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A Regularity Theorem for Curvature Flows

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1. Introduction

The regularity theory of minimal surfaces and minimizers of other elliptic functional is well known, due to DeGiorgi [DG], Federer and Fleming [FF], Reifenberg [R] and F. Almgren [A]. However the only regularity theory of evolutionary problem is due to K. Brakke [B] about mean curvature flow of unit density surfaces. Here we prove a regularity theory for general surface flows with a similar density condition (Section 2, Definition 9). The evolutions that we deal with do not necessarily come from a gradient flow of some functionals. This is the generalization of the paper [W2] where the corresponding elliptic problem was studied.

The equations that we consider are the following

$$V_t - F(\mathrm{II}, v) = 0\,, \tag{1.1}$$

where V_t is the normal velocity, II is the second fundamental form of a surface S and v is its normal. We assume that F is uniformly elliptic along the solution surface and Lipschitz in v and linear in II. Precise definitions of these terms are in Section 2.

The first example of this kind of equations is

$$V_t - F(\mathrm{II}, v) = V_t - tr\mathrm{II}\,, \tag{1.2}$$

whereas the corresponding solutions are the surfaces moving with their mean curvature.

The main result in this paper is to show that S is regular if it is flat enough and has density close to 1. One of the main difficulties is that the area of S does not make sense.

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2. Basic definitions

We will use the notations introduced in [CW]. We will repeat the definitions for completeness.

Let F be a function defined on symmetric $(n+1)\times(n+1)$ matrices and $R^{n+1}\backslash\{0\}$. Let S be a surface. Let T_v be the subspace perpendicular to v.

Definition 1 *Let II be a 2-form on T_v. Then an $(n+1)\times(n+1)$ matrix M is called a representation of II if*

$$v^T M v = II(v,v)$$

for any $v \perp v$.

Definition 2 *Let M be a representation of a second fundamental form, v be a normal vector and $(e_i,\cdots e_n, v)$ be an orthogonal coordinate system. If II is represented in $e_1,\cdots e_n, v$:*

$$\begin{pmatrix} M_{11} & \cdots & M_{1,n} & M_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ M_{n+1,1} & \cdots & M_{n+1,n} & M_{n+1,n+1} \end{pmatrix},$$

then we denote M^v, the matrix of the second fundamental form, the following matrix

$$\begin{pmatrix} M_{11} & \cdots & M_{1,n} \\ \vdots & \ddots & \vdots \\ M_{n,1} & \cdots & M_{n,n} \end{pmatrix},$$

and let

$$|M|^v = |M^v|.$$

We remark that M^v is the matrix of the second fundamental form with representation M and normal vector v, then $|M^v| = \inf\{|N| : N$ is a representation of the second fundamental form represented by $M\}$.

We will assume that F is quasilinear in II. Namely we assume that

$$F(II,v) = F(M,v) = tr\Phi(v)M + f(v),$$

where we will assume that $\Phi(v)$ and $f(v)$ is C^2 in its variables.

Definition 3 *We say a quasilinear F is geometric if the following two conditions hold:*

$$(\Phi(v)v, e) = 0\,, \tag{2.1}$$

for any v and e with $e \perp v$ and

$$F(II, av) = aF(II,v)\,. \tag{2.2}$$

Hence we have, for any unit vector v,

$$F(II, v) = tr\Phi(v)^v M^v + f(v).$$

Remark 1. Condition (2.1) means that F depends only on the tangential part of M, i.e., F depends only on the matrix of the second fundamental form.

Remark 2. Condition (2.2) is an artificial one since for a solution surface, (1.1) depends only on the unit normal vector. Hence condition (2.2) is a convenient extension of the equation to vectors in the normal directions. This extension simplifies our calculations.

Definition 4 *F is called uniformly elliptic if for each $a > 0, e \perp v, |e| = 1$,*

$$\Lambda a \geq F(M + ae \otimes e, v) - F(M, v) \geq \lambda a$$

for some fixed positive real numbers $\lambda, \Lambda > 0$.

Definition 5 *We say that F is Lipschitz in v if*

$$|F(M, v_1) - F(M, v_2)| \leq C|v_1 - v_2|(1 + |M^{v_1}| + |M^{v_2}|)$$

We will use the notations $cl(D)$, D^o and A^c to denote respectively the closure, interior and compliment of D.

Definition 6 *Let D be a bounded domain with non-empty interior. We say $S \subset \partial D$ is a subsolution of (1.1) if the distance function $d(t, x) = d(x, D(t))$ is a viscosity supersolution of*

$$d_t + F(-D^2 d, Dd) \leq 0$$

in a neighborhood of S (exclude S). We say S is a supersolution if the distance to its complement, $d_C(t, x) = d(x, C(D(t)))$ is a viscosity supersolution in a neighborhood of S. We say S is a solution if it is both a subsolution and a supersolution.

