

Morera type problems in Clifford analysis

Emilio Marmolejo Olea

Abstract. The Pompeiu and the Morera problems have been studied in many contexts and generality. For example in different spaces, with different groups, locally, without an invariant measure, etc. The variations obtained exhibit the fascination of these problems.

In this paper we present a new aspect: we study the case in which the functions have values over a Clifford Algebra. We show that in this context it is completely natural to consider the Morera problem and its variations. Specifically, we show the equivalence between the Morera problem in Clifford analysis and Pompeiu problem for surfaces in \mathbb{R}^n . We also show an invariance theorem. The non-commutativity of the Clifford algebras brings in some peculiarities.

Our main result is a theorem showing that the vanishing of the first moments of a Clifford valued function implies Clifford analyticity. The proof depends on results which show that a particular matrix system of convolution equations admits spectral synthesis.

0. Introduction.

The framework provided by Clifford Algebras has proven to be very useful to generalize many aspects of one variable complex analysis to \mathbb{R}^n . The subject has come to be known as Clifford Analysis. Unexpected links to classical harmonic analysis, several complex variables and representation theory have been discovered. Many books on the subject have recently appeared [11], [15], [16], [23], [24] and it has grown

to be an important area of research.

It is therefore completely natural to ask which aspects of the Morera problem in the complex plane are valid in this context. Let us point out that the non-commutativity of the Clifford Algebras brings many peculiarities to Clifford Analysis. In particular many familiar properties are not valid in this context. Nevertheless we will show a positive result for the Clifford Morera problem.

The plan of the paper is as follows. In the first section, we give a short survey on the Pompeiu problem and on the Morera problem. We include the results and examples that we will use later on. We also comment a little about the methods involved to prove this results.

In the second section, we set up the framework of Clifford analysis. We reproduce the most fundamental results for the Clifford holomorphic functions or regular functions. This includes the corresponding versions of the Stokes formula, the Cauchy representation formula and the Morera theorem. The Vahlen-Ahlfors representation of Moebius transformations in \mathbb{R}^n is also presented.

After these two preliminary sections we start our study properly. In the third section we present first the equivalence of the Morera problem and the Pompeiu problem for surfaces in \mathbb{R}^n . Although this is an easy fact to prove it has many consequences. We discuss these consequences in a sequence of corollaries. Then we show a non-invariant version of the Morera problem.

Section Four, our main contribution, deals with the statement and proof of a First Moments Theorem. Roughly speaking, this correspond to proving that a matrix system of convolution equations admits spectral synthesis. It turns out that the determinant minors of this matrix satisfy the Hörmander condition and the theorem follows. We note that in most Euclidean cases of the Pompeiu problem a reduction to the fundamental theorem of mean periodic function is made. This is not the case here.

Finally, in the last section, we discuss some problems for future research. The advantage of being able to carry specific calculations was important to prove the moments result but for generic surfaces we do not know how to proceed. The easy proof for one complex variable proof cannot be adapted to this context.

1. Preliminaries about the Pompeiu and Morera problems.

1.1. Notation.

As usual, let $\mathcal{E}(\mathbb{R}^n)$ denote the space of all infinitely differentiable functions on \mathbb{R}^n with the topology of uniform convergence of all derivatives on compact subsets of \mathbb{R}^n . Let $\mathcal{E}'(\mathbb{R}^n)$ be its dual space of distributions with compact support.

Also let $\mathcal{C}(\mathbb{R}^n)$ denote the space of all continuous functions on \mathbb{R}^n with the usual topology of uniform convergence on compact sets. We will denote the Fourier transform of a function or a distribution f by \widehat{f} or by $\mathcal{F}(f)$.

Let us also recall that the algebra $\widehat{\mathcal{E}}'(\mathbb{R}^n)$ can be characterized as the space of all holomorphic functions $F : \mathbb{C}^n \rightarrow \mathbb{C}$ satisfying the Paley-Wiener estimates: for some constants C, A, N greater than zero and all z in \mathbb{C}^n , $z = \operatorname{Re} z + i \operatorname{Im} z$

$$|F(z)| \leq C (1 + \|z\|)^N e^{A|\operatorname{Im} z|}.$$

1.2. The Pompeiu problem.

A general version of the Pompeiu problem can be formulate as follows [10]: Let X be a locally compact space, μ a non-negative Radon measure on X , $\{C_i\}_{i=1}^N$ a finite family of compact subsets of X , and G a topological group acting on X and keeping μ invariant. The Pompeiu map

$$P : C(X) \rightarrow (C(G))^N$$

is defined by

$$(P_i f)(g) := \int_{gC_i} f d\mu,$$

where P_i is the i th component of P and we denote by gx the action of the element $g \in G$ on the point $x \in X$.

We say that the family $\{C_i\}$ has the Pompeiu property if P is injective. The Pompeiu problem consists of deciding as explicitly as possible whether the family has the Pompeiu property. For a historical introduction to these problems as well as their ramifications, generalizations, progress and a complete bibliography we refer to [31], [30], [5],

[10], [28], [12]. In [10] a general method is explained and some theorems are proved for symmetric spaces of real rank 1.

When G is a separable unimodular Lie group, the Pompeiu map may be interpreted as a system of convolution equations on $\mathcal{E}'(G)$, the space of distributions of compact support on G . Further reduction is made rewriting the problem as a problem of spectral analysis. We illustrate this line of reasoning in the case when $X = \mathbb{R}^n$, $G = M(n)$, and $\mu = dx$, where $M(n)$ is the group of orientation preserving rigid motions, that is, the group generated by all translations and by all rotations in $SO(n)$, and dx is Lebesgue measure.

A translation invariant subspace \mathcal{M} of $\widehat{\mathcal{E}}'(\mathbb{R}^n)$ is said to admit spectral analysis if \mathcal{M} contains an exponential. If the exponential polynomials belonging to \mathcal{M} are dense in \mathcal{M} we say that \mathcal{M} admits spectral synthesis.

To decide whether the map P is injective one can assume by a standard approximation argument that f is a smooth function. Now for smooth f , we rewrite the conditions $Pf = 0$

$$\int_{gC_i} f dx = 0, \quad g \in M(n), \quad i = 0, \dots, N,$$

where $g(x) = \sigma x + y$ with $\sigma \in SO(n)$ and $y \in \mathbb{R}^n$, as the (infinite) system of convolution equations in $\mathcal{E}'(\mathbb{R}^n)$

$$\check{\chi}_{\sigma C_i} * f = 0, \quad \sigma \in SO(n), \quad i = 1, \dots, N.$$

where χ_C denotes the characteristic function on the set C and $\check{h}(x) = h(-x)$.

