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# On the $\frac{1}{2}$-Problem of Besicovitch: quasi-arcs do not contain sharp saw-teeth

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**Abstract**

In this paper we give an alternative proof of our recent result that totally unrectifiable 1-sets which satisfy a measure-theoretic flatness condition at almost every point and sufficiently small scales, satisfy Besicovitch's $\frac{1}{2}$-Conjecture which states that the lower spherical density for totally unrectifiable 1-sets should be bounded above by $\frac{1}{2}$ at almost every point. This is in contrast to rectifiable 1-sets which actually possess a density equal to unity at almost every point. Our present method is simpler and is of independent interest since it mainly relies on general properties of finite sets of points satisfying a scale-invariant flatness condition. For instance it shows that a quasi-arc of small constant cannot contain "sharp saw-teeth".

## 1. Introduction

Let us first recall that a set $E \subset \mathbb{R}^n$ is called a 1-set if $0 < \mathcal{H}^1(E) < \infty$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. Such a set $E$ splits naturally into two pieces. Namely $E = E' \cup E''$, where $\mathcal{H}^1(E' \cap E'') = 0$, and $E'$ is a rectifiable set, that is it can be covered (up to a subset of measure zero) by a countable number of Lipschitz (or $C^1$) curves. On the other hand, $E''$ is a totally unrectifiable set, that is $\mathcal{H}^1(E'' \cap \Gamma) = 0$ for every rectifiable curve $\Gamma$. Among the many contrasting properties between rectifiable and totally unrectifiable 1-sets are the spherical density properties. For a rectifiable 1-set $E$ in $\mathbb{R}^n$ (say), we have

$$\Theta^1(E, x) \equiv \lim_{r \to 0^+} \frac{\mathcal{H}^1(E \cap B(x, r))}{2r} = 1\,, \tag{1.1}$$

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for a.e. $x \in E$, where $B(x,r)$ is the closed ball with center at $x$, and radius $r$. This is originally due to Besicovitch [Be1], and for subsets of a metric space it is due to Kirchheim [Ki]. Totally unrectifiable 1-sets on the other hand have remarkably different behavior. In particular (for subsets of Euclidean space), the density $\Theta^1(E,x)$ does not exist for a.e. $x \in E$, as originally proved by Besicovitch [Be2]. Recall that $\sigma_1(\mathbb{R}^n)$ is defined to be the smallest number such that, if $E$ is a totally unrectifiable 1-set in $\mathbb{R}^n$, then the lower spherical density

$$\Theta^1_*(E,x) \equiv \lim_{r\to 0+}\inf \frac{\mathcal{H}^1(E \cap B(x,r))}{2r} \le \sigma_1(\mathbb{R}^n)\ , \tag{1.2}$$

for a.e. $x \in E$ (see e.g. [Pr], [Ma]). In 1928, Besicovitch [Be1] proved that

$$\sigma_1\left(\mathbb{R}^2\right) \le 1 - 10^{-2576}\,, \tag{1.3}$$

and thus established a characterization of rectifiability via the lower spherical density. In 1938 [Be2] he showed, by more natural methods, that

$$\sigma_1\left(\mathbb{R}^2\right) \le \frac{3}{4}\,. \tag{1.4}$$

He also gave examples showing that

$$\sigma_1\left(\mathbb{R}^2\right) \ge \frac{1}{2}\,, \tag{1.5}$$

and conjectured that $\sigma_1(\mathbb{R}^2) = \frac{1}{2}$. The upper estimates on $\sigma_1(\mathbb{R}^2)$ were shown to hold for $\sigma_1(\mathbb{R}^n)$ in [Mo], and to metric spaces in [PT]. In principle, one can hope that the conjecture is true even for subsets of a metric space. In [PT], the upper bound was slightly improved. Namely it is shown that for a metric space $M$,

$$\sigma_1(M) \le \frac{2+\sqrt{46}}{12} \approx 0.732\,. \tag{1.6}$$

Recently, in [Far2], we proved that $\frac{1}{2}$ is the correct upper bound on the lower spherical density for subsets of a Hilbert space, under the additional assumption that the set possesses a flatness property at sufficiently small scales, at almost every point. The present paper deals with the same kind of result but with a different approach. Both methods however were first attempts to probe the possibility of finding a systematic approach to attack this problem with no additional assumptions. Below, we will try to explain that whenever feasible. The reader can find treatments of Besicovitch's results in [Fal], and [Far1]. The present method of proof will quickly appear to be of independent nature. Indeed, once we insert the setup of the problem, and the hypothesis, the proof is then reduced to basically showing that a quasi-arc of small constant cannot contain sharp “saw-teeth”, that is (roughly speaking) a finite set of points which lie on the consecutive vertices of a piecewise linear graph of high slopes.

The rest of the paper is organized as follows:

- Section 2: we provide the appropriate definitions that we need to state the main results.
- Section 3: we state the main results.
- Section 4: we present a standard reduction of the problem using compactness results for measures and density properties for the one-dimensional Hausdorff measure. This gives the most general setup for the problem.
- Section 5: this is devoted to the proof of the main results.
- Section 6: we give some concluding remarks including the main motivation for our flatness hypothesis which actually came from the search for a general approach for a systematic attack of the problem.

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## 2. Definitions

First we recall the definition of the scale-invariant beta-numbers (see [Jo] where quadratic estimates of these numbers are used to characterize rectifiability):

**Definition 1** *For $E \subset \mathbb{R}^2$, $x \in \mathbb{R}^2, r > 0$, the Jones beta-number $\beta_E(x,r)$ is defined to be the smallest number such that there is a strip $S$ of width $r\beta_E(x,r)$, so that*

$$E \cap B(x,r) \subset S. \tag{2.1}$$

Now we make use of somewhat more general scale invariant quantities:

**Definition 2** *For $E \subset \mathbb{R}^2, x \in \mathbb{R}^2, \varepsilon \geq 0, r > 0$, we define $\gamma_E^1(x,r,\varepsilon)$ as the smallest number such that there is a strip $S$ of width $r\gamma_E^1(x,r,\varepsilon)$, so that*

$$\mathcal{H}^1\left(\left(E \cap B(x,r)\right) \backslash S\right) \leq \varepsilon. \tag{2.2}$$

*We now define* $\gamma_E^{*1}(x,\varepsilon)$ *and* $\gamma_E^{*1}(x)$ *via*

$$\gamma_E^{*1}(x,\varepsilon) \equiv \limsup_{r\to 0+} \gamma_E^1(x,r,\varepsilon)\,, \tag{2.3}$$

*and*

$$\gamma_E^{*1}(x) \equiv \sup_{\varepsilon>0} \gamma_E^1(x,\varepsilon)\,. \tag{2.4}$$

Note that for any set $E \subset \mathbb{R}^2$, all of the above scale-invariant numbers are bounded above by 2. We will refer to sets $E$ which have small values for $\gamma_E^{*1}(x)$ at almost every point as being *essentially flat.* It is for this type of sets that we prove the conjecture to hold. Note that in $\mathbb{R}^n$ or even a Banach space one can have similar definitions by using tubes.

For our purely geometrical results we will need the following definitions:

**Definition 3** *We call a finite collection of points* $\{x_1, y_1, x_2, \ldots, x_n, y_n\} \subset \mathbb{R}^2$ *saw-teeth with angle* $\Delta \in [0, 90^o]$ *if the line segments*

$$[x_1, y_1]\,, [y_1, x_2]\,, \ldots, [x_n, y_n]$$

*have arguments (with respect to some axis) which alternate in sign and whose absolute values lie in the interval* $[90^o - \Delta, 90^o + \Delta]$*, where arguments are measured in the interval* $(-180^o, 180^o]$.