Theorem 2.1 *For any bounded closed set G in R^{n+1}, the set $D =: \cap\{O : O$ is an open set such that $O \cap \{t = 0\}$ containing G and $\partial O - \partial G$ is a supersolution for $\{t \geq 0\}$ defines a solution (see [S]).*

We will work on orthogonal coordinate systems in R^{n+1}. In particular, a change of coordinates for the representations is simply: $G^T M G$ for some $(n+1) \times (n+1)$ orthogonal matrix G. Hence in a new coordinate system, the equation is

$$F_G(M, v) =: F(G^T M G, Gv)\,.$$

The following lemma is evident.

Lemma 2.2 *$F_G(M, v)$ is geometric, uniformly elliptic and Lipschitz in v. All the corresponding constants are the same as those for F.*

According to this lemma, we will drop the subindex G and not mention the specific coordinate system if no confusion should arise.

Lemma 2.3 *Suppose that S is a viscosity solution of (1.1). Let S_1 be a flow of surfaces such that S_1 is outside S for $t \le t_0$ and $X_0 \in S(t_0) \cap S_1(t_0)$ and S_1 is smooth near (t_0, X_0), then at (t_0, X_0), S_1 satisfies that*

$$V \ge F(II, v),$$

where v is the normal pointing toward the inside of $S(t_0)$, V is the normal velocity and II is the second fundamental form of S_1.

Proof. See Soner [S] or [W2].

3. The Main Theorem

We need more notations. Let $(e_1 \cdots e_n, e_{n+1})$ be an orthogonal coordinate system. We will use x to denote a variable in $(e_1 \cdots e_n)$ and y in the direction of e_{n+1}, t in the direction of time and X for (x, y). We will denote $T_r(t_0, x_0) = (t_0 - r^2, t_0] \times \{x : |x - x_0| < r\}$. $Q_r(t_0, x_0, y_0) = (t_0 - r^2, t_0] \times T_r(x_0) \times (-r + y_0, r + y_0)$. We will write T_r or Q_r if $(t_0, x_0) = 0$ or $(t_0, X_0) = 0$.

Definition 7 *Let $(e_1 \cdots e_n, e_{n+1})$ be a coordinate system. Let $S = \partial D \cap Q_1$. If $T_1 \times (-1, -\epsilon + y_0) \bigcap D = \emptyset$, $T_1 \times (\epsilon + y_0, 1) \subset D$, then we say S is ϵ-flat in Q_1 in the direction e_{n+1}.*

Definition 8 *Let e be a unit vector, then $\mathcal{P}_e$ denote the projection of R^{n+1} to $e^\perp$, the orthogonal compliment e.*

Definition 9 *(Multiplicity function and density) Let e_{n+1} be a unit vector. The multiplicity function of S is defined as, for a variable x in the perpendicular direction of e_{n+1},*

$$m(t, x) = \#\{y : (t, x, y) \in S \cap Q_1\}.$$

We say S has density less than $1 + \delta$ at 0 in Q_r if

$$\mathcal{H}_{n+1}(\{(t, x) : m(t, x) > 1\} \cap T_r) \le \delta \mathcal{H}_{n+1}(T_r). \tag{3.1}$$

If S has density less than $1 + \delta$ at 0 in Q_r for all $0 < r < 1$ we say it has density less than $1 + \delta$ at 0. Here we have used $\mathcal{H}_{n+1}$ as the $(n+1)-$dimensional Hausdorff measure. We define

$$\liminf_{r \to 0+} \frac{\mathcal{H}_{n+1}(S \cap Q_r)}{\mathcal{H}_{n+1}(T_r)},$$

the $n + 1$ dimensional lower density of S at x_0.

We see that S is flat and has density $1+\delta$ for some δ small mean that S is a single-valued graph except a small measure. We also remark that our definition of density depends on the choice of the coordinate. However in the case that S is a solution surface, the *density-1* is mutually comparable with different coordinates if the directions of the graph form a different angle.

Definition 10 *We say S that is $C^{1,\alpha}$ in Q_1 if it is $C^{1,\alpha}$ in x and $C^{\frac{1+\alpha}{2}}$ in time.*

Theorem 3.1 (Main Theorem) *If F is uniformly elliptic and C^2 in its variables, then there exists $\epsilon_0 > 0$ such that if $\epsilon \le \epsilon_0$, S is a solution of (1.1) in Q_1 with ϵ-flat in the direction e_{n+1}, and with density less than $1+\epsilon$ at 0 with some direction $\bar{e}$, $|e_{n+1} - \bar{e}| < C\epsilon^{2/3} \le \frac{\sqrt{2}}{2}$, then there are two $C^{1,\alpha}$ surfaces S_+ and S_-, with $C^{1,\alpha}$ norm bounded by $C\epsilon$, such that S_+ is above S and S_- is below S in the e_{n+1} direction and $\{S_+ = S_-\}$ has positive density at 0.*

Corollary 3.2 *If a solution S satisfies the conditions at each point of S in Q_1 then S is a $C^{1,\alpha}$ embedded surface in $Q_{1/2}$.*