Consider the convolution ideal \mathbb{I} in $\mathcal{E}'(\mathbb{R}^n)$ generated by the $\check{\chi}_{\sigma C_i}$. If \mathbb{I} is dense in $\mathcal{E}'(\mathbb{R}^n)$, then for any solution $f \in \mathcal{E}(\mathbb{R}^n)$ of the system and a generic element in \mathbb{I} , $\sum g_\alpha * \check{\chi}_{\sigma C_i}$, we have

$$\left(\sum g_\alpha * \check{\chi}_{\sigma C_i} \right) * f = \sum g_\alpha * (\check{\chi}_{\sigma C_i} * f) = 0,$$

thus by the density

$$f = \delta * f = 0.$$

A necessary condition for \mathbb{I} to be dense is that the Fourier transforms $\widehat{\chi}_{\sigma C_i}$ have no common zeroes. Moreover if x_0 is the common zero, then $f(x) = e^{ix \cdot x_0}$ is a non-zero solution of the system since

$$\check{\chi}_{\sigma C_i} * f = f \cdot \widehat{\chi}_{C_i}(x_0) = 0.$$

In the real case ($n = 1$) the condition is also sufficient. This result is a consequence of the Schwartz spectral synthesis theorem. Unfortunately the theorem is not true in $\mathbb{R}^n, n > 1$, [17]. Nevertheless, under certain symmetric conditions for the sets C_i , if their Fourier transforms $\widehat{\chi}_{C_i}$ have no common zeroes, a reduction to the Schwartz theorem can be made.

In the case of a single set C , the above discussion can be carried further, [12], to prove that C has the Pompeiu property if and only if $\widehat{\chi}_C$ does not vanish identically on any of the analytic varieties

$$C_\alpha = \{z \in \mathbb{C}^n : z_1^2 + z_2^2 + \dots + z_n^2 = \alpha\}, \quad \alpha \neq 0.$$

Note that no ball has the Pompeiu property [28]. We now state some of the known results [28], [30].

Theorem 1.1 (Two balls Theorem). *Let B_i denote the closed ball of radius r_i . Then $\{B_1, B_2\}$ has the Pompeiu property with respect to Lebesgue measure if and only if $r_1/r_2 \notin Z_n = \{\xi/\mu : \xi, \mu \text{ non zero roots of the Bessel equation } J_{n/2}(z) = 0\}$.*

Theorem 1.2 (Two spheres Theorem). *Let S_i denote a sphere of radius r_i . Then $\{S_1, S_2\}$ has the Pompeiu property with respect to surface measure if and only if $r_1/r_2 \notin Z_{n-2}$.*

In the case when X is a irreducible symmetric space of rank 1, there are analogues to the two balls and two spheres theorems above [10].

In the case we discuss below, a link to overdetermined problems is given in [27]. It has proven to be very important. When $C = \overline{\Omega}$, for Ω a bounded open set in R^n , if C^c is connected, then the failure of the Pompeiu property for C is equivalent to the existence of an eigenvalue for a overdetermined Neumann boundary value problem. Namely,

Theorem 1.3. *Let $C = \overline{\Omega}$, where Ω is a bounded open set, C^c is connected and ∂C is (at least) Lipschitz. Then C fails to have the Pompeiu property if and only if there is an eigenvalue α and a function u on Ω satisfying the overdetermined Neumann problem*

$$\begin{cases} \Delta u + \alpha u = 0, & \text{in } \Omega, \\ u = 1, \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

Theorem 1.4. *Let Ω be as above. If $\partial\Omega$ is Lipschitz but not real analytic everywhere then Ω has the Pompeiu property.*

1.3. The Morera problem.

There is already a discussion on Morera type theorems [6], but new results and different aspects keep appearing. We will mention only the results that we will try to generalize.

Let Γ be a Jordan curve in \mathbb{C} . We say that Γ has the Morera property if each continuous complex valued function f on \mathbb{C} which satisfies

$$\int_{\sigma(\Gamma)} f(z) dz = 0$$

for every rigid motion σ of \mathbb{C} is entire.

A similar definition holds for a family of Jordan curves $\{\Gamma_i\}$. The Morera problem is to decide as explicitly as possible whether the family has this property. We can also consider the hyperbolic case in which the function is defined only in the unit disk and the group is the Moebius group.

The Morera and Pompeiu problems are equivalent in the following situation [28], [12].

Theorem 1.5. *Suppose that $\{\Gamma_i\}$ is a family of Jordan curves and $\Omega_i = \text{int}(\Gamma_i)$ is a family of Jordan domains. Then the family $\{\Omega_i\}$ has the Pompeiu property if and only if the family $\{\Gamma_i\}$ has the Morera property.*

This theorem follows from the following version of the Green formula

$$\frac{d}{d\bar{z}} \chi_\Omega = \chi_{\partial\Omega} ,$$

taken in the distributional sense. Because of this equivalence and Theorem 1.4, many classes of curves satisfy the Morera property.

As the example of the circle shows, one single curve is not in general enough to solve the Morera problem. The following Theorem, [1], solves the problem of giving necessary and sufficient conditions for a single curve to determine holomorphy.

Theorem 1.6. (Moments Theorem). *Let $f \in C(\mathbb{C})$, and let Γ be a piecewise smooth Jordan curve. Then f is entire if and only if*

$$\int_{\sigma(\Gamma)} z^k f(z) dz = 0, \quad k = 0, 1, \dots,$$

for every rigid transformation σ of \mathbb{C} .

REMARK 1.7. 1. This result at first sight seem obvious, since for every σ the vanishing of the moment implies that the function can be extended holomorphically inside the region bounded by $\sigma(\Gamma)$. But we do not know that these extensions agree on overlaps.

2. The proof follows from an averaging argument and the argument principle.

3. A similar result is true in the unit disk \mathbb{D} .

4. The proof of 3 follows from the maximality of invariant algebras of functions in \mathbb{D} under Moebius transformations, [1].

5. Actually, it is enough to request that the moments do not grow too fast [29].

In the case of a circle only 2 moments are required [29].

Theorem 1.8 (Two Moments Theorem). *Let $f \in C(\mathbb{C})$ and let $r > 0$, $n > 1$ be fixed. Suppose that*

$$\int_{\partial B(z,r)} f(\zeta) d\zeta = \int_{\partial B(z,r)} (z - \zeta)^n f(\zeta) d\zeta = 0,$$

for all $z \in \mathbb{C}$. Then f is an entire function.

