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# Multiplier and weighted $\bar\partial$-estimates

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**Abstract**

We study estimates for the solution of the equation $\overline{\partial} u = f$ in one variable. The new ingredient is the use of holomorphic functions with precise growth restrictions in the construction of explicit solutions to the equation.

## 1. Introduction.

In the present paper we will consider the equation $\overline{\partial} u = f$ in one dimension. This equation plays a key role in the study of many problems in complex analysis and, for this reason has been extensively studied. It is of particular interest to have good estimates of the size of $u$ in terms of the size of $f$ (see [Ber94] for a survey on the state of the art of this problem). The purpose of this note is to show how a construction of holomorphic functions with very precise growth restrictions can yield estimates for the solutions to the $\overline{\partial}$-equation. With this tool we have been able to obtain new proofs of some well-known results and some new estimates as well.

The most basic estimate is given by Hörmander's theorem (see [Hör90, p. 92]):

**Theorem** (Hörmander)**.** *Let $\phi$ be a subharmonic function defined in a domain $\Omega \subseteq \mathbb{C}$ such that $\Delta\phi \geq \varepsilon$ for some $\varepsilon > 0$. Then there is a solution $u$ to the equation $\overline{\partial} u = f$ such that*

$$\|u\, e^{-\phi}\|_2 \lesssim \|f e^{-\phi}\|_2 \,.$$

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**Remark.** We write $f \lesssim g$ if there is a constant $K$ such that $f \leq Kg$, and $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$.

We will focus our attention on the case in which $\Omega$ is either the disk or the whole plane. When $\Omega = \mathbb{C}$, M. Christ has proved that the solution operator that solves the $\overline{\partial}$ equation with minimal weighted $L^2$ norm is also bounded on weighted $L^p$ norms, where $1 \leq p \leq \infty$ if we assume some regularity on the weight (see [Chr91]). His theorem is as follows:

**Theorem 1** (Christ)**.** *Let $\phi$ be a subharmonic function in $\mathbb{C}$ such that there is a radius $r > 0$ such that $\Delta\phi(D) \geq 1$ for any disk $D$ of radius $r$. Moreover we assume that $\Delta\phi$ is a doubling measure. Then there is a solution $u$ to the equation $\overline{\partial} u = f$ such that*

$$\|u\, e^{-\phi}\|_p \lesssim \|f e^{-\phi}\|_p \ ,$$

*for all $p \in [1, \infty]$.*

As M. Christ mentions, the doubling hypothesis on $\Delta\phi$ is not of an essential nature. It can be relaxed, but nevertheless one has to assume some regularity on $\phi$ apart from the strict subharmonicity if one wants to obtain $L^\infty$ estimates for instance. This is clearly seen in the following example, due to Berndtsson:

**Example.** Take

$$\phi(z) = \sum_{n \geq 3} \frac{1}{n^2} \log \left| z - \frac{1}{n} \right| .$$

This is a subharmonic function in $\mathbb{D}$ that is bounded (above and below) in $1/2 < |z| < 1$ and moreover $\phi(1/n) = -\infty$. Choose any smooth datum $f$ with support in a small disk lying inside the annulus $1/2 < |z| < 1$ and such that $\int_{\mathbb{D}} f(z)\, dm(z) \neq 0$.

If there is a solution $u$ to the equation $\overline{\partial} u = f$ in $\mathbb{D}$ with $\|u\, e^{-\phi}\|_\infty \lesssim \|f e^{-\phi}\|_\infty$, then $u(1/n) = 0$ since the right-hand side is finite. In addition $u$ is holomorphic outside the support of $f$. That means that $u$ is identically 0 in a neighborhood of $\partial\,\mathbb{D}$. This cannot be so, because $0 = \int_{\partial\mathbb{D}} u\, dz = \int_{\mathbb{D}} \overline{\partial} u\, dm(z) \neq 0$.

There are more sophisticated examples due to Fornæss and Sibony [FS91] that show that it is also impossible to have weighted $L^p$ estimates as in Hörmander's theorem for any $p > 2$ if we do not assume some regularity on the weight.

In another direction, it is possible to extend Hörmander's basic theorem to a larger class of weights including some non-subharmonic functions. This was done initially by Donnelly and Fefferman in [DF83] and many others afterwards (see [BC99] and the references therein). A variant of their theorem (in a particular case of a weight in the disk) is the following:

**Theorem.** *Let $\phi$ be a subharmonic function in the unit disk $\mathbb{D}$ such that its $(1-|z|^2)^2\,\Delta\phi > \varepsilon$ for some $\varepsilon > 0$. Then there is a solution $u$ to the equation $\overline{\partial} u = f$ with*

$$\int_{\mathbb{D}} \frac{|u(z)|^2}{1-|z|^2}\, e^{-\phi}\, dm(z) \lesssim \int_{\mathbb{D}} |f(z)|^2\, e^{-\phi}(1-|z|^2)\, dm(z)\,.$$

For a simple proof of this case see [BOC95].

If we assume some regularity on the weight, we can extend this result to $L^p$ norms. We require the Laplacian of the weight to be locally doubling (see Section 2 for the precise definition). We will prove the following:

**Theorem 2.** *Let $\phi$ be a subharmonic function in the unit disk $\mathbb{D}$ such that $\Delta\phi(D(z,r)) > 1$ for some $r > 0$ where $D(z,r)$ is any hyperbolic disk with center $z \in \mathbb{D}$ and radius $r$. Moreover we assume that $\Delta\phi$ is a locally doubling measure with respect to hyperbolic distance. Then there is a solution $u$ to the equation $\overline{\partial} u = f$ with*

$$\int_{\mathbb{D}} \frac{|u(z)|^p}{1-|z|^2}\, e^{-\phi}\, dm(z) \lesssim \int_{\mathbb{D}} \frac{|f(z)\,(1-|z|^2)|^p}{1-|z|^2}\, e^{-\phi}\, dm(z)\,,$$

*for any $p \in [1,+\infty)$ and*

$$\sup |u|\, e^{-\phi} \lesssim \sup |f(\zeta)\,(1-|\zeta|)|\, e^{-\phi(\zeta)}\,.$$

**Remark.** Observe that in the case $p \in [1,+\infty)$ we could have rewritten the statement of the theorem if we absorb the factor $1/(1-|z|)$ in the weight $\phi$. In this way it will look formally more similar to Hörmander's theorem, but we are allowing weights such that $(1-|z|^2)^2\Delta\phi > (-1+\varepsilon)$. In particular, it includes functions $\phi$ which are not subharmonic.

This is our main theorem, although the emphasis should be on the method of proof rather than the new estimates. For instance, it is also possible to show with the same type of proof that Theorem 1 holds when the measure $\Delta\phi$ is locally doubling instead of doubling.

Our main tool (the multiplier) is a holomorphic function with very precise growth restrictions. It is constructed in Section 3 and it may exist under a less restrictive hypothesis, as in [LM99]. Our construction yields a more precise result that it is needed when we want to obtain estimates for the $\overline{\partial}$ equation.

With the same technique we can deal with some degenerate cases when the weight $\phi$ is harmonic in large parts of the domain. In such a case one has to impose extra conditions on the data of the equation, as in the following theorem which may be of interest in the study of the so-called weighted Paley-Wiener spaces.

