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Outer and inner vanishing measures and division in $H^\infty + C$

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Abstract

Measures on the unit circle are well studied from the view of Fourier analysis. In this paper, we investigate measures from the view of Poisson integrals and of divisibility of singular inner functions in $H^\infty+C$. Especially, we study singular measures which have outer and inner vanishing measures. It is given two decompositions of a singular positive measure. As applications, it is studied division theorems in $H^\infty + C$.

1. Introduction

Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk D. We denote by $M(H^\infty)$ the maximal ideal space of H^∞, the space of nonzero multiplicative linear functionals of H^∞ with the weak*-topology. We view that $D \subset M(H^\infty)$ and D is an open subset of $M(H^\infty)$. By Carleson's corona theorem [2], D is dense in $M(H^\infty)$. Identifying a function in H^∞ with its Gelfand transform, we view that H^∞ is the closed subalgebra of $C(M(H^\infty))$, the space of continuous functions on $M(H^\infty)$.

We also identify a function in H^∞ with its boundary function and view that H^∞ is an (essentially) supremum norm closed subalgebra of L^∞, the usual Lebesgue space on the unit circle ∂D, see [4, 8, 9] for the study of the structure of H^∞ and $M(H^\infty)$. A closed subalgebra B of L^∞ containing H^∞ strictly is called a Douglas algebra. Then we view that its maximal ideal space $M(B)$ is a subset of $M(H^\infty)$, and $M(L^\infty)$ is the Shilov boundary of H^∞, see [3, 19] for the structure of Douglas algebras. In [20], Sarason proved that the smallest Douglas algebra is $H^\infty + C$, where C is the space

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of continuous functions on ∂D, and $M(H^\infty + C) = M(H^\infty) \setminus D$. For $f \in L^\infty \setminus H^\infty$, we denote by $H^\infty[f]$ the Douglas algebra generated by f. For a function f in H^∞, we put

$$\{|f| < 1\} = \{x \in M(H^\infty + C); |f(x)| < 1\}$$

and

$$Z(f) = \{x \in M(H^\infty + C); f(x) = 0\}.$$

We note that these sets are considered in $M(H^\infty) \setminus D$. A function f in H^∞ is called inner if $|f| = 1$ on $M(L^\infty)$. For a sequence $\{z_n\}_n$ in D satisfying $\sum_{n=1}^\infty 1 - |z_n| < \infty$, we have a Blaschke product

$$b(z) = \prod_{n=1}^\infty \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z}, \quad z \in D.$$

A Blaschke product is an inner function.

For a measurable subset E of ∂D, we denote by $|E|$ the value of the Lebesgue measure of E. Let $M(\partial D)$ be the Banach space of bounded Borel measures on ∂D with the total variation norm. Let M_s^+ be the set of positive singular measures in $M(\partial D)$ with respect to the Lebesgue measure on ∂D. We denote by $M_{s,c}^+$ and $M_{s,d}^+$ the sets of continuous and discrete measures in M_s^+, respectively. We use familiar notations in the measure theory like as "$\ll$" absolutely continuous and "$\perp$" mutually singular. For a finite signed measure μ, let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. For $\mu_1, \mu_2 \in M_s^+$, put $\mu_1 \vee \mu_2 = \mu_1 + (\mu_2 - \mu_1)^+$ and $\mu_1 \wedge \mu_2 = \mu_1 - (\mu_1 - \mu_2)^+$. Then $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are the least upper and the greatest lower bounds of μ_1 and μ_2, respectively. It is known that M_s^+ is a complete lattice. For a point $\zeta \in \partial D$, let δ_ζ be the unit point mass at ζ. For $\mu \in M_s^+$, we denote by $S(\mu)$ the closed support set of μ.

For each $\mu \in M_s^+$, let

$$\psi_\mu(z) = \exp\left(- \int_{\partial D} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(e^{i\theta}) \right), \quad z \in D.$$

Then ψ_μ is inner and called a singular inner function. We note that

$$- \log |\psi_\mu(z)| = \int_{\partial D} P_z(e^{i\theta}) \, d\mu(e^{i\theta}), \quad z \in D,$$

where P_z is the Poisson kernel, that is, $P_z(e^{i\theta}) = (1 - |z|^2)/|e^{i\theta} - z|^2$. Let

$$L^1_+(\mu) = \{\nu \in M_s^+; 0 \le \nu \ll \mu, \nu \ne 0\}.$$

Then we have a family of singular inner functions $\{\psi_\nu; \nu \in L^1_+(\mu)\}$ associated with μ. In [17], we call these functions singular inner functions of L^1-type for the measure μ, and we obtained results which are reminiscent of the results for Blaschke products in [13], see also [14, 15].