**Definition 4** *We say that an arc* $\Gamma$ *connecting* $z_1, z_2$ *is a quasi-arc of small constant* $0 < \delta << 1$*, if, for every* $x \in \Gamma$*,* $r > 0$*,* $\beta_\Gamma(x,r) \leq \delta$.

It is not hard to see (using the Pythagorean Theorem) that for small $\delta$ (for instance $\delta \leq \frac{1}{8}$ suffices), our definition agrees with the usual definition for a quasi-arc up to an adjustment of the constant.

## 3. The main results

The following are our two main theorems:

**Theorem 5** *Suppose* $E \subset \mathbb{R}^2$ *is a totally unrectifiable 1-set, and* $\gamma_E^{*1}(x) \leq \frac{1}{10}$ *for a.e.* $x \in E$*. Then* $\Theta_*^1(E,x) \leq \frac{1}{2}$ *for a.e.* $x \in E$*.*

A stronger version of Theorem 5 was achieved in [Far2] using a different (and much more complicated) method. Both methods also work in $\mathbb{R}^n$. Also, the bound "$\frac{1}{10}$" can be improved slightly (to about $\frac{1}{8}$) but our goal here is to mainly exhibit a different method because of its simplicity and its potential applications rather than proving the sharpest possible result using this method. In fact, as we will discuss in Sections 4 and 6, we intend to introduce a fundamentally different method in a forthcoming paper [Far3] which will systematize a general approach to the problem. In order to prove Theorem 5, we actually prove the following purely geometrical Theorem:

**Theorem 6** *Suppose* $\Gamma$ *is a quasi-arc of small constant* $\frac{1}{10}$. *Let* $x \in \Gamma, r > 0$, *and* $S$ *the optimal strip containing* $\Gamma \cap B(x,r)$. *Then* $\Gamma \cap B(x,r)$ *cannot contain sharp saw-teeth of angle* $\Delta \leq 11.53^o$, *measured with respect to the center-line of* $S$, *and such that* $z, z'$ *are the end points of the saw-teeth and* $\mathrm{Re}\{z' - z\} \geq 0.346\, r$.

# 4. A measure-theoretic reduction

## 4.1. The reduced problem with and without flatness

We here recall a standard reduction of the problem using compactness of measures. Another elementary approach can be found in [Fal], and [Far1]. The reduction here is similar to that in [Far2] and we only describe it briefly. The reader can find a closely related approach with details in [PT]. First it is helpful to recall a few standard facts for the one-dimensional Hausdorff measure. For references we refer the reader to [Fal], [Fe], and [Ma].

Recall that if $E$ is a 1-set, then the upper convex density $\overline{D}_c^1(E,x)$ of $E$ at $x \in \mathbb{R}^n$ (say) is defined via

$$\overline{D}_c^1(E,x) \equiv \limsup_{r \to 0+} \frac{\mathcal{H}^1(E \cap U)}{diam\,(U)} \tag{4.1}$$

where the supremum is taken over all convex sets $U$ with $x \in U$, and $diam\,(U) \leq r$. We now have the following standard density properties for a 1-set $E$:

**Proposition 7** $\overline{D}_c^1(E,x) = 0$ *for a.e.* $x \notin E$.

**Proposition 8** $\overline{D}_c^1(E,x) = 1$ *for a.e.* $x \in E$.

**Proposition 9** *Suppose* $\Theta_*^1(E,x) > \sigma > 0$ *on a set of positive measure, then there exist* $\varepsilon, \rho > 0$, *and a compact 1-set* $F \subset E$, *such that*

$$\mathcal{H}^1(E \cap B(x,r)) \geq (\sigma + \varepsilon)\, 2r\,, \quad \text{for all } x \in F,\ 0 \leq r \leq \rho. \tag{4.2}$$

Using compactness of measures, the above propositions, and total unrectifiability, we can reduce the assumption that $\Theta_*^1(E,x) > \frac{1}{2}$ on a set of positive measure (for a proof by contradiction) to the following setup which holds under very general conditions (in fact even for subsets of a metric space):

**Proposition 10** *Let $E$ be a totally unrectifiable 1-set such that $\Theta_*^1(E,x) > \frac{1}{2}$ on a set of positive measure. Then there exist $\varepsilon > 0$, and a measure $\mu$, such that $F \equiv spt\mu = F_1 \cup F_2$, for some closed sets $F_1$ and $F_2$, with $dist(F_1, F_2) > 0$, attained, and the following properties hold:*

$$\mu(B(x,r)) \geq (\varepsilon + 1)\, r, \tag{4.3}$$

*whenever $x \in F, r > 0$, and*

$$\mu(U) \leq diam(U), \tag{4.4}$$

*for any set $U$.*

This is the most general setup that one gets, but now our hypothesis that $\gamma_E^{*1}(x) \leq \frac{1}{10}$ for a.e. $x \in E$ translates into a third (geometrical) condition on $F = spt\mu$. Namely, we get

$$\beta_F(x,r) \leq \frac{1}{10} \tag{4.5}$$

for $x \in F, r > 0$. In the next section we show that no such measure $\mu$ can satisfy Properties (4.3)-(4.5). In the next subsection we briefly discuss how one may intuitively try to attack the general problem using only (4.3) and (4.4). We elaborate further on this in Section 6.

### 4.2. An overview of a general approach

This subsection is rather brief, and is mainly intended to give some motivation for our flatness hypothesis. In a forthcoming paper [Far3], we discuss this approach in greater detail. One of its consequences will be that it will allow significant generalizations of the results of this paper as far as the goal of trying to solve the $\frac{1}{2}$-problem is concerned. There is no real overlap in the methods however and hence our present paper is of independent interest as we have mentioned.

First we observe that (4.3) and (4.4) are basically all that we have to work with and therefore we have to expect that the geometry of $F$ (except for the fact that it has two positively separated pieces) is quite arbitrary and we have to deal with all possibilities. Let us now imagine that we can find, in one of the pieces of $F$, say $F_1$, a finite sequence of points in such a way that we can place disjoint balls, centered at these points so that the diameter of the union (intersected with $F$) is at most equal to the sum of the radii. Then clearly (4.3) and (4.4) give a contradiction. In fact, we wish to find these points with a specific geometric configuration. Namely, we wish to find these points as two distinct sequences $\{x_i\}_1^n, \{y_j\}_1^m$, such that $y_1 = x_1 \equiv X$, and $x_n = y_m \equiv Y$ , but that otherwise there are no mutual points among the sequences (see Figure (1)). Furthermore, we wish to have $|X - Y| = diam\left(\bigcup_i \{x_i\} \cup \bigcup_j \{y_j\}\right)$. In Section 6 we elaborate more on the geometry we expect for such sequences after the reader had a chance to see the present method.

