Liouville type theorems for φ-subharmonic functions

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Dedicated to the memory of Franca Burrone Rigoli

Abstract. In this paper we presents some Liouville type theorems for solutions of differential inequalities involving the φ-Laplacian. Our results, in particular, improve and generalize known results for the Laplacian and the p-Laplacian, and are new even in these cases. Phragmen-Lindeloff type results, and a weak form of the Omori-Yau maximum principle are also discussed.

0. Introduction.

Let $(M, \langle \cdot, \cdot \rangle)$ be a smooth, connected, non-compact, complete Riemannian manifold of dimension m. We fix an origin o, and denote by $r(x)$ the distance function from o, and by $B_t = \{x \in M : r(x) < t\}$ and $\partial B_t = \{x \in M : r(x) = t\}$ the geodesic ball and sphere of radius $t > 0$ centered at o.

To avoid inessential technical difficulties we will assume that ∂B_t is a regular hypersurface. This is certainly the case if o is a pole of M; in the general case one could overcome the problem using a Gaffney regularized distance instead of the Riemannian distance function $r(x)$.

We denote by $\mathrm{vol}\, B_t$ and $\mathrm{vol}\, \partial B_t$ the Riemannian measure of B_t and the induced measure of ∂B_t, respectively. Integrating in polar coordinates then gives

$$\operatorname{vol} B_t = \int_0^t \operatorname{vol} \partial B_s \, ds \, .$$

In this paper, we will always denote with φ a real valued function in $C^1((0, +\infty)) \cap C^o([0, +\infty))$ satisfying the following structural conditions

(0.1)
$$\text{i) } \varphi(0) = 0 \, ,$$
$$\text{ii) } \varphi(t) > 0 \, , \qquad \text{for all } t > 0 \, ,$$
$$\text{iii) } \varphi(t) \le A \, t^\delta \, , \qquad \text{for all } t \ge 0 \, ,$$

for some positive constants A and δ.

We will focus our attention on the differential operator defined for $u \in C^1(M)$ by

$$(0.2) \qquad \operatorname{div} \left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \right) ,$$

and which could be referred to as the φ-Laplacian. Of course, if the the vector field in brackets is not C^1, then the divergence in (0.2) must be considered in distributional sense. Note that the vector field in consideration may fail to be C^1 at the points where $\nabla u = 0$, even if u is assumed to be C^2.

We also note that the φ-Laplacian arises naturally when writing the Euler-Lagrange equation associated to the energy functional

$$\Lambda(u) = \int \Phi(|\nabla u|) \, ,$$

where $\Phi(t) = \int_0^t \varphi(s) \, ds$.

As important natural examples we mention:

1) the Laplace-Beltrami operator, Δu, corresponding to $\varphi(t) = t$;

2) or, more generally, the p-Laplacian, $\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right)$, $p > 1$, corresponding to $\varphi(t) = t^{p-1}$;

3) the generalized mean curvature operator, $\operatorname{div} \left(\nabla u / (1 + |\nabla u|^2)^\alpha \right)$, $\alpha > 0$, corresponding to $\varphi(t) = t/(1 + t^2)^\alpha$.

The general philosophy is to explore the mutual interactions between the behavior of solutions of differential equations/inequalities involving the φ-Laplacian, and geometric properties of the underlying manifold. As it will become clear in the sequel, many of the results we

present can be generalized to a slightly more general class of operators including, for instance, the $\mathcal{A}$-Laplace operators as defined in [HeKM]. For some related results in this setting we also refer to some recent work by I. Holopainen [Ho]. We have decided to concentrate on operators of the form given in (0.2) since all the main ideas appear already, and the techniques are more transparent in this setting. In any case, our results have interesting consequences in non-linear potential theory, and many of them appear to be new even for the Laplacian.

We introduce some notation. A function $u \in C^1(M)$ is said to be φ-subharmonic if

$$(0.3) \qquad \operatorname{div}\left(|\nabla u|^{-1}\varphi(|\nabla u|)\,\nabla u\right) \geq 0\,, \qquad \text{on } M\,.$$

Reversing the inequality, or replacing the inequality with an equality one obtains the definition of φ-super harmonic, and φ-harmonic function, respectively. Note that the notion of φ-(sub, super)harmonicity is unaffected by adding a constant to u. In accordance with what remarked after (0.2), if the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is not C^1, the inequality in (0.3) must be understood in weak sense. Explicitly, $u \in C^1$ is φ-subharmonic if

$$-\int \langle \nabla\psi, |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u \rangle \geq 0\,,$$

for all $0 \leq \psi \in C_c^\infty(M)$, or equivalently, for all nonnegative compactly supported Lipschitz functions on M.

One of the basic problems is to determine sufficient conditions so that (0.3) has only constant solutions. In the case of the Laplace-Beltrami operator, typically one considers the problem where u belongs to two main function classes: $\{u \in C^2(M) : \sup u < +\infty\}$ and $\{u \in C^2(M) : u \geq 0\} \cap L^q(M)$, $q > 1$. The fact that the only solutions of (0.3) in these two cases are constant amounts to the parabolicity of the manifold $(M, \langle \cdot, \cdot \rangle)$, and to an L^q-type Liouville property, respectively. It is well known, see for instance the recent survey paper by A. Grigor'yan, [Gr1], that

$$(0.4) \qquad \begin{aligned} &\text{i)} \ \frac{r}{\operatorname{vol}(B_r)} \notin L^1(+\infty)\,, \\[2em] &\text{ii)} \ \frac{r}{\displaystyle\int_{B_r} u^q} \notin L^1(+\infty)\,, \end{aligned}$$

are sufficient to guarantee parabolicity or the validity of an L^q-type Liouville property, respectively. However, both conditions are far from being necessary, as shown by a counterexample due to R. Greene and quoted in [V3].

On the other hand, it may be shown that parabolicity is equivalent to

$$(0.5) \qquad \frac{1}{\mathrm{vol}\,(\partial B_r)} \notin L^1(+\infty)$$

if the manifold $(M, \langle \cdot, \cdot \rangle)$ is a model in the sense of Greene and Wu, [GW], but the equivalence fails in general (see for instance [Gr1, example 7.9, p. 40]).

In this connection, we note that (0.5) is always implied by (0.4) i), for instance. Further, it is easy to construct examples of manifolds of exponential volume growth where (0.5) holds, while (0.4) i) obviously does not. We shall therefore concentrate on conditions involving $\mathrm{vol}\,\partial B_r$, as in (0.5), rather than $\mathrm{vol}\,B_r$ itself.

Following the classical terminology, we shall say that $(M, \langle \cdot, \cdot \rangle)$ is φ-parabolic if the only bounded above solutions of (0.3) are constant. As a consequence of the results presented in Section 1 below (see Theorem 1.5) we have:

Theorem A. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold, let φ and δ be as in (0.1) and assume that*

$$\left(\mathrm{vol}\,(\partial B_r)^{1/\delta}\right)^{-1} \notin L^1(+\infty)\,.$$

Then M is φ-parabolic.

Note that the same δ may correspond to different operators. For instance, $\delta = 1$ may be associated both to the Laplacian and to the mean curvature operator $\mathrm{div}\,(\nabla u/\sqrt{1 + |\nabla u|^2})$. It follows that if (0.5) holds, then M is parabolic both in the usual sense, and with respect to the mean curvature operator.

As for L^q-type Liouville Theorems we have:

Theorem B. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold, let φ and δ be as in (0.1) and let u be a C^1, non-negative φ-subharmonic function. If*

$$\left(\int_{\partial B_r} u^q\right)^{-1/\delta} \notin L^1(+\infty)\,,$$

for some $q > \delta$, then u is constant.

Theorem B generalizes two recent results for the Laplacian and the $\mathcal{A}$-Laplacian (see the above remark) due to K. T. Sturm, [St], and Holopainen, [Ho], respectively. In these papers, constancy of u is established assuming that the following stronger condition holds

$$\left(\frac{r}{\displaystyle\int_{B_r} u^q}\right)^{1/\delta} \notin L^1(+\infty).$$

More interestingly, our result extends to solutions of a large class of differential inequalities, see Theorem 2.2, and, in a different direction, Theorem 4.3. It should be pointed out that Sturm and Holopeinen have a version of Theorem B for nonnegative (p-)superharmonic functions satisfying the above growth condition with $q < \delta$. In Proposition 2.3 we show that our techniques are flexible enough to recover their result.

Note that the case $\delta = q$ is quite special. Indeed, for a rather long time it was not known whether an L^1-Liouville property was true on an arbitrary Riemannian manifold, even in the case of the Laplace-Beltrami operator. At the beginning of the '80's, reference was made to a preprint by L. O. Chung where the first example of a complete Riemannian manifold admitting a non-trivial integrable harmonic function u was constructed. A further example was published by P. Li and R. Schoen, [LS] in 1984. However, the constancy of integrable harmonic functions can be obtained provided we impose some further condition, for instance an appropriate bound on the growth. This was first observed by N. S. Nadirashvili, [N]. The following result may be viewed as a generalization and an improvement of [N, Theorem 2], even for the Laplace-Beltrami operator.

Theorem C. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold, and let φ and δ be as in (0.1). Let u be a C^1, non-negative φ-subharmonic function. If*

$$(0.6) \quad \text{i)} \quad \int_{\partial B_r} u^\delta \leq \frac{C}{r \log^b r} \qquad and \qquad \text{ii)} \quad u(x) \leq C \exp\left(r(x)^{1+1/\delta}\right),$$

for some positive constants b and C, and $r(x)$ sufficiently large, then u is constant.

We point out that the assumptions do not force u to belong to $L^\delta(M)$, and refer to Section 1 below for a more precise statement and a detailed discussion.

As mentioned above, many of our results extend and improve previous results valid for the Laplacian and the p-Laplacian. In many instances the latter have been obtained using capacity techniques. These techniques in general depend on the solvability of the Dirichlet problem (at least on annuli), and the fact that p-harmonic functions are minimizers of the appropriate energy integral. Underlying the method is the even more basic relationship between the energy density $\Phi(|\nabla u|)$ and the expression $\varphi(|\nabla u|)\,|\nabla u|$, which is crucial when applying the divergence theorem. In the case of the p-Laplacian the two expressions coincide and are equal to $|\nabla u|^p$. Since none of these facts holds in the general case of the φ-Laplacian, a capacity approach to φ-parabolicity appears to be infeasible, and alternative methods must be devised.

In the last section of the paper we show that, under suitable geometric assumptions, a weak version of the Omori-Yau maximum principle holds for the φ-Laplacian.

Theorem D. *Assume that*

$$(0.7) \qquad \liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{r^{1+\delta}} < +\infty \,,$$

and let u be a smooth function on M with $u^* = \sup u < +\infty$. Suppose further that the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is of class at least C^1. Then there exists a sequence $\{x_n\} \subset M$, $n = 1, 2, \ldots$, such that

$$(0.8) \qquad \begin{cases} u(x_n) \longrightarrow u^* \,, & as\ n \longrightarrow +\infty \,, \\[2mm] \operatorname{div}\left(|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\nabla u\right)(x_n) \leq \dfrac{1}{n} \,. \end{cases}$$

Observe that the regularity condition in the statement is certainly satisfied in the case of the Laplacian, of the p-Laplacian $(p \geq 2)$, or the generalized mean curvature operator, once u is assumed to be at least C^2.

We also note that in a recent paper, K. Takegoshi [T] asserted that, if (0.7) holds with $\delta = 1$, then the Laplacian satisfies the full strength Omori-Yau maximum principle, *i.e.*, there exists a sequence $\{x_n\}$ satisfying (0.8) and the additional condition $|\nabla u_n| \leq 1/n$. However, there seems to be a gap in his proof, which the present authors have not been able to fill.

The paper is organized as follows: in Section 1 we prove Theorem A, Theorem C and some related results. Section 2 is devoted to a generalization of Theorem B. In Section 3 we presents some extensions of Theorem A. Examples show that the main results in each of these sections are fairly sharp. In Section 4 we state some Phragmen-Lindelöff type results, and make some further comments. In Section 5 we discuss the weak Omori-Yau maximum principle, and some related topics.

1. Proof of theorems A, C and related results.

We keep the notation of the Introduction; in particular the constants A and δ refer to the structural conditions (0.1) satisfied by φ.

The following observation will be repeatedly used in the sequel. Assume that Ω is a bounded domain in M with smooth boundary $\partial\Omega$, and outward unit normal ν. Denote by $\rho(x)$ the distance function from $\partial\Omega$ (with the convention that $\rho(x)$ is > 0 if $x \in \Omega$ and < 0 if $x \notin \Omega$), so that ρ is the radial coordinate for the Fermi coordinates relative to $\partial\Omega$. By Gauss Lemma, $|\nabla\rho| = 1$ and, $\nabla\rho = -\nu$ on $\partial\Omega$. Finally, let $\Omega_\varepsilon = \{x \in \Omega : \rho(x) > \varepsilon\}$, and let ψ_ε be the Lipschitz function defined by

$$\psi_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in \Omega_\varepsilon, \\[2mm] \dfrac{1}{\varepsilon}\,\rho(x), & \text{if } x \in \Omega \setminus \Omega_\varepsilon, \\[2mm] 0, & \text{if } x \in \Omega^c. \end{cases}$$

Given a continuous vector field Z defined on $\overline{\Omega}$, the following version of the divergence theorem holds

$$(1.1) \qquad \lim_{\varepsilon \to 0+} \langle \operatorname{div} Z, \psi_\varepsilon \rangle = \int_{\partial\Omega} \langle Z, \nu \rangle.$$

Indeed, by definition of weak divergence, and by the co-area formula,

$$\langle \operatorname{div} Z, \psi_\varepsilon \rangle = -\frac{1}{\varepsilon} \int_{\Omega \setminus \Omega_\varepsilon} \langle Z, \nabla\rho \rangle = \frac{1}{\varepsilon} \int_0^\varepsilon dt \int_{\partial\Omega_t} \langle Z, \nabla\rho \rangle,$$

and (1.1) follows letting $\varepsilon \longrightarrow 0$. With slight abuse of notation, we will refer to (1.1) as the divergence theorem, and write, even in this case,

$$\int_\Omega \operatorname{div} Z = \int_{\partial\Omega} \langle Z, \nu \rangle.$$

Assume now that the differential inequality $\operatorname{div} Z \geq \lambda$ holds in weak sense for some real valued continuous function λ defined on $\overline{\Omega}$. Substituting into (1.1) yields

$$\int_{\Omega} \lambda = \lim_{\varepsilon \to 0} \int_{\Omega} \lambda \, \psi_{\varepsilon} \leq \lim_{\varepsilon \to 0} \langle \operatorname{div} Z, \psi_{\varepsilon} \rangle = \int_{\partial \Omega} \langle Z, \nu \rangle \,.$$

This observation will allow us to deal with continuous vector fields satisfying weak differential inequalities as if we were working with smooth vector fields satisfying pointwise inequalities.