Let $\mu \in M_s^+$. We say that μ has an *outer vanishing measure* if there exists $\nu \in L^1_+(\mu)$ such that $\{|\psi_\mu| < 1\} \subset Z(\psi_\nu)$, and ν is called an outer vanishing measure for μ. In this case, $\psi_\nu/\psi_\mu^n \in H^\infty + C$ for every positive integer n, see [7]. In [17, Theorem 2.2], it is actually proved that if $\mu \in M_s^+$ and $S(\mu) = \partial D$, then μ has an outer vanishing measure and L^∞ is generated by complex conjugate of singular inner functions of L^1-type for μ. Also in [17, Theorem 5.1], we give a characterization of $\mu \in M_{s,d}^+$ which has an outer vanishing measure. In [7, p. 181], Guillory and Sarason posed the following problem. Does there exist $\mu, \nu \in M_s^+, \mu \neq \nu$, such that ψ_μ and ψ_ν are codivisible in $H^\infty + C$? This problem is very interesting and remains unsolved.

In Section 2, we study measures which have outer vanishing measures. In Theorem 2.3, we prove that every $\mu \in M_s^+$ has a decomposition $\mu = \mu_a + \mu_b$, where μ_a has an outer vanishing measure, $\mu_a \perp \mu_b$, and there are no nonzero measures $\lambda \in L^1_+(\mu_b)$ which have outer vanishing measures.

In Section 3, we prove that if $\mu \in M_s^+$ and there is $\nu \in M_s^+$ such that $\mu \perp \nu$, and ψ_μ and ψ_ν are codivisible in $H^\infty + C$, then μ has an outer vanishing measure. So to attack Guillory and Sarason's problem, it is important to study measures which have outer vanishing measures.

In Section 4, we give some examples of measures. In Theorem 4.1, we prove the existence of $\mu \in M_{s,c}^+$ which does not have outer vanishing measures. This answers the problem posed in [17, Problem 5.1] negatively. Also in Theorem 4.2, we prove the existence of $\mu \in M_{s,c}^+$ which has an outer vanishing measure and $|S(\mu)| = 0$.

In Section 5, we study factorization in $H^\infty + C$. There are many factorization theorems in $H^\infty + C$, see [1, 6, 12, 21]. Let $f \in H^\infty + C$ and ψ be an inner function. In [7], Guillory and Sarason proved that $\{|\psi| < 1\} \subset Z(f)$ if and only if $f/\psi^n \in H^\infty + C$ for every positive integer n. In [16], it is proved that if $|f| \le |\psi|$ on $M(H^\infty + C)$, then $f^2/\psi \in H^\infty + C$. Let b be a Blaschke product. In [11], the author proved that for every $f \in H^\infty + C$ with $|f| \le |b|$ on $M(H^\infty + C)$, there exists a subproduct ψ of b such that $f/\psi \in H^\infty + C$ and $Z(\psi) = Z(b)$, and posed the problem whether there exists a subproduct ψ of b such that $f/\psi \in H^\infty + C$ and $|\psi| = |b|$ on $M(H^\infty + C)$. We study the same type of problem for a singular inner function.

For $\mu \in M_s^+$, we consider the following two conditions on μ, respectively;

(A) for every $f \in H^\infty + C$ satisfying $|f| \le |\psi_\mu|$ on $M(H^\infty + C)$, there exists $\nu \in M_s^+$ such that $\nu \le \mu, |\psi_\nu| = |\psi_\mu|$ on $M(H^\infty + C)$, and $f/\psi_\nu \in H^\infty + C$,

(B) for every inner function ψ satisfying $|\psi_\mu| \le |\psi|$ on $M(H^\infty + C)$, there exists $\nu \in L^1_+(\mu)$ such that $\mu \le \nu \le 2\mu, |\psi_\nu| = |\psi_\mu|$ on $M(H^\infty + C)$, and $\psi_\nu/\psi \in H^\infty + C$.

A measure $\nu \in L^1_+(\mu)$ with $0 \le \nu \le \mu$ is called an *inner vanishing measure* for μ if $\{|\psi_\nu| < 1\} \subset Z(\psi_\mu)$. Let μ_α be the upper band of inner vanishing measures for μ. Put $\mu_\beta = \mu - \mu_\alpha$. Then we have $\mu_\alpha \perp \mu_\beta$. In Theorem 5.1, we prove that conditions (A) and (B) are equivalent to the condition $Z(\psi_{\mu_\alpha}) = Z(\psi_\mu)$.