**Definition.** *A positive Borel measure $\mu$ in $\mathbb{C}$ is a two-sided Carleson measure whenever there is a constant $C > 0$ such that $\mu(D(x,r)) \leq C\, r$ for all disks of center $x \in \mathbb{R}$ and any positive radius $r$.*

**Theorem 3.** *Let $\phi$ be a subharmonic function in $\mathbb{C}$ such that the measure $\Delta\phi$ is a locally doubling measure supported in the real line and $\Delta\phi(I(x,r)) > 1$ for some $r > 0$ where $I(x,r)$ is the interval in $\mathbb{R}$ of center $x$ and radius $r$. Consider the equation $\overline{\partial} u = \mu$, where $\mu$ is a compactly supported measure such that $e^{-\phi}\, d|\mu|$ is a two-sided Carleson measure. Then there is a solution $u$ with*

$$\limsup_{z\to\infty} |u(z)|\, e^{-\phi(z)} = 0 \quad \textit{and} \quad |u(x)|\, e^{-\phi(x)} \leq C\Big(1 + \int_{|z-x|<1} \frac{d|\mu|(z)}{|x-z|}\Big),$$

*for any $x \in \mathbb{R}$, where $C$ does not depend on the support of $\mu$.*

The solution $u$ to the equation $f$ that we present is fairly explicit. It is *not* the canonical solution (*i.e.* the minimal $L^2$ weighted solution). For instance in the case of Theorem 1 our solution $u$ is given by an integral kernel

$$u(z) = \int_{\mathbb{C}} e^{\phi(z)-\phi(\zeta)}\, k(z,\zeta)\, f(\zeta)\, dm(\zeta)\,, \tag{1}$$

which behaves differently from the canonical one. The kernel for the canonical solution can sometimes be estimated. If the weight $\phi$ is of the form $\phi(z) = b(x)$ and $0 < c^{-1} < b''(x) < c$, then the kernel $k'$ of the canonical solution has at most an exponential decay, *i.e.* there is a constant $A$ such that $\limsup_{z\to\infty} |k'(z,0)| \exp{(A\,|z|)} = \infty$ ([Chr91, Proposition 1.18]). The kernel of our solution has a much faster decay, namely

**Proposition 1.** *Under the hypothesis of Theorem* 1 *there is a kernel* $k(z,\zeta)$ *such that the function* $u$ *given by* (1) *is a solution to the equation* $\overline{\partial} u = f$ *and for some* $\varepsilon > 0$,

$$|k(z,\zeta)| \simeq \frac{e^{-\varepsilon|z-\zeta|^2}}{|z-\zeta|}\,.$$

However, there are some instances in which the canonical kernel has a faster decay than our solution (when $\Delta\phi$ is very large).

The structure of the paper is the following. In Section 2 we will prove some basic results on locally doubling measures which will be needed later. In Section 3 we will construct our main technical tool, the so-called multiplier. We will do so in the disk and in the whole plane. The proof follows the same lines in both cases. Finally in Section 4 we will show how we can use the multipliers to prove Theorem 2 and a new proof of Theorem 1 in which the doubling condition on $\Delta\phi$ is replaced by the locally doubling condition. We will also sketch how the same ideas can be used to prove Theorem 3 and Proposition 1.

## 2. Locally doubling measures.

In this section we compile some basic facts we need on locally doubling measures. There are some intersections with the analysis of [Chr91]. Recall that we always work in a domain $\Omega$ which is either the plane or the disk. When the domain is $\mathbb{C}$ the natural distance is Euclidean; in the case of $\mathbb{D}$ we will work with hyperbolic distance.

**Definition.** *A measure* $\mu$ *in* $\Omega$ *is called a locally doubling measure whenever there is a constant* $C > 1$ *such that* $\mu(B) \le C\,\mu(B')$, *for all balls* $B \subset \Omega$ *of radius smaller than* 1, *where* $B$ *is the ball with the same center as* $B'$ *and twice its radius.*

**Example.** There are many locally doubling measures that are not doubling. They can grow faster, for instance $d\mu(z) = e^{|z|}\,dm(z)$ is a locally doubling measure in $\mathbb{C}$ equipped with Euclidean distance, while any doubling measure has at most polynomial growth. Moreover they do not need to satisfy any strong symmetric condition, for instance the measure $(\operatorname{Im} z)^3\,dm(z)$ for $\operatorname{Im} z > 0$ and $(\operatorname{Im} z)^2\,dm(z)$ for $\operatorname{Im} z < 0$ is locally doubling and it is not doubling.

We start with an elementary lemma which is in fact an alternative description of locally doubling measures.

**Lemma 1.** Let $\mu$ be a locally doubling measure in $\Omega$. Then there is a $\gamma > 0$ such that for any balls $B' \subset B$ of radius $r(B')$ and $r(B) < 1$ respectively, we have

$$\Big(\frac{\mu(B)}{\mu(B')}\Big)^{\gamma} \lesssim \frac{r(B)}{r(B')} \lesssim \Big(\frac{\mu(B)}{\mu(B')}\Big)^{1/\gamma}.$$

**Proof.** The left inequality is essentially [Chr91, Lemma 2.1] and the right inequality follows directly from the definition. The converse is also true. If a measure satisfies the inequalities with $B = 2B'$ then it is locally doubling.

As a consequence of this lemma any locally doubling measure has no atoms. But it is possible to prove more:

**Lemma 2.** *Given any segment $I \subset \Omega$ and any locally doubling measure $\mu$ in $\Omega$, then $\mu(I) = 0$.*

**Proof.** Assume that this is not the case. Then there is a subinterval $I' \subset I$ such that $\mu(I') > 0$ and such that the square of side length $|I'|$ that is bisected by $I'$ is inside $\Omega$ (see Figure 1). We define a doubling measure $\nu$ in the interval $J$ which is the base of the square that contains $I'$. The measure $\nu(A)$ of any set $A \subset J$ is defined as $\nu(A) = \mu(R_A)$, where $R_A$ is the set in the square that projects orthogonally onto $A$. Since $\mu$ is locally doubling, then $\nu$ is doubling, therefore it has no atoms. This implies that $\mu(I') = 0$.

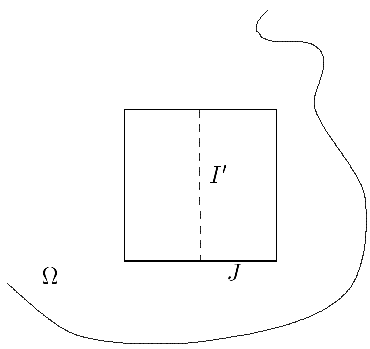


**Figure 1.**

Let us introduce some notations. Let $\mu$ be a locally doubling measure in $\Omega$ with $\mu(\Omega) = +\infty$.

**Definition.** *For any $z \in \Omega$, denote by $\rho(z)$ the radius such that*

$$\mu(B(z, \rho(z)) = 1\,.$$

This is always well defined since for any locally doubling measure in $\Omega$, the measure of any sphere is 0 (with the same proof as in Lemma 2). Thus the function $r \longrightarrow \mu(B(z, r))$ is continuous and strictly increasing.

Since the measures that we consider (in Theorems 1, 2 and 3) are all measures such that $\mu(B(z, r)) \geq 1$ for some $r$ uniformly in $z$, then $\rho(z)$ has an upper bound, but it can be very small.