Proof. From the main theorem, at each point on the solution surface there is a pair of $C^{1,\alpha}$ solutions. Clearly the lower envelop of all these upper $C^{1,\alpha}$ surfaces would also $C^{1,\alpha}$ and coincide with the solution surface in a set which has positive density everywhere. Hence the solution surface is the envelop and hence the solution is an embedded $C^{1,\alpha}$ surface. ■

Corollary 3.3 *If x_0 is a flat point of a mean curvature flow (i.e., $F(II, v) = trII$) $S(t)$ and $\mathcal{H}_{n+1}(S \cap T_1) \le (1+\epsilon_0)\mathcal{H}_{n+1}(T_1)$ then S is locally $C^{1,\alpha}$ near x_0.*

Proof. This conclusion follows from the monotonicity formula which states that the density is monotone increasing. ■

Now, let us define a deformation of S by polynomials.

Definition 11 *Let S_1 and S_2 be two sets. We define addition of them in e_{n+1} direction as*

$$S_1 \oplus S_2 = \{(t,x,y) : y = y_1 + y_2 \text{ for some } (x,y_1) \in S_1 \text{ and } (x,y_2) \in S_2\}.$$

Notice that if S_1 is the graph of a function $f_1(x)$ and S_2 is the graph of a function $f_2(x)$, then $S_1 \oplus S_2$ is the graph of $f_1 + f_2$.

Let P be a positive definite matrix with $\lambda Id \leq P \leq \Lambda Id$, and let

$$\begin{aligned} h_\delta(x,t) &= \delta(\frac{1}{2}x^T Px - t)\,, \\ H_\delta &= \{(t,x,y) : y \geq \delta(\frac{1}{2}x^T Px - t)\}\,, \\ H_\delta^\epsilon &= \{(t,x,y) : y \geq \epsilon + \delta(\frac{1}{2}x^T Px - t)\}\,. \end{aligned}$$

We will use the deformation $S \oplus G_\delta$ in our regularity theory.

Definition 12 *Let S be a graph and $v = (v', v_{n+1})$ be its normal vector. If $v_{n+1} \neq 0$, we define $\nabla S = -\frac{v'}{v_{n+1}}$ and call it the gradient vector. Moreover we define $GS = G(S) = (\nabla S, S - \nabla S \cdot X)$, the parabolic Gauss map. GS is a map from a subset of R^{n+2} to R^{n+2}. We will denote $GS(A)$ the image of GS of the set A.*

Definition 13 *We say a set in R^{n+2} with variables (x,y,t) is convex if it is convex in (x,y) and increasing in t. We say a function defined in R^{n+1} with variables x and t convex if it is convex in x and decreasing in t. Suppose T in R^{n+2} is a closed set. We denote $\Gamma(T)$ as its convex envelope, i.e. the smallest convex set containing T and call the points in $\{(t,x,y) : T = \Gamma(T)\}$ as the contact points. For simplicity of the exposition, we will denote $\Gamma(S) \cap (T_1 \times (0,1))$ as $\Gamma(S)$.*

Lemma 3.4 *Let $\delta > 2\epsilon$ and T be a convex set with $H_\delta \cap Q_1 \subset T \cap Q_1 \subset H_\delta^\epsilon \cap Q_1$, where H_δ and H_δ^ϵ are defined in the formula before Definition 12. Then*

$$(GH_\delta)(Q_{\frac{1}{2}}) \subset (GT)(Q_{\frac{1}{2}+\sqrt{\frac{2\epsilon}{\lambda\delta}}}),$$

where G is the Gauss map defined in Definition 12.

Proof. Let u be the defining function of T. Suppose that $(\xi, \eta) \in (GT)(Q_{\frac{1}{2}})$, that is, there is $(t_0, x_0) \in Q_{\frac{1}{2}}$ such that $L(x) = h + \xi \cdot (x - x_0)$ is a supporting plane of H_δ at (t_0, x_0, η). We see that $\eta = L(0)$. Noticing the fact that $\|u - h_\delta\|_{L^\infty} \leq \epsilon$, we have that $L + \epsilon'$ will be a global supporting plane of u, i.e.,

$$\begin{aligned} u &\geq L \quad \text{in } Q_{\frac{1}{2}}, \\ u(t_1, x_1) &= L + \epsilon' \end{aligned}$$

for some $t_1 \leq t_0$ and $\epsilon' \leq \epsilon$. From the ellipticity condition of the Hessian of h_δ, we see that $|x_1 - x_0| \leq \sqrt{\frac{2\epsilon}{\lambda\delta}}$ and $|t_0 - t_1| \leq \frac{\epsilon'}{\delta} \leq \frac{\epsilon}{\delta}$. Hence $(t_1, x_1) \in Q_{\sqrt{\frac{2\epsilon}{\lambda\delta}}}$. We also have $S - \nabla S \cdot x = \eta$ since $L(0) = \eta$ and $L + \epsilon'$ is a parallel translation of L.

