.1. Approximate solutions.

We introduce the following smooth approximations of the data

$$\begin{cases} u_{0,\varepsilon} \in C_0^\infty(\Omega)\,, & f_\varepsilon \in C_0^\infty(\Omega)\,, \\ u_{0,\varepsilon} \longrightarrow u_0 \text{ in } L^1(\Omega)\,, & f_\varepsilon \longrightarrow f \text{ in } L^1(Q)\,, \end{cases} \tag{3.1}$$

with

$$\|u_{0,\varepsilon}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}\,, \qquad \|f_\varepsilon\|_{L^1(Q)} \le \|f\|_{L^1(Q)}\,. \tag{3.2}$$

Moreover, we regularize the function g as follows

$$g_\varepsilon(u, \nabla u) = \frac{g(u, \nabla u)}{1 + \varepsilon\,|g(u, \nabla u)|}\,. \tag{3.3}$$

Note that g_ε belongs in $L^\infty(Q)$ and satisfy the sign condition (1.6) and the growth condition in (1.7). Then, classical results, see *e.g.* [15], [12], [7], (or, in the linear case, use a Galerkin method), provide the existence of a sequence $u_\varepsilon \in C^0(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, with $\partial_t u_\varepsilon \in L^2(0,T; H^{-1}(\Omega))$, of solutions of (1.1) where u_0, f and g are replaced by $u_{0,\varepsilon}, f_\varepsilon$ and g_ε respectively. We have

$$\begin{aligned} &\langle \partial_t u_\varepsilon, \phi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_\Omega A\nabla u_\varepsilon \cdot \nabla\phi\, dx \\ &\quad + \int_\Omega \left(g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) + b\cdot\nabla u_\varepsilon + d u_\varepsilon\right)\phi\, dx = \int_\Omega f_\varepsilon\, \phi\, dx\,, \end{aligned} \tag{3.4}$$

for all $T > 0$ and $\phi \in L^2(0,T; H_0^1(\Omega))$.

3.2. A priori estimates.

In this section, we are concerned with a priori estimates satisfied by the sequence u_ε of solutions of (3.4) which lead to compactness properties essential to the proof.

Proposition 1. *Let* A, b, d, g *satisfy* (1.3)-(1.7). *Then, there exist* $\beta > 0$, C_0 *and* C_1 *depending only on* $\|u_0\|_{L^1(\Omega)}$, $\|f\|_{L^1(Q)}$, $\|b\|_{L^\infty(Q)}$, $\|d\|_{L^\infty(Q)}$, $|\Omega|$ *and* T *such that the sequence* u_ε *of solutions of* (3.4) *satisfies*

$$\sup_{\substack{\varepsilon>0\\ t\in(0,T)}} \|u_\varepsilon(t)\|_{L^1(\Omega)} \leq \beta\,, \tag{3.5}$$

and

$$\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt \leq C_0 + C_1 \int_{E_n} |\nabla u_\varepsilon|\,dx\,dt\,. \tag{3.6}$$

In view of Lemma 1, we deduce immediately the following

Corollary 1. *Let* A, b, d, g *satisfy* (1.3)-(1.7). *Let* $1 \leq q < (N+2)/(N+1)$. *Then, there exists* $C > 0$ *depending only on the data, such that*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C\,. \tag{3.7}$$

PROOF OF PROPOSITION 1. Since T_k is a Lipschitz function and $u_\varepsilon \in L^2(0,T;H_0^1(\Omega))$, one has $T_k(u_\varepsilon) \in L^2(0,T;H_0^1(\Omega))$, see [19], [20], with, moreover,

$$\nabla T_k(u_\varepsilon) = \chi_{|u_\varepsilon|\leq k} \nabla u_\varepsilon\ ,$$

where $\chi_{|u_\varepsilon|\leq k}$ denotes the characteristic function of the set $\{(t,x)\in Q : |u_\varepsilon(t,x)| \leq k\}$. Thus, we choose $\phi = T_k(u_\varepsilon)$ as test function in (3.4). Writing $b\cdot\nabla u_\varepsilon = \nabla\cdot(b\,u_\varepsilon) - (\nabla\cdot b)\,u_\varepsilon$, one gets

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega S_k(u_\varepsilon)\,dx + \int_\Omega \chi_{|u_\varepsilon|\leq k} A\nabla u_\varepsilon\cdot\nabla u_\varepsilon\,dx\\
&+ \int_\Omega T_k(u_\varepsilon) g_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\,dx\\
&= \int_\Omega f_\varepsilon T_k(u_\varepsilon)\,dx + \int_\Omega \chi_{|u_\varepsilon|\leq k} u_\varepsilon b\cdot\nabla u_\varepsilon\,dx\\
&\quad + \int_\Omega ((\nabla\cdot b) - d)\,u_\varepsilon T_k(u_\varepsilon)\,dx\,.
\end{aligned} \tag{3.8}$$

By using Holder's and Young's inequalities, one obtains

$$\begin{aligned} &\Big| \int_\Omega \chi_{|u_\varepsilon|\le k} u_\varepsilon b \cdot \nabla u_\varepsilon \, dx \Big| \\ &\le \frac{\underline{a}}{2} \int_\Omega \chi_{|u_\varepsilon|\le k} |\nabla u_\varepsilon|^2 \, dx + \frac{1}{2\,\underline{a}} \, \|b\|^2_{L^\infty(Q)} \int_\Omega \chi_{|u_\varepsilon|\le k} |u_\varepsilon|^2 \, dx \,. \end{aligned} \tag{3.9}$$

Moreover, $u_\varepsilon T_k(u_\varepsilon)$ is non negative and we assume that d and $\nabla \cdot b$ belong to $L^\infty(Q)$. Hence, after integration of (3.8) with respect to t and using (1.5), we are led to

$$\begin{aligned} &\int_\Omega S_k(u_\varepsilon)(t) \, dx + \int_0^t \int_\Omega T_k(u_\varepsilon) g_\varepsilon \, dx \, ds \\ &+ \frac{\underline{a}}{2} \int_0^t \int_\Omega \chi_{|u_\varepsilon|\le k} |\nabla u|^2 \, dx \, ds \\ &\le \int_0^t \int_\Omega |f_\varepsilon T_k(u_\varepsilon)| \, dx \, ds + \int_\Omega S_k(u_{0,\varepsilon}) \, dx \\ &\quad + \frac{1}{2\,\underline{a}} \, \|b\|^2_{L^\infty(Q)} \int_0^t \int_\Omega \chi_{|u_\varepsilon|\le k} |u_\varepsilon|^2 \, dx \, ds \\ &\quad + \big(\|d\|_{L^\infty(Q)} + \|\nabla \cdot b\|_{L^\infty(Q)}\big) \int_0^t \int_\Omega u_\varepsilon T_k(u_\varepsilon) \, dx \, ds \,, \end{aligned} \tag{3.10}$$

where, by the sign assumption (1.6) and the definition of S_k, all the terms in the left hand side of (3.10) are non negative. Next, we observe that

$$\begin{aligned} 0 &\le z^2 \chi_{|z|\le k} \\ &\le z \, T_k(z) \\ &= z^2 \, \chi_{|z|\le k} + k \, |z| \, \chi_{|z|>k} \\ &\le z^2 \, \chi_{|z|\le k} + (2 \, k \, |z| - k^2) \, \chi_{|z|>k} \\ &= 2 \, S_k(z) \,, \end{aligned} \tag{3.11}$$

which yields

$$\begin{aligned} \int_\Omega S_k(u_\varepsilon)(t) \, dx \le &\int_0^t \int_\Omega |f_\varepsilon T_k(u_\varepsilon)| \, dx \, ds + \int_\Omega S_k(u_{0,\varepsilon}) \, dx \\ &+ C(b, d) \int_0^t \int_\Omega S_k(u_\varepsilon) \, dx \, ds \,, \end{aligned} \tag{3.12}$$

where $C(b,d)$ stands for

$$2\left(\|d\|_{L^\infty(Q)} + \|\nabla\cdot b\|_{L^\infty(Q)}\right) + \frac{1}{\underline{a}}\|b\|_{L^\infty(Q)}\,.$$

We set $z(t) = \int_\Omega S_k(u_\varepsilon)(t)\,dx$. Thus, dropping non negative terms, we have

$$0 \le z(t) \le z(0) + \int_0^t \int_\Omega |f_\varepsilon|\,|T_k(u_\varepsilon)|\,dx\,ds + C(b,d)\int_0^t z(s)\,ds$$

and we apply Gronwall's lemma to deduce that

$$z(t) \le e^{C(b,d)T}\Big(\int_\Omega S_k(u_{0,\varepsilon})\,dx + \int_Q |f_\varepsilon|\,|T_k(u_\varepsilon)|\,dx\,dt\Big) \tag{3.13}$$

holds.