REMARK 1.9. This result follows from rewriting the hypothesis as two convolution equations and appealing to the Schwartz spectral synthesis Theorem.

The last result is true if we consider functions defined in the unit disk but it is interesting that the following variation of the Morera Problems gives different results. Suppose $f \in C(\mathbb{D})$ satisfies

$$\int_{\Gamma} f(\sigma(z)) dz = 0$$

for all Moebius transformation σ in \mathbb{D} . Is it true that f is holomorphic in \mathbb{D} ?

Observe that the measure $dz|_{\Gamma}$ is not invariant under the action of the Moebius group $\mathcal{M} \simeq SU(1,1)$ also note that now we are moving the values of the function. The following Theorems [4] give the answer to this problem in the circular case and in the general case.

Theorem 1.10 (Circular Morera Theorem). *Let $r > 0$ and let $f \in C(\mathbb{D})$ satisfy*

$$\int_{\partial B(c,r)} f(\sigma(z)) dz = 0$$

for every Moebius transformation σ in \mathbb{D} .

a) *If $c \neq 0$ then f is holomorphic on \mathbb{D} .*

b) *If $c = 0$ then f is not necessarily holomorphic on \mathbb{D} (There are counterexamples).*

Theorem 1.11. *Let $\Omega \subset \mathbb{D}$ be a Jordan domain of class $C^{2,\varepsilon}$ for some $\varepsilon > 0$ and suppose that the Jordan curve $\Gamma = \partial\Omega$ is not real analytic. Assume $f \in C(\mathbb{D})$ satisfies*

$$\int_{\Gamma} f(\sigma(z)) dz = 0$$

for every $\sigma \in \mathcal{M}$. Then f is holomorphic on \mathbb{D} .

2. Rudiments of Clifford analysis.

2.1. Basic results.

The goal of this section is to present the basic definitions in Clifford Algebras and the basic concepts and results in Clifford Analysis as we will need them later on. For a complete development of the subject we refer to the books [11], [15], [24], [16].

We consider the real 2^n dimensional Clifford algebra \mathbb{A}_n generated out of \mathbb{R}^n as follows: let e_1, \dots, e_n be an orthonormal basis for \mathbb{R}^n . Then \mathbb{A}_n is defined by the anti-commutation relationship

$$e_i e_j + e_j e_i = -2 \delta_{ij} ,$$

where δ_{ij} is the Kronecker delta function. Consequently, the algebra \mathbb{A}_n has as basis elements

$$1, e_1, \dots, e_n, \dots, e_{j_1} \dots e_{j_r}, \dots, e_1 \dots e_n ,$$

where $j_1 < \dots < j_r$ and $1 \leq r \leq n$. Hence for an element $a \in \mathbb{A}_n$ we write

$$a = \sum_{\alpha} a_{\alpha} e_{\alpha} ,$$

where $a_{\alpha} \in \mathbb{R}$ and where we identify e_{α} with e_{j_1}, \dots, e_{j_r} for $\alpha = \{j_1, \dots, j_r\}$ and e_{\emptyset} with 1.

Note that if $x \in \mathbb{R}^n$ we have that $x^2 = -\|x\|^2$. It follows that every non zero $x \in \mathbb{R}^n$ is invertible with inverse $x^{-1} = -x/\|x\|^2$. Observe that $\mathbb{A}_1 = \mathbb{C}$, and $\mathbb{A}_2 = \mathbb{H}$, the quaternionic division algebra. For $n \geq 3$, \mathbb{A}_n is no longer a division algebra.

We will use the following two involutions. First the anti-automorphism defined by

$$\sim : \mathbb{A}_n \longrightarrow \mathbb{A}_n : e_{j_1} \dots e_{j_r} \longrightarrow e_{j_r} \dots e_{j_1} .$$

For an element $a \in \mathbb{A}_n$, we write \tilde{a} instead of $\sim(a)$. Second the anti-automorphism defined by

$$- : \mathbb{A}_n \longrightarrow \mathbb{A}_n : e_{j_1} \dots e_{j_r} \longrightarrow (-1)^r e_{j_r} \dots e_{j_1} .$$

Again we write \bar{a} for $-(a)$. This anti-automorphism is a generalization of complex conjugation.

The Clifford algebra \mathbb{A}_n becomes a Hilbert space and a Banach Algebra when the inner product on \mathbb{A}_n is defined by putting for any $a, b \in \mathbb{A}_n$,

$$\langle a, b \rangle = \sum_{\alpha} a_{\alpha} b_{\alpha} .$$

Note that for x, y vectors (*i.e.* $x, y \in \mathbb{R} \oplus \mathbb{R}^n$), we have $\langle x, y \rangle = (x \bar{y} + y \bar{x})/2$. In particular, $\|x\|^2 = x \bar{x}$ and $\|xy\| = \|x\| \|y\|$, but for general $a, b \in \mathbb{A}_n$, $\|a\|^2 \neq a \bar{a}$ and $\|ab\| \neq \|a\| \|b\|$.

We will consider the space $\mathcal{E}(\mathbb{R}^n, \mathbb{A}_n)$ of smooth \mathbb{A}_n valued functions, which is an \mathbb{A}_n module under pointwise multiplication. The topology we will consider in $\mathcal{E}(\mathbb{R}^n, \mathbb{A}_n)$ is the one of uniform convergence of all derivatives over compact subsets. Similar considerations are made for the space of continuous \mathbb{A}_n valued functions $\mathcal{C}(\mathbb{R}^n, \mathbb{A}_n)$.

Two basic definitions are

i) The *Dirac operator* is the differential operator

$$D = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i} .$$

ii) Let $f, g \in C^1(\mathbb{R}^n, \mathbb{A}_n)$ be differentiable functions. Then f is called *left regular* if

$$Df = \sum_{i=1}^n e_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^n \sum_{\alpha} e_i e_{\alpha} \frac{\partial f_{\alpha}}{\partial x_i} = 0 ,$$

and g is called *right regular* if

$$gD = \sum_{i=1}^n \frac{\partial g}{\partial x_i} e_i = \sum_{i=1}^n \sum_{\alpha} e_{\alpha} e_i \frac{\partial g_{\alpha}}{\partial x_i} = 0 .$$

In the literature left regular, left monogenic or left Clifford holomorphic are used indistinctly. Note that since $\tilde{D}f = -\tilde{f}D$, a function f is left regular if and only if \tilde{f} is right regular. Also note that if $f(x)$ is a left regular function then so is $f(x)a$ for any $a \in \mathbb{A}_n$ but not in general for $af(x)$.