4. The Main Estimates

We assume that S is an $\epsilon-$flat solution in Q_1 in the direction of e_{n+1}. In this section, we show that if S is flat enough, then S is flatter in a smaller neighborhood with respect to a tilted coordinate system. Before stating the main estimates, let us show the following lemma first.

Let us assume that, the smallness conditions,

$$|f(e_{n+1})| \leq \delta^2 \,, \tag{4.1}$$

$$\Big|\frac{\partial f(e_{n+1})}{\partial \nu}\Big| + \Big|\frac{\partial \Phi(e_{n+1})}{\partial \nu}\Big| \leq \delta \,, \tag{4.2}$$

i.e., the hyperplane $S = 0$ is almost a solution. We remark that this smallness is redundant after we blow up the coordinate. See Section 6.

Accordingly we may assume, from the ellipticity and smoothness of F,

$$F(M, v) = tr(F_1 M^{e_{n+1}}) + O(|M||v - e_{n+1}| + |v - e_{n+1}|^2),$$

where $F_1 = \Phi(e_{n+1})^{e_{n+1}}$ is a positive definite matrix with $\lambda Id \leq F_1 \leq \Lambda Id$. We will use the notation

$$L_0 u = tr(F_1 D^2 u) \,, \tag{4.3}$$

for the main part of the linearized operator of F.

Lemma 4.1 *$\Gamma(S \oplus G_\delta)$ is locally the graph of a $C^{1,1}$ function function from $(e_1, \ldots, e_n)$ to e_{n+1}:*

$$|D\Gamma(S \oplus H_\delta)| + \Big|\frac{\partial \Gamma(S \oplus H_\delta)}{\partial t}\Big| + |D^2\Gamma(S \oplus H_\delta)| \leq C\delta.$$

Moreover, if we take $H_\delta = \delta(\frac{1}{2}x^T F_1^{-1} x - t)$, then

$$-V_t + F(II_{S \oplus H_\delta}, v) \leq (n+1)\delta + C\delta^2.$$

Let $S^- = \Gamma(S \oplus H_\delta) - H_\delta$, we have

$$0 \geq -V_t + F(II_{S^-}, v).$$

Here C is a universal constant.

Proof. By the convexity, we only need to show the estimates for $w(t, x) =: \Gamma(S \oplus H_\delta)$ at the contact points. We only need to prove it at the point where the normal is e_{n+1} since the proof for other points are the same after adding a hyperplane to the graph in order to reduce the general case to this case. We also may suppose that the contact point is 0.

The estimates for the gradient come from the the convexity. Let $\rho(r) = \sup_{\partial Q_r} w$. We need only to prove $\rho(r) \le C\delta r^2$ for r sufficiently small.

Due to the convexity of w, we may suppose $\rho(r) = w(-r^2, re)$ for a vector e with $|e| = 1$. Let us suppose $e = e_n$. Hence, $w(t, x', x_n) \ge \rho(r)$ for any x', $x_n \ge r$ and $t \le -r^2$ from the convexity assumption on w.

We prove it in two steps.

$$(1) \qquad \inf_{|x|\le Mr} w(-r^2, x) < M\delta r^2 \text{ for } M = 2n\lambda + 1,$$

$$(2) \qquad \rho(r) \le C\delta r^2 \text{ for some } C.$$

We prove (1) by contradiction. Suppose $\inf_{|x|\le Mr} w(-r^2, x) \ge M\delta r^2$. Consider now a large domain $R = (-r^2, 0] \times B_{\sqrt{M}r}$ and the test function $h = -\delta Mt - \delta|x|^2$. It is easy to see that, for small r,

$$\begin{aligned} &(a) \qquad L^-(h) = h_t - F(II_h, v) < -\delta\,, \\ &(b) \qquad h \le 0 \quad \text{for } |x| = \sqrt{M}r\,, \\ &(c) \qquad h \le M\delta r^2 \quad \text{for } t = -r^2\,. \end{aligned}$$

Hence, $h \le w \le u$ on $\partial_p R$ and

$$L^-(h) < -\delta \le g\,.$$

However, this contradicts the definition of S since

$$\min(w - h) = (w - h)(0) = 0\,.$$

(2) is also proved by contradiction. Suppose $\rho \ge C\delta r^2$.

We need only to show that $w(-r^2, 0, -\sqrt{M}r) \ge \delta(M + 1)r^2$. By translating of the x-plane of the following test function, we show that $\inf w \ge (M + 1)\delta r^2$. Consider the domain

$$R = \left(-r^2 - Kr^2\frac{M+1}{\lambda}, -r^2\right) \times \left\{x' : |x'| \le r\sqrt{K\frac{\Lambda}{\lambda}}\right\} \times \left(-r\sqrt{M} - r, r\sqrt{M} + r\right).$$

Then, by taking K, C large and r small, the function

$$h = C\delta(x_n - (\sqrt{M} + 1)r)^2 + \delta(t - r^2) - \delta\frac{1}{(n-1)\Lambda}|x'|^2$$

will satisfy the same conditions as in (1). Hence $u \geq h$. So it follows that $w(-r^2, 0, -\sqrt{M}r) \geq h(-r^2, 0, -\sqrt{M}r) \geq (M+1)r^2$.