We set $k=1$ in (3.13). Remarking that $|T_1(z)| \le 1$ and $0 \le S_1(z) \le |z|$ leads to

$$\int_\Omega S_1(u_\varepsilon)(t)\,dx \le e^{C(b,d)T}\left(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)}\right)$$

by (3.2). Therefore, we end the proof of (3.5) with the following observation

$$\begin{aligned}\int_\Omega |u_\varepsilon|\,dx &= \int_{|u_\varepsilon|\le 1} |u_\varepsilon|\,dx + \int_{|u_\varepsilon|>1} |u_\varepsilon| dx\\ &\le \int_{|u_\varepsilon|\le 1} dx + \int_{|u_\varepsilon|>1}\Big(S_1(u_\varepsilon) + \frac{1}{2}\Big)\,dx\\ &\le \frac{3}{2}|\Omega| + e^{C(b,d)T}\left(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)}\right)\\ &= \beta\,.\end{aligned}$$

To achieve the proof of Proposition 1, we are left with the task of showing that (3.6) holds. According to [4], we introduce the function

$$\phi_n(z) = \begin{cases} 1\,, & \text{if } z \ge n+1\,,\\ z-k\,, & \text{if } n \le z < n+1\,,\\ 0\,, & \text{if } -n < z < n\,,\\ z+k\,, & \text{if } -n-1 < z \le -n\,,\\ -1\,, & \text{if } z \le -n-1\,,\end{cases} \tag{3.14}$$

and we set $\Psi_n(z) = \int_0^z \phi_n(\tau)\,d\tau$. We note that ϕ_n is a Lipschitz function. Thus, we have $\phi_n(u_\varepsilon) \in L^2(0,T;H_0^1(\Omega))$, see [19], [20] with

$$\nabla\phi_n(u_\varepsilon) = \chi_{B_n}\nabla u_\varepsilon\ ,$$

χ_{B_n} denoting the characteristic function of the set $B_n = \{(t,x) \in Q : n \le |u_\varepsilon(t,x)| \le n+1\}$. Then, taking $\phi = \phi_n(u_\varepsilon) \in L^2(0,T;H_0^1(\Omega))$ as test function in (3.4) gives

$$\frac{d}{dt}\int_\Omega \Psi_n(u_\varepsilon)\,dx + \int_\Omega \chi_{B_n} A\nabla u_\varepsilon\cdot\nabla u_\varepsilon\,dx + \int_\Omega g(u_\varepsilon,\nabla u_\varepsilon)\,\phi_n(u_\varepsilon)\,dx$$
$$= \int_\Omega f_\varepsilon\,\phi_n(u_\varepsilon)\,dx - \int_\Omega du_\varepsilon\,\phi_n(u_\varepsilon)\,dx - \int_\Omega b\cdot\nabla u_\varepsilon\,\phi_n(u_\varepsilon)\,dx\,.$$

Thus, integrating the above equation with respect to t, we have

$$\begin{aligned}
&\int_\Omega \Psi_n(u_\varepsilon)(t)\,dx + \int_0^t\int_\Omega \chi_{B_n} A\nabla u_\varepsilon\cdot\nabla u_\varepsilon\,dx\,dt \\
&+ \int_0^t\int_\Omega g(u_\varepsilon,\nabla u_\varepsilon)\,\phi_n(u_\varepsilon)\,dx\,dt \\
&= \int_\Omega \Psi_n(u_{0,\varepsilon})\,dx + \int_0^t\int_\Omega f_\varepsilon\phi_n(u_\varepsilon)\,dx\,dt \\
&- \int_0^t\int_\Omega du_\varepsilon\,\phi_n(u_\varepsilon)\,dx\,dt \\
&- \int_0^t\int_\Omega b\cdot\nabla u_\varepsilon\,\phi_n(u_\varepsilon)\,dx\,dt\,.
\end{aligned} \tag{3.15}$$

Since $|\phi_n(z)| \le 1$, and taking into account the estimate (3.5) we have

$$\left|\int_\Omega du_\varepsilon\,\phi_n(u_\varepsilon)\,dx\right| \le \beta\,\|d\|_{L^\infty(Q)}\,. \tag{3.16}$$

Furthermore, we remark that $u_\varepsilon\,\phi_n(u_\varepsilon) \ge 0$. Then, the third term in the left side is non negative. From the coercivity of A (see (1.5)), the positivity of $\Psi_n(\cdot)$ and (3.15) we deduce that

$$\begin{aligned}
\underline{a}\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt &\le \int_Q |\phi_n(u_\varepsilon)f_\varepsilon|\,dx\,dt + \int_\Omega \Psi_n(u_{0,\varepsilon})\,dx \\
&\quad + \beta\,\|d\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)}\int_Q |\nabla u_\varepsilon|\,|\phi_n(u_\varepsilon)|\,dx\,dt \\
&\le \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\,\|d\|_{L^\infty(Q)} \\
&\quad + \|b\|_{L^\infty(Q)}\int_Q |\nabla u_\varepsilon|\,|\phi_n(u_\varepsilon)|\,dx\,dt\,.
\end{aligned} \tag{3.17}$$

Let us split the last integral in (3.17) as follows

$$\begin{aligned}\int_Q |\nabla u_\varepsilon|\,|\phi_n(u_\varepsilon)|\,dx\,dt &= \int_{B_n} |\nabla u_\varepsilon|\,|\phi_n(u_\varepsilon)|\,dx\,dt + \int_{E_n} |\nabla u_\varepsilon|\,dx\,dt \\ &\le \int_{B_n} |\nabla u_\varepsilon|\,dx\,dt + \int_{E_n} |\nabla u_\varepsilon|\,dx\,dt\,, \end{aligned} \tag{3.18}$$

since $|\phi_n(u_\varepsilon)| = 1$ on $E_n = \{(t,x) \in Q : \ |u(t,x)| > n+1\}$ and $\phi_n(u_\varepsilon) = 0$ if $|u_\varepsilon(t,x)| < n$.

Using the fact $0 \le \Psi_n(z) \le |z|$ and (3.18), we deduce from (3.17) that

$$\begin{aligned}\underline{a}\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt &\le \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\,\|d\|_{L^\infty(Q)} \\ &\quad + \|b\|_{L^\infty(Q)}\Big(\int_{B_n} |\nabla u_\varepsilon|\,dx\,dt + \int_{E_n} |\nabla u_\varepsilon|\,dx\,dt\Big)\,.\end{aligned}$$

By using Holder's and Young's inequalities, we have

$$\begin{aligned}&\underline{a}\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt \\ &\quad\le C + \frac{\underline{a}}{2}\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt + \frac{1}{2\,\underline{a}}\,\|b\|^2_{L^\infty(Q)}\,T\,|\Omega| + \int_{E_n} |\nabla u_\varepsilon|\,dx\,dt\,,\end{aligned}$$

where $C = \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\,\|d\|_{L^\infty(Q)}$. This finishes the proof of (3.6) with C_0 and C_1 depending on $\|f\|_{L^1(Q)}$, $\|u_0\|_{L^1(Q)}$, $\|b\|_{L^\infty(Q)}$, $\|d\|_{L^\infty(Q)}$, $\underline{a}$, $|\Omega|$, T and the bound β.

Now, we are interested in the nonlinear term g_ε. We have

Lemma 2. *Suppose A, b, d, g satisfy* (1.3)-(1.7) *and let u_ε be a sequence of solutions of* (3.4). *Then, there exists $C > 0$ depending only on the data such that the sequence $g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ satisfies*

$$\sup_{\varepsilon>0} \|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1(Q)} \le C\,, \tag{3.19}$$

$$\lim_{k\to\infty}\ \sup_{\varepsilon>0} \int_{|u_\varepsilon|>k} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)|\,dx\,dt = 0\,. \tag{3.20}$$

PROOF. It is clear that

$$\begin{aligned}\int_{|u_\varepsilon|>n+1} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)|\, dx\, dt &= \int_{|u_\varepsilon|>n+1} \phi_n(u_\varepsilon)\, g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\, dx\, dt \\ &\leq \int_Q \phi_n(u_\varepsilon)\, g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\, dx\, dt\,,\end{aligned} \tag{3.21}$$

since we recall that $|\phi_n(z)| = 1$ when $|z| > n+1$ and $\phi_n(u_\varepsilon)\, g_\varepsilon$ is non negative by the sign condition (1.6). In the sequel, we will often write $g_\varepsilon = g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ when no confusion can arise. From the positivity of the first and the second terms in (3.15), we obtain

$$\begin{aligned}0 &\leq \int_Q \phi_n(u_\varepsilon)\, g_\varepsilon\, dx\, dt \\ &\leq \Big|\int_Q \phi_n(u_\varepsilon)\, f_\varepsilon\, dx\, dt\Big| + \Big|\int_\Omega \Psi_n(u_{0,\varepsilon})\, dx\Big| \\ &\quad + \Big|\int_Q d\,\phi_n(u_\varepsilon)\, u_\varepsilon\, dx\, dt\Big| + \Big|\int_Q \phi_n(u_\varepsilon) b\cdot \nabla u_\varepsilon\, dx\, dt\Big| \\ &\leq \|f_\varepsilon\|_{L^1(Q)} + \|u_{0,\varepsilon}\|_{L^1(\Omega)} + \|d\|_{L^\infty(Q)} \int_Q |u_\varepsilon|\, dx\, dt \\ &\quad + \|b\|_{L^\infty(Q)} \int_Q |\nabla u_\varepsilon|\, dx\, dt\end{aligned} \tag{3.22}$$

since $|\phi_n(z)| \leq 1$ and $0 \leq \Psi_n(z) \leq |z|$. By (3.2), (3.5) and the estimate (3.7) with $q = 1$, we deduce

$$\int_{|u_\varepsilon|>n+1} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)|\, dx\, dt \leq C\,. \tag{3.23}$$