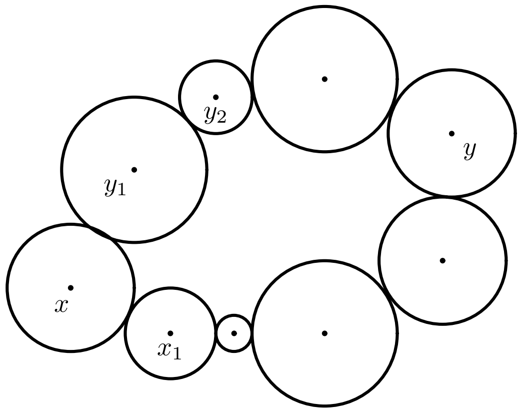


Figure 1

## 5. Proof of the main theorems

### 5.1. The idea behind the proof

In this subsection we give a rough idea of how to produce a sequence of points (which can in fact be viewed as two sequences in a natural way) using (4.3)-(4.5), and then use this sequence to produce a contradiction. We will leave the precise estimates to the next subsection. Our sequence is produced as follows. First we let $w' \in F_1, w'' \in F_2$ be points of minimum distance. Let us now rescale and set $|w' - w''| \equiv 1$. Now consider the ball $B\left(w', \frac{1}{4}\right)$. This ball is disjoint from $F_2$, and, by (4.3),

$$\mu\left(B\left(w', \frac{1}{4}\right)\right) \geq (1 + \varepsilon)\frac{1}{4}. \tag{5.1}$$

Hence, by (4.4), there must exist points $z_1, w_1 \in F_1 \cap B\left(w', \frac{1}{4}\right)$, so that

$$|w_1 - z_1| = diam\left(F \cap B\left(w', \frac{1}{4}\right)\right) \geq (1 + \varepsilon)\frac{1}{4}. \tag{5.2}$$

See Figure (2).

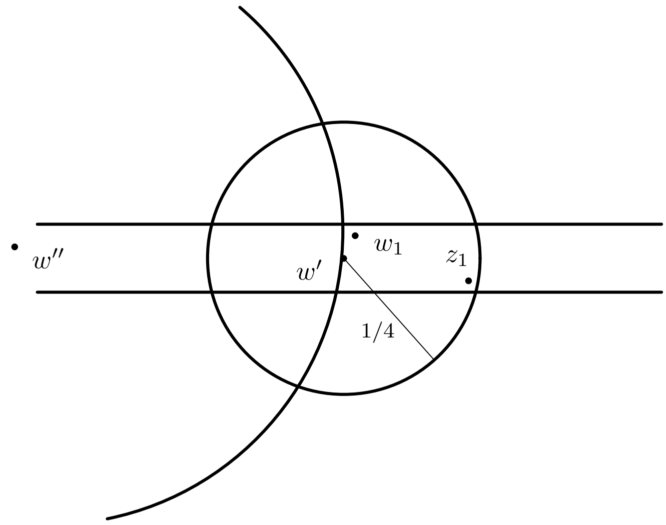


Figure 2

Now we consider the ball $B(w_1, |z_1 - w_1|)$, where (without loss of generality) we can assume $w_1$ is closer to $w'$ than $z_1$ is. Now we repeat the argument. Namely, (4.3), (4.4) imply that there exist a pair of points $z_2, w_2 \in B(w_1, |w_1 - z_1|) \cap F_1$ (with the same choice of labeling) so that

$$|z_2 - w_2| \geq (1 + \varepsilon) |z_1 - w_1| . \tag{5.3}$$

We can repeat inductively to find $z_i$ as long as $|z_{i-1} - w_{i-1}| < 1$, otherwise we pick up points from $F_1$. The sequence $\{z_i\}_1^N$ thus constructed is the required sequence. Suppose for instance that we were working in $\mathbb{R}^2$, then it is not hard to see from the geometrical picture that

$$\{z_i\}^N_{\substack{i=1, \\ i \text{ odd}}} , \quad \{z_i\}^N_{\substack{i=2, \\ i \text{ even}}} ,$$

has the resemblance (certainly not necessarily the properties) of the two sequences mentioned in Subsection 4.2.

In the method of this paper we deal with $\{z_i\}_{i=1}^N$ as one sequence. The main feature of this sequence is that it is of the sharp saw-teeth type. We will however have no control on the relative size of the "teeth" (of course it is trivial that (4.5) would be violated if the teeth were of equal size, as simple considerations would show). As soon as we construct the sequence $\{z_i\}$ using (4.3), (4.4), we can actually forget about the $\frac{1}{2}$-problem. The rest of the proof is established by proving Theorem 6 (in fact slightly more is proved).

### 5.2. The body of the proof

Below, we will denote the bound on $\beta_F(x,r)$ by $\beta$, and we will try to do the manipulations in terms of it whenever possible. Whenever an explicit computation is required however, we use $\beta = \frac{1}{10}$. Let us begin by recording some information regarding the sequence $\{z_i\}_{i=1}^N$ constructed in Subsection 5.1. Let $S$ be an infinite strip of width at most $\beta$, which contains $B(w_{N-1},1)\cap F_1$, oriented so that its center-line is parallel to the real-axis. Let the origin be placed at $w_{N-1}$. We will also measure arguments to lie in the interval $(-180^o, 180^o]$. This will be the most convenient convention. Geometric angles such as those inside a triangle will be considered positive unless declared otherwise. We start with the following lemma:

**Lemma 11** *Suppose $x, y, z \in \mathbb{R}^2$ are distinct points such that $|z-x| \leq |x-y| \leq |z-y|$, and so that all three points lie in a strip $S$ of width $\beta|x-y|$. Then $\sin(\widehat{xyz}) \leq \beta$.*

**Proof.** It suffices to prove this when $S$ is of minimal width. In that case however it is not hard to see that one of the boundary components of $S$ will contain $z, y$. Then by projecting the line segment $\overline{xy}$ on the orthogonal to the center-line of $S$, we can conclude the lemma. ■

**Lemma 12** *There exist sequences $Z \equiv \{z_i\}_1^N, W \equiv \{w_i\}_1^N \subset F_1$, with the following properties (let $r_i \equiv |z_i - w_i|$):*

1. *For $i \in \{2,\ldots,N\}$, $z_i, w_i \in B(w_{i-1}, r_{i-1}) \cap F_1$, and*

$$r_i = diam\,(F \cap B(w_{i-1}, r_{i-1}))\ ;$$

2. $z_i \in (B(w_{i-1}, r_{i-1}) \backslash B(w_{i-2}, r_{i-2})) \cap F_1$*;*

3. $r_i > r_{i-1}$ *(in fact* $r_i \geq (1+\varepsilon) r_{i-1}$*), also* $\frac{1}{2} \leq r_i < 1$ *for* $i \in \{1,\ldots,N-1\}$*, but* $r_N \geq 1$*, and* $r_1 \leq 0.504$*;*

4. $|z_j - z_i| > 0$*, for all* $i \neq j$*;*

5. $|\arg(z_{i+1}-z_i)| \in [90^o - \arcsin(2\beta), 90^o + \arcsin(2\beta)]$,
*with* $\arg(z_{i+1}-z_i)$ *alternating in sign;*

6. *If* $i > j$*, then* $|\arg(z_i - z_j)| \in [0^o, 90^o + \arcsin(2\beta)]$;

7. $Re(z_N) \geq 0.85$;

8. $\min_i \mathrm{Re}(z_i) \leq 0.504$.