The other assertions follow from Lemma 2.4.

Now, we go back to the proof of the main iteration.

Theorem 4.2 *Let $\delta = \epsilon^{\frac{2}{3}}$ and $A = \{(t,x) : \Gamma(S \oplus H_\delta) = S \oplus H_\delta\}$. Then*

$$\left| |A \cap Q_{\frac{1}{2}}| - |Q_{\frac{1}{2}}| \right| \leq C\epsilon^{\frac{1}{6}} .$$

Proof. Let $\mathcal{P}$ be the projection of R^{n+1} to $y = 0$. Let $h_\delta(t,x) = y = \delta(\frac{1}{2}x^T Px - t)$, where P is the inverse of the ellipticity matrix $\Phi(e_{n+1})$ of

$$F(M, e_{n+1}) = tr\Phi(v)M + f(v)$$

Hence we have

$$F(M, v) = tr(P^{-1}M^{e_{n+1}}) + C(|M||v - e_{n+1}| + |v - e_{n+1}|^2) .$$

Comparing the volume of the image of $Q_{\frac{1}{2}}$ under the map ∇G as in Theorem 4.2, we have,

$$G(H_\delta)(Q_{\frac{1}{2}}) \subset G(\Gamma(S \oplus H_\delta))(Q_{\frac{1}{2}+C\epsilon^{\frac{1}{6}}}).$$

Therefore,

$$\begin{aligned} \frac{1}{\det P}\frac{\delta^{n+1}}{2^{n+2}}|Q_1| = {} & \operatorname{vol}\Big(G(\Gamma_\delta)\Big) \\ & \leq \operatorname{vol}\big(G(\Gamma(S \oplus H_\delta)(Q_{\frac{1}{2}+C\epsilon^{\frac{1}{6}}}))\big) \\ & \leq \int_{\mathcal{P}(A)\bigcap Q_{\frac{1}{2}+C\epsilon^{\frac{1}{6}}}} (-\Gamma S \oplus H_\delta)_t \det D^2(\Gamma S \oplus H_\delta) \\ & \leq \int_{\mathcal{P}(A)\bigcap Q_{\frac{1}{2}+C\epsilon^{\frac{1}{6}}}} \frac{1}{\det P}\Big(\frac{trPD^2(\Gamma S \oplus H_\delta) - \Gamma(S \oplus H_\delta)_t}{n+1}\Big)^{n+1} \end{aligned}$$

(Here we used an inequality $\det B \cdot \det P \leq (\frac{trPB}{n+1})^{n+1}$ for $P, B \geq 0$)

$$\begin{aligned} & \leq \int_{\mathcal{P}(A)\bigcap Q_{\frac{1}{2}+C\epsilon^{\frac{1}{6}}}} \frac{1}{\det P}\Big(\frac{C\delta^2 + CD|\epsilon|^2 + (n+1)\delta}{n+1}\Big)^{n+1} \\ & = \frac{1}{\det P}|\mathcal{P}(A) \bigcap Q_{1/2+C\epsilon^{\frac{1}{6}}}|\big(\delta + C\delta^2\big)^{n+1} . \end{aligned}$$

Hence $\big||A \bigcap Q_{1/2}| - |Q_{\frac{1}{2}}|\big| \leq C|\epsilon|^{\frac{1}{6}}$. ∎

5. Approximation by linear equations

We will use the density condition in an essential way in order to show that the solution can be approximated by a linear function with some improvement and the solution will coincide with a $C^{1,1}$ function in a set a positive density.

Now, we have the crucial observation that $\Gamma \oplus -G_\delta$ is a supersolution on A, almost everywhere. This is due to the fact that $\Gamma \oplus -G_\delta$ is a supersolution and second order differentiable almost everywhere.

We cannot use fully nonlinear operator as an approximation here, which has been successful in [W1]. The reason is that we don't have a potential theory for fully nonlinear equations. However $\epsilon^{\frac{2}{3}}$ estimates on the derivatives ensure that the linear equation is a good approximation.

Lemma 5.1 *Suppose that we are given on $T_1(0)$ two continuous functions f^-, f^+ satisfying $-1 \le f^- \le f^+ \le 1, -f_t^- + \Delta f^- \le K, f_t^+ - \Delta f^+ \le K$ and such that $f^- = f^+$ except on a small set A with $|A| \le \eta$. Then there is a linear function L and $0 < \beta < 1$ such that*

$$\|f^\pm - L\|_{L^\infty(T_r)} \le C\eta^{\frac{1}{n+4}} + C(\beta, K)r^{1+\beta},$$

where $r \le r_1(\eta, \beta, K)$. In particular, $\|\nabla L\| \le 3$.