The next simple technical lemma, and its companion Lemma 2.1, are key ingredients in the proofs of our main results.

Lemma 1.1. *Let $f \in C^o(\mathbb{R})$, and let u be a non-constant C^1 solution of the differential inequality*

$$(1.2) \qquad \operatorname{div} \left(|\nabla u|^{-1} \, \varphi(|\nabla u|) \, \nabla u \right) \geq \varphi(|\nabla u|) \, |\nabla u| f(u) \,.$$

Assume that there are functions $\alpha \in C^1(I)$ and $\beta \in C^o(I)$ defined on an interval $I \supset u(M)$ such that

$$(1.3) \qquad\qquad\qquad \alpha(u) \geq 0 \,,$$

$$(1.4) \qquad\qquad \alpha'(u) + f(u)\,\alpha(u) \geq \beta(u) > 0 \,,$$

on M. Then there exist R_o depending only on u and a constant $C > 0$ independent of α and β, such that, for every $r > R \geq R_o$,

$$(1.5) \quad \left(\int_{B_R} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u| \right)^{-1} \geq C \left(\int_R^r \left(\int_{\partial B_t} \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}} \right)^{-1/\delta} \right)^{\delta} \,.$$

Proof. Let Z be the continuous vector field defined by

$$Z = \alpha(u)\,|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u \,.$$

Observing that $\alpha(u)$ is C^1, we compute the distributional divergence of Z, and use our assumptions on u, α, and β to obtain

$$\operatorname{div} Z \geq (\alpha(u)\,f(u) + \alpha'(u))\,\varphi(|\nabla u|)\,|\nabla u| \geq \beta(u)\,\varphi(|\nabla u|)\,|\nabla u| \,.$$

Integrating over B_t and applying the divergence theorem gives

$$(1.6) \qquad\qquad \int_{\partial B_t} \langle Z, \nabla r \rangle \geq \int_{B_t} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u| \,.$$

On the other hand, using Schwarz inequality, the assumed positivity of $\beta(u)$, Hölder inequality with conjugate exponents $1 + \delta$ and $1 + 1/\delta$, and the inequality $\varphi(t)^{1+1/\delta} \leq A^{1/\delta}\varphi(t)\,t$, we estimate

$$
\begin{aligned}
(1.7) \qquad \int_{\partial B_t} \langle Z, \nabla r\rangle &\leq \int_{\partial B_t} |Z| \\
&= \int_{\partial B_t} \alpha(u)\,\varphi(|\nabla u|) \\
&\leq A^{1/(1+\delta)} \Big(\int_{\partial B_t} \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \Big)^{1/(1+\delta)} \\
&\quad \cdot \Big(\int_{\partial B_t} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u| \Big)^{1+1/\delta}.
\end{aligned}
$$

Combining (1.6) and (1.7) yields

$$
\begin{aligned}
(1.8) \qquad \int_{B_t} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u| &\leq A^{1/(1+\delta)} \Big(\int_{\partial B_t} \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \Big)^{1/(1+\delta)} \\
&\quad \cdot \Big(\int_{\partial B_t} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u| \Big)^{1+1/\delta}.
\end{aligned}
$$

Denoting by $H(t)$ the left hand side of (1.8), and noting that, by the co-area formula

$$
H'(t) = \int_{\partial B_t} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u|,
$$

we may rewrite (1.8) in the form

$$
(1.9) \qquad H(t) \leq A^{1/(1+\delta)} \Big(\int_{\partial B_t} \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \Big)^{1/(1+\delta)} (H'(t))^{\delta/(1+\delta)}.
$$

Further, since u is non-constant, $\beta(u) > 0$ and $\varphi(t) > 0$ if $t > 0$, we deduce that there exists R_o such that $H(t) > 0$ for every $t \geq R_o$. It follows that the right hand side of (1.9) is also strictly positive for $t \geq R_o$. Rearranging we finally obtain

$$
H(t)^{-1-1/\delta} H'(t) \geq A^{-1/\delta} \Big(\int_{\partial B_t} \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \Big)^{-1/\delta}, \qquad t \geq R_o,
$$

whence, integrating between R and r, $R_o \leq R < r$, yields

$$H(R)^{-1/\delta} \geq H(R)^{-1/\delta} - H(r)^{-1/\delta} \geq \frac{1}{\delta A^{1/\delta}} \int_R^r \left(\int_{\partial B_t} \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \right)^{-1/\delta} dt \, ,$$

and (1.5) follows with $C = \delta^{-\delta} A^{-1}$.

Theorem 1.2. *Let $u \in C^1(M)$ be a non-negative φ-subharmonic function. If there exists $b > 0$ such that*

$$\left(\int_{\partial B_t} u^\delta \left(1 + \log\left(1+u\right)\right)^\delta \left(1 + \log^{b+\delta}\left(1 + \log\left(1+u\right)\right)\right) \right)^{-1/\delta}$$

$$(1.10) \hspace{8cm} \notin L^1(+\infty) \, ,$$

then u is constant.

PROOF. We argue by contradiction, and assume that u is not constant. For every integer $n \geq 1$, let α_n be the function defined for $t \geq 0$ by $\alpha_n(t) = \log^b \left(1 + \log\left(1 + 1/n + t\right)\right)$, and let

$$\beta_n = \alpha'_n(t) = \frac{b \log^{b-1}\left(1 + \log\left(1 + 1/n + t\right)\right)}{\left(1 + \log\left(1 + 1/n + t\right)\right)\left(1 + 1/n + t\right)} \, , \qquad \text{for all } t \geq 0 \, .$$

It is readily verified that α_n and β_n satisfy the conditions in the statement of Lemma 1.1, with $f \equiv 0$, so that

$$(1.11) \quad \left(\int_{B_R} \beta_n(u) \, \varphi(|\nabla u|) \, |\nabla u| \right)^{-1}$$
$$\geq C \left(\int_R^r \left(\int_{\partial B_t} \frac{\alpha_n(u)^{1+\delta}}{\beta_n(u)^\delta} \right)^{-1/\delta} \right)^\delta \, ,$$

with $C = \delta^{-\delta} A^{-1}$ independent of n. It is also easy to verify that there exists a positive constant γ which depends only on δ and b such that

$$(1+s)^\delta \log^{b+\delta}(1 + \log\left(1+s\right)) \leq \gamma s^\delta \left(1 + \log^{b+\delta}(1 + \log\left(1+s\right))\right) \, ,$$

for all $s \geq 0$, and therefore (using $s = 1/n + u$)

$$\frac{\alpha_n^{1+\delta}(u)}{\beta_n^\delta(u)} \leq \gamma \left(\frac{1}{n} + u\right)^\delta \left(1 + \log\left(1 + \frac{1}{n} + u\right)\right)^\delta$$
$$\cdot \left(1 + \log^{b+\delta}\left(1 + \log\left(1 + \frac{1}{n} + u\right)\right)\right) b^{-\delta} \, ,$$

on M. It follows that the expression in (1.11) is bounded below by a multiple of

$$\Big(\int_{R_o}^{r} \Big(\int_{\partial B_t} \Big(\frac{1}{n} + u \Big)^{\delta} \Big(1 + \log \Big(1 + \frac{1}{n} + u \Big) \Big)^{\delta}$$

$$\cdot \Big(1 + \log^{b+\delta} \Big(1 + \log \Big(1 + \frac{1}{n} + u \Big) \Big) \Big) \Big)^{-1/\delta} \Big)^{\delta} .$$

We substitute into (1.11), let n tend to infinity in the resulting inequality, and apply the monotone convergence, and dominated convergence theorems to conclude that

$$\Big(\int_{B_R} \frac{\varphi(|\nabla u|)\,|\nabla u|}{(1+u)\,(1+\log(1+u))\,\log^{1-b}(1+\log(1+u))} \Big)^{-1}$$

$$\geq C_1 \Big(\int_{R}^{r} \Big(\int_{\partial B_t} u^{\delta}\,(1+\log(1+u))^{\delta}$$

$$\cdot (1 + \log^{b+\delta}(1+\log(1+u))) \Big)^{-1/\delta} \Big)^{\delta} ,$$

with $C_1 = b^{1+\delta}/(A\,\gamma\,\delta^{\delta})$. Letting r tend to infinity, we contradict assumption (1.3).

PROOF (OF THEOREM C). We use conditions (0.6) to deduce that

$$\Big(\int_{\partial B_t} u^{\delta}(1+\log(1+u))^{\delta}\,(1+\log^{b+\delta}(1+\log(1+u))) \Big)^{-1/\delta} \geq \frac{C}{r\log r} ,$$

for large enough r. Thus (1.10) holds, and the conclusion follows from Theorem 1.2.

REMARK. We define, for $t \geq 0$, $L_1(t) = 1 + \log(1+t)$, and, for $k \geq 2$, $L_k(t) = 1 + \log L_{k-1}(t)$. It is a simple matter to verify that condition (1.10) in the statement of Theorem 1.2 can be replaced by the weaker

$$(1.12) \quad \Big(\int_{\partial B_t} u^{\delta} \Big(\prod_{k=1}^{n-1} L_k^{\delta}(t) \Big) (1 + \log^{b+\delta} L_{n-1}(u)) \Big)^{-1/\delta} \notin L^1(+\infty) ,$$

for some $n \geq 2$ and $b > 0$.

It follows that Theorem C may be correspondingly improved. Indeed, denoting by ℓ_k the kth-composition power of log, so that $\ell_k(t) =$

$\log{(\ell_{k-1}(t))}$, for sufficiently large $t > 0$, conditions (0.6) in the statement of Theorem C can be replaced by

$$(1.13) \quad \text{i)} \int_{\partial B_r} u^\delta \leq \frac{C}{r\, \ell_n^b(r)} \qquad \text{and} \qquad \text{ii)}\ u(x) \leq C \exp{(r(x)^{1+1/\delta})}\,,$$

for some integer $n \geq 1$, some positive constants b and C, and sufficiently large r.

The following example shows that Theorem C is rather sharp. For the sake of simplicity, we restrict our considerations to the case of the p-Laplacian, $p > 1$. This corresponds to the values $A = 1$ and $\delta = p - 1$ in (0.1).

Let $\sigma \in C^\infty([0, +\infty))$ be a positive function such that $\sigma(t) = t$ for $t \in [0, 1]$, and define

$$\langle \cdot, \cdot \rangle = dr^2 + \sigma^2(r)\, d\theta^2\,,$$

where (r, θ) are the polar coordinates on $\mathbb{R}^m \setminus \{0\} = (0, +\infty) \times S^{m-1}$, and $d\theta^2$ denotes the standard metric on S^{m-1}. Clearly, $\langle \cdot, \cdot \rangle$ extends to a smooth complete metric on $\mathbb{R}^m$. Next, let $a \in C^o([0, +\infty))$ be a non-negative function such that, for $t \in [0, 1]$

$$a(t) = \begin{cases} 1\,, & \text{if } 1 < p < 2\,, \\ t^{p-2}\,, & \text{if } p \geq 2\,. \end{cases}$$

We define the non-negative function

$$(1.14) \quad u(x) = \int_0^{r(x)} \sigma(t)^{-(m-1)/(p-1)} \left(\int_0^t a(s)\, \sigma(s)^{m-1}\, ds \right)^{1/(p-1)} dt\,,$$

where $r(x)$ denotes the distance function from 0. It is easily verified that u is C^2, and satisfies

$$\text{div}\,(|\nabla u|^{p-2}\, \nabla u)(x) = a(r(x))\,,$$

on $(\mathbb{R}^m, \langle \cdot, \cdot \rangle)$. Thus u is not constant and p-subharmonic. Since u is radial, for ease of notation we will write $u(r)$.

To construct the required example we fix $T_o > 1$, and choose the functions $a(t)$ and $\sigma(t)$ so as to satisfy the further conditions

$$(1.15) \quad a(t) = 0 \quad \text{and} \quad \sigma(t) = t^{-1/(m-1)} \exp{\left(-\frac{(p-1)\, t^{p/(p-1)}}{m-1} \right)}\,,$$

on $[T_o, +\infty)$. Inserting these in the definition of u, we deduce that there exist constants C_1, C_2 such that

$$u(r) = C_1 + C_2 \int_{T_o}^r \sigma^{-(m-1)/(p-1)}(t)\, dt$$

$$= C_1 + C_2 \int_{T_o}^r t^{1/(p-1)} \exp\left(t^{p/(p-1)}\right) dt\,.$$

Thus there exist constants $C_i > 0$ such that

$$u(r) \le C_3 \exp\left(r^{p/(p-1)}\right),$$

and

$$\int_{\partial B_r} u^{p-1} = C_4\, \sigma^{m-1}(r)\, u^{p-1}(r) \sim \frac{C_5}{r}\,, \qquad \text{as } r \longrightarrow +\infty\,,$$

showing that (0.6) ii) is satisfied, while (0.6) i) barely fails to hold.

On the other hand, let $\varepsilon > 0$ and choose

$$\sigma(t) = t^{-1/(m-1)} (\log t)^{-\varepsilon(p-1)/(m-1)} \exp\left(-\frac{(p-1)t^{p/(p-1)}(\log^\varepsilon t)}{m-1}\right),$$

on $[T_o, +\infty)$. Then

$$u(r) \sim C_6 \exp\left(r^{p/(p-1)} \log^\varepsilon r\right)$$

and

$$\int_{\partial B_r} u^{p-1} \sim \frac{C_7}{r \log^{\varepsilon(p-1)} r}\,,$$

as $r \longrightarrow +\infty$, so that, in this case, (0.6) i) holds, while (0.6) ii) does not.

We also observe that if $\varepsilon > 1/(p-1)$, then u belongs to $L^{p-1}(M)$. In particular, in the case of the Laplacian, where $p = 2$, this gives a further example, in the spirit of [LS] quoted in the Introduction, of an integrable non-negative subharmonic function. We note that in this case, the manifold $(M, \langle \cdot, \cdot \rangle)$ has finite volume.