**Proof.** See Figure (3).

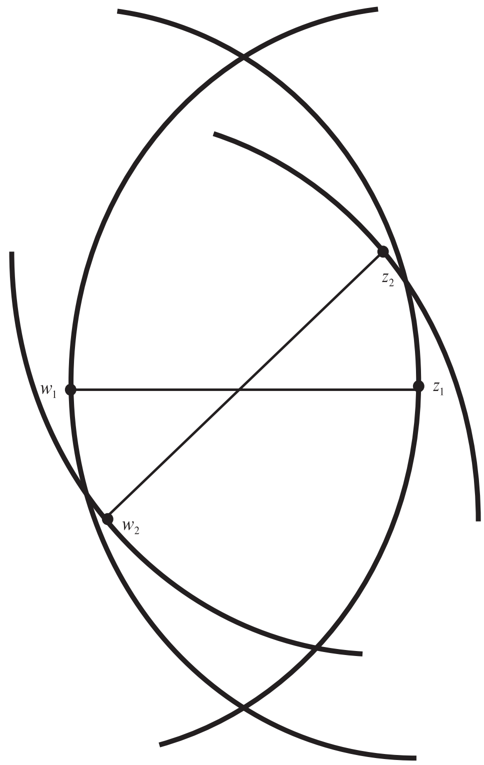


Figure 3

Properties (1)-(4) are immediate from the construction in Subsection 5.1 except for the upper and lower bounds on $r_1$ which can be proved as follows: we modify that construction of the sequence by starting the labeling $w_0, z_0$, as soon as we find that $r_0 < \frac{1}{2}$, whereas $r_1 \geq \frac{1}{2}$. Now we observe that, by Lemma 11, the worst case estimate occurs when $\sin\left(\widehat{w_0 z_0 w_1}\right) = \sin\left(\widehat{z_1 w_0 z_0}\right) = \beta$, in which case a computation using the law of cosines yields the stated upper bound on $r_1$. This also proves Property (8) by definition of $w_1, z_1$. To see Property (5), we observe first that the acute angle that any of the line segments $\overline{w_i z_i}$ makes with the horizontal cannot exceed $\arcsin\left(2\beta_F(0,1)\right) \leq \arcsin(2\beta)$. Since $z_{i+1} \in B(w_i, r_i)$, it is not hard to see then that

$$|\arg(z_{i+1} - z_i)| \geq 90^o - \arcsin(2\beta). \tag{5.4}$$

Similarly, it is also easy to see that $z_i \in B(w_{i+1}, r_{i+1})$, and hence

$$|\arg(z_{i+1} - z_i)| \leq 90^o + \arcsin(2\beta). \tag{5.5}$$

The alternation in sign can be seen from Figure (4).

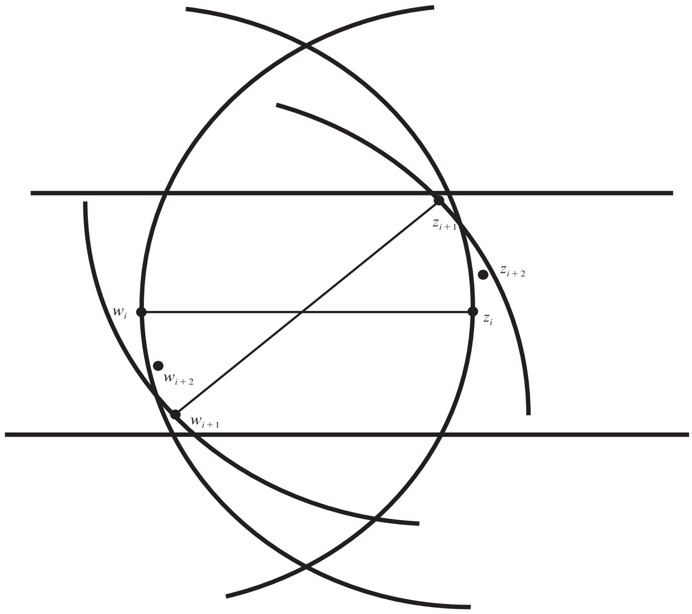


Figure 4

The proof of Property (6) is similar. To prove Property (7), we just make the crude observation that, by Property (5),

$$|z_N - z_{N-1}| \leq \frac{\beta}{\cos(\arcsin(2\beta))}. \tag{5.6}$$

By the triangle inequality,

$$r_{N-1} \geq r_N - |z_N - z_{N-1}|, \tag{5.7}$$

and then, by Property (5), and (22),

$$\operatorname{Re} z_N \geq r_{N-1} \cos(\arcsin(2\beta)) - 2\beta\, |z_N - z_{N-1}| \geq 0.85. \tag{5.8}$$

■

The proof of the main theorems will be completed by proving

**Proposition 13** *There exists a subsequence* $\{\eta_i\}_1^{\widetilde{N}}$ *of* $Z$, *such that*

$$\left|\arg\left(\eta_{\widetilde{N}} - \eta_1\right)\right| \geq 30^o, \tag{5.9}$$

*and*

$$\operatorname{Re}\left(\eta_{\widetilde{N}} - \eta_1\right) \geq 0.226. \tag{5.10}$$

This proposition contradicts that, since $Z \subset S$, the strip of width $\beta$, we must have

$$\left|\arg\left(\eta_{\tilde{N}} - \eta_1\right)\right| \leq \arctan\left(\frac{\beta}{0.226}\right) \leq 24^o.$$

The next three Lemmas will give a proof of Proposition 13. We start by refining the sequence $Z$ slightly to make it more manageable. For $z \in \mathbb{C}$, angles $\phi_1, \phi_2 \in (-180^o, 180^o]$, we let $S(z, \phi_1, \phi_2) \equiv \{\zeta \in \mathbb{C} : \phi_1 \leq \arg(\zeta - z) \leq \phi_2\}$.

**Lemma 14** *There exists a subsequence $Z' \equiv \{z_i'\}_1^{N'}$ of $Z = \{z_i\}_1^N$, with the following properties:*

1. $\min_i \operatorname{Re}(z_i') = \min_i \operatorname{Re}(z_i)$;
2. $\arg\left(z_{i+1}' - z_i'\right), \arg\left(z_{i+2}' - z_{i+1}'\right)$ *are of opposite signs;*
3. $\left|\arg\left(z_{i+1}' - z_i'\right)\right| \in [55^o, 90^o)$, *and $z_{i+1}'$ has maximum distance from $z_i'$ among the elements of $Z'$ which lie in the same sector ($S(z_i', 55^o, 90^o)$, or $S(z_i', -90^o, -55^o)$) as $z_{i+1}'$.*
4. $\operatorname{Re}(z_{N'}') \geq 0.83$

**Proof.** Let $z_{i_1} \in Z$ be the last element such that

$$\operatorname{Re} z_{i_1} = \min_i \operatorname{Re} z_i. \tag{5.12}$$

By Property (5) of Lemma 12, and for $\beta = \frac{1}{10}$, we get that

$$|\arg(z_{i_1+1} - z_{i_1})| \in [78.4^o, 90^o). \tag{5.13}$$

Set $z_1 = z_{i_1}$. Now let $z_{i_2} \in Z$ be the last element in $Z \cap (S(z_1', 55^o, 90^o) \cup S(z_1', -90^o, -55^o))$ with maximum distance to $z_1'$. Set $z_2' = z_{i_2}$. We have thus found two points in our subsequence but we certainly have not satisfied (4). Now suppose by induction that we have found $z_k'$ according to properties (1)-(3), and that (without loss of generality) $z_k' \in S\left(z_{k-1}', 55^o, 90^o\right)$. Let $z_k' = z_{i_k} \in Z$. By Property (5) of Lemma 12, and the maximality of $\left|z_k' - z_{k-1}'\right|$, we can conclude that

$$Z \cap S(z_k', 55^o, 90^o) = \phi. \tag{5.14}$$

We may also assume

$$Z \cap S(z_k', -90^o, -55^o) = \phi, \tag{5.15}$$

and that $\operatorname{Re}(z_k') < 0.83$ (otherwise we have the induction step). Combining these facts with Property (5) of Lemma 12, we conclude that

$$z_{i_{k+1}} \in cl\left(S(z_k', 90^o, 101.6^o) \cup S(z_k', -101.6^o, -90^o)\right). \tag{5.16}$$

Let $z_l$ be defined to be the last element in $Z \cap S\left(z_k', 90^o, 101.6^o\right)$ with $l \geq i_k$, and maximizing the distance to $z_k'$. Similarly, we let $z_{l'}$ be the corresponding element with the same properties for $S\left(z_k', -101.6^o, -90^o\right)$ (see Figure (5)).