The following claim is an immediate consequence of Lemma 1.

**Claim 1.** *Let $\mu$ be a locally doubling measure such that $\rho(z)$ has an upper bound. Then for any $K > 0$ there is a $C_K$ such that $1/C_K < \rho(z)/\rho(w) < C_K$ whenever*

$$d(z, w) \leq K \max\{\rho(z), \rho(w)\}\,.$$

Thus the radius of balls of measure one do not change very abruptly. The following estimate is basic in our analysis:

**Lemma 3.** *If $\mu$ is a locally doubling measure in $\Omega$, then there is an $m \in \mathbb{N}$ such that for any $\delta > 0$ ,*

$$\sup_{w\in\Omega} \int_{\delta\rho(w)\leq d(z,w)<1} \Big(\frac{\rho(z)}{d(z, w)}\Big)^m d\mu(z) < C_\delta < +\infty\,.$$

**Proof.** We split the integral into two. In the first we integrate over the region $\delta\rho(w) < d(z, w) < \rho(w)$. In this region $\rho(z) \simeq \rho(w)$, therefore the integral is bounded by $C'_\delta\mu(B(w, \rho(w))$. In the second we integrate over the region $\rho(w) \leq d(z, w) \leq 1$. We split this into rings of doubling size and we may estimate it by

$$\sum_{n=0}^{k} \int_{2^n<d(z,w)/\rho(w)<2^{n+1}} \Big(\frac{\rho(z)}{2^n\rho(w)}\Big)^m d\mu(z)\,,$$

where $k$ is such that $1 < 2^k\rho(w) \leq 2$.

Consider now the ball $B'$ of center $z$ and radius $\rho(z)$ and the ball $B$ of center $w$ and radius $C\,d(z,w) \simeq 2^n\rho(w)$. The constant $C$ is chosen in such a way that $C\,d(z,w) \geq \rho(z) + d(z,w)$. This is always possible, since $\rho(z)$ and $\rho(w)$ are equivalent whenever $z$ is close to $w$. Therefore $B' \subset B$, the radius of $B$ is smaller than 1 and we may apply Lemma 1. We estimate $\rho(z)/(2^n\rho(w))$ by $(C/\mu(B(w, 2^n\rho(w)))^\gamma$, and the integral is bounded by a constant times

$$\sum_{n\geq 0}^{k} \frac{1}{(\mu(B(w, 2^n\rho(w)))^{m\gamma-1}} = \sum_{n\geq 0}^{k} \frac{\mu(B(w,\rho(w)))^{m\gamma-1}}{(\mu(B(w, 2^n\rho(w)))^{m\gamma-1}} .$$

We apply Lemma 1 to this expression and compare the quotient of measures by the quotient of radii (we consider $1 = \mu(B') = \mu(B(w,\rho(w))$ as the numerator) and obtain

$$C \sum_{n=0}^{k} \Big(\frac{\rho(w)}{2^n\rho(w)}\Big)^{(m\gamma-1)/\gamma} < +\infty$$

provided that we choose $m$ so that $m\gamma > 1$.

## 3. The multipliers.

The main tool used to prove these results is the construction of the so-called multipliers. These are holomorphic functions that have very precise growth control. They have been used to solve some interpolation and sampling problems in several function spaces (see [OCS98], [LS94]) and also the zero sets as in the Beurling-Malliavin theorem (see also [Sei95]). They all boil down to an approximation of subharmonic functions by the logarithm of entire functions outside an exceptional set. The most general result of this type is due to Lyubarskiĭ and Malinnikova, [LM99], where they do not assume any regularity condition on the Laplacian of the subharmonic function. However we need a more precise description than theirs on the exceptional set in which the approximation need not hold.

The following theorem is a result by Lyubarskiĭ and Sodin which will serve as a model (see [LS94] for a proof).

**Theorem** (Lyubarskiĭ-Sodin). *Let $\phi$ be a subharmonic function in $\mathbb{C}$ such that its Laplacian satisfies $\Delta\phi \simeq 1$. Then there exists an entire function $f$ with a uniformly separated zero set $Z(f)$ (i.e. $\inf_{a,b\in Z(f), a\neq b} d(a,b) > 0$) such that*

$$|f(z)| \simeq e^{\phi(z)} ,$$

*when $|z-a| \geq \varepsilon$ for all $a \in Z(f)$.*

In the case of the disk the following theorem from Seip, [Sei95] is the analogue to the multiplier lemma of Lyubarskiĭ and Sodin,

**Theorem** (Seip). *Let $\psi$ be a subharmonic function in $\mathbb{D}$ such that its Laplacian verifies $(1-|z|^2)^2\Delta\psi \simeq 1$. Then there is a function $g \in \mathcal{H}(\mathbb{D})$, with a uniformly separated zero set $Z(g)$, and*

$$|g(z)| \simeq e^{\psi(z)} ,$$

*when $|z-a|/|1-\overline{a}\, z| \geq \varepsilon$ for all $a \in Z(g)$.*

We will need an analogous theorem for locally doubling measures in $\mathbb{C}$ and in $\mathbb{D}$.

**Theorem 4.** *Let $\psi$ be a subharmonic function in $\Omega$ such that its Laplacian $\Delta\psi$ is a locally doubling measure, with the property $\Delta\psi(D(z,R)) > 1$ for all disks of some large radius $R > 0$. Then there is a holomorphic function $h$ with zero set $Z(h) = \Lambda$ such that*

$$\frac{d(z,\Lambda)}{\rho(z)} \lesssim |h(z)|\, e^{-\psi(z)} \lesssim \Big(\frac{d(z,\Lambda)}{\rho(z)}\Big)^M ,$$

*for some fixed $M \in \mathbb{N}$, where $d(z,\Lambda)$ is the distance (in the appropriate metric) from $z$ to $\Lambda$.*

**Remark.** It will follow from the construction of $h$ that $d(z,\Lambda) \lesssim \rho(z)$, thus $|h| \simeq e^{\psi}$ outside a set $E_h$ composed of small disks around the zeros of $h$: $E_h = \cup_{\lambda\in\Lambda} D(\lambda, \varepsilon\rho(\lambda))$.

With a slight refinement of the construction it is possible to prove that the zero set $\Lambda$ can be chosen in such a way that

$$d(\lambda_i,\lambda_j) \geq \varepsilon \max\{\rho(\lambda_i), \rho(\lambda_j)\} ,$$

for some $\varepsilon > 0$ and $M$ can be chosen to be 1, but we won't need this here.

We will simultaneously prove Theorem 4 on the multipliers in the disk and in the plane, since we have to follow the same steps. To begin with, we need a partition of the domain into rectangles that is well adapted to the measure and the metric.

**Lemma 4.** *Assume that $\mu$ is a locally doubling measure in $\Omega$ with $\mu(D(z,R)) > 1$. Given any $N \in \mathbb{N}$ there is a partition of $\Omega$ into rectangles*

*$\{R_i\}_{i\in I}$ such that $\mu(R_i) = N$ and if we denote by $L_i$ the length of the longer side of $R_i$ and $l_i$ the length of the smaller side, then* $\sup_{i\in I} L_i/l_i = L(N) < +\infty$.

**Remark.** When $\Omega$ is a disk, one has to understand that by "rectangles" we mean rectangles in polar coordinates. This lemma is basically the partition theorem from [Yul85], but we include a proof, since the doubling assumption (which is not needed) makes it particularly easy.