We now show how to recover Theorem 2 of Nadirashvili, [N], from Theorem 1.2. For this, and for later comparison, we first state the following

Proposition 1.3. *Let* $(M, \langle \cdot, \cdot \rangle)$ *be a complete Riemannian manifold, let* $h \in C^o(M)$, $h \geq 0$, *and set*

$$v(t) = \int_{B_t} h$$

so that

$$v'(t) = \int_{\partial B_t} h \,.$$

Fix $R > 0$, *and let* $r > R$. *Then for any* $\delta > 0$,

$$(1.16) \qquad \int_R^r \left(\frac{t - R}{v(t)}\right)^{1/\delta} dt \leq C \int_R^r \frac{dt}{v'(t)^{1/\delta}} \,,$$

for some constant $C > 0$ *independent of* r. *In particular,*

$$(1.17) \qquad \left(\frac{t}{v(t)}\right)^{1/\delta} \notin L^1(+\infty) \qquad implies \qquad \frac{1}{v'(t)^{1/\delta}} \notin L^1(+\infty) \,.$$

We remark that the reverse implication in (1.17) does not hold in general. In some interesting cases, the two conditions can be equivalent. For instance, it was showed by Varopoulos, [V1], that if $(M, \langle \cdot, \cdot \rangle)$ is a regular cover of a compact manifold, then $r/\text{vol}\, B_r \notin L^1(+\infty)$ is equivalent to $1/\text{vol}\, \partial B_r \notin L^1(+\infty)$. The same is true if we impose curvature conditions, for instance if the Ricci curvature is non-negative (see [V2]). For further results in this direction, see [LT].

Proof. Proposition 1.3 is well known. We provide an elementary proof for completeness and the convenience of the reader. Fix $\varepsilon > 0$, and set

$$v_\varepsilon(t) = \int_{B_t} h + \varepsilon \,,$$

so that, by the co-area formula,

$$v_\varepsilon'(t) = \int_{\partial B_t} h + \varepsilon \,.$$

Applying Hölder inequality with conjugate exponents $1 + \delta$ and $1 + 1/\delta$ yields

$$\int_R^r \left(\frac{t - R}{v_\varepsilon(t)}\right)^{1/\delta} dt$$

$$(1.18)$$

$$\leq C \left(\int_R^r \left(\frac{t - R}{v_\varepsilon(t)}\right)^{1 + 1/\delta} v_\varepsilon'(t) \right)^{1/(1+\delta)} \left(\int_R^r \frac{dt}{v_\varepsilon'(t)^{1/\delta}} \right)^{\delta/(1+\delta)} \,.$$

Integrating by parts the first integral on the right hand side we get

$$\int_R^r \left(\frac{t-R}{v_\varepsilon(t)}\right)^{1+1/\delta} v'_\varepsilon(t) = -\delta \frac{(r-R)^{1+1/\delta}}{v_\varepsilon(r)^{1/\delta}} + (1+\delta) \int_R^r \left(\frac{t-R}{v_\varepsilon(t)}\right)^{1+1/\delta}$$

$$\leq (1+\delta) \int_R^r \left(\frac{t-R}{v_\varepsilon(t)}\right)^{1/\delta} dt \, ,$$

whence, substituting into (1.18),

$$(1.19) \qquad \int_R^r \left(\frac{t-R}{v_\varepsilon(t)}\right)^{1/\delta} dt \leq (1+\delta)^{1/\delta} \int_R^r \frac{dt}{v'_\varepsilon(t)^{1/\delta}} \, .$$

By dominated convergence, as $\varepsilon \longrightarrow 0$, v_ε and v'_ε decrease to v and v', respectively. Inequality (1.16) follows by applying the monotone convergence theorem to both sides of (1.19).

Since

$$\left(\frac{t-R}{v(t)}\right)^{1/\delta} \geq 2^{-1/\delta} \left(\frac{t}{v(t)}\right)^{1/\delta} , \qquad \text{for } t \geq 2R \, ,$$

it is clear that (1.17) follows from (1.16).

Proposition 1.3 shows that condition (1.10) in Theorem 1.2 may be replaced by the stronger

$$(1.20) \qquad \left(\frac{r}{\int_{B_t} u^\delta \left(1+\log\left(1+u\right)\right)^\delta \left(1+\log^{b+\delta}\left(1+\log\left(1+u\right)\right)\right)}\right)^{1/\delta} \notin L^1(+\infty) \, ,$$

for some $b > 0$.

Assume now that u is a non-negative φ-subharmonic function satisfying $u \in L^\delta(M)$ and $u(x) \leq C \exp\left(r(x)^{1+1/\delta-\varepsilon}\right)$, for some $\varepsilon > 0$ and $C > 0$, as in [N, Theorem 2]. It is easy to verify that the left hand side of (1.20) is bounded below by a multiple of $r^{-1+\varepsilon} \log^{-1-b/\delta}$, which is not integrable at infinity. In light of what remarked above, Theorem 1.2 applies and u is necessarily constant. This shows that Theorem 1.2 extends the work of Nadirashvili. The case where the assumption of non-negativity of u is replaced by the condition that there

exists $x_o \in M$ such that $u(x_o) > 0$, may be treated using similar techniques and will be taken up in Section 4 below (see Theorem 4.3 and the comment thereafter).

At this point, it also looks natural to consider the case of a non-negative φ-subharmonic function $u \in C^1(M)$ satisfying $u \in L^q(M)$, with $0 < q < \delta$. It turns out that to obtain constancy of u we need to impose some additional conditions on φ and on the geometry of M.

As far as φ is concerned, one could consider two kinds of conditions, namely that there exists $B > 0$ such that

$$(1.21) \qquad Bt^\delta \le \varphi(t) \,, \qquad \text{on } [0, +\infty) \,,$$

or that there exist constants c_o and c_1 such that

$$(1.22) \qquad c_o \le \frac{t \, \varphi'(t)}{\varphi(t)} \le c_1 \,.$$

Note that both condition are satisfied in the case of the p-Laplacian, while neither of them holds for the mean curvature operator.

We briefly consider the case where (1.21) is satisfied, leaving the case where (1.22) holds to the interested reader, who may refer to [Lb] for the general theory of operators satisfying this kind of conditions.

We are going to be sketchy since the arguments are standard. The starting point is the following Caccioppoli type inequality. Arguing as in the proof of Lemma 1.1 with the vector field

$$W = \psi^{1+\delta} \, (u + \varepsilon)^{q-\delta} |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \,,$$

one shows (no additional assumption on φ is needed here) that if ψ is a smooth, compactly supported function and u is a C^1, non-negative φ-subharmonic function, then, for every $\widetilde{q} > \delta$,

$$(1.23) \quad \int_{B_r} \psi^{1+\delta} \, u^{\widetilde{q}-\delta-1} \, \varphi\big(|\nabla u|\big)^{1+1/\delta}$$
$$\le \frac{A^{1+1/\delta} \, (1+\delta)^{1+\delta}}{(\widetilde{q}-\delta)^{1+\delta}} \int_{B_r} u^{\widetilde{q}} \, |\nabla \psi|^{1+\delta} \,.$$

Let $0 < \rho < r$ and apply (1.23) when ψ is a smooth cutoff function such that

$$(1.24) \qquad \psi = \begin{cases} 1 \,, & \text{on } B_\rho \,, \\ 0 \,, & \text{on } B_r \setminus B_\rho \,, \end{cases}$$
$$|\nabla \psi| \le \frac{C_o}{r - \rho} \,,$$

with C_o independent of r and ρ. Further, assume that (1.21) holds, and that the Sobolev inequality

$$(1.25) \quad \left(\int_{B_r(o)} |f|^{k(1+\delta)} \right)^{1/k(1+\delta)} \leq S_{k,1+\delta}(r) \left(\int_{B_r(o)} |\nabla f|^{1+\delta} \right)^{1/(1+\delta)}$$

is valid for some $k > 1$, and every $r > 0$ and $f \in C_o^2(B_r(o))$. Then one deduces the fundamental inequality

$$\left(\int_{B_\rho} u^{\widetilde{q}k} \right)^{1/k} \leq C S_{k,1+\delta}^{1+\delta}(r) \left(\left(\frac{A}{B} \right)^{1+1/\delta} \left(\frac{\widetilde{q}}{\widetilde{q}-\delta} \right)^\delta + 1 \right) (r-\rho)^{-1-\delta} \int_{B_r} u^{\widetilde{q}},$$

which holds for every $0 < \rho < r$ with a constant C that depends only δ and on the constant C_o in (1.24).

The Möser iteration procedure allows to deduce that for every $q > 0$ there exists a constant C which depends only on δ, k, q, A, B and C_o such that, for every $0 < \overline{R} < R$

$$(1.26) \quad \sup_{B_{\overline{R}}(o)} u \leq C \left(S_{k,1+\delta}(R) (R - \overline{R})^{-1} \right)^{k(1+\delta)/((k-1)q)} \left(\int_{B_R} u^q \right)^{1/q}.$$

Note now that if M satisfies the doubling condition

$$(1.27) \qquad\qquad \mathrm{vol}\,(B_{2r}(o)) \leq C\,\mathrm{vol}\,(B_r(o)),$$

for every $r > 0$ and $o \in M$, and the (weak) Poincaré inequality

$$(1.28) \quad \begin{aligned} &\fint_{B_r(o)} |f - f_{B_r(o)}| \\ &\qquad \leq C\, r\, \mathrm{vol}\,(B_{2r}(o))^{1-1/(1+\delta)} \left(\fint_{B_{2r}(o)} |\nabla f|^{1+\delta} \right)^{1/(1+\delta)}, \end{aligned}$$

for each $r > 0$, $o \in M$ and $f \in C^\infty(M)$, where $f_{B_r(o)}$ denotes the average of f over $B_r(o)$, then, by [HK, Theorem 1], the Sobolev inequality (1.25) holds for some $k > 1$ and for every $o \in M$ and $r > 0$, with

$$(1.29) \qquad S_{k,1+\delta}(r) \leq C\,(\mathrm{vol}\,(B_r(o)))^{-(k-1)/(k(1+\delta))},$$

and C depending only on δ, k and the constants in the doubling condition and in the weak Poincaré inequality.

We remark that (1.27) implies that $(M, \langle \cdot, \cdot \rangle)$ has at most polynomial growth.

Setting $\overline{R} = R - 1$, and inserting (1.29) into (1.26) yield the following

Theorem 1.4. *Assume that $\varphi \in C^1((0, +\infty)) \cap C^o([0, +\infty))$ satisfies the structural conditions*

$$\varphi(0) = 0, \qquad \text{and} \qquad B\, t^\delta \leq \varphi(t) \leq A\, t^\delta, \qquad \text{for all } t > 0,$$

for some $0 < B \leq A$. Let $M, \langle \cdot, \cdot \rangle$ be a complete Riemannian manifold satisfying the doubling condition (1.27) and the weak Poincaré inequality (1.28). Let $u \in C^1(M)$, be a non-negative φ-subharmonic function on M. Then, either $u \equiv 0$ or, for every $q > 0$,

$$\liminf_{r \to +\infty} \frac{1}{\mathrm{vol}\left(B_r(o)\right)} \int_{B_r(o)} u^q > 0.$$

We present now the following further application of Lemma 1.1, from which Theorem A follows immediately. Related, and somewhat stronger, results are presented in Section 3.

Theorem 1.5. *Let $u \in C^1(M)$ be a solution of the differential inequality*

$$\mathrm{div}\left(|\nabla u|^{-1}\, \varphi(|\nabla u|)\, \nabla u\right) \geq \varphi(|\nabla u|)\, |\nabla u|\, f(u),$$

where $f \in C^o(\mathbb{R})$ is such that

$$\inf_M f(u) > -\sigma,$$

for some $\sigma \in \mathbb{R}$. If

$$(1.30) \qquad \left(\int_{\partial B_t} e^{\sigma u}\right)^{-1/\delta} \notin L^1(+\infty),$$

then u is constant.

PROOF. If u were not constant, one could apply Lemma 1.1 with $\alpha(t) = e^{\sigma t}$ and $\beta(t) = \mu\, e^{\sigma t}$, $\mu = \inf_M f(u) + \sigma$, and contradict assumption (1.30).

We end this section observing that the conclusion (1.5) of Lemma 1.1 holds if M is a manifold with smooth boundary ∂M, with the only additional assumption that $\partial u/\partial \nu \leq 0$, where ν denotes the outward unit normal to ∂M. Correspondingly, one obtains a version of Theorem 1.5 for manifolds with boundary.

In analogy with the situation of the Laplacian, we may define a manifold with boundary M to be φ-parabolic if the only φ-subharmonic functions on M which are bounded above and satisfy $\partial u/\partial \nu \leq 0$ on ∂M are the constants. Applying the version of Theorem 1.5 for manifolds with boundary, we then conclude that if $\mathrm{vol}\,(\partial B_r)^{-1} \notin L^1(+\infty)$, then M is φ-parabolic.

2. Proof of Theorem B and related results.

The same reasoning used in the proof of Lemma 1.1 yields the following:

Lemma 2.1. *Let $f \in C^o(M)$ let u be a non-constant C^1 solution of the differential inequality*

$$(2.1) \qquad u \operatorname{div}\left(|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\nabla u\right) \geq \varphi(|\nabla u|)\,|\nabla u|\,f(u)\,.$$

Assume that for some functions $\alpha \in C^1(I)$ and $\beta \in C^o(I)$ defined in an interval $I \supset u(M)$

$$(2.2) \qquad\qquad\qquad \alpha(u) \geq 0\,,$$

$$(2.3) \qquad u\,\alpha'(u) + (1 + f(u))\,\alpha(u) \geq \beta(u) > 0\,,$$

on M. Then there exist R_o which depends only on u, and a constant $C > 0$ independent of α and β such that, for $r > R \geq R_o$ we have

$$(2.4) \qquad \left(\int_{B_R} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u|\right)^{-1}$$
$$\geq C\left(\int_R^r \left(\int_{\partial B_t} \frac{|u\,\alpha(u)|^{1+\delta}}{\beta(u)^\delta}\right)^{-1/\delta}\right)^\delta\,.$$

Remark. As in Lemma 1.1, if the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is not C^1 on M, the differential inequality (2.1) must be considered in the weak sense. Namely,

$$-\int_M \langle |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u, \nabla(u\,\psi)\rangle \geq \int_M \psi\,\varphi(|\nabla u|)\,|\nabla u|\,f(u)\,,$$

must hold for every non-negative, compactly supported Lipschitz continuous function ψ.