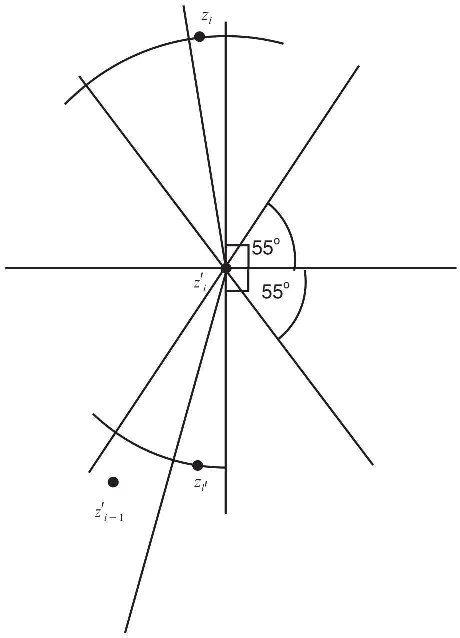


Figure 5

At this point we will not make use of points $z_j'$ with $j < i$, and there will be no loss of generality in assuming $\left|z_l - z_k'\right| \geq \left|z_{l'} - z_k'\right|$. We now show that we have a contradiction to (29), (30), by considering the first element $z_m \in Z$ with $m \geq i_k$, which leaves $B\left(z_k', \left|z_k' - z_l\right|\right)$. Let $S'$ be the infinite strip of width at most $\frac{1}{10}\left|z_l - z_k'\right|$, which contains $Z \cap B\left(z_k', \left|z_k' - z_l\right|\right)$. We parametrize the possible positions and orientations of this strip by two parameters, $\lambda$ being the positive angle that the center-line of $S'$ makes with the real-axis, and $\Delta$, the distance between $z_k'$ and the intersection of the "right" component of $\partial S'$ with the real-axis. First we consider the two worst cases that can happen if

$$90^o \leq \left|\arg\left(z_{m-1} - z_k'\right)\right| \leq 101.6^o. \tag{5.17}$$

Both cases are identical so assume $\arg(z_{m-1} - z_k') > 0^o$. The worst estimate then arises when

$$z_{m-1} = z_k' + i\left|z_l - z_k'\right|, \tag{5.18}$$

and

$$\arg\left(z_m - z_{m-1}\right) = -78.4^o. \tag{5.19}$$

Let $\widetilde{z}$ be the intersection of the half line starting at $z_{m-1}$ and containing $z_m$ with the half line starting at $z'_k$ making angle -55$^o$ with the real axis. It is easy to see that

$$|z'_k - z_m| \leq |z'_k - \widetilde{z}|\,. \tag{5.20}$$

On the other hand

$$\frac{|z'_k - \widetilde{z}|}{|z_l - z'_k|} \leq \frac{\sin(11.6^o)}{\sin(23.4^o)} \leq 0.51\,, \tag{5.21}$$

which contradicts the definition of $z_m$. Now we consider the case when

$$|\arg(z_{m-1} - z'_k)| \leq 55^o. \tag{5.22}$$

First we need some estimates on $\lambda$ and $\Delta$. The estimate on $\Delta$ is easily seen to satisfy

$$0 \leq \Delta \leq \frac{\beta}{\sin(\lambda)}. \tag{5.23}$$

The estimate on $\lambda$ is found by observing that the acute angle between the center-line of $S'$ and the line segment $\overline{z'_k z_l}$ cannot exceed $arcsin\,(\beta)$. On the other hand, we have

$$90^o \leq \arg(z_l - z'_k) \leq 101.6^o. \tag{5.24}$$

Hence, we have

$$84.2^o \leq \lambda \leq 107.4^o. \tag{5.25}$$

Now consider the triangle which has $z'_k$ as one vertex and two of its sides on the half lines $L_1 \equiv \{\zeta : \arg(\zeta - z'_k) = 55^o\}$, $L_2 \equiv \{\zeta : \arg(\zeta - z'_k) = -55^o\}$, and the third side on the "right" component of $\partial S'$. The worst cases can be easily seen to arise when $z_{m-1}$ lies on one of the other two vertices of this triangle and $\arg(z_m - z_{m-1}) = \pm 78.4^o$. Let us denote the top one (which lies on $L_1$) by $\zeta_1$, and the bottom one (which lies on $L_2$) by $\zeta_2$ (see Figure (6)).

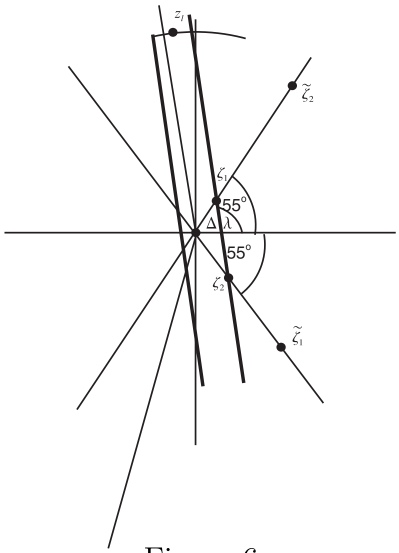


Figure 6

Let us first consider the case $z_{m-1} = \zeta_1$, and $\arg(z_m - z_{m-1}) = -78.4^o$. The worst estimate for this case is when $\Delta$ is a maximum, and $\lambda = 84.2^o$. This makes the distance between $\zeta_1, z'_k$ a maximum. This distance is easily estimated to satisfy

$$\text{(5.26)} \qquad \frac{|\zeta_1 - z'_k|}{|z_l - z'_k|} \le \frac{\sin(84.2^o)}{10\sin(84.2^o)\sin(35 - 5.8^o)} \le 0.21\,.$$

Let $\widetilde{\zeta_1}$ denote the intersection of $L_1$ with the half line $\{\zeta : \arg(\zeta - \zeta_1) = -78.4^o\}$ starting at $\zeta_1$. Clearly,

$$\text{(5.27)} \qquad |z_m - z'_k| \le \left|\widetilde{\zeta_1} - z'_k\right|.$$

On the other hand $\left|\widetilde{\zeta_1} - z'_k\right|$ satisfies

$$\text{(5.28)} \qquad \frac{\left|\widetilde{\zeta_1} - z'_k\right|}{|\zeta_1 - z'_k|} \le \frac{\sin(35 + 11.6^o)}{\sin(35^o - 11.6^o)} < 1.83\,.$$