**Proof.** We start by assuming that $N = 1$, the general case follows if we use the same construction with the measure $\sigma = \mu/N$ instead of the measure $\mu$. We will first find a partition into rectangles $\{\widetilde{R}_i\}_{i\in I}$ in such a way that $\mu(\widetilde{R}_i) \in \mathbb{N}$, $1 \le \mu(\widetilde{R}_i) \le C$ and with the ratio of side-lengths bounded and $C$ is the doubling constant. Later on, we will refine this partition in order to obtain rectangles of mass one.

Recall that there is some $R > 0$ such that $\mu(D(z,R)) > 1$ for all $z \in \Omega$. Let us partition the plane (see next paragraph for the disk) into parallel strips of width $R$. Then, we slice each strip in rectangles of mass a natural number (the sides of the rectangle have no mass because of Lemma 2). The length of any piece will be between $R$ and $2R$. Since any square of size $R \times R$ has mass at least 1, it is possible to slice the strip in such a way that the resulting rectangles have the ratio between the sides bounded by 2. We have no upper bound of the mass of these rectangles; we only know that it is a natural number.

In the case of the domain being the disk, one has to replace the strips by annuli centered at the origin of width between $R$ and $2R$ and in such a way that they all have mass which is a natural number. Now we split each annulus in rectangles of integer mass. The length of the sides will be between $R$ and $2R$, except possibly the last one which closes the circle and which has to been taken of side-length between $R$ and $3R$. In any case, the resulting rectangles have the ratio of lengths of sides bounded by 3 and again without control on the upper bound of the mass.

From now on the procedure in the disk and in the plane will be the same. We will divide each rectangle in two. All the resulting rectangles will still have integer mass and the ratio of the sides will always remain bounded by 3. We will bisect each rectangle until the mass is smaller than the doubling constant $C$ of the measure.

The bisection is done as follows: denote the original rectangle by $R = [a, a+w] \times [b, b+l]$ where $l \le w \le 3\,l$. Consider a smaller auxiliary rectangle $R' \subset R$ of the form $R' = [a+w/2-h/2, a+w/2+h/2] \times [b, b+l]$ with an auxiliary $h \le w$ so that $\mu(R') = 1$. The length $h$ of $R'$ cannot be very large. If $h > w/3$, then $R \subset 2R'$ and $\mu(R)$ is already smaller than the doubling constant $C$, thus we do not need to bisect $R$. Since $\mu(R') = 1$, there must be a $0 < t < h$ such that $\mu(R_1) \in \mathbb{N}$, where $R_1 = [a, a + w/2 - h/2 + t] \times [b, b + l]$. We denote by $R_2 = R \setminus R_1$. It also verifies $\mu(R_2) \in \mathbb{N}$. Finally it is easily checked that $l/3 \leq w/3 \leq w/2 - h/2 + t \leq w \leq 3\,l$, therefore the quotients of the sides-length of $R_1$ are bounded by 3 and similarly with $R_2$.

Thus far, the rectangles are not very deformed and all have a mass between 1 and $C$. In order to obtain rectangles of mass 1, we divide each of them into rectangles of mass one by cutting along the direction of the longest side. The local doubling condition ensures that all of them will be essentially of the same proportion (we use Lemma 1). At most we are dividing each rectangle in $C$ pieces, therefore the resulting rectangles have a bounded ratio of side-lengths as desired.

The family of rectangles that we have just constructed looks very much like squares, since the excentricity is bounded, but moreover the size of the rectangles changes very slowly, along with $\rho(z)$:

**Claim 2.** *The family of rectangles $\{R_i\}$ constructed in Lemma 4 has the following two properties*:

- *The ratio between the diameter of $R$ and $\rho(z)$ for any $z \in R$ is bounded above and below by two constants independent of $R$ and $z \in R$.*

- *For any $K > 0$ there is a constant $C_K > 0$ such that whenever $KR_i \cap KR_j \neq \varnothing$ the ratio between the diameters of $R_i$ and $R_j$ is bounded by $C_K$.*

**Proof.** The first assertion follows since $R$ has bounded excentricity and constant mass. The second one is an immediate consequence of Claim 1.

In order to construct the multiplier, we first select its zeros. We take a very large $N = m\,k$ (the same $m$ as given by Lemma 3 and $k \in \mathbb{N}$ that will be chosen in Lemma 5). We partition $\Omega$ in rectangles $\{R_i\}_{i\in I}$ of mass $N$ as in 4. For any $i \in I$, we will choose $N$ points $\{\lambda_1^i, \dots, \lambda_N^i\}$ which lie near $R_i$ and such that the moments of order $0, 1, 2, \dots m-1$ of the measure $\Delta\phi$ restricted to $R_i$ coincide with the corresponding moments of the measure $\sum_{j=1}^N \delta_{\lambda_j^i}$. The following lemma addresses this point.

**Lemma 5.** *Let $R$ be a rectangle with ratio of side-lengths bounded by $K$. Given any $m \in \mathbb{N}$ and any $C > 1$ there is a $k \in \mathbb{N}$ such that for any* *measure $\mu$ in a rectangle $R \subset \mathbb{C}$ of total mass $N = m\,k$, there are two sets of $N$ points $\Lambda(R) = \{\lambda_1, \dots, \lambda_N\}$ inside $R$ and $\kappa(R) = \{\kappa_1, \dots, \kappa_N\}$ inside $4CKR \setminus CR$ satisfying*

$$\int_R z^j \, d\mu(z) = \lambda_1^j + \cdots + \lambda_N^j = \kappa_1^j + \cdots + \kappa_N^j \,, \qquad j = 0, \dots, m-1 \,.$$

**Proof.** We want that

$$\int_R p(z)\, d\mu(z) = \sum_{i=1}^{N} p(\lambda_i) \,,$$

for all polynomials of degree less than or equal to $m-1$. We may take any Chebyshev quadrature formula with $k$ nodes in $R$ that is exact for polynomials of degree $m-1$. This can be done, eventually taking $k$ much larger than $m$ (see [Gau76] or [Kor94], for a survey on quadrature formulas with equal weights). These are the points that will be used in the construction of the multiplier; they will be in fact the zeros of it. Note that all the points $\lambda_j$ appear with a multiplicity $m$ since there are $N = km$ points with equal weights. For later use, it is convenient to have an alternative set of zeros $\kappa_1, \dots, \kappa_N$ at our disposal which are separated from the original ones (outside $CR$) and still have the same moments. This is easily done. It can be checked immediately that $m\, p(\lambda_j) = \sum_{l=0}^{m-1} p(\lambda_j + \tau e^{l2\pi i/m})$, for any $\tau \in \mathbb{C}$ and any polynomial of degree $m-1$. Thus, we could take as an alternative set $\kappa_{j,l} = \lambda_j + \tau e^{l2\pi i/m}$, $j = 1, \dots, k$, $l = 0, \dots, m-1$, where $\tau$ is chosen so that all $\kappa_j$ are outside $CR$ and inside $4CKR$.