It remains to evaluate the integral over $\{|u_\varepsilon| < n+1\}$. Assumption (1.6) yields

$$\begin{aligned}&\int_{|u_\varepsilon|<n+1} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)|\, dx\, dt \\ &\qquad \leq h\,(n+1) \int_{|u_\varepsilon|<n+1} (|\nabla u_\varepsilon|^\sigma + \gamma(t,x))\, dx\, dt\,,\end{aligned} \tag{3.24}$$

where we estimate as follows

$$\begin{aligned}\int_{|u_\varepsilon|<n+1} |\nabla u_\varepsilon|^\sigma \, dx\, dt &= \sum_{j=0}^{n} \int_{B_j} |\nabla u_\varepsilon|^\sigma \, dx\, dt \\ &\leq \sum_{j=0}^{n} |B_j|^{1-\sigma/2} \Big(\int_{B_j} |\nabla u_\varepsilon|^2 \, dx\, dt \Big)^{\sigma/2} .\end{aligned} \tag{3.25}$$

By using (3.6), (3.7) and Holder's inequality, we get

$$\begin{aligned}&\int_{|u_\varepsilon|<n+1} |\nabla u_\varepsilon|^\sigma \, dx\, dt \\ &\quad \leq (T\,|\Omega|)^{1-\sigma/2} \sum_{j=0}^{n} \Big(C_0 + C_1 \int_Q |\nabla u_\varepsilon| \, dx\, dt \Big)^{\sigma/2} \leq C \, .\end{aligned} \tag{3.26}$$

Combining (3.26) with (3.24) and (3.23), we conclude that g_ε is bounded in $L^1(Q)$ uniformly in ε.

We turn to the proof of (3.20). Obviously, one has

$$\int_{|u_\varepsilon|>k} |g_\varepsilon| \, dx\, dt \leq \frac{1}{k} \int_Q T_k(u_\varepsilon)\, g_\varepsilon \, dx\, dt \, . \tag{3.27}$$

Similarly, replacing $\phi_n(u_\varepsilon)$ by $T_k(u_\varepsilon)$ in (3.15) we obtain, similarly to (3.22), that

$$\begin{aligned} 0 &\leq \int_Q T_k(u_\varepsilon)\, g_\varepsilon \, dx\, dt \\ &\leq \int_Q |f_\varepsilon T_k(u_\varepsilon)| \, dx\, dt + \int_\Omega S_k(u_{0,\varepsilon}) \, dx \\ &\quad + \|d\|_{L^\infty(Q)} \int_Q T_k(u_\varepsilon)\, u_\varepsilon \, dx\, dt \\ &\quad + \|b\|_{L^\infty(Q)} \int_Q |T_k(u_\varepsilon)|\, |\nabla u_\varepsilon| \, dx\, dt \end{aligned} \tag{3.28}$$

holds. Let $M > 0$. According to [16], we use the following trick

$$\begin{cases} 0 \leq S_k(z) \leq M^2 + k\,|z|\, \chi_{|z|>M} \, , \\ |T_k(z)| \leq M + k\, \chi_{|z|>M} \, . \end{cases} \tag{3.29}$$

which gives

$$
\begin{aligned}
\int_{|u_\varepsilon|>k} |g_\varepsilon| \,dx\,dt \le{}& \frac{M}{k} \|f_\varepsilon\|_{L^1(Q)} + \int_{|u_\varepsilon|>M} |f_\varepsilon| \,dx\,dt \\
&+ \frac{M^2}{k} \|u_{0,\varepsilon}\|_{L^1(\Omega)} + \int_{|u_{0,\varepsilon}|>M} |u_{0,\varepsilon}| \,dx \\
&+ \frac{M}{k} \|d\|_{L^\infty(Q)} \|u_\varepsilon\|_{L^1(Q)} + \int_{|u_\varepsilon|>M} |u_\varepsilon| \,dx\,dt \\
&+ \frac{M}{k} \|b\|_{L^\infty(Q)} \|\nabla u_\varepsilon\|_{L^1(Q)} + \int_{|u_\varepsilon|>M} |\nabla u_\varepsilon| \,dx\,dt\,,
\end{aligned}
\tag{3.30}
$$

by (3.28) and (3.29). Since, on the one hand, u_ε is bounded in $L^q(0,T, W_0^{1,q}(\Omega))$ for some $q>1$ and f_ε, $u_{0,\varepsilon}$ are convergent sequences in $L^1(Q)$, $L^1(\Omega)$ respectively, and, on the other hand,

$$
\sup_{\varepsilon>0} \text{meas}\,\{(t,x)\in Q:\ |u_\varepsilon(t,x)|>M\} \le \frac{1}{M} \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^1(Q)} \le \frac{\beta T}{M}
$$

tends to 0 as M goes to ∞, we can choose M large enough so that the terms

$$
\sup_{\varepsilon>0} \int_{|u_\varepsilon|>M} |f_\varepsilon| \,dx\,dt\,,
$$

$$
\sup_{\varepsilon>0} \int_{|u_{0,\varepsilon}|>M} |u_{0,\varepsilon}| \,dx\,,
$$

$$
\sup_{\varepsilon>0} \int_{|u_\varepsilon|>M} |u_\varepsilon| \,dx\,dt\,,
$$

$$
\sup_{\varepsilon>0} \int_{|u_\varepsilon|>M} |\nabla u_\varepsilon| \,dx\,dt\,,
$$

are arbitrarily smalls, which, achieves the proof of (3.20).

Let the assumptions of Proposition 1 be fulfilled. Then, u_ε is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$, g_ε is bounded in $L^1(Q)$ which imply, in view of the equation satisfied by u_ε that $\partial_t u_\varepsilon$ is bounded in $L^1(0,T; W^{-1,q}(\Omega)) + L^1(Q)$. Therefore, possibly at the cost of extracting subsequences, see *e.g.* [18], [20] we can assume that

$$(3.31)\qquad \begin{cases} u_\varepsilon \longrightarrow u\,, & \text{strongly in } L^q(Q) \\ & \text{and almost everywhere in } Q\,, \\ |u_\varepsilon(t,x)| \le \Gamma(t,x)\,, & \text{almost everywhere in } Q, \\ & \text{with } \Gamma \in L^q(Q)\,, \\ \nabla u_\varepsilon \rightharpoonup \nabla u\,, & \text{weakly in } L^q(Q)\,. \end{cases}$$

3.3. Convergence almost everywhere of the gradients.

The weak convergence of the gradients is clearly insufficient to pass to the limit when $\varepsilon \longrightarrow 0$ in nonlinear terms. Then, we claim

Lemma 3. *Let the assumptions of Proposition* 1 *be fulfilled and let* u_ε *satisfy* (3.31). *Then, the sequence* $\{\nabla u_\varepsilon\}_\varepsilon$ *converges to* ∇u *almost everywhere as* ε *goes to zero.*

PROOF. It suffices to show that $\{\nabla u_\varepsilon\}_\varepsilon$ is a Cauchy sequence in measure, see [8], *i.e.* for all $\mu > 0$

$$(3.32)\qquad \operatorname{meas}\{(t,x) \in Q :\ |\nabla u_{\varepsilon'} - \nabla u_\varepsilon| > \mu\} \longrightarrow 0\,,$$

as $\varepsilon', \varepsilon \longrightarrow 0$. Let us denote by $\mathcal{A}$ the subset of Q involved in (3.32). Let $k > 0$ and $\delta > 0$. Following [17], we remark that

$$(3.33)\qquad \mathcal{A} \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4\ ,$$

where

$$(3.34)\qquad \begin{aligned} \mathcal{A}_1 &= \{(t,x) \in Q :\ |\nabla u_\varepsilon| \ge k\}\,, \\ \mathcal{A}_2 &= \{(t,x) \in Q :\ |\nabla u_{\varepsilon'}| \ge k\}\,, \\ \mathcal{A}_3 &= \{(t,x) \in Q :\ |u_\varepsilon - u_{\varepsilon'}| \ge \delta\}\,, \\ \mathcal{A}_4 &= \{(t,x) \in Q :\ |\nabla u_\varepsilon - \nabla u_{\varepsilon'}| \ge \mu :\ |\nabla u_\varepsilon| \le k, \\ &\qquad |\nabla u_{\varepsilon'}| \le k,\ |u_\varepsilon - u_{\varepsilon'}| \le \delta\}\,. \end{aligned}$$