An important property is that $D^2 = -\Delta$, the Laplacian over \mathbb{R}^n , hence, each component of a left or right regular function is harmonic. The function

$$G(x) = \frac{1}{\omega_n} \frac{-x}{\|x\|^n} = \frac{1}{\omega_n} \frac{x^{-1}}{\|x\|^{n+2}} ,$$

where ω_n is the surface area of the unit sphere in \mathbb{R}^n is left and right regular. This function $G(x)$ plays the role of the Cauchy kernel.

The Green Formula can be formulated in the framework of Clifford algebra valued functions as follows [11], [15].

Theorem 2.1. *Let f and g be Clifford algebra valued functions defined in a domain $U \subset \mathbb{R}^n$ and let M be a bounded domain in U with Lipschitz boundary. Then*

$$\int_{\partial M} g(x) n(x) f(x) dS(x) = \int_M ((gD)(x)f(x) + g(x)(Df)(x)) dv(x) .$$

Note that, here and in the following theorems, dS is the canonical surface measure, $n(x)$ stands for the outward unit normal to ∂M regarded as a Clifford algebra-valued function, dv is the volumen element, and the integrands are interpreted in the sense of Clifford algebra multiplication.

The Borel-Pompeiu formula for Clifford valued functions is the following.

Theorem 2.2. *Let M be a bounded domain with Lipschitz boundary. Then for $f \in C^1(U, \mathbb{A}_n)$ and $x \in M$,*

$$f(x) = \int_{\partial M} G(y-x) n(y) f(y) dS(y) - \int_M G(y-x) Df(y) dv(y).$$

The Cauchy integral formula is given by the following theorem.

Theorem 2.3. *Let M be a bounded domain in U with Lipschitz boundary. If f is a left regular function on U , then for each x in M ,*

$$f(x) = \int_{\partial M} G(y-x) n(y) f(y) dS(y).$$

We also have the Morera theorem.

Theorem 2.4. *If f is a Clifford algebra valued continuous function on the domain U such that*

$$\int_{\partial M} n(y) f(y) dS(y) = 0,$$

for every bounded domain M in U with Lipschitz boundary, then f is left regular.

Of course there are similar versions of this theorems for right regular functions. Taylor series where the polynomial are regular functions are also possible [11]. In this paper, we will use only the polynomials

$$P_i(x) = x_i e_1 + x_1 e_i, \quad i = 2, \dots, n,$$

which are a basis for both the right (left) module of homogeneous left (right) regular polynomials of degree 1.

2.2. Vahlen matrices.

We now introduce the Vahlen matrices. The collection of all products of non-zero vectors in \mathbb{R}^n form a group \mathbb{A}_n^* lying in \mathbb{A}_n . Let $\mathbb{V}(n)$ be the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

- i) $a, b, c, d \in \mathbb{A}_n^*$.
- ii) $a\tilde{c}, c\tilde{d}, \tilde{d}\tilde{b}$ and $d\tilde{a} \in \mathbb{R}^n$.
- iii) $a\tilde{d} - b\tilde{c} = \pm 1$.

A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{V}(n)$ is called a Vahlen matrix. The usefulness of this concept is given by the following theorem [2].

Theorem 2.5. *Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{V}(n)$. Then the function $\phi(x) = (ax + b)(cx + d)^{-1}$ defines a Moebius transformation over $\mathbb{R}^n \cup \{\infty\}$. Moreover this representation gives a surjective group homomorphism from $\mathbb{V}(n)$ with matrix multiplication to the orientation preserving Moebius group over $\mathbb{R}^n \cup \{\infty\}$ with kernel $\pm \mathbb{I}$.*

A computation shows that the Jacobian of $\phi(x) = (ax + b)(cx + d)^{-1}$ is given by

$$\text{Jac}(\phi(x)) = \frac{1}{\|cx + d\|^{2n}}.$$

The following theorem can be seen as a change of variable for Clifford valued functions under Moebius transformations [22].

Theorem 2.6. *Suppose that $y = \phi(x) = (ax + b)(cx + d)^{-1}$ is a Moebius transformation and f and g are Clifford valued functions. If S is a closed, bounded and oriented surface then*

$$\begin{aligned} \int_S g(y) n(y) f(y) dS(y) \\ = \int_{\phi^{-1}(S)} g(\phi(x)) \widetilde{J(\phi, x)} n(x) J(\phi, x) f(\phi(x)) dS(x), \end{aligned}$$

where

$$J(\phi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^n}.$$

The factor $J(\phi, x)$ is called the covariance of $\phi(x)$.

The Dirac operator and the covariance are intertwined as follows (see for example [22]).

Theorem 2.7. *Let f be a Clifford valued function and $\phi(x) = (ax + b)(cx + d)^{-1}$ a Moebius transformation. Then*

$$DJ(\phi, x)f(\phi(x)) = J_{-1}(\phi, x)Df(\phi(x)),$$

where

$$J_{-1}(\phi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^{n+2}}.$$

As the composition or product of regular functions is not regular, the following theorem provides a kind of substitute [22].

Theorem 2.8. *Let $y = \phi(x)$ be a Moebius transformation and $f(y)$ a Clifford valued function. Then $f(y)$ is left regular if and only if $J(\phi, x)f(\phi(x))$ is left regular.*

Finally note that

$$\widetilde{J}_{-1}(\phi, x)J(\phi, x) = \text{Jac}(\phi(x)).$$

This ends our summary on the basic facts in Clifford Analysis. We are now ready to start our study properly.

3. First results.

3.1. Equivalence of Morera and Pompeiu.

In this section we give the results which are easy to prove and similar to the complex case.

By a *Jordan surface* S we will mean a Lipschitz embedding of the $(n - 1)$ -sphere in \mathbb{R}^n (i.e. S is homeomorphic to the $(n - 1)$ -sphere by a Lipschitz function). Let $M = \overline{\text{int } S}$. We say that a Jordan surface S in \mathbb{R}^n (or a collection of them $\{S_j\}$), has the *Morera property* if any $f \in \mathcal{C}(\mathbb{R}^n, \mathbb{A}_n)$ satisfying

$$(1) \quad \int_{\sigma S} n(x) f(x) dS(x) = 0,$$

for every rigid motion $\sigma \in M(n)$ is left regular. Note that here as in the rest of the section, the integrals and product are considered in the Clifford analysis setting.

The Morera problem consist of deciding as explicitly as possible whether a surface (or a family of them) has the Morera property.

More generally we can state the Morera problem on a different space or with a different group or with a more general surface. For example we can take the space as the unit ball in \mathbb{R}^n and the group as the group of Moebius transformation of the ball.