Now we take a holomorphic function $h$ that vanishes at all the points $\{\lambda_j^i\}_{i\in I, j=1,\dots,N}$, where $\{\lambda_j^i\} \subset R_i$ (the rectangles defined in Lemma 4). This function is defined up to a factor of the form $e^g$, with $g \in \mathcal{H}(\Omega)$. We choose this $g$ in such a way that

$$\log|h| = \psi - \frac{1}{2\pi} \int_{\mathbb{C}} \log|z-\zeta| \Big(\Delta\psi - \sum \delta_{\lambda_j^i}\Big) \,,$$

in the case of $\Omega = \mathbb{C}$ and

$$\log|h| = \psi - \frac{1}{2\pi} \int_{\mathbb{D}} \log\Big|\frac{z-\zeta}{1-\bar{\zeta} z}\Big| \Big(\Delta\psi - \sum \delta_{\lambda_j^i}\Big)$$

in the case of $\Omega = \mathbb{D}$. In both cases $\psi$ is the subharmonic function in the statement of Theorem 3. Thus the problem has been reduced to show that

$$(2) \quad M \log \frac{d(z,\Lambda)}{\rho(z)} + C \le \int_{\mathbb{C}} \log|z-\zeta| \Big(\Delta\psi - \sum \delta_{\lambda_j^i}\Big) \le \log \frac{d(z,\Lambda)}{\rho(z)} + C \,,$$

in the case of $\Omega = \mathbb{C}$, and when $\Omega = \mathbb{D}$, we have to obtain

$$(3)\quad M \log \frac{d(z,\Lambda)}{\rho(z)} + C \le \int_{\mathbb{D}} \log\Big|\frac{z-\zeta}{1-\bar\zeta z}\Big|\Big(\Delta\psi - \sum \delta_{\lambda_j^i}\Big) \le \log \frac{d(z,\Lambda)}{\rho(z)}\ .$$

The integral (2) is split as

$$\sum_{i\in I} \int_{\mathbb{C}} \log|z-\zeta|\Big(\chi_{R_i}(\zeta)\,\Delta\psi(\zeta) - \sum_{j=1}^{N} \delta_{\lambda_j^i}(\zeta)\Big)\,.$$

In any of these integrals we can subtract any polynomial of degree $m-1$ to the logarithm since the moments up to order $m-1$ of $\chi_{R_i}(\zeta)\Delta\psi(\zeta)$ and $\sum_{j=1}^N \delta_{\lambda_j^i}(\zeta)$ are the same. For any $R_i$ far from $z$ (we exclude the rectangle $R_j$ to which $z$ belongs and its immediate neighbors) we take a polynomial $p$ of degree $m-1$, which is the Taylor expansion of $\log|z-\zeta|$ about some point $\lambda_0^i \in R_i$.

The difference between $|\log|z-\zeta| - p(\zeta)|$ is bounded by

$$\frac{C}{|z-w|^m}\,|\zeta-\lambda_0^i|^m\,,$$

where $w$ is some point in $R_i$. Since $z$ does not belong to $R_i$ or any of its immediate neighbors and $\zeta \in R_i$, it follows that $|z-w| \simeq |z-\zeta|$ and $|\zeta - \lambda_0^i| \lesssim \rho(\zeta)$ by Claim 2. Thus the integral is bounded by a constant times

$$\int_{R_i} \frac{\rho(\zeta)^m}{|z-\zeta|^m}\Delta\psi(\zeta) + N\frac{\rho(\lambda_0^i)^m}{|z-\lambda_0^i|^m}\ .$$

Both the integral and the sum are comparable since $\rho(\zeta) \simeq \rho(\lambda_0^i)$, $|z-\zeta| \simeq |z-\lambda_0^i|$ and the mass of the rectangle is $N$. This estimate is true for all $R_i$ except the one that contains $z$ and its neighbors. There is a $\delta > 0$ (uniform in $z$ because of Claim 1), such that the sum over all such rectangles is bounded by

$$\int_{|z-\zeta|\ge\delta\rho(z)} \frac{\rho(\zeta)^m}{|z-\zeta|^m}\,\Delta\psi(\zeta)\,.$$

Call $M = 2\sup_z \rho(z)$. If we integrate in the region $\delta\rho(z) \le |z-\zeta| \le M$ we may apply Lemma 3. If we integrate in the region $|z-\zeta| \ge M$, we may estimate the integral by

$$\int_{|z-\zeta|\ge M} K\frac{\rho(\zeta)^2}{|z-\zeta|^3}\,\Delta\psi(\zeta)\,.$$

We use that $\rho(\zeta)^2 \simeq \int_{|\zeta-w|\leq\rho(\zeta)} dm(w)$ and then Fubini's theorem to obtain

$$\int_{|z-w|\geq M/2} K\frac{1}{|z-w|^3}\,dm(w) < +\infty\,.$$

There are at most a finite number of immediate neighboring rectangles (uniformly in $z\in\mathbb{C}$) to the rectangle that contains $z$ because all of them have size comparable to $\rho(z)$. In each of them the integral is bounded by

$$\int_{R_i}\log\frac{|z-\zeta|}{\rho(z)}\,\Delta\psi(\zeta)+\sum_{j=1}^{N}\log\frac{|\lambda_j-z|}{\rho(z)}\;.$$

The integral is bounded whenever $\Delta\psi$ is locally doubling. This is [Chr91, Lemma 2.3] which is in turn a direct consequence of Lemma 1. The sum accounts for the term

$$\Big(\frac{d(z,\Lambda)}{\rho(z)}\Big)^M$$

in the statement of the theorem.

We will to estimate now the integral (3), which can be expressed as

$$\sum_{i\in I}\int_{\mathbb{D}}\log\frac{|z-\zeta|}{|1-\overline{\zeta}z|}\Big(\chi_{R_i}(\zeta)\Delta\psi(\zeta)-\sum_{j=1}^{N}\delta_{\lambda_j^i}(\zeta)\Big).$$

As before we can subtract a Taylor polynomial of degree $m-1$ at a point $\lambda_0^i\in R_i$. Now, since

$$\Big|\nabla_\zeta^m\log\frac{|z-\zeta|}{|1-\overline{\zeta}z|}\Big|\lesssim\frac{1-|z|^2}{|1-\overline{\zeta}z|\,|z-\zeta|^m}\;,$$

the integral is bounded by

$$\begin{aligned}&C\int_{\zeta\notin\delta D(z,\rho(z))}\frac{(1-|z|^2)\,(1-|\zeta|^2)^m\rho(\zeta)^m}{|1-\overline{\zeta}z|\,|z-\zeta|^m}\Delta\psi(\zeta)\\ &\quad+\sum\log\frac{|z-\lambda_i|}{\rho(z)\,|1-\overline{\lambda}_i z|}\;,\end{aligned}\tag{4}$$

where the sum is over all $\lambda_i$ that are in the rectangle $R_i$ which contains $z$ and its immediate neighbors.