By Corollary 1 and (3.31), we conclude easily for the three first sets. Indeed, one has

$$|\mathcal{A}_1| \le \frac{1}{k}\,\|\nabla u_\varepsilon\|_{L^1(Q)} \le \frac{C}{k}$$

and an analoguous estimate holds for $\mathcal{A}_2$. Hence, by choosing k large enough, $|\mathcal{A}_1| + |\mathcal{A}_2|$ is arbitrarily small. Similarly, one gets

$$|\mathcal{A}_3| \le \frac{1}{\delta} \|u_\varepsilon - u_{\varepsilon'}\|_{L^1(Q)}$$

which, for $\delta > 0$ fixed, tends to 0 when $\varepsilon, \varepsilon' \longrightarrow 0$ since, by (3.31), u_ε is a Cauchy sequence in $L^1(Q)$. Then, the proof is completed by choosing δ so that $|\mathcal{A}_4|$ is given arbitrarily small, uniformly with respect to $\varepsilon, \varepsilon'$. To this end, we shall use the equations satisfied by u_ε and $u_{\varepsilon'}$. Indeed, we observe that

$$\begin{aligned} |\mathcal{A}_4| &\le \frac{1}{\mu^2} \int_{\mathcal{A}_4} |\nabla u_\varepsilon - \nabla u_{\varepsilon'}|^2 \,dx\,dt \\ &\le \frac{1}{\mu^2} \int_{|u_\varepsilon - u_{\varepsilon'}| \le \delta} |\nabla u_\varepsilon - \nabla u_{\varepsilon'}|^2 \,dx\,dt \\ &= \frac{1}{\mu^2} \int_Q |\nabla (T_\delta(u_\varepsilon - u_{\varepsilon'}))|^2 \,dx\,dt\,. \end{aligned} \tag{3.35}$$

Substracting the relations obtained with $\phi = T_\delta(u_\varepsilon - u_{\varepsilon'})$ as test funtion in equation (3.4) satisfied successively by u_ε and $u_{\varepsilon'}$ leads to

$$\begin{aligned} &\frac{d}{dt} \int_\Omega S_\delta(u_\varepsilon - u_{\varepsilon'})\,dx + \int_\Omega A(\nabla u_\varepsilon - \nabla u_{\varepsilon'})\,\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx \\ &\quad= \int_\Omega (f_\varepsilon - f_{\varepsilon'})\,T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx \\ &\quad\quad- \int_\Omega (d(u_\varepsilon - u_{\varepsilon'}) - b\,\nabla(u_\varepsilon - u_{\varepsilon'}))\,T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx \\ &\quad\quad- \int_\Omega (g_\varepsilon - g_{\varepsilon'})\,T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx\,. \end{aligned} \tag{3.36}$$

Since $|T_\delta(z)| \le \delta$ and $0 \le S_\delta(z) \le \delta\,|z|$, integrating (3.36) with respect to t and using the coercivity of A (see (1.5)) yield

$$\begin{aligned} &\underline{a} \int_Q |\nabla(T_\delta(u_\varepsilon - u_{\varepsilon'}))|^2 \,dx\,dt \\ &\le \delta\,(\|f_\varepsilon - f_{\varepsilon'}\|_{L^1(Q)} + \|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_{L^1(\Omega)} \\ &\quad + \|d\|_{L^\infty(Q)} \|u_\varepsilon - u_{\varepsilon'}\|_{L^1(Q)} + \|b\|_{L^\infty(Q)} \|\nabla(u_\varepsilon - u_{\varepsilon'})\|_{L^1(Q)} \\ &\quad + \|g_\varepsilon - g_{\varepsilon'}\|_{L^1(Q)})\,. \end{aligned} \tag{3.37}$$

Therefore, by using (3.2) and the bounds (3.7), uniform in ε, on $\|u_\varepsilon\|_{L^1(Q)}$, $\|\nabla u_\varepsilon\|_{L^1(Q)}$ and on $\|g_\varepsilon\|_{L^1(Q)}$, we deduce from (3.37) that

$$\underline{a}\int_Q |\nabla(T_\delta(u_\varepsilon - u_{\varepsilon'}))|^2\,dx\,dt \le 2\,\delta\,(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}) \\ + 2\,\delta\,C\,(1 + \|b\|_{L^\infty(Q)} + \|d\|_{L^\infty(Q)})\,, \tag{3.38}$$

goes to zero as δ goes to zero, uniformly in $\varepsilon, \varepsilon'$. This completes the proof of Lemma 3.

Having disposed of the proof of Lemma 3, let us consider the behaviour of g_ε as ε goes to 0, when it is assumed that $0 \le \sigma < 2$.

Corollary 2. *Let the assumptions of Proposition* 1 *be fulfilled and let* u_ε *satisfy* (3.31). *Then,* (*up to subsequences*) *the sequence* $\{g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\}_\varepsilon$ *converges to* $g(u, \nabla u)$ *almost everywhere in* Q *and strongly in* $L^1(Q)$.

PROOF. This result is similar to those obtained in [9] in the context of elliptic problems. For the sake of completeness, we sketch the proof. By combining Lemma 3 and (3.31), it is clear that

$$g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \longrightarrow g(u, \nabla u)$$

almost everywhere in Q as ε tends to 0, since $g(t, x, \lambda, \xi)$ is a continuous function with respect to $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^N$. Thus, by classical results, see *e.g.* [8], the sequence g_ε will be actually strongly convergent in $L^1(Q)$ if one shows that g_ε lies in weakly compact set in $L^1(Q)$. This property follows from (3.20) since $0 \le \sigma < 2$. Indeed, let $\mathcal{A}$ be a measurable set in Q. We split

$$\int_{\mathcal{A}} |g_\varepsilon|\,dx\,dt = \int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} |g_\varepsilon|\,dx\,dt + \int_{\mathcal{A}\cap\{|u_\varepsilon|>k\}} |g_\varepsilon|\,dx\,dt\,, \tag{3.39}$$

where it is clear that

$$\int_{\mathcal{A}\cap\{|u_\varepsilon|>k\}} |g_\varepsilon|\,dx\,dt \le \int_{|u_\varepsilon|>k} |g_\varepsilon|\,dx\,dt$$

tends to 0, uniformly in ε as $k \longrightarrow \infty$, by (3.20). Moreover, the growth condition (1.6) and Holder's inequality yield

$$
\begin{aligned}
&\int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} |g_\varepsilon|\,dx\,dt\\
&\le h(k)\Big(\int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} |\nabla u_\varepsilon|^\sigma\,dx\,dt + \int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} \gamma(t,x)\,dx\,dt\Big)\\
&\le h(k)\Big(\int_{|u_\varepsilon|\le k} |\nabla u_\varepsilon|^2\,dx\,dt\Big)^{\sigma/2} |\mathcal{A}|^{(2-\sigma)/2} + h(k)\int_{\mathcal{A}} \gamma(t,x)\,dx\,dt\\
&\le h(k) C_k |\mathcal{A}|^{(2-\sigma)/2} + h(k)\int_{\mathcal{A}} \gamma(t,x)\,dx\,dt
\end{aligned}
\tag{3.40}
$$

by using (3.6) and (3.7) as in (3.26). Since $\sigma < 2$ and $\gamma \in L^1(Q)$, the right hand side of this last inequality goes to 0 as $|\mathcal{A}| \longrightarrow 0$. We conclude that

$$\lim_{|\mathcal{A}|\to 0} \sup_{\varepsilon>0} \int_{\mathcal{A}} |g_\varepsilon|\,dx\,dt = 0\,,$$

which completes the proof of Corollary 2.

3.4. Cauchy property in $C^0(0,T;L^1(\Omega))$ and passage to the limit.

We end our review of the properties of the sequence u_ε with the following result.

Lemma 4. *Let the assumptions of Proposition* 1 *be fulfilled. We assume that the sequence* $\{u_\varepsilon\}_\varepsilon$ *satisfies* (3.31). *Then,* $\{u_\varepsilon\}_\varepsilon$ *is a Cauchy sequence in* $C^0(0,T;L^1(\Omega))$.

PROOF. We set $w_{\varepsilon,\varepsilon'} = u_\varepsilon - u_{\varepsilon'}$, $F_{\varepsilon,\varepsilon'} = f_\varepsilon - f_{\varepsilon'}$ and $G_{\varepsilon,\varepsilon'} = g_\varepsilon - g_{\varepsilon'}$. We multiply the equations (3.4) satisfied respectively by u_ε and $u_{\varepsilon'}$ by $T_1(w_{\varepsilon,\varepsilon'})$. Substracting the obtained relations yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega S_1(w_{\varepsilon,\varepsilon'})\,dx + \int_{|w_{\varepsilon,\varepsilon'}|\le 1} A\nabla w_{\varepsilon,\varepsilon'}\cdot\nabla w_{\varepsilon,\varepsilon'}\,dx\\
&= \int_\Omega F_{\varepsilon,\varepsilon'} T_1(w_{\varepsilon,\varepsilon'})\,dx - \int_\Omega G_{\varepsilon,\varepsilon'} T_1(w_{\varepsilon,\varepsilon'})\,dx\\
&\quad - \int_\Omega (b\cdot\nabla w_{\varepsilon,\varepsilon'} + d\,w_{\varepsilon,\varepsilon'})\,T_1(w_{\varepsilon,\varepsilon'})\,dx\,.
\end{aligned}
\tag{3.41}
$$