2. Outer vanishing measures

For $\mu, \nu \in M_s^+$, we have $|\psi_{(a\mu+b\nu)}| = |\psi_\mu|^a|\psi_\nu|^b, \{|\psi_{(a\mu+b\nu)}| < 1\} = \{|\psi_\mu| < 1\} \cup \{|\psi_\nu| < 1\}$, and $Z(\psi_{(a\mu+b\nu)}) = Z(\psi_\mu) \cup Z(\psi_\nu)$ for every positive numbers a, b. We also have $\{|\psi_{\mu\vee\nu}| < 1\} = \{|\psi_\mu| < 1\} \cup \{|\psi_\nu| < 1\}$. For measures μ, ν in M_s^+ such that $\nu \le \mu$, we have $|\psi_\mu| \le |\psi_\nu|$ on $M(H^\infty)$ and $Z(\psi_\nu) \subset Z(\psi_\mu)$. We use these facts frequently without mention.

The following theorem gives a sufficient condition on $\mu \in M_s^+$ which has an outer vanishing measure.

Theorem 2.1 *Let $\mu \in M_s^+$ and let $\{\mu_n\}_n$ be a sequence in M_s^+ such that $\mu = \sum_{n=1}^\infty \mu_n$. Let $\{\nu_n\}_n$ be a sequence in M_s^+ satisfying the following conditions;*

(i) $\nu_n \in L^1_+(\mu)$ *for every n,*

(ii) $\{|\psi_{\mu_n}| < 1\} \subset Z(\psi_{\nu_n})$ *for every n.*

Then μ has an outer vanishing measure.

To prove our theorem, we need some lemmas.

A Blaschke product with zeros $\{z_n\}_n$ is called interpolating if for every bounded sequence of complex numbers $\{a_n\}_n$ there exists $f \in H^\infty$ such that $f(z_n) = a_n$ for every n. The following is proved essentially in [22], see also [18].

Lemma 2.1 *Let ψ be an inner function. Then there is an interpolating Blaschke product b such that $\{|b| < 1\} = \{|\psi| < 1\}$.*

The following lemma is proved in [7, p. 176].

Lemma 2.2 *Let $f \in H^\infty + C$ and let ψ be an inner function. Then $\{|\psi| < 1\} \subset Z(f)$ if and only if $f/\psi^n \in H^\infty + C$ for every positive integer n. And in this case, $|f/\psi| = |f|$ on $M(H^\infty + C)$.*

For $x, y \in M(H^\infty)$, let $\rho(x,y) = \sup\{|f(y)|; f \in H^\infty, f(x) = 0, \|f\| \leq 1\}$. Put $P(x) = \{\zeta \in M(H^\infty); \rho(x,\zeta) < 1\}$. When $P(x) \neq \{x\}$, $P(x)$ is called a non-trivial Gleason part. In [9], Hoffman proved that if $P(x)$ is non-trivial, then there exists a continuous one to one map L_x from D onto $P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. Also he proved that $P(x) \neq \{x\}$ if and only if $b(x) = 0$ for some interpolating Blaschke product b.

Lemma 2.3 *Let b be an interpolating Blaschke product with zeros $\{z_n\}_n$. Let $\mu \in M_s^+$ such that $\psi_\mu(z_n) \to 0$ as $n \to \infty$. Then $\{|b| < 1\} \subset Z(\psi_\mu)$, $\psi_\mu/b \in H^\infty + C$, and $|\psi_\mu/b| = |\psi_\mu|$ on $M(H^\infty + C)$.*

Proof. Since $Z(b) = cl\,\{z_n\}_n \setminus \{z_n\}_n$ [8, p. 205], where $cl\,\{z_n\}_n$ is the closure of $\{z_n\}_n$ in $M(H^\infty)$, we have $\psi_\mu = 0$ on $Z(b)$. Since $(\psi_\mu)^{1/k} = \psi_{\mu/k} = 0$ on $Z(b)$ for every positive integer k, $\psi_\mu = 0$ on $P(x)$ for every $x \in Z(b)$. Then by [1, 6], we have $\psi_\mu/b^n \in H^\infty + C$ for every positive integer n. By Lemma 2.2, we have our assertion. ■

Proof of Theorem 2.1. Since $\sum_{n=1}^\infty \|\mu_n\| = \|\mu\| < \infty$, there exists a sequence of positive numbers $\{p_n\}_n$ such that $\sum_{n=1}^\infty p_n\|\mu_n\| < \infty$,

$$p_n \leq p_{n+1} \quad \text{for every } n, \tag{2.1}$$

and $p_n \to \infty$ as $n \to \infty$. Since $\{a_n\nu_n\}_n$ also satisfies conditions (i) and (ii) in Theorem 2.1 for every sequence of positive numbers $\{a_n\}_n$, we may assume that $\sum_{n=1}^\infty \|\nu_n\| < \infty$. Put