Proof. The proof follows the lines of that of Lemma 1.1. Applying the divergence theorem to the continuous vector field

$$Z = u\,\alpha(u)\,|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\,,$$

and using Hölder inequality we deduce that

$$H(t) \leq A^{1/(1+\delta)}\left(\int_{\partial B_t} \frac{|u\,\alpha(u)|^{1+\delta}}{\beta(u)^\delta}\right)^{1/(1+\delta)}(H'(t))^{\delta/(1+\delta)}\,,$$

where, as in Section 1, we have set

$$H(t) = \int_{B_t} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u|\,.$$

Since u is not constant, there exists R_o which depends only on u, such that $H(t) > 0$ for $t \geq R_o$. Thus we also have

$$\int_{B_t} \frac{|u\,\alpha(u)|^{1+\delta}}{\beta(u)^\delta} > 0 \qquad \text{and} \qquad H'(t) > 0\,,$$

for $t \geq R_o$. Rearranging and integrating the resulting differential inequality yield the required conclusion.

Theorem 2.2. *Let $f \in C^o(\mathbb{R})$, and let $u \in C^1(M)$ be a solution of the differential inequality (2.1) on M, with*

$$(2.5) \qquad\qquad \inf_M f(u) > -1\,.$$

Let $q \in \mathbb{R}$ be such that

$$(2.6) \qquad\qquad q > \delta - \inf_M f(u)\,.$$

If

$$(2.7) \qquad \left(\int_{\partial B_r} |u|^q \right)^{-1/\delta} \notin L^1(+\infty) \, ,$$

then u is constant. If $u > 0$, the same conclusion holds without assuming (2.5).

PROOF. Assume by contradiction that u is not constant. For every integer $n \geq 1$, let $\alpha_n(t) = (t^2 + 1/n)^{(q-\delta-1)/2}$. Then

$$u\,\alpha_n'(u) + (1 + f(u))\,\alpha_n(u)$$
$$= \left(u^2 + \frac{1}{n} \right)^{(q-\delta-1)/2 - 1} \left((q - \delta + f(u))\,u^2 + \frac{1}{n}\,(1 + f(u)) \right),$$

and assumptions (2.5) and (2.6) imply that the second factor on the right hand side is bounded below by

$$\left(q - \delta + \inf_M f(u) \right) u^2 + \frac{1}{n}\left(1 + \inf_M f(u) \right) \geq C\left(u^2 + \frac{1}{n} \right),$$

with $C = \min\{ q - \delta - \inf_M f(u),\, 1 + \inf_M f(u) \}$. We therefore conclude that

$$u\,\alpha_n'(u) + (1 + f(u))\,\alpha_n(u) \geq \beta_n(u) > 0$$

with

$$\beta_n(t) = C\left(t^2 + \frac{1}{n} \right)^{(q-\delta-1)/\delta} .$$

We apply Lemma 2.1 and deduce that there exist $R_o > 0$ independent of n and $C_1 > 0$ independent of n and r such that, for every $r > R_o$

$$\left(\int_{B_{R_o}} \left(u^2 + \frac{1}{n} \right)^{(q-\delta-1)/2} \varphi(|\nabla u|)\,|\nabla u| \right)^{-1}$$
$$\geq C_1 \left(\int_{R_o}^r \left(\int_{\partial B_t} |u|^{1+\delta} \left(u^2 + \frac{1}{n} \right)^{(q-\delta-1)/2} \right)^{-1/\delta} \right)^\delta .$$

Letting $n \longrightarrow +\infty$, and using the dominated and monotone convergence theorems we conclude that for every $r > R_o$

$$\left(\int_{B_{R_o}} |u|^{(q-\delta-1)/2} \varphi(|\nabla u|)\,|\nabla u| \right)^{-1} \geq C_1 \left(\int_{R_o}^r \left(\int_{\partial B_t} |u|^q \right)^{-1/\delta} \right)^\delta ,$$

which contradicts (2.7).

If we assume that $u > 0$, we can repeat the reasoning using $\alpha(t) = t^{q-\delta-1}$ and $\beta(t) = C\, t^{q-\delta-1}$, with $C = q - \delta + \inf_M f(u)$.

Theorem B in the Introduction is an immediate consequence of Theorem 2.2. We also note that in the case of subharmonic and p-subharmonic functions, we can compare with L. Karp, [K1] and Holopainen, [Ho], respectively. Indeed, using Proposition 1.3, assumption (2.7) can be replaced by any of the following

$$(2.8) \qquad \left(\frac{t}{\displaystyle\int_{B_r} |u|^q} \right)^{1/\delta} \notin L^1(+\infty) \,,$$

$$(2.9) \qquad \liminf_{r \to +\infty} \frac{1}{r^{1+\delta}} \int_{B_r} |u|^q < +\infty \,,$$

$$(2.10) \qquad \limsup_{r \to +\infty} \frac{1}{r^{1+\delta} F(r)^\delta} \int_{B_r} |u|^q < +\infty \,,$$

where $F(t)$ is a positive function defined for sufficiently large values of t, and such that $1/(t\, F(t))$ is not integrable at infinity, the remaining assumptions of Theorem 2.2 being unchanged. It is easily verified that both (2.9) and (2.10) imply (2.8).

Lemma 2.1 also allows to obtain the following Liouville type result for p-superharmonic function, which compares with Sturm, [St], in the case of the Laplacian, and with Holopainen, [Ho], in the case of the $\mathcal{A}$-Laplacian. This is also an instance of a situation where the differential inequality (2.1) arises naturally.

Proposition 2.3. *Let* $u \in C^1(M)$ *be p-superharmonic and non-negative on* M. *If*

$$(2.11) \qquad \left(\int_{\partial B_r} u^q \right)^{-1/(p-1)} \notin L^1(+\infty) \,,$$

for some $q \in \mathbb{R}$, $q < p - 1$, *then* u *is constant.*

PROOF. For every integer $n \geq 1$, let $v_n = (u + 1/n)^{-1}$. Then $\nabla v_n = -v_n^2\, \nabla u$ and

$$\operatorname{div}\left(|\nabla v_n|^{p-2}\, |\nabla v_n|\right) = -v_n^{2(p-1)} \operatorname{div}\left(|\nabla u|^{p-2}|\nabla u|\right) + 2\,(p-1)\, v_n^{-1}\, |\nabla v_n|^p \,.$$

Since u is p-superharmonic and $v_n > 0$, it follows that

$$v_n \operatorname{div}\left(|\nabla v_n|^{p-2}\nabla v_n\right) \geq 2\,(p-1)\,|\nabla v_n|^p\,,$$

showing that v_n satisfies (2.1) with $\varphi(t) = t^{p-1}$, and $f(t) \equiv 2\,(p-1)$.

The proof now follows the lines of that of Theorem 2.2. If we set $\alpha(t) = t^{-p-q}$, we have

$$t\,\alpha'(t) + (1 + f(t))\,\alpha(t) = (p-1-q)\,t^{-p-q}\,, \qquad \text{for all } t > 0\,,$$

so that (2.3) is verified with $\beta(t) = (p-1-q)\,t^{-p-q}$.

Assume by contradiction that u is not constant. By Lemma 2.1 we conclude that there exist C and $R_o > 0$, such that

$$\left(\int_{B_{R_o}} v_n^{-p-q}\,|\nabla v_n|^p\right)^{-1} \geq C\left(\int_{R_o}^r \left(\int_{\partial B_t} v_n^{-q}\right)^{-1/(p-1)} dt\right)^{p-1}\,,$$

for every $r > R_o$. Note that both C and R_o are independent of n, as it can be easily verified from the proof of the Lemma. Indeed, C depends only on the structural constants in (0.1), in the case at hand, $A = 1$ and $\delta = p - 1$, while R_o is the infimum of the values t such that the function

$$H_n(t) = \int_{B_t} v_n^{-p-q}\,|\nabla v_n|^p = \int_{B_t} \left(u + \frac{1}{n}\right)^{p+q}|\nabla u|^p$$

is positive. It is clear that the right hand side is bounded below by $H_1(t)$.

Rewriting the main inequality in terms of u, letting $n \longrightarrow +\infty$, and using the monotone and dominated convergence theorems we obtain

$$\left(\int_{B_{R_o}} u^{p+q}\,\varphi(|\nabla u|)\,|\nabla u|^p\right)^{-1} \geq C\left(\int_R^r \left(\int_{\partial B_t} u^q\right)^{-1/(p-1)} dt\right)^{p-1}\,.$$

Letting $r \longrightarrow +\infty$ we contradict (2.11).

The following easy consequence of Theorem 2.2 will be useful to show its sharpness.

Corollary 2.4. *Assume that*

(2.12) $$\operatorname{vol}\partial B_r \leq C\,r^{\eta-1}\,,$$

for some $\eta \geq 0$, $C > 0$, and sufficiently large r. Let $u \in C^1(M)$ be a non-negative φ-subharmonic function on M. If there exist $q > \delta$ and a constant $C_1 > 0$ such that

$$(2.13) \qquad u(x)^q \leq C_1\, r(x)^{\delta - \eta + 1} \log^{\delta}(r(x))\,,$$

for $r(x)$ sufficiently large, then u is constant.

Remark. Assumption (2.13) deserves some further comment. Indeed, if $\delta - \eta + 1 < 0$, then u tends to zero at infinity and the validity of the maximum principle would force u to vanish identically, with no need for (2.12). Therefore, this begs the question: when does the φ-Laplacian, $\mathrm{div}\,(|\nabla u|^{-1}\, \varphi(|\nabla u|)\, \nabla u)$, satisfy a maximum, or at least a comparison principle? The following elementary result answers in the affirmative if φ is non-decreasing (see also [PSZ]).

Proposition 2.5. *Let φ satisfy conditions (0.1) i) and ii), i.e., $\varphi(0) = 0$ and $\varphi(t) > 0$ if $t > 0$, and assume moreover that φ is non-decreasing on $[0, +\infty)$. Let Ω be a bounded domain with smooth boundary $\partial\Omega$, and let u and $v \in C^1(\overline{\Omega})$ satisfy*

$$\mathrm{div}\,(|\nabla u|^{-1}\, \varphi(|\nabla u|)\, \nabla u) \geq \mathrm{div}\,(|\nabla v|^{-1}\, \varphi(|\nabla v|)\, \nabla v)\,, \qquad \text{on } \Omega\,,$$

$$u \leq v\,, \qquad \text{on } \partial\Omega\,.$$

Then $u \leq v$ on Ω.

Proof. We choose $\alpha \in C^1(\mathbb{R})$ such that

$$\text{i) } \alpha(t) = 0 \text{ on } (-\infty, 0], \qquad \text{ii) } \alpha'(t) > 0 \text{ on } (0, +\infty)\,,$$

and consider the vector field W defined on $\overline{\Omega}$ by

$$W = \alpha(u - v)\,(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v)\,.$$

A computation that uses the properties of u, v and α, shows that

$$\mathrm{div}\, W \geq \alpha'(u - v)\, h\,, \qquad \text{on } \overline{\Omega}\,,$$

where

$$h(x) = \langle |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v, \nabla u - \nabla v\rangle(x)\,.$$

Applying the divergence theorem (see the observation at the beginning of Section 1) and noting that $u \leq v$ on $\partial\Omega$ implies $\alpha(u - v) = 0$ there, we obtain

$$(2.14) \qquad \int_\Omega \alpha'(u - v) \, h \leq 0 \, .$$

Observe now that a simple computation shows that $h(x)$ is equal to

$$\left(\varphi(|\nabla u|) - \varphi(|\nabla v|)\right) \left(|\nabla u| - |\nabla v|\right)(x)$$
$$+ \left(|\nabla u|^{-1}\varphi(|\nabla u|) + |\nabla v|^{-1}\varphi(|\nabla v|)\right) \left(|\nabla u||\nabla v| - \langle\nabla u, \nabla v\rangle\right)(x) \, .$$

Since φ is non-decreasing, we deduce from Schwarz inequality that $h(x) \geq 0$ for every x, with equality if and only if $\nabla u(x) = \nabla v(x)$. Therefore, it follows from (2.14) that $\alpha'(u - v) \, h$ vanishes identically on Ω.

Next, we assume by contradiction that

$$\mathcal{O} = \{x \in \Omega : \ u(x) > v(x)\} \neq \varnothing \, .$$

Since $\alpha'(u - v) > 0$ on $\mathcal{O}$, we must have $\nabla u = \nabla v$ on $\mathcal{O}$, so that $u - v$ is constant on each connected component of $\mathcal{O}$. But $u \leq v$ on $\partial\mathcal{O}$ (indeed, $u(z) = v(z)$ if $z \in \partial\mathcal{O} \cap \Omega$ by definition of $\mathcal{O}$, while $u(z) \leq v(z)$ by assumption if $z \in \partial\mathcal{O} \cap \partial\Omega$) and therefore $u \leq v$ on $\mathcal{O}$, contradicting the definition of $\mathcal{O}$.

We explicitly observe that the structural condition (0.1) iii) was not used in Proposition 2.5. Since constants are φ-harmonic, the Proposition easily implies that if φ is non-decreasing, then a φ-subharmonic function on Ω attains its maximum on $\partial\Omega$. In particular, a nonnegative, φ-subharmonic function on M that vanishes at infinity is necessarily identically zero. Indeed, under the further assumption that $\liminf_{t \to 0+} t\, \varphi'(t)/\varphi(t) > 0$, a slight modification of the proof of [PW, Theorem 5, pp. 61-64] shows that the usual strong maximum principle holds, namely, u cannot attain an interior maximum unless it is constant ([P]). For a version of the strong maximum principle valid under slightly different, and somewhat weaker, assumptions see also [PSZ, Theorem 1].

To show that Corollary 2.4 is sharp we proceed as in Section 1. We keep the notation used there, and consider the case of the p-Laplacian. Here $\varphi(t) = t^{p-1}$, $p > 1$, is increasing, and therefore we only need to consider the case where assumption (2.13) holds with $p \geq \eta \geq 0$.

Given any $q > p - 1$, choose $a(t)$ as in (1.15), and

$$\sigma(t) = t^{\mu(p-1)/(m-1)} (\log t)^{\nu(p-1)/(m-1)},$$

on $[T_o, +\infty)$, with constants μ and ν to be specified later. Then

$$\operatorname{vol} \partial B_r = C_m \, \sigma^{m-1}(r) = C_m \, r^{\mu(p-1)} (\log t)^{\nu(p-1)},$$

for $r \geq T_o$. Proceeding as in Section 1 it is easy to verify that if u is defined in (1.14) then

$$u(r) = C_1 + C_2 \int_{T_o}^{r} t^{-\mu} (\log t)^{-\nu} \, dt$$

$$\sim C \begin{cases} r^{1-\mu} (\log r)^{-\nu}, & \text{if } \mu < 1, \\ (\log r)^{-\nu+1}, & \text{if } \mu = 1, \ \nu < 1, \\ \log(\log r), & \text{if } \mu = 1 = \nu, \end{cases}$$

as $r \longrightarrow +\infty$.