Since $0 \le z\,T_1(z) = z^2\,\chi_{|z|\le 1} + |z|\,\chi_{|z|>1} \le z^2\,\chi_{|z|\le 1} + (2\,|z|-1)\,\chi_{|z|>1} = 2\,S_1(w_{\varepsilon,\varepsilon'})$, one gets

$$\Big|\int_\Omega d\,w_{\varepsilon,\varepsilon'} T_1(w_{\varepsilon,\varepsilon'})\,dx\Big| \le 2\,\|d\|_{L^\infty(Q)} \int_\Omega S_1(w_{\varepsilon,\varepsilon'})\,dx\,.$$

Moreover, one has $|T_1(w_{\varepsilon,\varepsilon'})| \le 1$. Then, integrating (3.41) between 0 and t and from the positivity of A, it follows

$$\begin{aligned}
&\int_\Omega S_1(w_{\varepsilon,\varepsilon'})(t)\,dx \\
&\le \int_\Omega S_1(w^0_{\varepsilon,\varepsilon'})\,dx + \int_0^t\int_\Omega |F_{\varepsilon,\varepsilon'}|dx\,ds + \int_0^t\int_\Omega |G_{\varepsilon,\varepsilon'}|\,dx\,ds \\
&\quad + \|b\|_{L^\infty(Q)}\int_0^t\int_\Omega |\nabla w_{\varepsilon,\varepsilon'}|\,dx\,ds + 2\,\|d\|_{L^\infty(Q)}\int_0^t\int_\Omega S_1(w_{\varepsilon,\varepsilon'})\,dx\,ds\,,
\end{aligned} \tag{3.42}$$

where $w^0_{\varepsilon,\varepsilon'} = u_{0,\varepsilon} - u_{0,\varepsilon'}$. Hence, Gronwall's lemma implies that

$$\int_\Omega S_1(w_{\varepsilon,\varepsilon'})\,dx \le a_{\varepsilon,\varepsilon'}\,, \tag{3.43}$$

where $a_{\varepsilon,\varepsilon'}$ stands for

$$\begin{aligned}
a_{\varepsilon,\varepsilon'} &= e^{CT}\Big(\int_\Omega S_1(w^0_{\varepsilon,\varepsilon'})\,dx + \int_Q |F_{\varepsilon,\varepsilon'}|\,dx\,dt \\
&\qquad + \int_Q |G_{\varepsilon,\varepsilon'}|\,dx\,dt + \int_Q |\nabla w_{\varepsilon,\varepsilon'}|\,dx\,dt\Big) \\
&\le e^{CT}\,(\|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_{L^1(\Omega)} + \|f_\varepsilon - f_{\varepsilon'}\|_{L^1(Q)} \\
&\qquad + \|g_\varepsilon - g_{\varepsilon'}\|_{L^1(Q)} + \|\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|_{L^1(Q)})\,,
\end{aligned}$$

since $S_1(z) \le |z|$. By (3.1), $u_{0,\varepsilon}$ and f_ε are convergent sequences in $L^1(\Omega)$ and $L^1(Q)$, respectively and by Corollary 2, g_ε is a convergent sequence in $L^1(Q)$. Furthermore, by Corollary 1 and Lemma 3, ∇u_ε is both bounded in $L^q(Q)$ and almost everywhere in Q convergent, which implies that ∇u_ε is actually strongly convergent in $L^p(Q)$ for $1 \le p < q$, and in particular in $L^1(Q)$. Hence, it is clear that $a_{\varepsilon,\varepsilon'}$ tends to 0 as

$\varepsilon, \varepsilon' \longrightarrow 0$. Finally, by Holder's inequality, we have

$$\begin{aligned}
&\int_\Omega |w_{\varepsilon,\varepsilon'}|\,dx \\
&= \int_{|w_{\varepsilon,\varepsilon'}|\le 1} |w_{\varepsilon,\varepsilon'}|\,dx + \int_{|w_{\varepsilon,\varepsilon'}|>1} |w_{\varepsilon,\varepsilon'}|\,dx \\
&\le \Big(\int_{|w_{\varepsilon,\varepsilon'}|\le 1} |w_{\varepsilon,\varepsilon'}|^2\,dx\Big)^{1/2}\Big(\int_{|w_{\varepsilon,\varepsilon'}|\le 1} \mathbf{1}\,dx\Big)^{1/2} + \int_{|w_{\varepsilon,\varepsilon'}|>1} |w_{\varepsilon,\varepsilon'}|\,dx \\
&\le \sqrt{|\Omega|}\,\Big(\int_{|w_{\varepsilon,\varepsilon'}|\le 1} 2\,S_1(w_{\varepsilon,\varepsilon'})\,dx\Big)^{1/2} + \int_{|w_{\varepsilon,\varepsilon'}|>1} 2\,S_1(w_{\varepsilon,\varepsilon'})\,dx\,,
\end{aligned}$$

since

$$\frac{|z|}{2}\,\chi_{|z|>1} \le \Big(\frac{|z|}{2} + \frac{|z|-1}{2}\Big)\,\chi_{|z|>1} = S_1(w_{\varepsilon,\varepsilon'})\,\chi_{|z|>1}$$

and

$$\frac{|z|^2}{2}\,\chi_{|z|\le 1} = S_1(z)\,\chi_{|z|\le 1}\;.$$

By (3.43), we deduce that

$$\int_\Omega |u_\varepsilon - u_{\varepsilon'}|\,dx = \int_\Omega |w_{\varepsilon,\varepsilon'}|\,dx \le \sqrt{2\,|\Omega|}\,\sqrt{a_{\varepsilon,\varepsilon'}} + 2\,a_{\varepsilon,\varepsilon'}$$

tends to 0 as $\varepsilon, \varepsilon' \longrightarrow 0$ which proves that u_ε is a Cauchy sequence in $C^0(0,T;L^1(Q))$.

Finally, we achieve the proof of Theorem 1 by passing easily to the limit $\varepsilon \longrightarrow 0$ in the following weak formulation

$$\begin{aligned}
&\int_\Omega u_\varepsilon \phi(t)\,dx - \int_\Omega u_{0,\varepsilon}\phi(0,x)\,dx \\
&- \int_0^t \int_\Omega u_\varepsilon \partial_t \phi\,dx\,dt + \int_0^t \int_\Omega A\nabla u_\varepsilon \cdot \nabla\phi\,dx\,dt \\
&+ \int_0^t \int_\Omega \Big(b\cdot\nabla u_\varepsilon + d\,u_\varepsilon + g_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\Big)\phi\,dx\,dt \\
&= \int_0^t \int_\Omega f_\varepsilon \phi\,dx\,dt\,,
\end{aligned} \tag{3.44}$$

with $\phi \in C^0(0,T,W_0^{1,q'}(\Omega)) \cap C^1(0,T,L^{q'}(\Omega))$, obtaining in this way that the limit u is a solution of (1.1) in the sense of (1.9).

REMARK 1. We point out the fact that the assumption on the derivatives of the coefficient b is useful uniquely to obtain the uniform bound (3.5) in $L^\infty(0,T;L^1(\Omega))$.

REMARK 2. A similar existence result may be obtained if the strong convergences in (3.1) are replaced by weak L^1 convergences.

4. Entropic solutions: End of proof of Theorem 2.

In this Section, we assume $g = 0$. First, we prove that, besides the weak "natural" formulation (1.9), the limit u of the sequence of approximate solutions u_ε also satisfies the entropic relation (1.11). Having disposed of the existence of such a solution, we show that u is unique in the class of entropic solutions.

4.1. Existence of entropic solution.

Let us recall the convergence properties obtained in Section 3 on the sequence u_ε, after suitable extraction of subsequences. First, u_ε converges to u strongly in $L^q(Q)$, with $1 \le q < (N+2)/(N+1)$, in $C^0(0,T;L^1(\Omega))$, almost everywhere in Q and is dominated. Moreover, ∇u_ε is bounded in $L^q(Q)$ and converges almost everywhere in Q to ∇u; thus, the convergence actually holds strongly in $L^p(Q)$, for $1 \le p < q$ and in particular in $L^1(Q)$. We can also assume that ∇u_ε is dominated. Let $k > 0$. Since T_k is continuous and bounded by k, $T_k(u_\varepsilon)$ converges almost everywhere in Q and, by Lebesgue's theorem, strongly in $L^2(Q)$ to $T_k(u)$. Furthermore, from (3.10) it is easy to see that $\nabla T_k(u_\varepsilon)$ is bounded in $L^2(Q)$ (uniformly in ε, the bound depending on k). Therefore, we may suppose that $\nabla T_k(u_\varepsilon) \rightharpoonup \nabla T_k(u)$ weakly in $L^2(Q)$.