Proof. Letting $u = f^+ - f^-$, we have

$$u_t - \Delta u \le 2K$$

and for any $r \ge \eta^{\frac{1}{n+2}}$ we have by Fubini theorem,

$$\frac{1}{l}\int_{\partial_x Q_l} u + \int_{t=-l^2} u \le C\eta$$

for some $r > l > \frac{r}{2}$. Let v be the solution of the backward heat equation,

$$\begin{aligned} -v_t - \Delta v &= d((t,x), \partial Q_1) \\ v &= 0 \quad \text{on } \partial_x Q_1 \quad \text{and on } \{t = 0\}. \end{aligned}$$

Then v is $C^{2,\alpha}$ and $v > 0$ in Q_1. In particular, we have $|\nabla v| \le C$. Letting $v_l(x,t) = v(l^{-1}x, l^{-2}t)$, we have

$$\int_{Q_l} v(u_t - \Delta u)dxdt = \int_{Q_l}(v_t - \Delta v)udxdt - \int_{\{t=-l^2\}\cap Q_1} vudx + \int_{-l^2}^0 \int_{\partial B_l} u\frac{\partial v}{\partial n}dsdt$$

Hence,

$$\frac{1}{l^2}\int_{Q_l} d((t,x), \partial Q_1)u \le C(\frac{1}{l}\int_{\partial_x Q_l} u + \int_{t=-l^2} u) + l^{n+2}2CK.$$

Therefore, we have

$$\fint_{Q_{l/2}} u \le \frac{C\eta}{l^{n+2}} + Cr^2 K.$$

Now we use the local maximum estimates on u, we have

$$\sup_{Q_{r/4}} u \le \frac{C\eta}{r^{n+2}} + Cr^2 K.$$

Taking $r = \eta^{\frac{1}{n+4}}$, we have

$$\sup_{Q_{r/4}} u \le C\eta^{\frac{2}{n+4}} K.$$

By a covering of Q_1 we have

$$u \le C\eta^{\frac{2}{n+4}} K$$

in $Q_{1-\eta^{\frac{1}{n+4}}}$.

Let w be the solution of

$$\begin{aligned} -w_t + \Delta w &= \max(-f_t^- + \Delta f^-, -K) \\ w &= f^- \quad \text{on } \partial Q_{\frac{1}{2}}. \end{aligned}$$

We have $-K \le -w_t + \Delta w \le K \le -f_t^+ + \Delta f^+ + 2K$. By the maximum principle, we have $w \ge f^-$. We also observe that

$$\sup_{Q_{\frac{1}{2}}} |w - f^+| \le \sup_{\partial_p Q_{\frac{1}{2}}} |w - f^+| + C\left(\int_{Q_{\frac{1}{2}}} |(w - f^+)_t - \Delta(w - f^-)|^{n+1}\right)^{\frac{1}{n+1}}.$$

Hence we have,

$$\sup_{Q_{\frac{1}{2}}} |w - f^+| \le C\eta^{\frac{1}{n+4}} + CK\eta^{\frac{1}{n+1}}.$$

The lemma follows from the interior estimates for w,

$$|w - L| \le Cr^{1+\alpha}.$$

■

Corollary 5.2 *The same conclusion of the above lemma holds in $T_{\frac{1}{2}}$ if the Laplace operator of $f^{\pm}$ is replaced to a uniformly elliptic operator with constant coefficients.*

Theorem 5.3 (Main Regularity) *Suppose that S is ϵ_1 flat in Q_1 in the direction of e_{n+1} and has $1+\epsilon_2$ density at 0 with respect to $\bar{e}$ and $|\bar{e}-e_{n+1}| \le \frac{\sqrt{2}}{2}$. Then there are ϵ_0, $0<r_1<1$, $0<\alpha<1$ and surfaces S^+ and S^- such that S is, in the coordinate system e, $\epsilon r_1^{1+\alpha}$-flat in Q_{r_1} with respect to e direction for some $|e-e_{n+1}| \le C\epsilon$ and $S=S^+=S^-$ in $Q_{\frac{1}{2}}$ except a set of measure $C(\epsilon^{\frac{1}{6}}+\epsilon_2)$.*

Proof. Let $S^- = \Gamma(S \oplus G_\delta) - G_\delta$ and $S^+ = -(\Gamma((-S)\oplus G_\delta) - G_\delta)$. Let

$$f^-(x) = \frac{\Gamma(S \oplus G_\delta) - G_\delta}{\epsilon}, \quad f^+(x) = -\frac{\Gamma((-S)\oplus G_\delta) - G_\delta}{\epsilon}.$$