REMARK 3.1. The Morera problem is stated for the case in which the function is continuous but it is equivalent to the case in which the function is smooth. This follows from a standard smoothing argument. We reproduce it in here for the sake of completeness.

Suppose that $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{A}_n)$, satisfying (1) implies that f is left regular. Let $g \in \mathcal{C}(\mathbb{R}^n, \mathbb{A}_n)$ satisfies (1). Let ϕ be a (real value) approximate identify of compact support. Then $g * \phi \in \mathcal{E}(\mathbb{R}^n, \mathbb{A}_n)$ and satisfies (1). Therefore $g * \phi$ is left regular. Now since

$$g * \phi_{\varepsilon_n} \longrightarrow g$$

uniformly on compact sets as $\varepsilon_n \longrightarrow 0$, we conclude that g is left regular. Therefore, we will assume from now on that the function is smooth.

Let $\{S_j\}$ be a collection of Jordan surfaces and let $M_j = \overline{\text{int } S_j}$. As in the complex case we have:

Theorem 3.2. *$\{S_j\}$ has the Morera property in \mathbb{A}_n if and only if $\{M_j\}$ has the Pompeiu property in \mathbb{R}^n .*

PROOF. Let $g \in C^1(\mathbb{R}^n, \mathbb{R})$. Then there is a Clifford valued function f such that f solves the Dirac equation $Df = g$ ([11, Theorem 19.2]). Then by the Green formula (Section 2, Theorem 2.1), for every rigid motion $\sigma \in M(n)$, we have

$$\int_{\sigma(M)} g(x) dv(x) = \int_{\text{int}(\sigma(M))} n(x) f(x) dS(x) = \int_{\sigma(S)} n(x) f(x) dS(x).$$

Hence, if S satisfy 3.1 then M has the Pompeiu property.

Conversely if $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{A}_n)$, then by Stokes Theorem and the Pompeiu property for M we have that $Df \equiv 0$, so f is left regular and S has the Morera property.

This equivalence has several consequences. Using the results of Section 2 we get at once the following corollaries.

Corollary 3.3.

1. *No sphere has the Morera property.*
2. *Two spheres have the Morera property if and only if their radii r_1, r_2 satisfy the condition in Theorem 1.1 of Section 1, namely $r_1/r_2 \notin Z_n = \{\xi/\mu : \xi, \mu \text{ non zero roots of the Bessel equation } J_{n/2}(z) = 0\}$.*
3. *We have a condition for the Morera property in terms of the Fourier transform of the characteristic function of M .*

Note the difference with the two spheres Theorem of Section 1.

Among the concrete examples for which the Morera property holds are ellipsoids, tori, and some surfaces of revolution, [13].

Corollary 3.4. *If the Jordan surface S is Lipschitz but not real analytic everywhere then S has the Morera property.*

It follows that polygonal surfaces have the Morera property, *e.g.* n -cubes, polyhedra, etc.

Another corollary to the equivalence of Morera and Pompeiu problem is the study of the local situation. This is what can we say if the function is defined only on a domain $D \subset \mathbb{R}^n$ and the vanishing of the integrals is required only when $\sigma S \subset D$. It turns out that the local Pompeiu problem is a harder question [7]. As before we get the following corollary.

Corollary 3.5. *Let $r_1, r_2 > 0$ be such $r_1/r_2 \notin Z_n$, and let $R > r_1 + r_2$. If $f \in \mathcal{C}(B(R, 0), \mathbb{A}_n)$ satisfies*

$$\int_{\partial B(y, r_i)} n(x) f(x) dS(x) = 0, \quad i = 1, 2,$$

for all $y \in \mathbb{R}^n$ such that $\partial B(y, r_i) \subset B(R, 0)$, then f is left regular. Moreover the condition is sharp.

3.2. Non-invariant measures.

We now study a non-invariant measure variant of the Morera problem. Using the result of section 2, we can state the problem as follows. Let $\phi \in \mathcal{M}$, where \mathcal{M} is the group of Möbius transformations of the unit ball \mathbb{B} in R^n . We know that $\phi(x) = (ax + b)(cx + d)^{-1}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{V}(n)$ a Vahlen matrix. If f is a regular function defined in \mathbb{B} , then $J(\phi, x) f(\phi(x))$ is also left regular in \mathbb{B} . Therefore by the Cauchy Theorem,

$$\int_S J(\tilde{\phi}, x) n(x) J(\phi, x) f(\phi(x)) dS(x) = 0,$$

for every surface S in \mathbb{B} . The problem is to determine whether for a fixed surface S and a continuous function f the above condition implies that f is left regular.

The next proposition shows that the above problem could be reduced to the Pompeiu Problem for the unit ball and the Möbius group.

Proposition 3.6. *Let f be a continuous Clifford valued function defined in the unit ball \mathbb{B} in R^n and let S be a Jordan surface in \mathbb{B} . If*

$$\int_S J(\tilde{\phi}, x) n(x) J(\phi, x) f(\phi(x)) dS(x) = 0,$$

for every $\phi \in \mathcal{M}$, where \mathcal{M} is the group of Möbius transformations of the ball, then f is left regular if and only if $M = \overline{\text{int } S}$ has the Pompeiu property with respect to \mathcal{M} .

PROOF. By the Clifford algebra version of Stokes Theorem we have that

$$\begin{aligned} \int_S J(\tilde{\phi}, x) n(x) J(\phi, x) f(\phi(x)) dS(x) \\ = \int_M J(\tilde{\phi}, x) D(J(\phi, x) f(\phi(x))) dv. \end{aligned}$$

Now using Theorem 2.2 of Section 2 we get that the integral is equal to

$$\int_M J(\tilde{\phi}, x) J_{-1}(\phi, x) Df(\phi(x)) dv.$$

By using that $J_{-1}(\phi, x) J(\phi, x) = \text{Jac}(\phi(x))$ we get that the last integral is equal to

$$\int_M \text{Jac}(\phi(x)) Df(\phi(x)) dv.$$

Now by a change of variable this integral is equal to

$$\int_{\phi^{-1}(M)} Df(y) dv(y).$$

Using that $d\mu = dv/(1 - \|y\|^2)^2$ is the hyperbolic measure for the ball the last integral is equal to

$$\int_{\phi^{-1}(M)} Df(y) (1 - \|y\|^2)^2 d\mu(y).$$

Hence that we get the Pompeiu problem for the function $Df(y) (1 - \|y\|^2)$ in the ball \mathbb{B} with the group \mathcal{M} . The conclusion follows.