Again this gives a contradiction to the definition of $z_m$. Finally the case $z_{m-1} = \zeta_2$, and $\arg(z_m - z_{m-1}) = 78.4^o$ is similar but has worse estimates. Namely,

$$\text{(5.29)} \qquad \frac{|\zeta_2 - z'_k|}{|z'_k - z_l|} \le \frac{1}{10\sin(35^o - 17.4^o)} \le 0.34\,.$$

Also, if $\widetilde{\zeta_2}$ is the corresponding point on $L_1$, then

$$\text{(5.30)} \qquad \frac{\left|\widetilde{\zeta_2} - z'_k\right|}{|\zeta_2 - z'_k|} \le \frac{\sin(35^o + 11.6^o)}{\sin(35^o - 11.6^o)} < 1.83\,.$$

Combined with the fact that in this case $|z_m - z'_k| \le \left|\widetilde{\zeta}_2 - z'_k\right|$, we get a contradiction to the definition of $z_m$. The induction step is thus established, unless we have reached a point $z'_{N'}$ that lies close to $z_N$ (the end of the sequence $Z$). The worst case for that is certainly when

$$\operatorname{Re} z_N \ge \operatorname{Re} z_{N'}. \tag{5.31}$$

According to our analysis above, this happens when $z_N$ coincides with the point $\zeta_2$. A very crude estimate can be obtained from

$$|z_l - z_{N'}| \le \frac{\beta}{\cos(11.6^o)}, \tag{5.32}$$

and then, by (5.30),

$$\operatorname{Re}(z_N - z'_k) \le \frac{0.34\beta\cos(55^o)}{\cos(11.6^o)} < 0.02\,. \tag{5.33}$$

By Property (7) of Lemma 12, we get

$$\operatorname{Re} z_{N'} \ge 0.83. \tag{5.34}$$

■

The main problem now is to control the behavior of the sequence $Z'$. The fact that the size and the small angles of the "teeth" is allowed to change makes the situation more complicated as one may have some accumulations of such segments near each other followed by a sudden change of scale without violating the flatness imposed by the beta-numbers. The next lemma will produce yet another subsequence $Z''$ such that such behavior will be eliminated.

**Lemma 15** *There exists a subsequence $Z'' \equiv \{z''_1, \ldots, z''_{N''}\}$ of $Z'$ with the following properties:*

1. $\arg\left(z''_i - z''_{i-1}\right)$, $\arg\left(z''_{i+1} - z'_i\right)$ *are of opposite sign;*
2. *For all* $i \in \{1, \ldots, N''-1\}$, $|\arg(z_{i+1} - z_i)| \in [45^o, 50.6^o]$;
3. $\operatorname{Re} z''_{N''} \ge \operatorname{Re} z'_{N'} - \beta \ge 0.73$;
4. $\min_i(\operatorname{Re} z''_i) = \min_j\left(\operatorname{Re} z'_j\right)$.

**Proof.** Set $z_1'' = z_1'$ (this immediately satisfies (4) since $\operatorname{Re} z_1'$ is the minimum for $Z'$), and let $z_l' \in S(z_1'', 45^o, 90^o) \cap Z'$ be the element that maximizes the distance from $z_1''$. Similarly, let $z_{l'}'$ be the corresponding element for $S(z_1'', -90^o, -45^o) \cap Z'$. Assume (without loss of generality) that $|z_l' - z_1''| \geq |z_{l'}' - z_1''|$. In particular, by Lemma 12, we have $|z_l' - z_1''| > 0$. We now show that $\arg(z_l' - z_1'') \in [45^o, 50.6^o]$. Suppose not, then $\arg(z_l' - z_1'') \in (50.6^o, 90^o]$. We will consider the element $z_m' \in Z'$ defined to be the first element in $Z'$ which leaves the ball $B(z_1'', |z_l' - z_1''|)$. By choice of $z_l'$, we must have

$$|\arg(z_m' - z_1'')| \leq 45^o. \tag{5.35}$$

Let us first eliminate the possibility that $55^o \leq \arg\left(z_m' - z_{m-1}'\right) < 90^o$. This case is technically easier but it is what dictates the choice of the uncertainty interval $[45^o, 50.6^o]$. To eliminate this case, one merely has to consider the strip $S''$ of width $\beta|z_1'' - z_l'|$ which contains $Z' \cap B(z_1'', |z_1'' - z_l'|)$. We certainly have $z_{m-1} \in S''$. It is easy to see that in this case $z_m'$ has to lie in $B(z_l', |z_{l'} - z_1''|)$. But this cannot happen since $\arcsin\left(\frac{1}{10}\right) \leq 5.6^o$ which is the width of the uncertainty window. See Figure (7).

We now consider the case $-90^o < \arg\left(z_m' - z_{m-1}'\right) \leq -55^o$. The same argument above also eliminates the case $z_m' \in B(z_l', |z_{l'} - z_1''|)$. Now let $\zeta_1 \in \mathbf{C}$ be the point such that

$$|\zeta_1 - z_1''| = |z_l' - z_1''|, \quad \text{and } \arg(\zeta_1 - z_1'') = 45^o. \tag{5.36}$$

Let $L$ be the half line $\arg(\zeta - \zeta_1) = -55^o$, and its intersection with the half line $\arg(\zeta - z_1'') = -45^o$ be $\zeta_2$ (see Figure (8)).

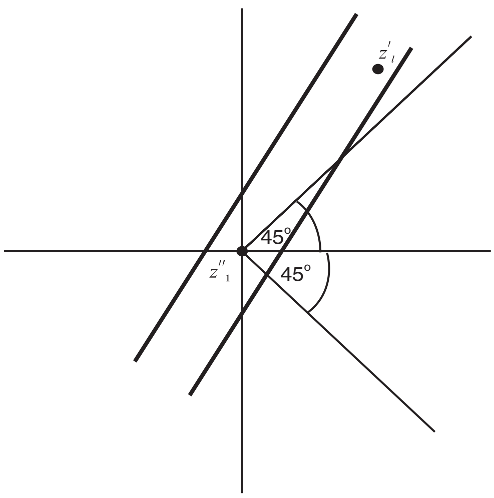


Figure 7

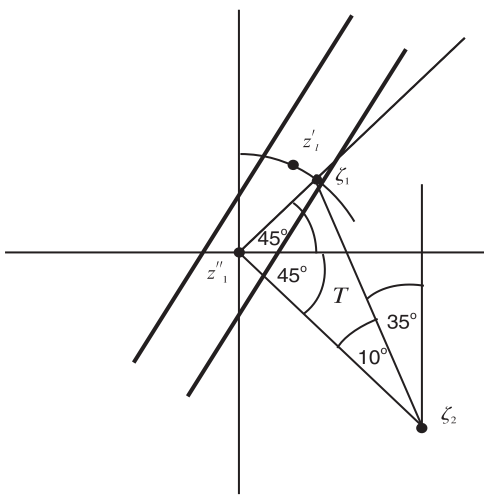


Figure 8

Let $T$ be the triangle with vertices $z_1'', \zeta_1, \zeta_2$. It is not hard to see now that $z_m'$ must lie in the triangle $T$, and we will show that this violates the bound $\left(\frac{1}{10}\right)$ on the beta-numbers. To wee that we merely observe that, for any point $\zeta \in T \backslash \left( B\left(z_l', |z_{l'} - z_1''|\right) \cup B\left(z_1'', |z_1'' - z_l'|\right)\right)$, the geometric (positive) acute angle $\widehat{z_l' \zeta z_1''}$ satisfies

$$\widehat{z_l' \zeta z_1''} \geq \phi \equiv \left|\arg\left(z_l' - \zeta_2\right) - \arg\left(z_1'' - \zeta_2\right)\right|. \tag{5.37}$$