Fix $k > 0$ and let $\psi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ with $\partial_t\psi \in L^2(0,T;H^{-1}(\Omega))$. We set $P = \|\psi\|_{L^\infty(Q)}$. It is clear that $|T_k(u_\varepsilon - \psi)| \le k$ and

$$(4.1)\quad |\nabla T_k(u_\varepsilon - \psi)| = \chi_{|u_\varepsilon - \psi| \le k} \, |\nabla(u_\varepsilon - \psi)| \le \chi_{|u_\varepsilon| \le k+P} \, |\nabla u_\varepsilon| + |\nabla \psi|\,,$$

which implies that $T_k(u_\varepsilon - \psi)$ belongs to (a bounded set in) $L^2(0,T;$

$H_0^1(\Omega))$. Then, plugging $\phi = T_k(u_\varepsilon - \psi)$ in (3.4) gives

$$\begin{aligned}
&\int_\Omega S_k(u_\varepsilon - \psi)(T)\,dx - \int_\Omega S_k(u_{0,\varepsilon} - \psi(0,\cdot))\,dx \\
&+ \int_0^T \langle \partial_t \psi, T_k(u_\varepsilon - \psi)\rangle\,ds \\
&+ \int_Q A\nabla u_\varepsilon \nabla T_k(u_\varepsilon - \psi)\,dx\,ds \\
&+ \int_Q (b\cdot\nabla u_\varepsilon + d\,u_\varepsilon)\,T_k(u_\varepsilon - \psi)\,dx\,ds \\
&= \int_Q f_\varepsilon T_k(u_\varepsilon - \psi)\,dx\,ds\,.
\end{aligned} \tag{4.2}$$

We shall study the behaviour of (4.2) when we let ε go to 0. Since S_k is k-Lipschitz, one has

$$\Big|\int_\Omega S_k(u_\varepsilon - \psi) - S_k(u - \psi)\,dx\Big| \le k\int_\Omega |u_\varepsilon - u|\,dx\,,$$

for all $t \in [0,T]$ where the right hand side tends to 0 as $\varepsilon \longrightarrow 0$. Next, since we have assumed that $\partial_t\psi$ lies in $L^2(0,T;H^{-1}(\Omega))$, we have to prove that

$$T_k(u_\varepsilon - \psi) \rightharpoonup T_k(u - \psi)\,, \qquad \text{in } L^2(0,T;H_0^1(\Omega))\,. \tag{4.3}$$

Obviously, this convergence holds in $L^2(Q)$ since u_ε converges to u almost everywhere in Q and T_k is continuous and bounded by k. Derivating $T_k(u_\varepsilon - \psi)$ leads to

$$\begin{aligned}
\nabla T_k(u_\varepsilon - \psi) &= \nabla T_k(T_{k+P}(u_\varepsilon) - \psi) \\
&= \chi_{|T_{k+P}(u_\varepsilon)-\psi|\le k}(\nabla T_{k+P}(u_\varepsilon) - \nabla\psi)\,,
\end{aligned} \tag{4.4}$$

where, by the above mentioned convergences, $\nabla T_{k+P}(u_\varepsilon)$ converges weakly in $L^2(Q)$ to $\nabla T_{k+P}(u)$ which proves (4.3). We also deal easily with the terms involving b, d and f_ε since it appears in these integrals a product of the sequence $T_k(u_\varepsilon - \psi)$ which converges almost everywhwere in Q and is bounded in $L^\infty(Q)$ with a sequence which converges at least weakly in $L^1(Q)$. Finally, it remains to show that

$$\int_Q A\nabla u\cdot\nabla T_k(u-\psi)\,dx\,ds \le \liminf_{\varepsilon\to 0}\int_Q A\nabla u_\varepsilon\cdot\nabla T_k(u_\varepsilon-\psi)\,dx\,ds\,. \tag{4.5}$$

By using (4.4), we split the integral in the right hand side as follows

$$\begin{aligned}
&\int_Q A\nabla u_\varepsilon \cdot \nabla T_k(u_\varepsilon - \psi)\, dx\, ds \\
&\qquad = \int_Q \chi_{|T_{k+P}(u_\varepsilon)-\psi|\le k} A\nabla u_\varepsilon \cdot \nabla T_{k+P}(u_\varepsilon)\, dx\, ds \\
&\qquad\quad - \int_Q \chi_{|T_{k+P}(u_\varepsilon)-\psi|\le k} A\nabla u_\varepsilon \cdot \nabla \psi\, dx\, ds \\
&\qquad = A_\varepsilon - B_\varepsilon\,,
\end{aligned}$$

where, by the same argument as above, we have

$$\lim_{\varepsilon\to 0} B_\varepsilon = \int_Q \chi_{|T_{k+P}(u)-\psi|\le k} A\nabla u \cdot \nabla\psi\, dx\, ds\,.$$

Therefore (4.5) is a consequence of Fatou's lemma, applied by combining (4.3) with T_{k+P} and the positiveness property (1.5). Finally, letting $\varepsilon \longrightarrow 0$ in (4.2), one gets (1.11).

4.2. Uniqueness.

Let v be an entropic solution. To obtain the uniqueness, we will show that $v = u$, u still being the solution obtained by approximation. To this end, it would be natural to choose $T_h(u_\varepsilon)$ as test function ψ in (1.11). However, as pointed out in [17], T_h is not regular enough which leads to difficulties in order to write the term involving the time derivative of the test function. Then, it is necessary to regularize the truncation. Let $\nu > 0$. We introduce $T_h^\nu \in C^2(\mathbb{R},\mathbb{R})$ satisfying

$$\text{(4.6)} \qquad \begin{cases} (T_h^\nu)'(z) = 0\,, & \text{if } |z| \ge h\,,\\ (T_h^\nu)'(z) = 1\,, & \text{if } |z| \le h-\nu\,,\\ 0 \le (T_h^\nu)'(z) \le (T_h)'(z) \le 1\,. \end{cases}$$

Note that $|T_h^\nu(z)| \le |T_h(z)|$, and $(T_h^\nu)"(z) = 0$ when $|z| \ge h$ or $|z| \le h-\nu$.

In the sequel, let us denote

$$L(f,u) = f - b\cdot\nabla u - d\,u\,.$$

We take $\psi = T_h^\nu(u_\varepsilon)$ as test function in the entropic formulation (1.11) satisfied by v, we have

$$\begin{aligned}
&\Big[\int_\Omega S_k(v - T_h^\nu(u_\varepsilon))\,dx\Big]_0^t \\
&+ \int_0^t \langle \partial_t u_\varepsilon, (T_h^\nu)'(u_\varepsilon)\,T_k(v - T_h^\nu(u_\varepsilon))\rangle\,ds \\
&+ \int_0^t\int_\Omega A\nabla v\nabla T_k(v - T_h^\nu(u_\varepsilon))\,dx\,ds \\
&\le \int_0^t\int_\Omega L(f,v)\,(T_h^\nu)'(u_\varepsilon)\,T_k(v - T_h^\nu(u_\varepsilon))\,dx\,ds\,.
\end{aligned} \tag{4.7}$$

By using (3.4), we write the term involving the time derivative of the test function as follows

$$\int_0^t \langle \partial_t u_\varepsilon, \phi\rangle\,ds = \int_0^t\int_\Omega (L(f_\varepsilon,u_\varepsilon)\phi - A\nabla u_\varepsilon\nabla\phi)\,dx\,ds\,, \tag{4.8}$$

where $\phi = (T_h^\nu)'(u_\varepsilon)\,T_k(v - T_h^\nu(u_\varepsilon))$ and, consequently,

$$\nabla\phi = \nabla u_\varepsilon (T_h^\nu)''(u_\varepsilon)\,T_k(v - T_h^\nu(u_\varepsilon)) + (T_h^\nu)'(u_\varepsilon)\nabla(T_k(v - T_h^\nu(u_\varepsilon)))\,.$$

By (4.8), the entropic formulation (4.7) is equivalent to

$$\begin{aligned}
&\Big[\int_\Omega S_k(v - T_h^\nu(u_\varepsilon))\,dx\Big]_0^t \\
&+ \int_0^t\int_\Omega A(\nabla v - (T_h^\nu)'(u_\varepsilon)\nabla u_\varepsilon)\nabla T_k(v - T_h^\nu(u_\varepsilon))\,dx\,ds \\
&- \int_0^t\int_\Omega A\nabla u_\varepsilon\nabla u_\varepsilon (T_h^\nu)''(u_\varepsilon)\,T_k(v - T_h^\nu(u_\varepsilon))\,dx\,ds \\
&\le \int_0^t\int_\Omega (L(f,v) - L(f_\varepsilon,u_\varepsilon)\,(T_h^\nu)'(u_\varepsilon))\,T_k(v - T_h^\nu(u_\varepsilon))\,dx\,ds\,.
\end{aligned} \tag{4.9}$$

Now, according to [17], let successively $\nu \longrightarrow 0$, $\varepsilon \longrightarrow 0$ and $h \longrightarrow \infty$. Difficulties only arise from the third integal in the left hand, denoted by I_ν which involves the second derivative of T_h^ν; the remaining integrals being treated by using the Lebesgue theorem. Indeed, it is clear that

$$|S_k(v - T_h^\nu(u_\varepsilon))| \le k\,|v| + k\,h\,,$$

$$|T_k(v - T_h^\nu(u_\varepsilon))| \le k\,,$$

$$|(T_h^\nu)'(u_\varepsilon)\,T_k(v - T_h^\nu(u_\varepsilon))| \le k\,,$$

and

$$|\nabla T_k(v - T_h^\nu(u_\varepsilon))| \le (|\nabla T_{k+h}(v)| + |\nabla T_h(u_\varepsilon)|) .$$

Next, we wish to obtain an estimate on I_ν. Following [17] (see also [2]), we define another C^2 function R_h^ν, satisfying for $z \ge 0$: $(R_h^\nu)'(z) = 1 - (T_h^\nu)'(z)$, $R_h^\nu(0) = 0$, $R_h^\nu(-z) = R_h^\nu(z)$.