4. The moment condition for Clifford valued functions.

4.1. Introduction.

In this section we show that a sphere has the Morera property if we add the first Clifford moments. Namely we show that a continuous function in \mathbb{R}^n with values in the Clifford Algebra \mathbb{A}_n , whose first moments over all spheres of fixed radius r vanish is a regular function. We prove this result by first reducing the problem to a overdetermined matrix system of convolution equations in \mathbb{R}^n . Then we need to see than spectral synthesis holds for this kind of system.

It turns out that the determinants of the maximal minors of this matrix of convolution operators satisfy the Hörmander conditions and thus spectral synthesis holds.

4.2. Statement of the problem.

We saw in Section 3 than no sphere has the Morera property. The natural question is to look for the extra conditions needed. In the spirit of the Two Moment theorem of Section 1 we found the following Theorem.

Theorem 4.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{A}_n$ be a continuous function with Clifford values. Let $r > 0$ be fixed. If for each $x \in \mathbb{R}^n$,*

$$\int_{\partial B(x,r)} n(y) f(y) dS(y) = 0,$$

and

$$\int_{\partial B(x,r)} P_i(y-x) n(y) f(y) dS(y) = 0,$$

for

$$P_i(x) = e_1 x_i + e_i x_1, \quad i = 2, \dots, n,$$

then f is a left regular function.

Of course the integrals are understood in the sense of Section 2.

PROOF. As before we can assume that f is smooth. Applying the Clifford version of the Green formula (2.1 in Section 2) and using that P_i is right regular, we get that

$$\int_{B(x,r)} Df(y) dV(y) = 0$$

and

$$\int_{B(x,r)} P_i(y-x) Df(y) dV(y) = 0,$$

for each x in \mathbb{R}^n . Let $g = Df$. Then the above conditions can be rewritten as

$$\chi_r * \check{g} = 0$$

and

$$P_i \chi_r * \check{g} = 0, \quad i = 2, \dots, n,$$

where χ_r denotes the characteristic function on the ball of radius r and the (Clifford) convolutions are understood in the natural way.

We have a system of convolution equations for Clifford valued functions. We want to show that $g = 0$ is the only solution to the system. In order to do that we first need to have a short discussion about general systems of convolution equations and present some properties of Bessel functions. We will do that in the next two sections, and then come back to the system.

4.3. Spectral synthesis for modules.

Given an r -tuple of functions $F_1, \dots, F_r \in \widehat{\mathcal{E}}'(\mathbb{R}^n)$, the Hörmander condition, [18], gives a necessary and sufficient condition to guarantee

Theorem 4.2. *Let F be a $m \times N$ matrix with coefficients in the ring $\widehat{\mathcal{E}}'(\mathbb{R}^n)$. If the $N \times N$ minors of F generate $\widehat{\mathcal{E}}'(\mathbb{R}^n)$, then there exists a solution M of the Bezout equation $MF = \mathbb{I}_N$.*

4.4. Some lemmas about Bessel functions.

Here we collect some properties of the Bessel functions and its zeros that will be used further on. Our references are [26], [14]. We assume $v > 1$.

For the Bessel function $J_v(z)$ of real order v , we consider its normalized function $j_v(z) = J_v(z)/z^v$. Note that $j_v(z)$ is an entire even function and that $z = 0$ is not a zero of $j_v(z)$. For $z \in \mathbb{C}^n$ we write $z^2 = z_1^2 + z_2^2 + \cdots + z_n^2$.

Lemma 4.3. *Let $Q(x)$ be a homogeneous, harmonic polynomial of degree k in \mathbb{R}^n . Then the complexified Fourier transform of $Q\chi_r$ is given by*

$$\mathcal{F}(Q\chi_r)(z) = \kappa r^{2\mu} Q(z) j_{n/2+k}(r\sqrt{z^2}),$$

where κ is a constant depending only on k and n ; and $\mu = n/2 + k$.

PROOF. The proof follows from [25, Theorem 3.10, p. 158] and a simple computation.

REMARK 4.4.

1. The Macmahon's asymptotic development of the positive zeros $\alpha_{k,v}$ of $J_v(z)$:

$$0 < \alpha_{1,v} < \alpha_{2,v} < \cdots,$$

is given by

$$\alpha_{k,v} = (2k+1)\frac{\pi}{2} + (2v+1)\frac{\pi}{4} + O\left(\frac{1}{k}\right).$$

2. The positive zeros of $J_v(z)$ are interlaced with those of $J_{v+1}(z)$

$$0 < \alpha_{1,v} < \alpha_{1,v+1} < \alpha_{2,v} < \alpha_{2,v+1} < \cdots,$$

The next lemma will estimate the growth of $j_v(z)$ away from its zeroes V_v . Let $d(z, V) = \min\{1, \text{dist}(z, V)\}$.

Lemma 4.5. *Let $\varepsilon > 0$ be given. If $d(z, V_v) > \varepsilon$ and $|z|$ is big enough, then*

$$|j_v(z)| \geq \frac{e^{|\operatorname{Im} z|}}{8\pi e \sqrt{2\pi} |z|^{v+3/2}} .$$

PROOF. We use the following asymptotic development of the Bessel function $J_v(z)$ (see [14]),

$$\left| J_v(z) - \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{\pi}{4} (2v + 1) \right) \right| \leq \frac{3e}{8} \sqrt{\frac{\pi}{2}} (4v^2 - 1) \frac{e^{|\operatorname{Im} z|}}{|z|^{3/2}}$$

which is valid when $|z| \geq (\pi/8)(4v^2 - 1)$. On the other hand, the cosine satisfies the Lojasiewicz inequality

$$|\cos z| \geq \frac{1}{\pi e} d(z, V) e^{|\operatorname{Im} z|} ,$$

where $V = \{(2l + 1)\pi/2 : l \in \mathbb{Z}\}$.

It follows that if $d(z, V_v) > \varepsilon$, then

$$\left| \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{\pi}{4} (2v + 1) \right) \right| \geq \sqrt{\frac{2}{\pi}} \frac{1}{\pi e} \frac{\varepsilon e^{|\operatorname{Im} z|}}{|z|^{1/2}} .$$

After subtracting the bounds above and taking $|z|$ big enough, we get the desired conclusion.

4.5. Proof of the Moments Theorem.

Let us recall that we want to solve the system

$$\chi_r * \check{g} = 0$$

and

$$P_i \chi_r * \check{g} = 0, \quad i = 2, \dots, n ,$$

where the P_i are the regular polynomials and g is a Clifford valued function.