Consider first the case $p > \eta$. Let $q(1 - \mu) = p - \eta$ and $-\nu q = p - 1$, i.e., $\mu = (q - p + \eta)/q < 1$ and $\nu = -(p - 1)/q$. Then the non-constant p-subharmonic function u satisfies

$$u(r)^q \sim C \, r^{p-\eta} (\log r)^{p-1}, \qquad \text{as } r \longrightarrow +\infty.$$

and condition (2.13) is met. On the other hand,

$$\operatorname{vol} \partial B_r = C \, r^{(p-1)(\eta-p+q)/q} (\log r)^{-(p-1)^2/q}.$$

The exponent of r on the right hand side is greater than $\eta - 1$ for every $q > p - 1$, and tends to $\eta - 1$ as q tends $p - 1$, showing that (2.12) barely fails.

Turning things around, if we take $\mu = (\eta - 1)/(p - 1) < 1$ and $\nu = 0$, then (2.12) is satisfied, while

$$u(r)^q \sim C \, r^{q(p-\eta)/(p-1)}, \qquad \text{as } r \longrightarrow +\infty.$$

Again, the exponent of r on the right hand side is greater than $p - \eta$ for every $q > p - 1$, and tends to $p - \eta$ as q tends to $p - 1$, showing that the non-constant p-subharmonic function narrowly fails to satisfy (2.13).

The case $p = \eta$ is dealt with similarly. To show that if (2.12) fails, then there are non-constant p-subharmonic functions satisfying (2.13), it suffices to take $\mu = 1$, and $\nu = (q - p + 1)/q$. Then

$$u(r)^q \sim C \log^{p-1} r \,, \qquad \text{as } r \longrightarrow +\infty \,,$$

while

$$\operatorname{vol} \partial B_r = C_m \, r^{p-1} \, (\log r)^{(p-1)(q-p+1)/q} \,, \qquad r \geq T_o \,,$$

so that (2.12) is off only by a logarithmic term. On the other hand, if we take $\mu = 1$ and $\nu = 0$, then (2.12) holds, while

$$u(r)^q \sim C \log^q r \,, \qquad \text{as } r \longrightarrow +\infty \,,$$

so that (2.13) is not satisfied for every $q > p - 1$.

3. Further results.

Lemmas 1.1 and 2.1 give estimates from above for the quantity

$$H(t) = \int_{B_R} \beta(u) \, \varphi(|\nabla u|) \, |\nabla u| \,.$$

The next lemma provides an estimate from below. By combining the two estimates, we will obtain new results.

Lemma 3.1. *Let* $f \in C^o(\mathbb{R})$, *and let* $u \in C^1(M)$ *be a solution of the differential inequality*

$$(3.1) \qquad u \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \, \nabla u\right) \geq \varphi(|\nabla u|) \, |\nabla u| \, f(u) \,, \qquad on \ M \,.$$

Assume that there exist functions $\rho \in C^1(I)$ *and* $\beta \in C^o(I)$ *defined in an interval* $I \supset u(M)$ *such that*

$$(3.2) \qquad\qquad\qquad \beta(u) > 0 \,,$$

$$(3.3) \qquad\qquad\qquad \rho(u) \geq 0 \,,$$

$$(3.4) \qquad\qquad\qquad \frac{|u \, \rho(u)|}{\beta(u)} \leq L < +\infty \,,$$

$$(3.5) \qquad\qquad\qquad u \, \rho'(u) + (1 + f(u)) \, \rho(u) > 0 \,,$$

on M. Then there exist $R_o > 0$ and a constant $C > 0$ such that, for every $r > R \geq R_o$,

$$(3.6) \qquad \int_{B_r \setminus B_R} \beta(u)\, \varphi(|\nabla u|)\, |\nabla u| \geq C \int_R^r \left(\int_{\partial B_t} \beta(u)^\delta \right)^{-1/\delta} dt\,.$$

Proof. Note first of all that, using the structural condition $\varphi(t) \leq A\, t^\delta$, we may estimate

$$\int_{\partial B_t} \beta(u)\, \varphi(|\nabla u|)\, |\nabla u| \geq A^{-1/\delta} \int_{\partial B_t} \beta(u)\, \varphi(|\nabla u|)^{1+1/\delta}\,.$$

Now, by Hölder inequality with conjugate exponents $1 + \delta$ and $1 + 1/\delta$, we have

$$\int_{\partial B_t} \beta(u)\, \varphi(|\nabla u|)$$

$$\leq \left(\int_{\partial B_t} \beta(u) \right)^{1/(1+\delta)} \left(\int_{\partial B_t} \beta(u)\, \varphi(|\nabla u|)^{1+1/\delta} \right)^{\delta/(1+\delta)},$$

whence, using $\beta(u) > 0$, rearranging and substituting, we obtain

$$(3.7) \qquad \begin{aligned} &\int_{\partial B_t} \beta(u)\, \varphi(|\nabla u|)\, |\nabla u| \\ &\qquad \geq A^{-1/\delta} \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} \left(\int_{\partial B_t} \beta(u)\, \varphi(|\nabla u|) \right)^{1+1/\delta}. \end{aligned}$$

Next, we consider the continuous vector field X defined by

$$X = u\, \rho(u)\, |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\,,$$

and set

$$\gamma(t) = \int_{\partial B_t} \langle X, \nabla r \rangle\,,$$

so that, by Schwarz inequality and assumptions (3.2) and (3.4) we get

$$(3.8) \qquad \gamma(t) \leq \int_{\partial B_t} |u\, \rho(u)|\, \varphi(|\nabla u|) \leq L \int_{\partial B_t} \beta(u)\, \varphi(|\nabla u|)\,.$$

On the other hand, computing the divergence of X and using the assumption $\rho(u) \geq 0$ we estimate

$$\operatorname{div} X \geq (u\, \rho'(u) + (1 + f(u))\, \rho(u))\, \varphi(|\nabla u|)\, |\nabla u|\,,$$

so that, by the divergence theorem,

$$\gamma(t) = \int_{B_t} \operatorname{div} X \geq \int_{B_t} \left(u\,\rho'(u) + (1 + f(u))\,\rho(u) \right) \varphi(|\nabla u|)\,|\nabla u|\,.$$

Since u is not constant, and (3.5) holds, there exist R_o and a constant $C_o > 0$, both depending on ρ and f only through the quantity $u\,\rho'(u) + (1 + f(u))\,\rho(u)$, such that

$$\gamma(t) \geq C_o\,, \qquad \text{for all } t \geq R_o\,.$$

Combining this with (3.8) and inserting into (3.7) yield

$$\int_{\partial B_t} \beta(u)\,\varphi(|\nabla u|)\,|\nabla u| \geq A^{-1/\delta} \Big(\int_{\partial B_t} \beta(u) \Big)^{-1/\delta} (L^{-1}\gamma(t))^{1+1/\delta}$$

$$\geq C \Big(\int_{\partial B_t} \beta(u) \Big)^{-1/\delta}\,,$$

with $C = A^{-1/\delta}\,(C_o/L)^{1+1/\delta}$. Integrating over $[R, r]$, $R_o \leq R < r$, and using the co-area formula we obtain (3.6).

REMARK. In some applications it is crucial to avoid the explicit dependence on β and ρ of the quantity R_o and the constant C in (3.6). It is clear from the above proof that this may be achieved if we assume that L is independent of β and ρ and replace (3.5) with

$$(3.9) \qquad u\,\rho'(u) + (1 + f(u))\,\rho(u) \geq \varepsilon\,,$$

for some absolute constant $\varepsilon > 0$.

Putting together the estimate from below just obtained with the estimate from above provided by Lemma 2.1 we obtain

Lemma 3.2. *Let $f \in C^o(\mathbb{R})$, and let $u \in C^1(M)$ be a solution of the differential inequality (3.1). Assume that there exist functions $\beta \in C^o(I)$ and $\alpha, \rho \in C^1(I)$ defined in an interval $I \supset u(M)$ such that*

$$(3.10) \qquad \beta(u) > 0\,, \qquad \alpha(u), \rho(u) \geq 0\,,$$

$$(3.11) \qquad u\,\alpha'(u) + (1 + f(u))\,\alpha(u) \geq \beta(u)\,,$$

$$(3.12) \qquad u\,\rho'(u) + (1 + f(u))\,\rho(u) > 0\,,$$

$$(3.13) \qquad \frac{|u\,\rho(u)|}{\beta(u)} \leq L < +\infty\,,$$

on M. Then there exist $R_o > 0$ and a constant $C > 0$ such that, for every $r > R \geq R_o$,

$$(3.14) \qquad \frac{1}{\sup_{B_r} \left| \dfrac{u \, \alpha(u)}{\beta(u)} \right|} \int_R^r \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} dt \leq C \,.$$

REMARK. As it will become clear from the proof below, if we can guarantee that the constants appearing in the conclusion (3.6) of Lemma 3.1 do not depend explicitly on β and ρ, then the quantity R_o and the constants C above do not depend explicitly on α, β and ρ. In particular, this is the case if we assume that L is independent of β and ρ and replace (3.12) with (3.9). This will be used in Theorem 3.6 below.

PROOF. The assumptions of Lemma 2.1 and Lemma 3.1 are satisfied, so there exist R_o and constants $C_1, C_2 > 0$ such that for every $r > R \geq R_o$ (2.4) and (3.6) hold with constant C_1 and C_2, respectively.
    Denote as above

$$H(t) = \int_{B_t} \beta(u) \, \varphi(|\nabla u|) \, |\nabla u| \,,$$

and let $r > \overline{R} \geq R \geq R_o$. It follows from (2.4) that

$$H(\overline{R})^{-1} \geq \frac{C_1}{\sup_{B_r} \left| \dfrac{u \, \alpha(u)}{\beta(u)} \right|^{1+\delta}} \left(\int_{\overline{R}}^r \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} dt \right)^{\delta} \,,$$

while (3.6) yields

$$H(\overline{R}) - H(R) \geq C_2 \int_R^{\overline{R}} \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} dt \,.$$

Combining the two inequalities we deduce that

$$1 \geq \frac{H(\overline{R}) - H(R)}{H(\overline{R})}$$

$$\geq C_3 \left(\sup_{B_r} \left| \frac{u \, \alpha(u)}{\beta(u)} \right| \right)^{-(1+\delta)}$$

$$\cdot \left(\int_{\overline{R}}^r \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} dt \right)^{\delta} \int_R^{\overline{R}} \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} dt \,,$$

for every $r > \overline{R} \geq R \geq R_o$, with $C_3 = C_1 C_2$. We claim that we can choose $\overline{R}$ in such a way that the product of the two integrals on the right hand side is equal to

$$\frac{\delta^{\delta}}{(1+\delta)^{\delta}} \left(\int_R^r \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} dt \right)^{1+\delta} .$$

Indeed, having set

$$B = \int_R^r \left(\int_{\partial B_t} \beta(u) \right)^{-\delta} , \qquad x = \int_{\overline{R}}^r \left(\int_{\partial B_t} \beta(u) \right)^{-\delta} ,$$

the claim amounts to finding a solution $x_o \in (0, B)$ to the equation

$$x^{\delta} (B - x) = \frac{\delta^{\delta}}{(1+\delta)^{\delta}} B^{1+\delta} ,$$

and it is easily verified that the (unique) solution x_o in $(0, B)$ to the given equation is

$$x_o = \frac{\delta}{1+\delta} B .$$

We conclude that, for every $r > R \geq R_o$

$$1 \geq C_4 \left(\sup_{B_r} \left| \frac{u\,\alpha(u)}{\beta(u)} \right| \right)^{-(1+\delta)} \left(\int_{\overline{R}}^r \left(\int_{\partial B_t} \beta(u) \right)^{-1/\delta} dt \right)^{1+\delta}$$

with

$$C_4 = C_3 \frac{\delta^{\delta}}{(1+\delta)^{1+\delta}} ,$$

whence, rearranging, we obtain (3.14).

As a first consequence of Lemma 3.2 we have

Theorem 3.3. *Let $u \in C^1(M)$ be a solution of the differential inequality*

(3.15) $$u \operatorname{div} \left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \right) \geq 0 ,$$

on M. If

(3.16) $$\liminf_{r \to +\infty} \frac{\sup_{B_r} |u|}{\displaystyle\int_R^r (\operatorname{vol} \partial B_t)^{-1/\delta} \, dt} = 0 ,$$

for some $R > 0$ sufficiently large, then u is constant.

Proof. Assume by contradiction that u is not constant. We apply Lemma 3.2 with $\alpha = \beta = 1$, $\rho(t) = (1 + t^2)^{-1/2}$ (and $f \equiv 0$) to conclude that there exist $R_o > 0$ and a constant $C > 0$ such that for every $r > R \geq R_o$,

$$\frac{1}{\sup_{B_r} |u|} \int_R^r (\operatorname{vol} \partial B_t)^{-1/\delta} \, dt \geq C \, .$$

It is clear that this contradicts our assumption (3.16).

Remark. If u is non-negative we can replace (3.16) with

$$(3.17) \qquad \liminf_{r \to +\infty} \frac{\sup_{B_r} u}{\displaystyle\int_R^r (\operatorname{vol} \partial B_t)^{-1/\delta} dt} = 0 \, ,$$

for some $R > 0$ sufficiently large. Observe that, if $u \not\equiv 0$, then (3.17) implies that

$$(3.18) \qquad (\operatorname{vol} \partial B_t)^{-1/\delta} \notin L^1(+\infty) \, .$$

This in particular implies that a non-negative φ-subharmonic function u satisfying (3.17) is necessarily constant. In this connection we remark that in [RSV, Theorem 3], it was shown, with a different proof, that the same conclusion holds without any sign condition on u if (3.17) and (3.18) hold. This also follows from the results presented in Section 4 below. We note however that the proof in [RSV] does not seem to adapt to the case of solutions of the differential inequality (3.15), and therefore does not yield the further consequences of Lemma 3.2 presented below.

The following corollary is the companion of Corollary 2.4 and will be useful to show the sharpness of Theorem 3.3.