The graph of ϵf^- coincides with S except a measure of $C\epsilon^{\frac{1}{6}}$. Notice that the graph of $f^\pm$ is also a graph in the direction of $\bar{e}$ since $|e-e_{n+1}| \le C\epsilon$. We can see that the projection of the set $\{S = \text{graph of } \epsilon f^\pm\}$ to $e^\perp$ has measure less than $\cos(angle(e_{n+1},e))C\epsilon^{\frac{1}{6}}$. By the density condition that S is a single valued graph in the direction e except a set of measure ϵ_2, we have that S =graph of ϵf^+ = graph of ϵf^- except a set of measure

$$(\cos(angle(e_{n+1},e)))^{-1}C[\epsilon^{\frac{1}{6}}+\epsilon_2 r_1].$$

Therefore $f^-(x) = f^+(x)$ on a set in $T_{\frac{1}{2}}$ except a set of measure $C\epsilon^{\frac{1}{6}}+\epsilon_2$. Hence f^- and f^+ satisfy the conditions of Corollary 5.3. Therefore there exists a linear function L such that

$$\|f^\pm - L\|_{L^\infty(T_r)} \le C(\epsilon^{\frac{1}{6}}+\epsilon_2)^{\frac{1}{n+2}} + Cr^{1+\beta}.$$

That is,

$$\|\epsilon f^\pm - \epsilon L\|_{L^\infty(T_r)} \le C\epsilon(\epsilon^{\frac{1}{6}}+\epsilon_2)^{\frac{1}{n+2}} + C\epsilon r^{1+\beta}$$

Let e be the direction perpendicular to the graph of ϵL and let $(\bar{e_1},...,\bar{e_n},\bar{e})$ be a coordinate system. Scaling back, in this coordinate system, for each $(\bar{x},\bar{y})$ on the graph of $S^\pm$ we have

$$|\bar{y}| \le C\epsilon r^{1+\beta} + C\epsilon(\epsilon^{\frac{1}{6}}+\epsilon_2)^{\frac{1}{n+2}} \le \frac{1}{4}r_1\epsilon = r_0^{1+\alpha}\epsilon$$

for r_1 sufficiently small and $\alpha = -\frac{ln4}{\ln r_1}$. Clearly, we can take ϵ and ϵ_2 small enough such that $S^+ = S^- = S$ in T_{r_1} in $\frac{1}{2}|T_{r_1}|$, i.e., density $\frac{1}{2}$. Then S is $r_0^{1+\alpha}\epsilon$ flat in the direction e. The theorem follows. ∎

6. Final Iteration

Applying the main regularity theorem, we obtain that S is $C^{1,\alpha}$ at 0 in the sense that, for some $r > 0$, there is a coordinate system such that S is flatter in Q_r in that coordinate system. Moreover S coincides with a $C^{1,\alpha}$ graph in every neighborhood with positive density at 0.

We will repeat the construction of S^+ and S^- in a small scale in T_{r_1}. We show, in Lemma 6.4, that the new approximate surfaces of S, in a smaller cube, has larger contact set, which actually contains the contact set of a large scale.

Lemma 6.1 *Suppose that S is ϵ flat. Then, for $|x| \le \frac{2}{3}$, (x, y) is a contact point of $\Gamma(S \oplus G_\delta)$ iff there is a hyperplane L such that S is above $L \oplus (-G_\delta)$ and tangent to $L \oplus (-G_\delta)$ at (x, y).*

This is evident.

Lemma 6.2 *Under the conditions of theorem 5.5, for integers $k = 0, 1, 2, \ldots$ there are linear functions L_k, unit vectors e_k, surfaces S_k^- and S_k^+ defined in $Q_{r_1^k}(0)$ such that*

$$|e_k - e_{k+1}| \le C r_1^\alpha \epsilon,$$

$$S_k = S_{k+1}, \text{ where } S = S_k \text{ in } Q_{r_1^k},$$

$$S \text{ is } \epsilon r_1^{k(1+\alpha)}\text{-flat in } e_k \text{ direction.}$$

Moreover,

$$S_k^- = S_k^+ = S \text{ in } Q_{r_1^k}$$

except a set of measure less than $\frac{1}{2}|T_{r_1^k}|$.

Proof. The existence of $S_1^\pm$ follows from Theorem 5.5. We prove the rest by induction. Suppose that the lemma is true for k.

Let $\tilde{S} = T_k S$, where $T_k(t, x, y) = (r_1^{2k} t, r_1^k x, r_1^k y)$. Now $\tilde{S}$ satisfies a scaled equation

$$V_t - \widetilde{F}(\mathrm{II}, v) = 0,$$

where the scaled equation is given by $\widetilde{F}(M, v) = r_1^k F(\frac{M}{r_1^k}, v)$. We can check directly that $\tilde{F}(0, e_{n+1}) = r_1^k F(0, e_{n+1})$ and $\partial \widetilde{F}(0, e_{n+1}) v = r_1^k \partial F(0, e_{n+1}) v$. So the smallness conditions are satisfied.