In order to do that first we consider \mathbb{A}_n as the matrix subalgebra of $M(2^n \times 2^n, \mathbb{R})$. In this way we will see the system of Clifford valued convolution equations as an overdetermined matrix system of convolution equations.

First we view \mathbb{A}_n as a matrix subalgebra of $M(2^n \times 2^n, \mathbb{R})$ as follows [20], [15]. Consider the matrices $e_j := E_j^n$, $j = 1, \dots, n$, where for each $1 \leq k \leq n$, $\{E_j^k\}_{j=1}^k$ are inductively defined by

$$E_1^1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and in general for $1 \leq k \leq n - 1$, and $1 \leq j \leq k$

$$E_j^{k+1} := \begin{pmatrix} E_j^k & 0 \\ 0 & -E_j^k \end{pmatrix}$$

and

$$E_{k+1}^{k+1} := \begin{pmatrix} 0 & -\mathbb{I}_{2^k} \\ \mathbb{I}_{2^k} & 0 \end{pmatrix}.$$

Then it is easy to check that the generator relations hold. Thus \mathbb{A}_n is isomorphic to the subalgebra of $M(2^n \times 2^n, \mathbb{R})$ consisting of all matrices generated by the E_j^n .

It is important to note that under this representation, the Clifford conjugation corresponds to the transposition of matrices. In particular, if $a \in \mathbb{A}_n$ is such that $a\bar{a} \in \mathbb{R}$ (for example for vectors), then the determinant of the corresponding matrix A is given by

$$\text{Det}(A) = (a\bar{a})^{2^{n-1}}.$$

It follows using the representation in $M(2^n \times 2^n, \mathbb{R})$ that the system of Clifford valued convolution equations is equivalent to a matrix system of convolution equations in $\mathcal{E}'(\mathbb{R}^n)$. Indeed, let T be the $(n 2^n) \times 2^n$ matrix of convolution operators whose blocks T_i are the matrices corresponding to the distributions χ_r ($i = 1$) and $P_i \chi_r$ for $i = 2, \dots, n$. Let G be the matrix corresponding to g . Thus we can write the system as

$$TG = 0,$$

where $T \in M((n 2^n) \times 2^n, \mathcal{E}'(\mathbb{R}^n))$ and $G \in M(2^n \times 2^n, \mathcal{E}(\mathbb{R}^n))$.

Let F be the $(n 2^n) \times 2^n$ matrix obtained from T via Fourier Transform in each entry. We will show that the minors of F generate $\widehat{\mathcal{E}}'(\mathbb{R}^n)$.

Note that the blocks F_i correspond to the Fourier transform of the matrix representation of $P_i \chi_r$. Then from the form of P_i , Lemma 4.3 and the note above about determinants, we get that

$$\text{Det}(F_i) = (\lambda(z_1^2 + z_i^2) j_{n/2+1}(r\sqrt{z^2}))^{2^{n-1}},$$

for $i = 2, \dots, n$, where λ is a constant depending only on r and n . Similarly for the distribution χ_r the determinant of the matrix F_1 is given by

$$\text{Det}(F_1) = (\lambda j_{n/2}(r \sqrt{z^2}))^{2^{n-1}}$$

for a constant λ as above.

So far we have obtained the determinant of n minors of T , we will need only one more. Note that taking a minor of T is equivalent to taking a linear combination of the $P_i \chi_r$. In other words, since the P_i are a basis of the left regular homogeneous polynomial of degree 1, any left regular homogeneous polynomials of degree 1 can be obtained as a minor of T . Hence, we can repeat the argument used for the $P_i \chi_r$ said for $q \chi_r$ with $q = e_2 x_3 + e_3 x_2$. We then get that the determinant of this minor F_{n+1} is given by

$$\text{Det}(F_{n+1}) = (\lambda (z_2^2 + z_3^2) j_{n/2+1}(r \sqrt{z^2}))^{2^{n-1}}.$$

We will drop the exponent 2^{n-1} from these functions as they are not relevant.

It is clear that the functions $f_i := \text{Det}(F_i)$ for $i = 1, \dots, n + 1$, have no common zeros because the two Bessel functions which appear have no common, zeros and the polynomials have no common zeros. Moreover, we claim that the set $\{f_i\}$ generate $\widehat{\mathcal{E}}'(\mathbb{R}^n)$.

Since the zeroes of $j_{n/2}$ and $j_{n/2+1}$ interlace, and they are separated from each other by a fixed number (see Remark 4.4), we can find an estimate as in Lemma 4.5 that works for the sum of the two functions. Thus for all $w \in \mathbb{C}$,

$$|j_{n/2}(r w)| + |j_{n/2+1}(r w)| \geq \frac{\kappa e^{-|\text{Im } w|}}{(1 + |w|)^{n+5/2}},$$

where κ is a positive constant. Now note that for $z \in \mathbb{C}^n$,

$$|\text{Im } \sqrt{z^2}| \leq |\text{Im } z|.$$

It follows that for all $z \in \mathbb{C}^n$,

$$|j_{n/2}(r \sqrt{z^2})| + |j_{n/2+1}(r \sqrt{z^2})| \geq \frac{\kappa e^{-|\text{Im } z|}}{(1 + \|z\|)^{n+5/2}}.$$

Now for a set of polynomials, the Hörmander condition (4.3) is equivalent to that the polynomials have no common zeros. In that case, we

can take $B = 0$. It then follows from this and the above inequality that the set of functions $\{f_i\}$ satisfies the Hörmander condition.

Applying Theorem 4.2, the proof is completed.

REMARK 4.6. We need all the first moments in the theorem. Indeed if we have less of the $P_i \chi_r$, the respective convolution system will have a non-zero solution.

REMARK 4.7. It follows from the proof of the theorem that for the moments of order greater than one, what we need is that the corresponding minors have no common zeros. This will follow from dimensionality.

5. Conclusions.

There are many directions for which the type of problems we have considered could be investigated. This includes the study in other spaces, other operators of Dirac type or more concrete surfaces.

As we showed in Section Four some of the results in the plane generalize to the Clifford analysis setting but the proofs are more involved than the ones for the case of the plane. Hence some difficulties are expected for the other variations. Of course, it would not be possible to recover all the results in the plane in part because there is no Riemann Mapping Theorem when $n > 2$. For instance, for the case of higher order moments we can only offer the following remarks.