We split the integral in two pieces. In the first we integrate over the domain $\Omega_1=\{\zeta\in\mathbb{D},\ d(z,\zeta)<1,\zeta\notin\delta D(z,\rho(z))\}$, and we use Lemma 3 to obtain

$$\int_{\Omega_1}\frac{\rho(\zeta)^m}{d(z,\zeta)^m}\,\Delta\psi(\zeta)<\infty\,.$$

In $\Omega_2$ we have that $d(z,\zeta) > 1$, and (4) is bounded by

$$\int_{\Omega_2} \frac{(1-|z|^2)\,(1-|\zeta|^2)^2\rho(\zeta)^2}{|1-\overline{\zeta}z|^3}\,\Delta\psi(\zeta)\,.$$

We may think of $(1-|\zeta|^2)^2\rho(\zeta)^2$ as $\int_{d(w,\zeta)<\rho(\zeta)} dm(w)$ and apply Fubini's theorem to obtain

$$\int_{\Omega_2} \frac{(1-|z|^2)\,(1-|\zeta|^2)^2\rho(\zeta)^2}{|1-\overline{\zeta}z|^3}\,\Delta\psi(\zeta) \le \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-z\,\overline{w}|^3}\,dm(w) < +\infty\,.$$

Theorem 4 is not yet what we need for the estimates to the $\overline{\partial}$-equation because the exceptional set of the multiplier introduces a technical difficulty. We need to approximate the weight by a holomorphic function *everywhere*. This obstruction can be avoided using several multipliers simultaneously as described in the next proposition:

**Proposition 2.** *Given $\psi$ as in the statement of Theorem* 4 *there is a collection of multipliers $h_1,\dots,h_n$ satisfying the conclusion of Theorem* 4*. Moreover their exceptional sets* (*see the remark after Theorem* 4) *are disjoint*; *i.e.* $E_{h_1}\cap\cdots\cap E_{h_n}=\varnothing$.

**Proof.** Take the partition of $\Omega$ in rectangles given by Lemma 4. We distribute the rectangles in a finite number of families of rectangles $\Omega = \cup_{l=1}^n(\cup_{i\in I_l}R_i^l)$ with the property that any two rectangles of the same family $R_i^l$, $R_j^l$ are very far apart (*i.e.* $MR_i^l\cap MR_j^l=\varnothing$, for some large constant $M$). This is possible with the Besicovitch covering lemma. Now for each family $\{R_i^l\}_{i\in I_l}$ we can construct a multiplier $h_l$ in such a way that it has no zeros in any of the rectangles of the family $R_i^l$ nor in their immediate neighbors. The way to proceed to construct $h_l$ is the following: For any rectangle $R$ that is neither from the family $\{R_i^l\}_{i\in I_l}$ nor one of its immediate neighbors we take the set of points $\lambda(R)$ given by Lemma 5. For the rectangles $R$ from the family or its adjacent rectangles we use the alternative set of points $\kappa(R)$ also defined in Lemma 5. We build as before a multiplier $h_l$ with zeros at the selected points. It has the right growth and the additional property that it has no zeros in the rectangles from the family $\{R_i^l\}_{i\in I_l}$ and its adjacents. This is clear because we can choose a constant $C$ in Lemma 5 in such a way that the points $\kappa(R)$ are neither in $R$ nor in its immediate neighbors. Moreover they are not so far apart from $R$ that they reach another rectangle from the family (this can be prevented by choosing a very large $M$ in the splitting of the rectangles into families).

Thus the exceptional set for $h_l$ does not include any rectangle from the family $\{R_i^l\}_{i\in I_l}$.

## 4. The $\overline{\partial}$-estimates.

This section contains three parts. In the first one, we will see how the weights that we consider can be regularized without losing generality. In the second subsection we prove the $L^p$ weighted $\overline{\partial}$-estimates in the plane and the disk. Finally in the last part we indicate how Theorem 3 can be proved.

### 4.1. The regularization of $\phi$.

In the hypothesis of the theorem we assume that for some large radius $r > 0$, $\Delta\phi(D(z, r)) > 1$ at any point $z \in \Omega$. This is a condition that ensures that $\phi$ is "strictly subharmonic". It will be more convenient for us to assume that $\Delta\phi > \varepsilon\, dm(z)$. This means that the measure is more regular since there are no "holes" with zero measure. The following proposition allows us to do so:

**Lemma 6.** *If the measure $\Delta\phi$ is a locally doubling measure in $\Omega$ and $\Delta\phi(D(z, r)) > 1$ for some large radius $r > 0$ and any point $z \in \Omega$ then there is a subharmonic weight $\psi$ equivalent to the original, i.e.* $\sup_\Omega |\phi - \psi| < +\infty$, *such that $\Delta\psi$ is a locally doubling measure and moreover $\Delta\psi > \varepsilon\, dm(z)$ for some $\varepsilon > 0$.*

**Proof.** We will split $\Delta\phi$ in two measures $\mu_1 + \mu_2$. To describe the measure $\mu_1$, let us tile the plane into squares $Q_j$ of diameter $R > 0$ (dyadic squares in the case of the disk) in such a way that $\Delta\phi(Q_j) > 2$ for all $Q_j$. This is feasible because of the hypothesis on the measure. The measure $\mu_1$ is defined as

$$\mu_1|_{Q_j} = \frac{1}{\Delta\phi(Q_j)} \Delta\phi\, .$$

The measure $\mu_2$ is $\mu - \mu_1$. It follows from the definition that

$$\frac{1}{2}\, \Delta\phi \le \mu_2 \le \Delta\phi\, ,$$

therefore $\mu_2$ is a locally doubling measure. It is also true that $\mu_1$ is locally doubling because $\Delta\phi(Q_j)$ does not change abruptly in neighboring squares and moreover $\mu_1(Q_j) = 1$.

We will regularize the measure $\mu_1$ by taking the convolution (the invariant convolution when $\Omega$ is a disk) of it with the normalized characteristic function of a very large disk

$$\widetilde{\mu}_1 = \mu_1 \star \frac{\chi_{D(0,2R)}}{|D(0,2R)|} \,.$$

The measure $\widetilde{\mu}_1$ in the plane satisfies $\varepsilon\, dm(z) < \widetilde{\mu}_1 < K dm(z)$ (when $\Omega = \mathbb{D}$, it satisfies $\varepsilon < (1-|z|^2)^2\, \widetilde{\mu}_1 < K$).

It is clear from their definition that $\mu_1(D(0,r)) \lesssim r^2$ in $\mathbb{C}$ and $\mu_1(D(0,r)) \lesssim (1-r)^{-2}$ in the disk. The same is true for $\widetilde{\mu}_1$. We introduce integral operators $K[\mu_1]$ and $K[\widetilde{\mu}_1]$ that solve the Poisson equation $\Delta K[\nu] = \nu$. The operator may be defined as

$$K[\mu] = \int_\Omega k(z,\zeta)\, d\nu(\zeta)\,.$$

In the case of $\Omega = \mathbb{C}$ we choose

$$k(z,\zeta) = \frac{1}{4\pi} \log|z-\zeta|^2 - \frac{1}{2\pi}\,(1-\chi_{D(0,1)}(\zeta))\,\mathrm{Re}\Big(\ln|\zeta| - \frac{z}{\zeta} + \frac{1}{2}\,\frac{z^2}{\zeta^2}\Big)\,,$$

which makes the integrals defining $K[\mu_1]$ and $K[\widetilde{\mu}_1]$ convergent. In the case of the disk, set

$$k(z,\zeta) = \frac{1}{4\pi}\Big(\log\Big|\frac{z-\zeta}{1-\overline{\zeta}\,z}\Big|^2 + (1-|\zeta|^2)\Big(\frac{1}{(1-\overline{z}\,\zeta)} + \frac{1}{(1-z\,\overline{\zeta})} - 1\Big)\Big)\,.$$

Andersson [And85] and Pascuas [Pas88] estimate

$$|k(z,\zeta)| \lesssim \Big(\frac{1-|\zeta|^2}{|1-\overline{\zeta}\,z|}\Big)^2\Big(1+\log\Big|\frac{1-\overline{\zeta}\,z}{z-\zeta}\Big|\Big)\,,$$

therefore the integrals defining $K[\mu_1]$ and $K[\widetilde{\mu}_1]$ are convergent.