It thus suffices to show that the angle $\phi$ violates the maximum angle allowed (namely, $\arcsin(\beta)$). To do that we first need an estimate on $|\zeta_2 - z_1''|$, which can be found via (the law of sines)

$$\frac{|\zeta_2 - z_1''|}{|z_l' - z_1''|} \leq \frac{\sin(80^o)}{\sin(10^o)} < 5.7. \tag{5.38}$$

Hence, by the law of sines again, we have

$$\frac{|z_l' - z_1''|}{\sin(\phi)} = \frac{|z_1'' - \zeta_2|}{\sin(135^o + \phi)}, \tag{5.39}$$

which leads to

$$\phi > 6.2^o. \tag{5.40}$$

Thus we conclude that

$$\arg\left(z_l' - z_1''\right) \in [45^o, 50.6^o]. \tag{5.41}$$

Now let $z_2'' = z_l'$, and repeat inductively (now the previous step already determines which sector will contain the element of maximum distance from $z_2''$ etc.). This procedure can continue at least till we reach a $z_{N''}''$ which is close to $z_{N'}'$. The worst case estimate can be crudely seen to be better than when $z_{N'}', z_{N''}''$ are like the elements $\zeta_1, z_1''$ in the proof above, which leads to

$$\mathrm{Re}\, z_{N''}'' \geq \mathrm{Re}\, z_{N'}' - \beta \geq 0.73. \tag{5.42}$$

■

The next lemma will show that $Z''$ has properties conflicting with flatness.

**Lemma 16** *Let $Z'' = \{z_1'', \ldots, z_{N''}''\}$ be as in Lemma 14. We have for $i \in \{2, \ldots, N''\}$,*

$$\left|\arg\left(z_i'' - z_1''\right)\right| \in [30^o, 50.6^o]. \tag{5.43}$$

**Proof.** For convenience we will change the notation as follows. We set $x_1 = z''_1, y_1 = z''_2$, and in general $x_i = z''_{2i-1}$, and $y_i = z''_{2i}$ etc. If $N''$is odd, $N'' = 2n+1$, then we define $y_{n+1} \equiv x_{n+1} = z''_{N''}$. Also, set $L_i =|x_i - y_i|, M_i = |y_i - x_{i+1}|$ (see Figure(9)),

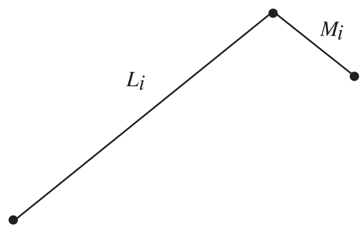


Figure 9

and let $\phi_i$ be the acute angle (measured positive only) between the line segments $\overline{x_i y_i}$, and $\overline{y_i x_{i+1}}$. We first observe that, by construction of $Z''$, we have

$$78.8^o \le \phi_i \le 90^o. \tag{5.44}$$

For each $i$, it is then easily seen that

$$\min(L_i, M_i) \approx \beta \max(L_i, M_i). \tag{5.45}$$

Since $\arg(y_i - x_i) \in [45^o, 50.6^o]$, then, by Lemma 15, we can also conclude that

$$|\arg(x_{i+1} - x_i)| \in [39.4^o, 50.6^o]. \tag{5.46}$$

Let us assume (without loss of generality) that

$$\arg(y_1 - x_1) \in [45^o, 50.6^o]. \tag{5.47}$$

The statement of the Lemma is certainly true if we replace $z''_{N''}$ by $y_1$. Suppose now that the lemma is true for a sequence having the properties of $Z''$ but with $m$ pairs$\{x_i, y_i\}$ of points, we will show that we can prove the induction step. In particular, suppose

$$|\arg(x_m - x_1)| \in [30^o, 50.6^o]. \tag{5.48}$$

If either $L_{m+1} < M_{m+1}$, $\arg(x_m - x_1) \in [-50.6^o, -30^o]$, or $L_{m+1} > M_{m+1}$, $\arg(x_m - x_1) \in [30^o, 50.6^o]$, then it is easily seen that we're done (see Figure (10)).

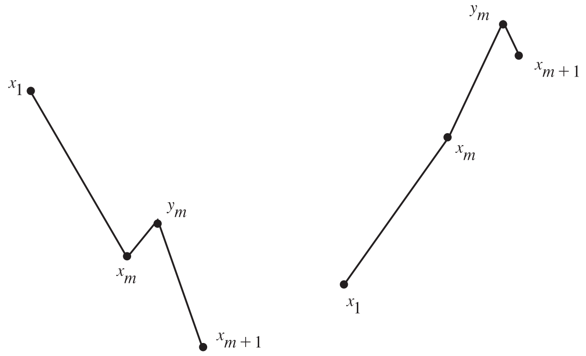


Figure 10

A similar argument also takes care of the cases $M_{m+1} > \max(L_{m+1}, |x_1 - x_m|)$, or $L_{m+1} > \max(M_{m+1}, |x_1 - x_m|)$ . See Figure (11).

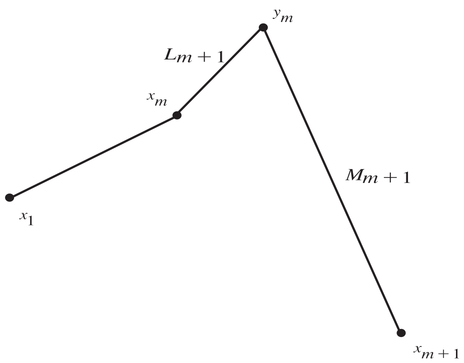


Figure 11

It therefore suffices to consider the following two cases: $M_{m+1} < L_{m+1} < |x_1 - x_m|, \arg(x_m - x_1) \in [-50.6^o, -30^o]$ (see Figure (12)), and $L_{m+1} < M_{m+1} < |x_1 - x_m|, \arg(x_m - x_1) \in [30^o, 50.6^o]$ (see Figure (13)).

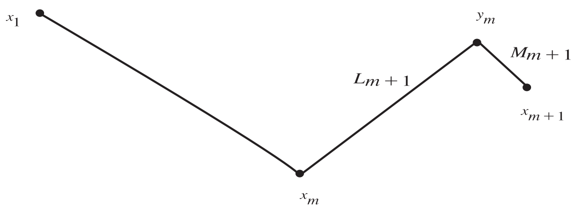


Figure 12

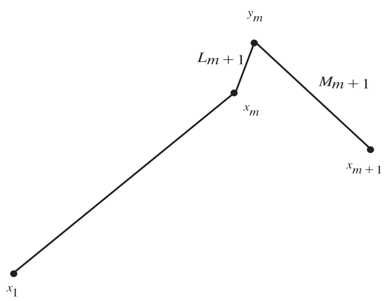


Figure 13

Both cases have identical treatments, so we will only discuss the first.

Certainly $\arg(x_{m+1} - x_1) < 0^o$, but of course we wish to show that in fact it is less than $-30^o$. See Figure (14).