Using the Premelj formulas [21] and the Taylor series expansion for regular functions (see [11]), it is easy to show that if S is a Jordan surface and f is a continuous function defined on S with Clifford values then f can be extended to a left regular function inside S if and only if

$$\int_S V_{l_1, \dots, l_k}(x) n(x) f(x) dS(x) = 0,$$

for every k and for every homogeneous regular polynomial $V_{l_1, \dots, l_k}(x)$ of degree k . This means that a function could be extended to be left regular inside a surface if and only if all its Clifford moments vanish. Using this we can formulate the general version of the moments problem as follows.

Let S be a Jordan surface and let $f \in \mathcal{C}(\mathbb{R}^n, \mathbb{A}_n)$. Suppose that for every $\sigma \in M(n)$, f can be extended to be left regular inside σS . Does

it follow that f is left regular? that is if

$$\int_{\sigma S} V_{l_1, \dots, l_k}(x) n(x) f(x) dS(x) = 0,$$

for every k , and for every homogeneous regular polynomial $V_{l_1, \dots, l_k}(x)$ of degree k , and for every $\sigma \in M(n)$ is then f left regular? As we mention in Section One, the proof for the complex case relies on the argument principle. But in Clifford analysis there is no argument principle.

It is shown in [3] that we do not need vanishing of moments but only that they do not grow too fast. Whether or not this is true in the situation of Section Four is another interesting problem.

Acknowledgments. The author would like to express his gratitude to Professor Carlos A. Berenstein for the suggestion of the problem and guidance. He would like also to thank Professor John Ryan for the numerous talks they had and to thank the referee for the careful reading of the paper and valuable suggestions.

References.

- [1] Agranovsky, M., Valskii, R. E., Maximality of invariant algebras of functions. *Sibirsk Mat. Zh.* **12** (1971), 3-12.
- [2] Ahlfors, L. V., Moebius transformations in \mathbb{R}^n expressed through 2×2 matrices of Clifford numbers. *Complex variables* **45** (1986), 215-224.
- [3] Aizenberg, L., Variations on the theme of the Morera theorem and the Pompeiu problem. *Dokl. Akad. Nauk.* **337** (1994), 709-712.
- [4] Berenstein, C., Pacuas, D., Morera and mean value type theorems. *Israel J. Math.* **86** (1994), 61-106.
- [5] Berenstein, C. A., The Pompeiu problem. What's new? Complex analysis, harmonic analysis and applications. R. Deville et al., Ed. *Pitman Research Notes in Mathematics* **347** (1996), 1-11.
- [6] Berenstein, C. A., Chang, D. C., Pascuas, D., Zalzman, L., Variations on the theorem of Morera. *Contemporary Math.* **137** (1992), 63-78.
- [7] Berenstein, C. A., Gay, R., A local version of the two circle theorem. *Israel J. Math.* **55** (1986), 267-288.
- [8] Berenstein, C. A., Struppa, D. C., On explicit solutions to the Bezout equation. *Systems Control Lett.* **4** (1984), 33-39.

- [9] Berenstein, C. A., Struppa, D. C., Complex analysis and convolution equations. *Several complex variables*, V. G. M. Henkin, Ed. Encyclopaedia of Mathematical Sciences. **54** Springer-Verlag, 1993, 1-108.
- [10] Berenstein, C. A., Zalcman, L., Pompeiu's problem on symmetric spaces. *Comm. Math. Helv.* **55** (1980), 593-621.
- [11] Brackx, F., Delanghe, R., Sommen, F., *Clifford Analysis*. Pitman Research Notes in Mathematics **76**, 1982.
- [12] Brown, L., Schreiber, B., Taylor, B. A., Spectral synthesis and the Pompeiu problem. *Ann. Inst. Fourier* **23** (1973), 125-154.
- [13] Dalmasso, R., A new result on the Pompeiu problem. *Trans. Amer. Math. Soc.* **6** (2000), 2723-2736.
- [14] Gatteschi, L., On the zeros of certain functions with application to Bessel functions. *Indag. Math.* **14** (1952), 224-229.
- [15] Gilbert, J., Murray, M., *Clifford Algebras and Dirac Operators in Harmonic Analysis*. Cambridge University Press, 1991.
- [16] Guerlebeck, K., Sproessig, W., *Quaternionic Analysis and Elliptic Boundary Value Problems*. Birkhäuser, 1990.
- [17] Gurevich, D. I., Counterexamples to a problem of L. Schwartz. *Funct. Anal. Appl.* **9** (1975), 116-120.
- [18] Hörmander, L., Generators for some ring of analytic functions. *Bull. Amer. Math. Soc.* **73** (1967), 943-949.
- [19] Kelleher, J. J., Taylor, B. A., Finitely generated ideals in rings of analytic functions. *Math. Ann.* **193** (1971), 225-237.
- [20] Mitrea, M., *Clifford Wavelets, Singular Integrals and Hardy Spaces*. Lecture Notes in Math. **1575** Springer-Verlag, 1994.
- [21] Ryan, J., Plemelj formulae and transformation associated to plane wave decompositions in complex Clifford analysis. *Proc. London Math. Soc.* **64** (1992), 70-94.
- [22] Ryan, J., Conformally covariant operators in Clifford analysis. *J. Anal. Appl.* **14** (1995), 677-704.
- [23] Ryan, J., (Ed.), *Clifford Algebras in Analysis and Related Topics*. CRC Press, 1996.
- [24] Ryan, J., *Clifford Analysis Forcoming Book*. Enciclopaedia of Mathematics and Applications. Cambridge University Press, 1999.
- [25] Stein, E. M., Weiss, G., *Fourier Analysis in Euclidean Spaces*. Princeton University Press, 1971.
- [26] Watson, G. N., *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 1962.
- [27] Williams, S. A., A partial solution to the Pompeiu problem. *Math. Ann.* **223** (1976), 183-190.

- [28] Zalcman, L., Analyticity and the Pompeiu problem. *Arch. Rational Mech. Anal.* **47** (1972), 237-254.
- [29] Zalcman, L., Mean values and differential equations. *Israel J. Math.* **14** (1973), 339-352.
- [30] Zalcman, L., Offbeat integral geometry. *Amer. Math. Monthly* **87** (1980), 161-175.
- [31] Zalcman, L., A bibliographic survey of the Pompeiu problem. In *Approximation by Solutions of Partial Differential Equations*. B. Fuglede et al., Ed. (1992), 185-194.

Recibido: 1 de febrero de 2.000

Revisado: 8 de mayo de 2.000

Emilio Marmolejo Olea
Instituto de Matematicas
Unidad Cuernavaca
A.P. 273-3, Cuernavaca Morelos
C.P. 65251, MEXICO
emilio@matcuer.unam.mx