Corollary 3.4. *Assume that*

$$(3.19) \qquad \operatorname{vol} \partial B_r \leq C \, r^{\eta - 1}$$

for some $\eta \geq 0$, $C > 0$, and sufficiently large r. Let $u \in C^1(M)$ be a non-negative φ-subharmonic function on M. If

$$(3.20) \qquad u(x)^\delta = o(r(x)^{\delta - \eta + 1}) \, , \qquad \text{if } \delta > \eta - 1 \, ,$$

$$(3.21) \qquad u(x) = o(\log r(x)) \, , \qquad \text{if } \delta = \eta - 1 \, ,$$

as $r(x) \longrightarrow +\infty$, then u is constant.

As in the examples in sections 1 and 2, we consider the case of the p-Laplacian, and keep the notation used there. In particular, $\delta = p - 1$, $a(t)$ is defined in (1.15), and u is the p-subharmonic function defined in (1.14). As in Section 2, it suffices to consider the case $p \geq \eta$. The construction done there provides examples of manifolds satisfying (3.19), and admitting non-constant p-subharmonic functions which barely fail to satisfy (3.20) or (3.21) respectively in the case $p > \eta$ and $p = \eta$.

On the other hand, if we choose

$$\sigma(t) = \begin{cases} t^{(\eta-1)/(m-1)} \left(\log t\right)^{\nu(p-1)/(m-1)}, & \text{if } p > \eta, \\ t^{(\eta-1)/(m-1)} \left(\log\log t\right)^{\nu(\eta-1)/(m-1)}, & \text{if } p = \eta, \end{cases}$$

on $[T_o, +\infty)$, for some $\nu > 0$, then we have

$$\operatorname{vol}\partial B_r = C_m \begin{cases} r^{\eta-1} \left(\log r\right)^{\nu(p-1)}, & \text{if } p > \eta, \\ r^{\eta-1} \left(\log\log r\right)^{\nu(p-1)}, & \text{if } p = \eta, \end{cases}$$

and

$$u(r) \sim C \begin{cases} r^{(p-\eta)/(p-1)} \left(\log r\right)^{-\nu}, & \text{if } p > \eta, \\ \log r \left(\log\log r\right)^{-\nu}, & \text{if } p = \eta, \end{cases}$$

as $r \longrightarrow +\infty$. Thus u satisfies condition (3.20) or (3.21), respectively, while (3.19) barely fails to hold.

Lemma 3.2 also yields the following

Theorem 3.5. *Let $f \in C^o(\mathbb{R})$, and let $u \in C^1(M)$ be a solution of the differential inequality (3.1) on M satisfying*

$$u > 0, \qquad \inf_M f(u) > -\gamma,$$

for some $\gamma > 0$. If

$$(3.22) \qquad \liminf_{r \to +\infty} \frac{(\sup_{B_r} u)^{1+\gamma/\delta}}{\displaystyle\int_R^r (\operatorname{vol}\partial B_t)^{-1/\delta}\, dt} = 0,$$

for some $R > 0$ sufficiently large, then u is constant.

Proof. Assuming by contradiction that u is not constant, we set

$$\mu = \gamma + \inf_M f(u) > 0$$

and let α, β, ρ be the functions defined on $(0, +\infty)$ by $\alpha(t) = t^\gamma$, $\beta(t) = \mu\, t^\gamma$ and $\rho(t) = t^{\gamma-1}$. It is easy to verify that the assumptions of Lemma 3.2 are satisfied, and we deduce that there exist $R_o > 0$ and a constant $C > 0$ such that for every $r > R \geq R_o$

$$\frac{\mu^{1+\gamma/\delta}}{\sup_{B_r} u} \int_R^r \left(\int_{\partial B_t} u^\gamma \right)^{-1/\delta} dt \leq C\,,$$

which contradicts (3.22).

When u is not assumed to be positive, we have the following version of the above result.

Theorem 3.6. Let $f \in C^o(\mathbb{R})$, and let $u \in C^1(M)$ be a solution of the differential inequality (3.1) on M satisfying

$$\inf_M f(u) > -1\,.$$

If

$$(3.23) \qquad \liminf_{r \to +\infty} \frac{(\sup_{B_r} |u|)^{1+1/\delta}}{\displaystyle\int_R^r (\operatorname{vol} \partial B_t)^{-1/\delta}\, dt} = 0\,,$$

for some $R > 0$ sufficiently large, then u is constant.

Proof. Again, we assume by contradiction that u is not constant. Let $S = 1 + \inf_M f(u) > 0$, and, for every integer $n \geq 1$, define $\alpha_n(t) = (t^2 + 1/n)^{1/2}$, $\beta_n = S\alpha_n$ and $\rho_n(t) = \rho(t) \equiv 1$. Then

$$u\, \alpha_n'(u) + (1 + f(u))\, \alpha_n(u) \geq \beta_n(u)\,,$$
$$u\, \rho'(u) + (1 + f(u))\, \rho(u) = 1 + f(u) \geq S > 0\,.$$

Moreover

$$\frac{|u\, \rho(u)|}{\beta_n(u)} = \frac{|u|}{S(u^2 + 1/n)^{1/2}} \leq \frac{1}{S}\,, \qquad \text{on } M\,,$$

independently of n. By Lemma 3.2 and the remark thereafter, there exist $R_o > 0$ and a constant $C > 0$ independent of n such that, for every $r > R \geq R_o$,

$$\frac{1}{\sup_{B_r} |u|} \int_R^r \left(\int_{\partial B_t} \left(u^2 + \frac{1}{n} \right)^{1/2} \right)^{-1/\delta} dt \leq C \ .$$

Letting $n \longrightarrow +\infty$, and using the monotone and dominated convergence theorems we deduce that, for every $r > R \geq R_o$,

$$\frac{1}{\sup_{B_r} |u|} \int_R^r \left(\int_{\partial B_t} |u| \right)^{-1/\delta} dt \leq C \ ,$$

and this contradicts assumption (3.23).

4. Phragmen-Lindelöff type results.

Lemma 4.1. *Let $u \in C^1(\Omega) \cap C^o(\overline{\Omega})$ be a φ-subharmonic function on an unbounded domain $\Omega \subset M$, and assume that $u \leq \Gamma$ on $\partial\Omega$, for some $\Gamma \in \mathbb{R}$. Given $B > \Gamma$, define*

$$\Omega_B = \left\{ x \in \Omega : \ u(x) > B \right\},$$

and suppose that Ω_B is not empty with boundary $\partial\Omega_B$. Let $\alpha \in C^1$ and $\beta \in C^o$ be defined in $[B, +\infty)$ and such that $\alpha(u) \geq 0$, $\alpha'(u) \geq \beta(u) > 0$ on $\overline{\Omega}_B$. Let also $\lambda \in C^1(\mathbb{R})$ be such that $\lambda(t) = 0$ for $t \leq B$, $\lambda(t) > 0$ for $t > B$ and $\lambda'(t) \geq 0$. Then there exist R_o and a constant $C > 0$ independent of α and β and λ such that, for every $r > R \geq R_o$

$$\left(\int_{B_R \cap \Omega_B} \lambda(u) \, \beta(u) \, \varphi(|\nabla u|) \, |\nabla u| \right)^{-1}$$

$$\geq C \left(\int_R^r \left(\int_{\partial B_t \cap \Omega_B} \lambda(u) \, \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \right)^{-1/\delta} \right)^\delta .$$

PROOF. Observe first of all that since $B > \Gamma$, then $\overline{\Omega}_B \subset \Omega$. Thus $u = B$ on $\partial\Omega_B$, and it follows that u cannot be constant on any component of Ω_B. In particular ∇u does not vanish identically on Ω_B.

The argument now follows the lines of the proof of Lemma 1.1. Let $\widetilde{Z}$ be the vector field on $\overline{\Omega}_B$ defined by

$$\widetilde{Z} = \lambda(u) \, \alpha(u) \, |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \ .$$

Note that $\widetilde{Z}$ can be extended to a continuous vector field on M by setting it equal to 0 on Ω_B^c. Similarly, we can and will similarly extend to all of M every product containing a factor $\lambda(u)$. Set also

$$\widetilde{H}(t) = \int_{B_t} \lambda(u)\,\beta(u)\,\varphi(|\nabla u|)\,|\nabla u|\,,$$

so that, by the co-area formula,

$$\widetilde{H}'(t) = \int_{\partial B_t} \lambda(u)\,\beta(u)\,\varphi(|\nabla u|)\,|\nabla u|\,.$$

Since

$$\operatorname{div} Z \geq \lambda(u)\,\beta(u)\,\varphi(|\nabla u|)\,|\nabla u|\,, \qquad \text{on } M\,,$$

integrating over B_t, applying the divergence theorem, Hölder inequality with exponents $1 + \delta$ and $1 + 1/\delta$, and using the structural condition $\varphi(t)^{1/\delta} \leq A^{1/\delta}\,t$, we obtain

$$(4.1) \quad \widetilde{H}(t) \leq A^{1/(1+\delta)} \Big(\int_{\partial B_t} \lambda(u)\,\frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \Big)^{1/(1+\delta)} (\widetilde{H}'(t))^{\delta/(1+\delta)}\,.$$

Since ∇u is not identically zero on Ω_B, and $\beta(u)$ and $\lambda(u)$ are there strictly positive, there exists $R_o > 0$ (independent of α, β and λ) such that $\widetilde{H}(t) > 0$ if $t \geq R_o$. It follows that the right hand side of (4.1) is also strictly positive for $t \geq R_o$. In particular, Ω_B is necessarily unbounded.

Rearranging and integrating between R and r, $R_o \leq R < r$, we obtain

$$\widetilde{H}(R)^{-1/\delta} \geq \frac{1}{\delta\,A^{1/\delta}} \int_R^r \Big(\int_{\partial B_t} \lambda(u)\,\frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} \Big)^{-1/\delta} dt\,.$$

To conclude we only have to observe that, since $\lambda(u) = 0$ off Ω_B, the integrals over B_t and ∂B_t may be replaced with integrals over $B_t \cap \Omega_B$ and $\partial B_t \cap \Omega_B$, respectively.

We remark that if in the above proof Ω is assumed to be bounded, then for t sufficiently large (4.1) leads to a contradiction. This in turn, forces $\Omega_B = \varnothing$, and we conclude that $u \leq \Gamma$ on Ω. In other words, if u is C^1 in a bounded domain Ω, continuous up to the boundary, and

φ-subharmonic in Ω, then u attains its maximum on $\partial\Omega$. Of course, if φ is non-decreasing, then this follows also from Proposition 2.5.

As a consequence of Lemma 4.1 we have the following Phragmen-Lindelöff type result:

Theorem 4.2. *Let Ω be an unbounded domain in M and let $u \in C^1(\Omega) \cap C^o(\overline{\Omega})$, be a non-negative φ-subharmonic function on Ω such that $u \leq \Gamma$ on $\partial\Omega$. Assume that, for some $q > \delta$,*

$$(4.2) \qquad \left(\int_{\partial B_r \cap \Omega} u^q \right)^{-1/\delta} \notin L^1(+\infty) .$$

Then $u \leq \Gamma$ on Ω.

PROOF. Assume by contradiction that $\{x \in \Omega : u(x) > \Gamma\} \neq \varnothing$, and choose $B > \Gamma \geq 0$ sufficiently close to Γ that $\Omega_B = \{x \in \Omega : u(x) > B\} \neq \varnothing$. We apply Lemma 4.1 with the choices

$$\alpha(t) = t^{q-\delta} , \qquad \beta(t) = \alpha'(t) = (q - \delta)\, t^{q-\delta-1} , \qquad t \geq B ,$$

and $\lambda \in C^1(\mathbb{R})$ satisfying the conditions in the statement of the lemma, and $\sup_{\mathbb{R}} \lambda(t) = 1$. It follows that there exist R_o and $C > 0$ such that, for every $r > R \geq R_o$,

$$\left(\int_{B_R \cap \Omega_B} \lambda(u)\, u^{q-\delta-1}\, \varphi(|\nabla u|)\, |\nabla u| \right)^{-1} \geq C \left(\int_R^r \left(\int_{\partial B_t \cap \Omega} u^q \right)^{-1/\delta} \right)^{\delta} .$$

By virtue of (4.2), letting $r \longrightarrow +\infty$ this yields the required contradiction.

Lemma 4.1 also allows us to prove the following, slightly more general version of Theorem B.

Theorem 4.3. *Let $u \in C^1(M)$ be a φ-subharmonic function on M. Assume that there exists $x_o \in M$ such that $u(x_o) > 0$ and let $u_+(x) = \max\{0, u(x)\}$. If there exists $q > \delta$ such that*

$$\left(\int_{\partial B_r} u_+^q \right)^{-1/\delta} \notin L^1(+\infty) ,$$

then u is constant on M.

Proof. Arguing as above, assume by contradiction that u is not constant, and let $B > 0$ be sufficiently small that $\Omega_B = \{x \in M : u(x) > B\}$ is a non-empty set with boundary $\partial \Omega_B$. Applying Lemma 4.1 with the same choice of α, β and λ as in Theorem 4.2, we obtain

$$\left(\int_{B_R \cap \Omega_B} \lambda(u)\, u^{q-\delta-1}\, \varphi(|\nabla u|)\, |\nabla u| \right)^{-1} \geq C \left(\int_R^r \left(\int_{\partial B_t \cap \Omega_B} u^q \right)^{-1/\delta} \right)^\delta ,$$

for every $r > R \geq R_o$. Since the surface integral on the right hand side is bounded above by $\int_{\partial B_t} u_+^q$, a contradiction is reached letting $r \longrightarrow +\infty$.

Similarly, applying Lemma 4.1 with $\alpha(t) = \log^b(1 + \log(1+t))$ and $\beta(t) = \alpha'(t)$, and arguing as above, one proves a version of Theorem 1.2 valid for functions of arbitrary sign. Namely, if u is a φ-subharmonic function such that $u(x_o) > 0$ for some $x_o \in M$ and (1.10) holds with u_+ instead of u then u is necessarily constant. This, in turn, yields a version of Theorem C valid for functions of arbitrary sign.

The next lemma is a version of Lemma 3.2 on a domain.