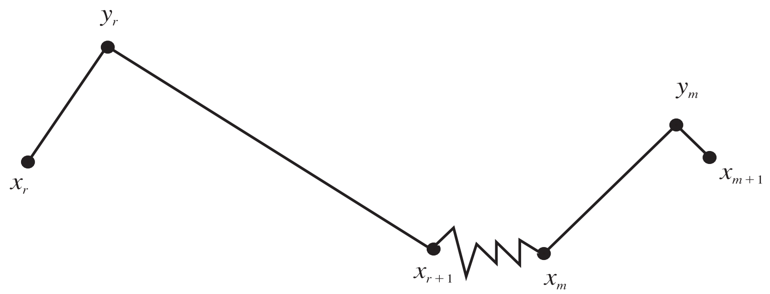


Figure 14

Let $r \in \{1, \ldots, m\}$ be the largest integer for which

$$\arg(x_{m+1} - x_{r+1}) < 0^o. \tag{5.49}$$

By the induction hypothesis,

$$\arg(x_{m+1} - x_{r+1}) \in [30^o, 50.6^o]. \tag{5.50}$$

By choice of $r$, we must have that $|x_{r+1} - x_{m+1}| \approx \beta |x_r - x_{r+1}|$. Since $\beta_F(x_r, |x_r - x_{r+1}|) \leq \frac{1}{10}$, we conclude that

$$\arg(x_{m+1} - y_r) \in [-50.6^o, -39.4^o]. \tag{5.51}$$

We now have two possibilities to consider. The first is when $\arg(x_r - x_1) > 0^o$. In this case, since we already know that $\arg(x_{m+1} - x_1) < 0^o$, we must have that

$$|x_1 - x_r| \approx \beta |y_r - x_{r+1}|. \tag{5.52}$$

Now an application of $\beta_F(y_r, |y_r - x_{r+1}|) \leq \frac{1}{10}$, gives

$$\arg(x_{m+1} - x_1) \in [-50.6^o, -33.8^o]. \tag{5.53}$$

If $\arg(x_r - x_1) < 0^o$ however, then using the induction hypothesis with (68), gives

$$\arg(x_{m+1} - x_1) \in [-50.6^o, -30^o]. \tag{5.54}$$

In either case we have established the induction hypothesis, and the proof of the lemma is complete. ■

The proof of Proposition 13 is now also complete, and so are Theorems 5 and 6.

## 6. Concluding remarks

The reader may have noticed that we did not push the method above to its limits. That tends to complicate the estimates which, although elementary, tend to get quite involved. Although it is possible to push the method a bit more, our goal was more to prove some general properties of finite sets possessing some flatness condition and then applying the results to the $\frac{1}{2}$-problem. One would also imagine that our method would have other applications as well. We refer the reader to [Far2] for a method that was mainly intended for the $\frac{1}{2}$-problem (and does allow a stronger result). We also wish to remark (this remark is due to the referee), that for sufficiently small beta, the "saw-teeth" Lemma (Lemma 12) implies the $\frac{1}{2}$-bound as follows: By flatness, the projection of $Q_i \equiv F_1 \cap B\left(z_i, |z_i - z_{i+1}|\right)$ is contained in an interval $J_i$ of length $\varepsilon\left|z_i - z_{i+1}\right|$. Since $J_i$ covers $\left[\frac{1}{2}, \frac{3}{4}\right]$ say, there is a disjoint union of some of them of measure at least $\frac{1}{8}$, and we get that the measure of the union is at least $\frac{1}{8\varepsilon}$. This measure however is contained in a region of bounded diameter, so for $\varepsilon$ small enough we have a contradiction.

Projection arguments however fail rather quickly for this problem. The most promising approach is the one aluded to in Section 4, where we gave some remarks on the two sequences which we would like to construct. Better yet, we would like these sequences to be such that we can place balls $B\left(x_i, r_i\right), B\left(y_j, s_j\right)$, so that $B\left(x_i, r_i\right) \cap B\left(x_{i+1}, r_{i+1}\right)$ is a singleton, and the same for the other chain of balls. Also, we require that there is no overlap between the two chains except for the first and last balls on each chain (these are chosen to be common). The picture now should look like a *doubly connected loop* (see Figure (1)), i.e. two chains of balls connecting to the end points without overlap among them except for the beginning and the end. It is not hard to see now (by the triangle inequality and (4.3)) that a configuration like that would give

$$\mu(\text{union of the balls}) \geq (1+\varepsilon)\left|X - Y\right|. \tag{6.1}$$

Suppose further that we can even guarantee that

$$\textit{diam}\left(F \cap \text{union of the balls }\right) = \left|X - Y\right|. \tag{6.2}$$

Then, we would get a contradiction to (4.4). Why one should be able to construct any such loop at all is difficult to describe here but let us remark that the reader can find a construction of this type in [Far1]. In [Far3], we pursue this idea much further. Note that it is possible to find that $F_1$(say) contains a loop that "closes at $\infty$" By that we mean two chains of the type described above but they start from $w'$ (say $w' \in F_1, w'' \in F_2$ are points of minimum distance) and leave a ball of radius $\frac{1}{\varepsilon^2}$ before coming together. In such a case (4.4) does not necessarily fail. The idea then is to work in $F_2$. Suppose that the same situation takes place in $F_2$. Then it is easy to see that (4.4) fails in $B\left(w', \frac{1}{\varepsilon^2}\right)$ provided that we only use balls of radius strictly less than $dist\,(F_1, F_2)$ (this just guarantees that there is no overlap between the chains of balls in each piece). In such a case, by (4.3),

$$\mu\left(B\left(w', \frac{1}{\varepsilon^2}\right)\right) \geq \frac{2}{\varepsilon^2}(1 - O\left(\varepsilon^2\right))\left(1+\varepsilon\right) > \frac{2}{\varepsilon^2}\,, \tag{6.3}$$

which contradicts (4.4). If, on the other hand, we find instead a closing loop in $F_2$, then we have the first type of contradiction described before.

Using the methods of this paper we cannot show that one can get such loops under reasonable conditions, but interestingly enough we did show that there are two naturally identified sequences which have to exist within $F_1$ (say), when we imposed $\beta_F\left(w', dist\,(F_1, F_2)\right) \leq \frac{1}{10}$ for example. We could not guarantee that the sequences have the required properties but instead we argued directly that $\beta_F\left(x, r\right) \leq \frac{1}{10}$ failed at some other point and scale, using a general property of finite sets possessing a flatness condition like the one we imposed on $F$. In fact, flatness goes against the possibility of constructing such loops, and considering this type of hypothesis was intended to show whether the loop idea was at all achievable. Thus the fact that flatness allowed another type of contradiction directly is especially reassuring.

In a forthcoming paper [Far3], we establish the foundation for a new and fundamentally different and *systematic* method of attack on the $\frac{1}{2}$-problem. Besides being a method which should extend quite generally, one of the immediate by-products of the method is that the results of [Far1], [Far2], and the present paper will be significantly generalized. More specifically, the requirement of flatness at almost every point and all sufficiently small scales will be replaced by a weaker property that is required to hold at only a countable number of points and one scale (for each such point). Among the many sets which can then be subjected to such a method will be the (known) examples of totally unrectifiable 1-sets $E$ which actually achieve $\Theta^1_*\left(E, x\right) = \frac{1}{2}$, at almost every point. Such sets tend to have a certain structure, and even that structure can be reasonably understood from within such a method.

**Note added in proof**: The author has recently succeeded in constructing the loops described in Section 6 with all but one property built in, without any assumptions on the geometry. This will appear in the updated version of [Far3].

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