Take $(R_h^\nu)'(u_\varepsilon)$ as test function in (3.4). By using the positivity of R_h^ν and the fact that $(R_h^\nu)''(z) = |(T_h^\nu)''(z)|$, we obtain according to [17] the following estimate

$$\begin{aligned} |I_\nu| \le k &\int_0^t \int_\Omega |L(f_\varepsilon, u_\varepsilon)|\, \chi_{|u_\varepsilon|>h-\nu}\, dx\, ds \\ &+ k \int_\Omega |u_{0,\varepsilon}|\, \chi_{|u_{0,\varepsilon}|>h-\nu}\, dx . \end{aligned} \tag{4.10}$$

By Lebesgue's theorem, we can pass to the limit $\nu \longrightarrow 0$ in the right hand side of (4.10), obtaining without difficulties

$$\limsup_{\nu\to 0} |I_\nu| \le k \int_0^t \int_\Omega |L(f_\varepsilon, u_\varepsilon)|\, \chi_{|u_\varepsilon|>h}\, dx\, ds + k \int_\Omega |u_{0,\varepsilon}|\, \chi_{|u_{0,\varepsilon}|>h}\, dx .$$

Collecting these results, we get from (4.9) the following estimate

$$\begin{aligned} &\Big[\int_\Omega S_k(v - T_h(u_\varepsilon))\, dx\Big]_0^t \\ &+ \int_0^t \int_\Omega A\nabla(v - T_h(u_\varepsilon))\, \nabla T_k(v - T_h(u_\varepsilon))\, dx\, ds \\ &\le \int_0^t \int_\Omega (L(f, v) - L(f_\varepsilon, u_\varepsilon)\, (T_h)'(u_\varepsilon))\, T_k(v - T_h(u_\varepsilon))\, dx\, ds \\ &\quad + k \int_0^t \int_\Omega |L(f_\varepsilon, u_\varepsilon)|\, \chi_{|u_\varepsilon|>h}\, dx\, ds + k \int_\Omega |u_{0,\varepsilon}|\, \chi_{|u_{0,\varepsilon}|>h}\, dx . \end{aligned} \tag{4.11}$$

The assumptions on the sequence of data and the properties of u_ε recalled above allow us to apply the Lebesgue theorem to pass to the limit as $\varepsilon \longrightarrow 0$ in the first term of the left hand side as well as in the right hand side. In addition, the coercivity of A (see (1.5)) and the following almost everywhere convergence

$$\begin{aligned} \nabla T_k(v - T_h(u_\varepsilon)) = \chi_{|v - T_h(u_\varepsilon)| \le k} &(\nabla v - \chi_{|u_\varepsilon| \le h} \nabla u_\varepsilon) \\ &\longrightarrow \chi_{|v - T_h(u)| \le k} (\nabla v - \chi_{|u| \le h} \nabla u) , \end{aligned}$$

permit us to apply Fatou's lemma on the second term in the left hand side.

It remains to deal with $h \longrightarrow \infty$ in the following relation

$$\begin{aligned}
&\Big[\int_\Omega S_k(v-T_h(u))dx\Big]_0^t \\
&+\int_0^t\int_\Omega A\nabla T_k(v-T_h(u))\nabla T_k(v-T_h(u))\,dx\,ds \\
&\leq \int_0^t\int_\Omega \left(L(f,v)-L(f,u)(T_h)'(u)\right)T_k(v-T_h(u))\,dx\,ds \\
&\quad + k\,\mathcal{O}(h)\,,
\end{aligned} \tag{4.12}$$

where $\mathcal{O}(h)$ stands for

$$\int_0^t\int_\Omega \left(|f|+\|b\|_{L^\infty(Q)}|\nabla u|+\|d\|_{L^\infty(Q)}|u|\right)\chi_{|u|>h}\,dx\,ds + \int_\Omega |u_0|\,\chi_{|u_0|>h}\,dx\,.$$

which goes to 0 as $h \longrightarrow \infty$ because $f, u, \nabla u$ belong to $L^1(Q)$.

We search for another expression of the integral in the right hand side of (4.12). We write, on the one hand,

$$\begin{aligned}
&L(f,v)-L(f,u)(T_h)'(u) \\
&\qquad = f-b\cdot\nabla v-d\,v-(f-b\cdot\nabla u-d\,u)\,(T_h)'(u) \\
&\qquad = L(f,u)\,(1-(T_h)'(u))-b\cdot\nabla(v-u)-d\,(v-u)
\end{aligned} \tag{4.13}$$

and, on the other hand

$$\begin{aligned}
&\int_0^t\int_\Omega (b\cdot\nabla(v-u))\,T_k(v-T_h(u))\,dx\,ds \\
&\qquad = -\int_0^t\int_\Omega (v-u)b\cdot\nabla T_k(v-T_h(u)) \\
&\qquad\qquad + (\nabla\cdot b)(v-u)\,T_k(v-T_h(u))\,dx\,ds\,.
\end{aligned} \tag{4.14}$$

By (4.13)-(4.14), inequality (4.12) becomes

$$
\begin{aligned}
&\Big[\int_\Omega S_k(v-T_h(u))\,dx\Big]_0^t \\
&+\int_0^t\int_\Omega A\nabla T_k(v-T_h(u))\nabla T_k(v-T_h(u))\,dx\,ds \\
&\le \int_0^t\int_\Omega |(v-u)\,b\cdot\nabla T_k(v-T_h(u))|\,dx\,ds \\
&\quad+\int_0^t\int_\Omega |(d-(\nabla\cdot b))\,(v-u)T_k(v-T_h(u))|\,dx\,ds \\
&\quad+\int_0^t\int_\Omega |L(f,u)(1-(T_h)'(u))T_k(v-T_h(u))|\,dx\,ds \\
&\quad+k\,\mathcal{O}(h)\,.
\end{aligned}
\tag{4.15}
$$

We remark that $1-(T_h)'(u)$ tends to 0 as $h\longrightarrow\infty$, and by Lebegue's theorem the third term of the right hand side can be included in the general expression $k\mathcal{O}(h)$ which tends to 0 as $h\longrightarrow\infty$.

Proceeding as in Section 4 leads to

$$
\begin{aligned}
&\int_\Omega S_k(v-T_h(u))(t)\,dx \\
&+\frac{\underline{a}}{2}\int_0^t\int_\Omega |\nabla T_k(v-T_h(u))|^2\,dx\,ds \\
&\le \int_\Omega S_k(v-T_h(u))(0)\,dx \\
&\quad+\frac{1}{\underline{a}}\,\|b\|_{L^\infty(Q)}\int_0^t\int_\Omega \chi_{|v-T_h(u)|<k}\,|v-u|^2\,dx\,ds \\
&\quad+(\|d\|_{L^\infty(Q)}+\|\nabla\cdot b\|_{L^\infty(Q)}) \\
&\qquad\cdot\int_0^t\int_\Omega |(v-u)\,T_k(v-T_h(u))|\,dx\,ds \\
&\quad+k\,\mathcal{O}(h)\,.
\end{aligned}
\tag{4.16}
$$

In classical way, by Lebesgue's theorem and Fatou's lemma, letting h go to ∞, we are led to inequality (4.16) where $T_h(u)$ is replaced by u and the last term in the right hand side vanishes. By using $0 \leq z\, T_k(z) \leq 2\, S_k(z)$, $0 \leq z^2\, \chi_{|z|\leq k} \leq 2\, S_k(z)$, we deduce as in Section 2 that

$$\begin{aligned}\int_\Omega S_k(v-u)(t)\,dx + \frac{\underline{a}}{2}\int_0^t\int_\Omega |\nabla T_k(v-u)|^2\,dx\,ds \\ \leq \int_\Omega S_k(v-u)(0)\,dx + C(b,d)\int_0^t\int_\Omega S_k(v-u)\,dx\,ds\,,\end{aligned} \tag{4.17}$$

where

$$C(b,d) = 2\,\|\nabla b\|_{L^\infty(Q)} + 2\,\|d\|_{L^\infty(Q)} + \frac{1}{\underline{a}}\,\|b\|_{L^\infty(Q)}\,.$$

Therefore, it suffices to apply Gronwall's lemma to deduce that

$$\int_\Omega S_k(v-u)(t)\,dx = 0\,,$$

since $v_0 = u_0$, which gives $v = u$.

5. Lower regularity requirement on b.

Our aim in this section is to weaken the regularity requirement on b, replacing the $L^\infty(Q)$ condition by $b \in L^s(Q)$ for $s > q'$; precisely one has

Theorem 3. *Let A, d, g satisfy* (1.3)-(1.5) *and let $b \in L^s(Q)$ with $s > q' = q/(q-1)$ (recall that $1 \leq q < (N+2)/(N+1)$) and $\nabla\cdot b \in L^\infty(Q)$. Then, there exists a weak solution of* (1.1) *in the sense of Definition* 1.