Lemma 4.4. *Let $f \in C^o(\mathbb{R})$. Let $\Omega \subset M$ be an unbounded domain, and let $u \in C^1(\Omega) \cap C^o(\overline{\Omega})$, satisfy*

$$u \operatorname{div}\left(|\nabla u|^{-1}\, \varphi(|\nabla u|)\, \nabla u \right) \geq \varphi(|\nabla u|)\, |\nabla u|\, f(u), \qquad on\ \Omega,$$

and assume that $u \leq \Gamma$ on $\partial \Omega$. Given $B > \Gamma$, define $\Omega_B = \{x \in \Omega : u(x) > B\}$, and suppose that Ω_B is not empty with boundary $\partial \Omega_B$. Assume that there exist functions $\beta \in C^o([B, +\infty))$ and $\alpha, \rho \in C^1([B, +\infty))$ such that

$$\beta(u) > 0, \qquad \alpha(u), \rho(u) \geq 0,$$

$$u\, \alpha'(u) + (1 + f(u))\, \alpha(u) \geq \beta(u),$$

$$u\rho'(u) + (1 + f(u))\, \rho(u) > 0,$$

$$\frac{|u\, \rho(u)|}{\beta(u)} \leq L < +\infty,$$

on Ω_B. Finally, let $\lambda \in C^1(\mathbb{R})$ be such that $\lambda(t) = 0$ for $t \leq B$, $\lambda(t) > 0$ for $t > B$ and $\lambda'(t) \geq 0$. Then there exist R_o and a constant $C > 0$

such that, for every $r > R \geq R_o,$

$$\frac{1}{\displaystyle\sup_{B_r \cap \Omega_B} \frac{|u\,\alpha(u)|}{\beta(u)}} \int_R^r \left(\int_{\partial B_t \cap \Omega_B} \lambda(u)\,\beta(u) \right)^{-1/\delta} dt \leq C \, .$$

REMARK. As in Lemma 3.2, if we assume that L is independent of β and ρ, and that $u\,\rho'(u) + (1 + f(u))\,\rho(u) \geq \varepsilon > 0$ on Ω_B, then R_o and C do not depend explicitly on α, β, λ and ρ.

PROOF. The proof is modeled after that of Lemma 3.2. Set

$$\widehat{H}(t) = \int_{B_R \cap \Omega_B} \lambda(u)\,\beta(u)\,\varphi(|\nabla u|)\,|\nabla u| \, .$$

Arguing as in Lemma 2.1, one shows that there exist R_1 and $C_1 > 0$ (independent of α, β and λ) such that, for every $r > \overline{R} \geq R_1,$

$$\widehat{H}(\overline{R})^{-1} \geq C_1 \left(\int_{\overline{R}}^r \left(\int_{\partial B_t \cap \Omega_B} \lambda(u)\,\frac{|u\,\alpha(u)|^{1+\delta}}{\beta(u)^\delta} \right)^{-1/\delta} \right)^\delta$$

$$\geq \frac{C_1}{\displaystyle\sup_{B_r \cap \Omega_B} \left(\frac{|u\,\alpha(u)|}{\beta(u)} \right)^{1+\delta}} \left(\int_{\overline{R}}^r \left(\int_{\partial B_t \cap \Omega_B} \lambda(u)\,\beta(u) \right)^{-1/\delta} \right)^\delta \, .$$

On the other hand , a minor modification of the proof of Lemma 3.1 shows that there exist $R_o \geq R_1$ and $C_2 > 0$ such that for every $\overline{R} > R \geq R_o,$

$$\widehat{H}(\overline{R}) - \widehat{H}(R) \geq C_2 \int_R^{\overline{R}} \left(\int_{\partial B_t} \lambda(u)\,\beta(u) \right)^{-1/\delta} dt \, .$$

The required conclusion follows as in the final part of the proof of Lemma 3.2.

Theorem 4.5. *Let* Ω *be an unbounded domain in* M*, and let* $u \in C^1(\Omega) \cap C^o(\overline{\Omega})$ *be a* φ*-subharmonic function on* Ω *such that* $u \leq \Gamma$ *on* $\partial\Omega$*. Assume that*

$$\mathrm{vol}\,(\partial B_r \cap \Omega)^{-1/\delta} \notin L^1(+\infty) \, ,$$

and

$$(4.3) \qquad \liminf_{r \to +\infty} \frac{\sup_{B_r \cap \Omega} u}{\displaystyle\int_R^r \mathrm{vol}\,(\partial B_r \cap \Omega)^{-1/\delta}} = 0\,,$$

for some $R > 0$ sufficiently large. Then $u \leq \Gamma$ on Ω.

PROOF. Note first that if C is a constant and $v = u + C$, then v is φ-subharmonic on Ω, $v \leq \Gamma + C$ on $\partial\Omega$, and v satisfies (4.3). Clearly, $u \leq \Gamma$ on Ω if and only if $v \leq \Gamma + C$ on Ω. Without loss of generality, we can therefore assume that $\Gamma > 0$.

Assume by contradiction that $\{x \in \Omega : u(x) > \Gamma\} \neq \varnothing$, and choose $B > \Gamma$ close enough to Γ that $\Omega_B = \{x \in \Omega : u(x) > B\}$ is not empty. We apply Lemma 4.4 with the choices

$$\alpha(t) = 1\,, \qquad \beta(t) = 1\,, \qquad \text{and} \qquad \rho(t) = (1 + t^2)^{-1/2}\,,$$

and with λ satisfying the further condition $\sup_{\mathbb{R}} \lambda = 1$. We conclude that there exist R_o and $C > 0$ such that, for $r > R \geq R_o$,

$$\frac{1}{\displaystyle\sup_{B_r \cap \Omega_B} u} \int_R^r \mathrm{vol}\,(\partial B_t \cap \Omega_B)^{-1/\delta}\, dt \leq C\,.$$

Since $\mathrm{vol}\,(\partial B_t \cap \Omega)^{-1/\delta} \leq \mathrm{vol}\,(\partial B_t \cap \Omega_B)^{-1/\delta}$ and $\sup_{B_r \cap \Omega} u = \sup_{B_r \cap \Omega_B} u$, this clearly contradicts (4.3).

We observe that if φ is non-decreasing, by Proposition 2.5, $\sup_{B_r \cap \Omega} u$ may be replaced by $\sup_{\partial B_r \cap \Omega} u$.

We conclude this section by showing how Theorem 4.5 allows us to recover the conclusion of [RSV, Theorem 3], quoted in Section 3.

Corollary 4.6. *Assume that*

$$\mathrm{vol}\,(\partial B_t)^{-1/\delta} \notin L^1(+\infty)\,,$$

and let $u \in C^1(M)$ be a φ-subharmonic function on M. If

$$\liminf_{r \to +\infty} \frac{\sup_{B_r} u}{\displaystyle\int_R^r \mathrm{vol}\,(\partial B_r)^{-1/\delta}} = 0\,,$$

then u is constant on M.

PROOF. Assume that u is not constant, and choose $\Gamma < \sup u$ in such a way that $\varnothing \neq \Omega = \{x : u(x) > \Gamma\}$ and $\partial\Omega$ is of class C^1. Since $\mathrm{vol}\,(\partial B_t)^{-1/\delta} \leq \mathrm{vol}\,(\partial B_t \cap \Omega)^{-1/\delta}$ and $\sup_{B_r \cap \Omega} u = \sup_{B_r \cap \Omega} u$, both the assumptions of Theorem 4.5 hold, and we conclude that $u \leq \Gamma$ on Ω, contradicting the definition of Ω.

5. A weak maximum principle.

We begin by proving a weak maximum principle asserting that, under suitable volume growth conditions, given a smooth function u which is bounded above on M, the set where u is close to its supremum and $\mathrm{div}\,(|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\nabla u)$ is less that any given positive constant is nonempty. A special case of this result for the Laplacian was proved by Karp in [K2, Theorem 2.3]. His proof made use of the stochastic completeness of the underlying manifold. In our general setting, such an approach is clearly not feasible. The proof presented below is direct and based on elementary considerations. We recall that A and δ are the constants that appear in the structural condition (0.1) iii). Throughout this section it will be assumed that the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is of class at least C^1. As mentioned in the Introduction, if u is C^2, this is certainly the case for the Laplacian, or the p-Laplacian with $p \geq 2$ and for the (generalized) mean curvature operators.

Theorem 5.1. *Let $u \in C^2(M)$ be such that $u^* = \sup_M u < +\infty$, and assume that the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is of class at least C^1. Given $\alpha < u^*$, let $\Omega_\alpha = \{x \in M : u(x) > \alpha\}$, and assume that*

$$(5.1) \qquad \liminf_{r \to +\infty} \frac{\log \mathrm{vol}\,(B_r \cap \Omega_\alpha)}{r^{1+\delta}} < +\infty \,.$$

Then,

$$(5.2) \qquad \inf_{\Omega_\alpha} \mathrm{div}\,(|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\nabla u) \leq 0 \,.$$

PROOF. According to (5.1), there exist $0 < \gamma < +\infty$, a constant $C_o > 0$, and a sequence $R_k \nearrow +\infty$ such that

$$(5.3) \qquad \mathrm{vol}\,(B_{R_k} \cap \Omega_\alpha) \leq C_o\, e^{\gamma R_k^{1+\delta}} \,,$$

for every $k = 1, 2, \ldots$. Fix $\eta > 0$ and define $v = u - u^* + \eta$, so that

$$\Omega_\alpha = \left\{ x \in M : \ v(x) > \alpha - u^* + \eta \right\}.$$

We are going to prove that

$$\begin{aligned}
\inf_{\Omega_\alpha} \operatorname{div} \left(|\nabla u|^{-1} \, \varphi(|\nabla u|) \, \nabla u \right) &= \inf_{\Omega_\alpha} \operatorname{div} \left(|\nabla v|^{-1} \, \varphi(|\nabla v|) \, \nabla v \right) \\
&\leq A \, \gamma \, \delta^\delta \, 2^{1+\delta} \, \eta^\delta \,,
\end{aligned}$$
(5.4)

whence the required conclusion follows letting $\eta \longrightarrow 0^+$.

To prove (5.4), we assume by contradiction that for some $\rho > 0$

$$\text{(5.5)} \quad \operatorname{div} \left(|\nabla u|^{-1} \, \varphi(|\nabla u|) \, \nabla u \right) \geq A \, \gamma \, \delta^\delta \, 2^{1+\delta} \, \eta^\delta \, (1 + \rho) = B \,, \quad \text{on } \Omega_\alpha \,.$$

We choose a C^1 function λ such that $\lambda(t) = 0$ if $t \leq \alpha - u^* + \eta$, $\lambda(t) > 0$ if $t > \alpha - u^* + \eta$, $\lambda' \geq 0$, and $\sup \lambda(t) = 1$. Fix $\varepsilon > 0$, and let $\sigma = \gamma / \eta$, $\zeta = 2\gamma + \varepsilon$ and $\theta = (\gamma / \gamma + \varepsilon)^{1/(1+\delta)}$, so that $0 < \theta < 1$. Finally, choose a smooth cutoff function $h = h_k$ such that $h = 1$ on $B_{\theta R_k}$, $h = 0$ off B_{R_k} and $|\nabla h| \leq C_1 / ((1 - \theta) R_k)$, for some $C_1 > 0$ independent of k, and define the vector field

$$W = h^{1+\delta} \, \lambda(v) \, e^{(\sigma v - \zeta) r^{1+\delta}} \, |\nabla v|^{-1} \varphi(|\nabla v|) \nabla v \,.$$

We compute the divergence of W, and use (5.5), $\lambda' \geq 0$, Schwarz inequality, $|\nabla r| = 1$, the inequality $\sigma \, v - \zeta \leq 0$, and the structural condition $|\nabla v| \geq A^{-1/\delta} \varphi(|\nabla v|)^{1/\delta}$, to obtain, after some computations,

$$\begin{aligned}
\operatorname{div} W \\
\geq - (1+\delta) \, h^\delta \, \lambda(v) \, e^{(\sigma v - \zeta) r^{1+\delta}} \, \varphi(|\nabla v|) \, |\nabla h| + h^{1+\delta} \, \lambda(v) \, e^{(\sigma v - \zeta) r^{1+\delta}} \\
\cdot \left(B - (1+\delta) \, \zeta r^\delta \, \varphi(|\nabla v|) + \sigma \, A^{-1/\delta} \, r^{1+\delta} \, \varphi(|\nabla v|)^{1+1/\delta} \right).
\end{aligned}$$

We claim that, if $\varepsilon > 0$ is small enough,

$$\begin{aligned}
B - (1+\delta) \, \zeta \, r^\delta \varphi(|\nabla v|) + \sigma \, A^{-1/\delta} \, r^{1+\delta} \, \varphi(|\nabla v|)^{1+1/\delta} \\
\geq \Lambda \, r^{1+\delta} \varphi(|\nabla v|)^{1+1/\delta} \,,
\end{aligned}$$
(5.6)

with $\Lambda = \Lambda(\varepsilon) > 0$.

Postponing the proof of the claim, we insert (5.6) into the above inequality, integrate over $\Omega_\alpha \cap B_{R_k}$, and apply the divergence theorem.

Since every factor containing $\lambda(v)$ vanishes off Ω_α, while every product containing h vanishes off B_{R_k} (so that we may equivalently integrate over B_{R_k} thus avoiding possible problems due to the non-smoothness of the boundary of $\Omega_\alpha \cap B_{R_k}$), we obtain

$$\int_{\Omega_\alpha \cap B_{R_k}} h^{1+\delta}\, \lambda(v)\, e^{(\sigma v - \zeta)r^{1+\delta}}\, r^{1+\delta}\, \varphi(|\nabla v|)^{1+1/\delta}$$

$$\leq \frac{1+\delta}{\Lambda} \int_{\Omega_\alpha \cap B_{R_k}} h^\delta\, \lambda(v)\, e^{(\sigma v - \zeta)r^{1+\delta}}\, \varphi(|\nabla v|)\, |\nabla h|\,.$$

Applying Hölder inequality with conjugate exponents $1+\delta$ and $1+1/\delta$ to estimate the right hand side, rearranging and using the properties of the cutoff function h and the inequality $\lambda(v) \leq 1$, we conclude that

$$(5.7) \qquad \int_{\Omega_\alpha \cap B_{\theta R_k}} r^{1+\delta}\, e^{(\sigma v - \zeta)r^{1+\delta}}\, \lambda(v)\, \varphi(|\nabla v|)^{1+1/\delta}$$

$$\leq \frac{C_2}{R_k^{(1+\delta)^2}} \int_{B_{R_k} \setminus B_{\theta R_k}} e^{(\sigma v - \zeta)r^{1+\delta}}\,,$$

with $C_2 = ((1+\delta)\,C_1/(\Lambda(1-\theta)\,\theta^\delta))^{1+\delta}$. Now, using the definitions of σ, γ, ζ, and $\sup v = \eta$, we have $\sigma\, v - \zeta \leq -(\gamma + \varepsilon)$. Thus

$$\int_{B_{R_k} \setminus B_{\theta R_k}} e^{(\sigma v - \zeta)r^{1+\delta}} \leq \int_{B_{R_k} \setminus B_{\theta R_k}} e^{-(\gamma+\varepsilon)r^{1+\delta}}$$

$$\leq e^{-(\gamma+\varepsilon)(\theta R_k)^{1+\delta}}\, \mathrm{vol}\, B_{R_k}\,,$$

and it follows from the definition of θ and (5.3) that that the right hand side is bounded above by $C_o > 0$. Inserting into (5.7), we deduce that

$$\int_{\Omega_\alpha \cap B_{\theta R_k}} r^{1+\delta}\, e^{(\sigma v - \zeta)r^{1+\delta}}\, \lambda(v)\, \varphi(|\nabla v|)^{1+1/\delta} \leq \frac{C_3}{R_k^{(1+\delta)^2}}\,.$$

Letting $k \longrightarrow +\infty$, we conclude that the integrand vanishes identically in Ω_α. Since $\lambda(v) > 0$ in Ω_α, and $\varphi(t) > 0$ if $t > 0$, this implies that v is constant on every connected component of Ω_α, and this contradicts assumption (5.5).