Hence $\tilde{S}$ satisfies the condition of Theorem 5.4, such that there are $\tilde{S}^\pm$ and $\tilde{L}$ such that $\tilde{S}$ is $r_1^{k\alpha}\epsilon$-flat in Q_r with respect to $\bar{e}$ direction for some $|\bar{e} - e_{n+1}| \le C\epsilon$ and $\tilde{S} = \tilde{S}^+ = \tilde{S}^-$ in $Q_{\frac{1}{2}}$ except a set of measure $C(r_1^{k\alpha}\epsilon)^{\frac{1}{6}} + \epsilon_2$. Let $S_{k+1}^\pm = T_k^{-1} \tilde{S}^\pm$. The lemma follows. ■

Corollary 6.3 *There is a direction $\bar{e}$ such that $|e_{n+1} - \bar{e}| \leq C\epsilon$ and S is $C\epsilon r_1^{k(1+\alpha)}$ in $Q_{r_1^{k(1+\alpha)}}(0)$ in this coordinate system.*

Proof. Clearly e_k converges to a unit vector $\bar{e}$ with $|e_k - \bar{e}| \leq C\epsilon r_1^{k(1+\alpha)}$. The flatness follows from this. ■

Lemma 6.4 *Suppose that $(t, x, y) \in Q_{\frac{2}{3}}$ is a contact point for $\Gamma(S \oplus G_{\epsilon^{\frac{2}{3}}})$. Then (x, y) is also a contact point for $\Gamma(S \oplus_e G_{\frac{\epsilon^{\frac{2}{3}}}{r_1^\alpha}})$, provided $|e_{n+1} - e| \leq C\epsilon^{\frac{2}{3}}$ is sufficiently small. That is (x, y) is also a contact point for the next smaller scale.*

Proof. From Lemma 6.3, we need only to show that, if S is above $L + \frac{1}{2}\epsilon^{\frac{2}{3}} x^T P x$ for $|\nabla L| \leq C\epsilon^{\frac{2}{3}}$ then we have that S is above

$$L' + \frac{\delta}{2r_1^\alpha} x'^T P' x'$$

for some L' and

$$\frac{1}{\Lambda} Id \leq P' \leq \frac{1}{\lambda} Id$$

and the coordinate system $(e_1', ..., e_{n+1}')$ and $e_{n+1}' = e$. However it is clear by tilting the coordinate a little bit and by noticing that the curvature of the polynomial is bigger than that in the unit scale. ■

Lemma 6.5 (Gluing of barriers) *There are two surfaces $S_\pm$ such that $S_\pm = S_k^\pm$ in $Q_{\frac{1}{4}r_1^k} - Q_{\frac{1}{2}r_1^{k+1}}$ and and $S_\pm$ is in between $S_k^\pm$ and $S_{k+1}^\pm$ wherever they are both defined. Moreover the sets of $\{S = S_\pm\}$ is bigger than the contact set of $S_k^\pm$ in each $Q_{r_1^k}$ and $S_\pm$ is $C^{1,\alpha}$ at 0.*

Proof. The proof is elementary as soon as we observe from Lemma 6.4 that S_k^- is below S_{k+1}^- in Q_{r_1}. We only give the construction in $Q_1 \backslash Q_{r_1^2}$. We can do the same for other scales. Let η be a smooth function such that $0 \leq \eta \leq 1, \eta = 1$ in $Q_1 \backslash Q_{\frac{r_1^2}{2}}$ and $\eta = 0$ in $Q_{\frac{1}{4}r_1^2}$. Now let

$$\tilde{S}_2^\pm = \eta S_1^\pm + (1 - \eta) S_2^\pm.$$

We can see that $S_\pm = S_1$ in $Q_{\frac{2}{3}} \backslash Q_{r_1^2}$ at the contact set of $S_1^\pm$ from Lemma 6.4. The $C^{1,\alpha}$ regularity of $S_\pm$ follows from the convergence of the coordinate system. ■

Proof of the main theorem. First we may assume 4.1 and 4.2 hold by expending the coordinates. Consider $\tilde{S} = T_0 S$, where

$$T_0(x, y) = (\epsilon^{\frac{8}{7}} t, \epsilon^{\frac{4}{7}} x, \epsilon^{\frac{4}{7}} y)\,.$$

Now $\tilde{S}$ satisfies the conditions, with a different ϵ, of Lemma 6.1 in Q_1 with the equation

$$V - \tilde{F}(\mathrm{II}, v) = 0,$$

where $\tilde{F}(M, v) = r_1^k F(\frac{M}{r_1^k}, v)$. The main theorem follows by taking $S_- = \tilde{S_k^-}$ for $Q_{r_1^k} \backslash Q_{r_1^{k+1}}$ and $S_+ = \tilde{S_k^+}$ for $Q_{r_1^k} \backslash Q_{r_1^{k+1}}$, where $\tilde{S}_k$ is the modification given by Lemma 6.3. ■

Remark on further regularity. As soon as we know S is a local $C^{1,\alpha}$ graph, further regularity follows from [W1].

We also remark that, similar regularity theorem hold if F depends on X. We will prove a similar evolutionary equations in a forthcoming paper.

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