PROOF. The outline of the proof is the same of Theorem 1. Consider the approximate solution of (3.4). In the first step, we show, according to (3.5), that

$$u_\varepsilon \text{ is uniformly bounded in } L^\infty(0,T;L^1(\Omega))\,. \tag{5.1}$$

Reproducing the proof of Proposition 1, we take $\phi = T_k(u_\varepsilon)$ as test function in (3.4), and we find (3.8). All terms are treated as above except those involving $u_\varepsilon b\cdot\nabla T_k(u_\varepsilon)$ which becomes

$$\begin{aligned}\Big|\int_Q u_\varepsilon b\cdot\nabla T_k(u_\varepsilon)\,dx\,dt\Big| \\ \leq \frac{\underline{a}}{2}\int_Q |\nabla T_k(u_\varepsilon)|^2\,dx\,dt + \frac{1}{2\,\underline{a}}\int_Q \chi_{|u_\varepsilon|\leq k}|b\,u_\varepsilon|^2\,dx\,dt\,,\end{aligned}$$

by using Holder's and Young's inequalities. Since $s > 2$ the last integral is bounded by $(1/2\,\underline{a})\,k^2\,(T\,|\Omega|)^{s/(s-2)}\,\|b\|^2_{L^s(Q)}$. Then, from (3.12), we deduce that

$$\int_\Omega S_k(u_\varepsilon)(t)\,dx \le \alpha_0 + \alpha_1 \int_0^t \int_\Omega S_k(u_\varepsilon)\,dx\,ds$$

holds where α_0 depends on $\|f\|_{L^1(Q)}, \|u_0\|_{L^1(\Omega)}$ and $\|b\|_{L^s(Q)}$ and α_1 depends on $\|d\|_{L^\infty(Q)}$, $\|\nabla b\|_{L^\infty(Q)}$. Gronwall's Lemma permits us to conclude as in Proposition 1 and leads to (5.1).

To establish an estimate on the solutions in $L^q(0,T,W_0^{1,q}(\Omega))$, we follow step by step the proofs of estimate (3.6) and of Lemma 1 which need to be adapted. For that, take $\phi = \phi_n(u_\varepsilon)$ in (3.4). We deduce from (3.15)

$$\begin{aligned}\underline{a}\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt &\le \int_Q |\phi_n(u_\varepsilon) f_\varepsilon|\,dx\,dt + \int_\Omega \Psi_n(u_{0,\varepsilon})\,dx \\ &\qquad + \beta\,\|d\|_{L^\infty(Q)} + \int_{E_n} |b\cdot\nabla u_\varepsilon|\,|\phi_n(u_\varepsilon)|\,dx\,dt \\ &\le \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\,\|d\|_{L^\infty(Q)} \\ &\qquad + \Big(\int_{E_n} |b|^{q'}\,dx\,dt\Big)^{1/q'} \Big(\int_Q |\nabla u_\varepsilon|^q\,dx\,dt\Big)^{1/q}.\end{aligned}$$

Since $s > q'$, using Holder's inequality, with exponents s/q' and $s/(s-q')$, yields the following substitute to (3.6)

$$\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt \le C_0 + C_1 \|\nabla u_\varepsilon\|_{L^q(Q)}\,|E_n|^{(s-q')/(sq')}, \tag{5.2}$$

where C_0 stands for

$$\frac{1}{\underline{a}}\,(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\,\|d\|_{L^\infty(Q)})$$

and $C_1 = \|b\|_{L^s(Q)}$.

Recall that $q < 2$. Therefore, Holder's inequality yields

$$\begin{aligned}\int_{B_n} |\nabla u_\varepsilon|^q\,dx\,dt &\le |B_n|^{(2-q)/2}\Big(\int_{B_n} |\nabla u_\varepsilon|^2\,dx\,dt\Big)^{q/2} \\ &\le |B_n|^{(2-q)/2} \qquad\qquad (5.3) \\ &\qquad \cdot (C_0^{q/2} + C_1^{q/2}\,\|\nabla u_\varepsilon\|^{q/2}_{L^q(Q)}|E_n|^{((s-q')/(sq'))(q/2)}).\end{aligned}$$

Let r and K as in Lemma 1. By using (2.6), one gets

$$\begin{aligned}\int_{B_n} |\nabla u_\varepsilon|^q \, dx\, dt \\ &\le C_0^{q/2} \frac{1}{n^{r(2-q)/2}} \Big(\int_{B_n} |u_\varepsilon|^r \, dx\, dt \Big)^{(2-q)/2} \\ &\quad + C_1^{q/2} \, \|\nabla u_\varepsilon\|_{L^q(Q)}^{q/2} \, \|u_\varepsilon\|_{L^r(Q)}^{r((s-q')/(sq'))(q/2)} \\ &\quad \cdot \frac{1}{n^{r((s-q')/(sq'))(q/2)+r(2-q)/2}} \Big(\int_{B_n} |u_\varepsilon|^r \, dx\, dt \Big)^{(2-q)/2}.\end{aligned}$$

Repeated use of Holder's inequality, as in (2.11), implies

$$\begin{aligned}\sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u_\varepsilon|^q \, dx\, dt \\ &\le C_0^{q/2} \Big(\sum_{n=K+1}^{\infty} \frac{1}{n^{r(2-q)/q}} \Big)^{q/2} \|u_\varepsilon\|_{L^r(Q)}^{r(2-q)/2} \\ &\quad + C_1^{q/2} \, \|\nabla u_\varepsilon\|_{L^q(Q)}^{q/2} \, \|u_\varepsilon\|_{L^r(Q)}^{r(((s-q')/(sq'))(q/2)+(2-q)/2)} \\ &\quad \cdot \Big(\sum_{n=K+1}^{\infty} \frac{1}{n^{r(((s-q')/(sq'))+(2-q)/q)}} \Big)^{q/2}.\end{aligned} \tag{5.4}$$

The conditions

$$r\,\frac{2-q}{q} > 1 \qquad \text{and} \qquad r\Big(\frac{s-q'}{s\,q'} + \frac{2-q}{q}\Big) > 1 \tag{5.5}$$

ensure the convergence of the series. As in Section 2, we deduce from (2.8), that

$$\begin{aligned}&\|\nabla u_\varepsilon\|_{L^q(Q)}^q \\ &\le C(K) \\ &\quad + \delta(K) \left(\|u_\varepsilon\|_{L^r(Q)}^{r(2-q)/2} + \|\nabla u_\varepsilon\|_{L^q(Q)}^{q/2} \|u_\varepsilon\|_{L^r(Q)}^{r(((s-q')/(sq'))(q/2)+(2-q)/2)} \right),\end{aligned}$$

holds where $\delta(K)$ tends to zero as K goes to infinity. Therefore, Young's inequality yields

$$\begin{aligned}\|\nabla u_\varepsilon\|_{L^q(Q)}^q &\le C(K) \\ &\quad + \delta(K) \left(\|u_\varepsilon\|_{L^r(Q)}^{r(2-q)/2} + \|u_\varepsilon\|_{L^r(Q)}^{r((s-q')/(sq')q+2-q)} \right).\end{aligned} \tag{5.6}$$

If we choose $r = q\,(N+1)/N$, estimate (2.17) becomes

$$\|u_\varepsilon\|^r_{L^r(Q)} \le C\,\|u_\varepsilon\|^q_{L^q(0,T;L^{q^\star}(\Omega))}\,.$$

Using Sobolev's theorem, as in Section 2, we derive the following estimate on u_ε in $L^q(0,T;L^{q^\star}(\Omega))$

$$\begin{aligned}&\|u_\varepsilon\|^q_{L^q(0,T;L^{q^\star}(\Omega))}\\ &\quad\le C(K) + \delta(K)\left(\|u_\varepsilon\|^{q(2-q)/2}_{L^q(0,T;L^{q^\star}(\Omega))} + \|u_\varepsilon\|^{q((s-q')/(sq')q+2-q)}_{L^q(0,T;L^{q^\star}(\Omega))}\right).\end{aligned}$$

Since $(2-q)/2 < 1$ and $q\,(s-q')/(s\,q') + 2 - q < 1$, we can use again Young's inequality which, choosing K large enough, leads to a bound on u_ε in $L^q(0,T;L^{q^\star}(\Omega))$ and, thus, in $L^q(0,T;W^{1,q}_0(\Omega))$. Finally, let us verify the compatibility of conditions (5.5). For

$$r = q\,\frac{N+1}{N}\,,$$

the first condition is equivalent to

$$1 \le q < \frac{N+2}{N+1}$$

and the second condition means that

$$s > \frac{(N+1)\,q'}{q'-1}$$

which is clearly satisfied since it is yet required $s > q'$.

Finally, one can easily verify that Lemma 2, Lemma 3, Corollary 2 and Lemma 4 are valid in the context of Theorem 3 and the proof follows.

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Recibido: 16 de julio de 1.999
Revisado: 4 de febrero de 2.000

Thierry Goudon
Laboratoire J. A. Dieudonné
Université Nice-Sophia Antipolis
Parc Valrose
06108 Nice cedex 02 FRANCE.
goudon@math.unice.fr

and

Mazen Saad
Mathématiques Appliquées de Bordeaux
Université Bordeaux I
351, cours de la Libération
33405 Talence Cedex, FRANCE
saad@math.u-bordeaux.fr