We take as $\psi = \phi + K[\widetilde{\mu}_1] - K[\mu_1]$. The Laplacian of $\psi$ is $\widetilde{\mu}_1 + \mu_2$ which has the desired properties. Moreover

$$|\phi - \psi| = |K[\mu_1] - K[\widetilde{\mu}_1]| = \Big|K[\mu_1] - K[\mu_1] \star \frac{\chi_{D(0,2R)}}{|D(0,2R)|}\Big|\,.$$

This difference is bounded by

$$\int_{D(z,2R)} \log\frac{2R}{d(z,\zeta)}\, d\mu_1(\zeta)\,.$$

This integral is bounded by a constant times $\mu_1(D(z,2R))$, whenever $\mu_1$ a locally doubling measure. This is [Chr91, Lemma 2.3]. The disk $D(z,2R)$ is covered by a bounded number of cubes $Q_j$, therefore the difference between $\psi$ and $\phi$ is bounded as claimed.

## 4.2. Proofs of Theorems 1 and 2.

Let us start with Theorem 2. There are some weights that are particularly simple. These are the standard radial weights $\phi(z) = \alpha \log 1/(1-|z|^2)$. The following lemma deals with this situation.

**Lemma 7.** *For any $\alpha \in (0,1)$ and $p \in [1,+\infty)$, the solution*

$$u(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{1-|\zeta|^2}{1-\bar\zeta z} \, \frac{f(\zeta)}{z-\zeta} \, dm(\zeta)$$

*to the equation $\overline{\partial} u = f$ in $\mathbb{D}$ satisfies the estimate*

$$\int_{\mathbb{D}} |u(z)|^p \, (1-|z|)^{\alpha-1} \, dm(z) \lesssim \int_{\mathbb{D}} |f(z)\,(1-|z|)|^p \, (1-|z|)^{\alpha-1} \, dm(z) \, .$$

*Moreover,*

$$\sup_{\mathbb{D}} |u(z)| \, (1-|z|)^{\alpha} \lesssim \sup_{\mathbb{D}} |f(z)| \, (1-|z|)^{1+\alpha} \, .$$

**Proof.** This is an immediate consequence of Hölder's inequality.

We take an arbitrary weight $\phi$ under the hypothesis of Theorem 2, that is $(1-|z|^2)^2 \Delta\phi > \varepsilon$ and $\Delta\phi$ is a locally doubling measure with respect to the hyperbolic measure. Consider the auxiliary subharmonic function $\psi = \phi - (\varepsilon/2)\log(1-|z|^2)$. By hypothesis $(1-|z|^2)^2 \Delta\psi > \varepsilon/2$ and still $\Delta\psi$ is locally doubling. Using Theorem 4, we can construct a a holomorphic function $g$ such that

$$\frac{d(z, Z(g))}{\rho(z)} \lesssim |g| \, e^{-\psi} \lesssim \frac{d(z, Z(g))^M}{\rho(z)^M} \, .$$

To begin, let us assume that the support of $f$ is far from the zero set of the multiplier $g$. That is, there is some $\delta > 0$ such that

$$\frac{d(z, Z(g))}{\rho(z)} \geq \delta \, .$$

Instead of solving the equation $\overline{\partial} u = f$, we consider the auxiliary equation $\overline{\partial} v = f/g$. We take as a solution $v$ the function provided by Lemma 7 (we take $\alpha = \varepsilon/2$). Then, since $\overline{\partial} g = 0$, the function $u = v\, g$ is a solution to $\overline{\partial} u = f$. Moreover, because of Lemma 7, we know that for any $1 \le p < \infty$

$$\int_{\mathbb{D}} \frac{|u(z)/g(z)|^p}{(1-|z|)} (1-|z|)^{\varepsilon/2}\, dm(z) \lesssim \int_{\mathbb{D}} \frac{|f(z)/g(z)\,(1-|z|)|^p}{(1-|z|)} (1-|z|)^{\varepsilon/2}\, dm(z)\,.$$

We always have that $|g| \lesssim e^{\psi}$, thus

$$\int_{\mathbb{D}} \frac{|u(z)|^p}{(1-|z|)} e^{-\phi(z)}\, dm(z) \lesssim \int_{\mathbb{D}} \frac{|u(z)/g(z)|^p}{(1-|z|)} (1-|z|)^{\varepsilon/2}\, dm(z)\,,$$

and since the support of $f$ is far from the zero sets of $g$, then

$$\int_{\mathbb{D}} \frac{|f(z)/g(z)\,(1-|z|)|^p}{(1-|z|)} (1-|z|)^{\varepsilon/2}\, dm(z) \simeq \int_{\mathbb{D}} \frac{|f(z)\,(1-|z|)|^p}{(1-|z|)} e^{-\phi(z)}\, dm(z)\,.$$

The case $p = \infty$ follows with the same scheme.

Now, we must overcome the restriction on the support of $f$. We denote as above $\psi = \phi - \varepsilon/2 \log(1-|z|^2)$. For this subharmonic function we take the set of multipliers $h_i$ given by Proposition 2 and its corresponding exceptional sets $E_{h_i}$.

We split the domain into disjoint pieces

$$\Omega = \big(\Omega \setminus E_{h_1}\big) \cup \big(E_{h_1} \setminus E_{h_2}\big) \cup \big((E_{h_1} \cap E_{h_2}) \setminus E_{h_3}\big) \cup \cdots \cup \big((E_{h_1} \cap \cdots \cap E_{h_{n-1}}) \setminus E_{h_n}\big)\,.$$

For the sake of simplicity we denote this partition by $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$. In each $\Omega_i$ the multiplier $|h_i| \simeq e^{\psi}$. We can take as a solution to the equation $\overline{\partial} u = f$ the function

$$\begin{aligned} u(z) &= \int_{\mathbb{D}} \Big( \sum_{i=1}^{n} \frac{h_i(z)\, \chi_{\Omega_i(\zeta)}}{h_i(\zeta)} \Big) \frac{1}{\pi}\, \frac{1-|\zeta|^2}{1-\overline{\zeta} z}\, \frac{f(\zeta)}{z-\zeta}\, dm(\zeta) \\ &= \int_{\mathbb{D}} \kappa(z,\zeta)\, f(\zeta)\, dm(\zeta)\,. \end{aligned}$$

Thus,

$$|\kappa(z,\zeta)| \simeq \frac{(1-|\zeta|^2)}{|1-\overline{\zeta}z|\,|\zeta-z|}\,\frac{(1-|\zeta|)^{\varepsilon/2}\,e^{\phi(z)}}{(1-|z|)^{\varepsilon/2}\,e^{\phi(\zeta)}}\,.$$

From this estimate the $L^p$ boundedness of the solution follows. This proves Theorem 2.