To conclude it remains to prove (5.6). Setting $x^{-1} = r^\delta \varphi(|\nabla v|)$ this amounts to showing that

$$\Lambda = \inf_{x>0} \{B\, x^{1+1/\delta} - (1+\delta)\,\zeta\, x^{1/\delta} + \sigma\, A^{-1/\delta}\} > 0\,.$$

It is easily verified that the function in braces attains its minimum on $(0, +\infty)$ at $x_o = \mu(B(1+\delta))^{-1}$ where it is equal to

$$\frac{\sigma}{A^{1/\delta}} - \frac{\delta \, \zeta^{1+1/\delta}}{B^{1/\delta}} \; .$$

Recalling the definitions of the quantities involved, an easy computation shows that the latter quantity is equal to

$$\frac{\gamma}{A^{1/d} \, \eta} \left(1 - \frac{(1 + \varepsilon/2\,\gamma)^{1+1/\delta}}{(1+\rho)^{1/\delta}} \right) ,$$

which is strictly positive if ε is small enough.

Theorem 5.1 immediately yields the following weak version of the Omori-Yau maximum principle for the φ-Laplacian.

Corollary 5.2. *Assume that*

$$(5.8) \qquad\qquad \liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{r^{1+\delta}} < +\infty ,$$

and let u be a smooth function on M with $u^* = \sup u < +\infty$, such that $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least C^1. For every n the set

$$Z_n = \left\{ y \in M : \; u(y) > u^* - \frac{1}{n} \; , \; \operatorname{div} \left(|\nabla u|^{-1} \, \varphi(|\nabla u|) \, \nabla u \right) < \frac{1}{n} \right\} \neq \varnothing .$$

In particular, this yields Theorem D in the Introduction.

Proof. We define the sets $A_n = \{y : \; u(y) > u^* - 1/n\}$ and $B_n = \{y : \operatorname{div} \left(|\nabla u|^{-1} \, \varphi(|\nabla u|) \, \nabla u \right)(y) < 1/n\}$ so that $Z_n = A_n \cap B_n$.

Clearly, $A_n \neq \varnothing$, and, by Theorem 5.1 we also have $B_n \neq \varnothing$. We may therefore define $u_n^* = \sup_{B_n} u$, and the required conclusion follows from $u_n^* = u^*$.

Indeed, assume by contradiction that $u_n^* < u^*$, and let $\Omega_{u_n^*} = \{x : u(x) > u_n^*\} \neq \varnothing$. By definition, $\Omega_{u_n^*} \subset B_n^c$, and therefore

$$\operatorname{div} \left(|\nabla u|^{-1} \, \varphi(|\nabla u|) \, \nabla u \right) \geq \frac{1}{n} , \qquad \text{on } \Omega_{u_n^*} ,$$

and this contradicts Theorem 5.1.

In a forthcoming paper we shall see how the validity of this weak form of the maximum principle for the Laplacian is tightly related to stochastic completeness (see [PRS]).

The next theorem provides a version of the weak maximum principle when the boundedness of the function is replaced by a suitable condition on the growth at infinity. We note that Theorem 5.1 corresponds to the limit case $b = 1 + \delta$. For the case of the Laplacian, our result generalizes [K2, Theorem 2.2] (where it is considered the case $b = 1$).

Theorem 5.3. *Assume that, for some $1 \leq b < 1 + \delta$,*

$$\liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{r^b} = \gamma_o < +\infty \,.$$

Let $u \in C^2(M)$ be such that the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is of class at least C^1, and assume that

$$(5.9) \qquad \limsup_{r(x) \to +\infty} \frac{u(x)}{r(x)^{(1+\delta-b)/\delta}} \leq a_o \,,$$

for some $a_o \geq 0$. Then

$$(5.10) \qquad \inf_M \operatorname{div}\left(|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\nabla u\right) \leq \frac{A\,\gamma_o(a_o\delta)^\delta\,(2\,b)^{1+\delta}}{(1+\delta)^{1+\delta}} \,.$$

PROOF. The proof is a modification of that of Theorem 5.1. Let $a > a_o$, and $\gamma > \gamma_o$. We are going to show that

$$(5.11) \qquad \inf_M \operatorname{div}\left(|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\nabla u\right) \leq \frac{A\,\gamma(a\delta)^\delta\,(2\,b)^{1+\delta}}{(1+\delta)^{1+\delta}} \,,$$

whence the required conclusion follows letting $a \longrightarrow a_o$ and $\gamma \longrightarrow \gamma_o$.

By adding a suitable constant to u, it may be assumed that

$$(5.12) \qquad \text{i)} \ \frac{u(x)}{(1 + r(x))^{(1+\delta-b)/\delta}} < a\,, \qquad \text{on } M \,,$$

$$\text{ii) there exists } x_o \in M \text{ such that } u(x_o) > 0 \,.$$

Clearly it suffices to show that (5.11) holds when the infimum is taken over the set $\Omega = \{x : \ u(x) > 0\}$ instead of M. To prove this, we assume by contradiction that for some $\rho > 0$,

$$(5.13) \ \operatorname{div}\left(|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\nabla u\right) \geq \frac{A\,\gamma\,(a\,\delta)^\delta\,(2\,b)^{1+\delta}\,(1+\rho)}{(1+\delta)^{1+\delta}} = B > 0 \,,$$

on Ω. Let $\lambda \in C^1(\mathbb{R})$ such that $\lambda(t) = 0$ if $t \leq 0$, $\lambda(t) > 0$ if $t > 0$, $\lambda'(t) \geq 0$ and $\sup \lambda = 1$. Let also $\varepsilon > 0$, and define

$$\sigma = \frac{\gamma}{a}, \qquad \theta = \Big(\frac{\gamma}{\gamma + \varepsilon}\Big)^{1/b}, \qquad \text{and} \qquad \zeta = 2\gamma + \varepsilon\,.$$

By the volume growth assumption and the inequality $\gamma > \gamma_o$, there exist a sequence $R_k \nearrow +\infty$ and a constant $C_o > 0$ such that

(5.14) $$\operatorname{vol} B_{R_k} \leq C_o\, e^{\gamma R_k^b}\,.$$

For every k, we let $h = h_k$ be a smooth cutoff function such that $h = 1$ on $B_{\theta R_k}$, $h = 0$ off B_{R_k}, and $|\nabla h| \leq C_1/((1-\theta)\,R_k)$ with $C_1 > 0$ independent of k. Finally, let W be the vector field defined by

$$W = h^{1+\delta}\lambda(u)\,\Xi(u)\,|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\,,$$

where we have set, for notational convenience,

$$\Xi(u) = \exp\Big((1+r)^b\Big(\frac{\sigma\,u}{(1+r)^{(1+\delta-b)/\delta}} - \zeta\Big)\Big)\,.$$

A computation that uses (5.13), $\lambda' \geq 0$, and Schwarz inequality, shows that

$$\operatorname{div} W \geq h^{1+\delta}\lambda(u)\,\Xi(u)\,B - (1+\delta)\,h^\delta\,\lambda(u)\,\Xi(u)\,\varphi(|\nabla u|)\,|\nabla h|$$
$$+ \sigma\,(1+r)^{(b-1)(1+1/\delta)}h^{1+\delta}\,\lambda(u)\,\Xi(u)\,\varphi(|\nabla u|)\,|\nabla u|$$
$$+ h^{1+\delta}\lambda(u)\,\Xi(u)\,|\nabla u|^{-1}\,\varphi(|\nabla u|)\,\langle\nabla u, \nabla r\rangle$$
$$\cdot (1+r)^{b-1}\Big((b-1)\Big(1+\frac{1}{\delta}\Big)\frac{\sigma\,u}{(1+r)^{(1+\delta-b)/\delta}} - b\zeta\Big)\,.$$

Using (5.12) i), it is easily verified that the quantity in braces on the right hand side is negative on Ω. It follows that the last term in the above inequality is bounded below by

$$h^{1+\delta}\lambda(u)\,\Xi(u)\,\varphi(|\nabla u|)\,(1+r)^{b-1}\,(\cdots)$$
$$\geq -h^{1+\delta}\lambda(u)\,\Xi(u)\,\varphi(|\nabla u|)\,(1+r)^{b-1}\,b\zeta\,, \qquad \text{on }\Omega\,,$$

where we have used the fact that $u \geq 0$ on Ω. Substituting, and using the structural condition (0.1) iii), we obtain

div W

$$
\begin{aligned}
\geq &- (1 + \delta)\, h^{\delta} \lambda(u)\, \Xi(u)\, \varphi(|\nabla u|)\, |\nabla h| \\
&+ h^{1+\delta}\, \lambda(u)\, \Xi(u)\, \big(B - b\, \zeta\, (1 + r)^{b-1}\, \varphi(|\nabla u|) \\
&\qquad + \sigma A^{-1/\delta} \varphi(|\nabla u|)^{(\delta+1)/\delta}\, (1 + r)^{(b-1)(1+1/\delta)}\big)\,.
\end{aligned}
$$

Setting $x^{-1} = (1 + r)^{b-1}\, \varphi(|\nabla u|)$ and arguing exactly as in the proof of the claim in Theorem 5.1 one shows that for ε sufficiently small the quantity in braces on the right hand side is bounded from below by

$$
\Lambda\, (1 + r)^{(b-1)(1+1/\delta)}\, \varphi(|\nabla u|)^{1+1/\delta}\,.
$$

At this point the proof proceeds as in Theorem 5.1. Integrating div W over $\Omega \cap B_{R_k}$, applying the divergence theorem, Hölder inequality with exponents $1 + \delta$ and $1 + 1/\delta$, and using the properties of the cutoff function h and $\sup \lambda = 1$ we obtain

(5.15)
$$
\int_{\Omega \cap B_{\theta R_k}} \lambda(u)\, \Xi(u)\, (1 + r)^{(b-1)(1+1/\delta)} \varphi(|\nabla u|)^{1+1/\delta}
$$
$$
\leq \frac{C_2}{R_k^{b(1+\delta)}} \int_{B_{R_k} \setminus B_{\theta R_k}} \Xi(u)\,,
$$

for some constant $C_2 > 0$ independent of k.

It follows from (5.12) i), and from the definition of the quantities involved that

$$
\Xi(u) \leq \exp\left((1 + r)^b\, (\sigma\, a - \zeta)\right) = \exp\left(-(\gamma + \varepsilon)\, (1 + r)^b\right),
$$

so that, using the volume estimate (5.14), we deduce that the integral on the right hand side of (5.15) is bounded above by C_o for every k. Thus

$$
\int_{\Omega \cap B_{\theta R_k}} \lambda(u)\, \Xi(u)\, (1 + r)^{(b-1)(1+1/\delta)} \varphi(|\nabla u|)^{1+1/\delta} \leq \frac{C_3}{R_k^{b(1+\delta)}}\,,
$$

with C_3 independent of k. Letting $k \longrightarrow +\infty$ we conclude that the integrand must be identically equal to zero on Ω, and therefore that u

is constant on every connected component of Ω. But this contradicts (5.13), as required to finish the proof.

We conclude by showing that Theorems 5.1 and 5.3 are sharp with respect to the volume growth conditions in their statement. We consider the p-Laplacian, and keep the notation introduced in Section 1. For $1 \leq b \leq p$, let $\sigma(t)$ satisfy

$$\sigma(t) = \exp\left(t^b \log^\nu t\right), \qquad \text{for all } t \geq T_o\,,$$

for some $T_o > 1$, and let u be the p-subharmonic function defined in (1.14), with $a(t) = 1$ for every t. Then $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 1$ on M, and there exist constants C_1 and C_2 such that, for $r > T_o$,

$$u(r) = C_1 + \int_{T_o}^r \sigma(t)^{-(m-1)/(p-1)}\left(C_2 + \int_{T_o}^t \sigma(s)^{m-1}\,ds\right)^{1/(p-1)} dt\,.$$

It is easy to verify that

$$\int_{T_o}^t \sigma(s)^{m-1}\,ds \sim C\,t^{1-b}\,(\log t)^{-\nu}\exp\left((m-1)\,t^b \log^\nu t\right), \quad \text{as } t \longrightarrow +\infty\,,$$

and therefore

$$\frac{\log \operatorname{vol} B_r}{r^b} \sim C\,(\log r)^\nu\,, \qquad \text{as } r \longrightarrow +\infty\,.$$

Furthermore,

$$\sigma(t)^{-(m-1)/(p-1)}\left(C_2 + \int_{T_o}^t \sigma(s)^{m-1}ds\right)^{1/(p-1)}$$

$$\sim \frac{C}{t^{(b-1)/(p-1)}\,(\log t)^{\nu/(p-1)}}\,.$$

To show that Theorem 5.1 is sharp, we choose $b = p$ and $\nu > p - 1$. Then u is bounded, and the conclusion of the theorem clearly does not hold. Since $u(r)$ is increasing, the set Ω_α $(\alpha < \sup u)$ is a ball, and $\log \operatorname{vol}\left(\Omega_\alpha \cap B_r\right) \sim \log \operatorname{vol} B_r$, showing that the volume growth condition in the statement of the theorem fails by a log factor.

On the other hand, if we take $1 \leq b < p$, and $\nu > 0$ then, the volume growth condition (5.8) fails (again by a log term). In this case

$$u(r) \sim C\,r^{1+1/(p-1)-b/(p-1)}\,, \qquad \text{as } r \longrightarrow +\infty\,,$$

so that assumption (5.9) is satisfied with $a_o = 0$, while (5.10) clearly is not.

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