The same construction proves Theorem 1. We replace Lemma 7 by the following one which is also a direct consequence of Hölder's inequality:

**Lemma 8.** *For any $\alpha > 0$ and $p \in [1,+\infty]$, the solution*

$$u(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{e^{2\alpha(\overline{\zeta}z-|\zeta|^2)}}{z-\zeta}\,f(\zeta)\,dm(\zeta)$$

*to the equation $\overline{\partial}u = f$ in $\mathbb{C}$ satisfies the estimate $\|u(z)\,e^{-\alpha|z|^2}\|_p \lesssim \|f(\zeta)\,e^{-\alpha|\zeta|^2}\|_p$ for any $p \in [1,\infty]$.*

In this case the auxiliary subharmonic function $\psi$ is $\phi - \varepsilon/2|z|^2$. We take as a solution to the $\overline{\partial}$ equation the function

$$\int_{\mathbb{C}} \Big(\sum_{i=1}^{n} \frac{h_i(z)\chi_{\Omega_i(\zeta)}}{h_i(\zeta)}\Big)\frac{1}{\pi}\,\frac{e^{2\varepsilon(\overline{\zeta}z-|\zeta|^2)}}{z-\zeta}\,f(\zeta)\,dm(\zeta) = \int_{\mathbb{C}} \kappa'(z,\zeta)\,f(\zeta)\,dm(\zeta)\,.$$

Therefore,

$$|\kappa'(z,\zeta)| \simeq \frac{e^{\phi(z)}\,e^{-\varepsilon|z-\zeta|^2}}{e^{\phi(\zeta)}\,|z-\zeta|}\,.$$

This estimate proves Proposition 1 and Theorem 1.

## 4.3. The degenerate weight.

We can prove this $\overline{\partial}$ estimate along the same lines. We need two ingredients, a multiplier theorem and some $\overline{\partial}$ estimates when the weight $\phi$ is of the form $\alpha\,|\mathrm{Im}\,z|$ for some $\alpha > 0$. This is the multiplier theorem that we need:

**Theorem 5.** *Let $\phi$ be a subharmonic function in $\mathbb{C}$ such that the measure $\Delta\phi$ is a locally doubling measure supported in the real line and $\Delta\phi(I(x,r)) > 1$ for some $r > 0$ where $I(x,r)$ is the interval in $\mathbb{R}$ of center $x$ and radius*

*$r$. There is a holomorphic function $f$ with zero set $\Lambda$ contained in $\mathbb{R}$ such that for any $\varepsilon > 0$, $|f(z)| \simeq e^{\phi(z)}$, for all $z$ such that $|z - \lambda_n| \geq \varepsilon\, \rho(\lambda_n)$ for all $\lambda_n \in Z(f)$.*

**Proof.** The proof of this theorem is the same as in Theorem 4 when $\Omega = \mathbb{C}$, except that at some points it is easier. For instance, it is trivial to split the real line into intervals all of mass $N$.

On the other hand the $\overline{\partial}$-estimate that we need in the flat case, *i.e.* when $\phi = \alpha\,|\mathrm{Im}\, z|$, is not as easy as in the disk or the plane; we need the following theorem, a proof of which can be found in [OCS99]:

**Theorem.** *Consider the equation $\overline{\partial} u = \mu$, where $\mu$ is a compactly supported measure such that $e^{-\alpha|\mathrm{Im}\, z|}\, d|\mu|$ is a two-sided Carleson measure for some $\alpha > 0$. Then there is a solution $u$ with*

$$\limsup_{z\to\infty} |u(z)|\, e^{-\alpha|\mathrm{Im}\, z|} = 0 \qquad \textit{and} \qquad |u(x)| \leq C\Big(1 + \int_{|z-x|<1} \frac{d|\mu|(z)}{|x-z|}\Big),$$

*for any $x \in \mathbb{R}$, where $C$ only depends on the Carleson constant of*

$$e^{-\alpha|\mathrm{Im}\, z|}\, d|\mu|\,.$$

These two ingredients together prove Theorem 3 in the same way as we proved Theorem 1 and Theorem 2.

**Added in proof.** Richard Rochberg has informed me of un unpublished manuscript of Tom Wolff from 1988 where some of the ideas concerning the multiplier (Theorem 3) are already present.

## References.


[And85] Andersson, M., Solution formulas for the $\partial\overline{\partial}$-equation and weighted Nevanlinna classes in the polydisc. *Bull. Sci. Math.* **109** (1985), 135-154.

[BC99] Berndtsson, B., Charpentier, P., A Sobolev mapping property of the Bergman kernel. *Math. Z.* **235** (2000), 401-413.

[Ber94] Berndtsson, B., *Some recent results on estimates for the $\overline{\partial}$ equation.* Contributions to Complex Analysis and Analytic Geometry (Braunschweig) (H. Skoda and J.-M. Trepreau, eds.). Vieweg, 1994, 27-42.

[BOC95] Berndtsson, B., Ortega Cerdà, J., On interpolation and sampling in Hilbert spaces of analytic functions. *J. Reine Angew. Math.* **464** (1995), 109-128.

[Chr91] Christ, M., On the $\overline{\partial}$ equation in weighted $L^2$ norms in $\mathbb{C}^1$. *J. Geom. Anal.* **1** (1991), 193-230.

[DF83] Donnelly, H., Fefferman, C., $L^2$-cohomology and index theorem for the Bergman metric. *Ann. of Math.* **118** (1983), 593-618.

[FS91] Fornæss, J. E., Sibony, N., *On $L^p$ estimates for $\overline{\partial}$.* Several complex variables and complex geometry, Part 3. Amer. Math. Soc., 1991, 129-163.

[Gau76] Gautschi, W., *Advances in Chebyshev quadrature.* Numerical analysis (Proc. 6th Biennial Dundee Conf. Univ. Dundee, 1975). Lecture Notes in Math. **506** (1976), 100-121. Springer.

[Hör90] Hörmander, L., *An introduction to complex analysis in several variables.* North-Holland, 1990.

[Kor94] Korevaar, J., *Chebyshev-type quadratures: use of complex analysis and potential theory.* Complex potential theory (Montreal, PQ, 1993). Kluwer Acad. Publ., 1994. Notes by Arno B. J. Kuijlaars, 325-364.

[LM99] Lyubarskiĭ, Y. I., Malinnikova, E., On approximation of subharmonic functions. *J. Anal. Math.* **33** (2001), 121-149.

[LS94] Lyubarskiĭ, Y. I., Seip,K., Sampling and interpolation of entire functions and exponential systems in convex domains. *Ark. Mat.* **32** (1994), 157-193.

[OCS98] Ortega-Cerdà, J., Seip, K., Beurling-type density theorems for weighted $L^p$ spaces of entire functions. *J. Anal. Math.* **75** (1998), 247-266.

[OCS99] Ortega-Cerdà, J., Seip, K., Multipliers for entire functions and an interpolation problem of Beurling. *J. Funct. Anal.* **162** (1999), 400-415.

[Pas88] Pascuas, D., Zeros i interpolació en espais de funcions holomorfes del disc unitat. Ph. D. Thesis. Universitat Autònoma de Barcelona, 1988.

[Sei95] Seip, K., On Korenblum's density condition for the zero sequences of $A^{-\alpha}$. *J. Anal. Math.* **67** (1995), 307-322.

[Yul85] Yulmukhametov, R. S., Approximation of subharmonic functions. *Anal. Math.* **11** (1985), 257-282.

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