$$\nu = \sum_{n=1}^\infty (p_n\mu_n + \nu_n). \tag{2.2}$$

Then by (i), $\nu \in L^1_+(\mu)$. We shall prove that $\{|\psi_\mu| < 1\} \subset Z(\psi_\nu)$. By Lemma 2.1, there is an interpolating Blaschke product b with zeros $\{z_k\}_k$ such that $\{|b| < 1\} = \{|\psi_\mu| < 1\}$. Then

$$A = \sup_k |\psi_\mu(z_k)| < 1. \tag{2.3}$$

By Lemma 2.3, it is sufficient to show that

$$\psi_\nu(z_k) \to 0 \quad \text{as } k \to \infty. \tag{2.4}$$

To prove this, suppose not. Then there exist a positive number a and a subsequence $\{z_{k_j}\}_j$ of $\{z_k\}_k$ such that

$$|\psi_\nu(z_{k_j})| \to a > 0 \quad \text{as } j \to \infty. \tag{2.5}$$

Here we have

$$|\psi_{\mu_n}(z_{k_j})| \to 1 \quad \text{as } j \to \infty \text{ for every } n. \tag{2.6}$$

To prove this, suppose not. Considering further subsequence, we may assume that $|\psi_{\mu_n}(z_{k_j})| < r$ for every j for some positive integer n and $0 < r < 1$. By condition (ii), we have $\psi_{\nu_n}(z_{k_j}) \to 0$ as $j \to \infty$. Hence by (2.2),

$$\limsup_{j\to\infty} |\psi_\nu(z_{k_j})| \le \limsup_{j\to\infty} |\psi_{\nu_n}(z_{k_j})| = 0.$$

This contradicts (2.5). Thus we get (2.6).

Now for each positive integer N, we have

$$\begin{aligned} \limsup_{j\to\infty} |\psi_\nu(z_{k_j})| &\le \limsup_{j\to\infty} \prod_{n=1}^{\infty} |\psi_{\mu_n}(z_{k_j})|^{p_n} \quad \text{by (2.2)} \\ &\le \limsup_{j\to\infty} \prod_{n=N}^{\infty} |\psi_{\mu_n}(z_{k_j})|^{p_N} \quad \text{by (2.1) and (2.6)} \\ &= \limsup_{j\to\infty} |\psi_\mu(z_{k_j})|^{p_N} \quad \text{by (2.6) and } \mu = \sum_{n=1}^{\infty} \mu_n \\ &\le A^{p_N} \quad \text{by (2.3).} \end{aligned}$$

Since $0 \le A < 1$ and $p_n \to \infty$ as $n \to \infty$, we obtain $\psi_\nu(z_{k_j}) \to 0$ as $j \to \infty$. This contradicts (2.5). Thus we get (2.4). This completes the proof. ■

Corollary 2.1 *Let $\mu \in M_s^+$ and let $\{\mu_n\}_n$ be a sequence in M_s^+ such that $\mu = \sum_{n=1}^{\infty} \mu_n$. If μ_n has an outer vanishing measure for every n, then μ has an outer vanishing measure.*

For a subset E of the complex plane, we denote by $\overline{E}$ the closure of E. In [17, Proposition 5.1], it is proved that if $\mu \in M_{s,c}^+$ and $S(\mu) = \overline{J}$ for an open subarc J of ∂D, then μ has an outer vanishing measure. For a subset E of ∂D, we denote by $int\, E$ the interior of E.

Corollary 2.2 *Let $\mu \in M_s^+$ such that $\|\mu\| = \mu(int\, S(\mu))$. Then μ has an outer vanishing measure.*

Proof. There is a set of countably many disjoint open subarcs $\{J_n\}_n$ of ∂D such that $int\, S(\mu) = \bigcup_{n=1}^{\infty} J_n$. Put $\mu_n = \mu_{|J_n}$. Then $\mu = \sum_{n=1}^{\infty} \mu_n$, $S(\mu_n) = \overline{J}_n$, and $\|\mu_n\| = \mu_n(J_n)$. By the fact mentioned above, μ_n has an outer vanishing measure. By Corollary 2.1, we get our assertion. ■

Corollary 2.3 *Suppose that $\mu \in M_s^+$ has an outer vanishing measure. Let $\lambda \in M_s^+$ such that $\mu \ll \lambda \ll \mu$. Then λ has an outer vanishing measure.*

Proof. By our assumption, there exists $\nu \in L^1_+(\mu)$ such that $\{|\psi_\mu| < 1\} \subset Z(\psi_\nu)$. We have $\nu \in L^1_+(\mu) = L^1_+(\lambda)$. By Radon-Nikodym's theorem, $d\lambda = f d\mu$, where $f > 0$ a.e. $d\mu$. Put

$$E_n = \{e^{i\theta} \in \partial D; n-1 < f(e^{i\theta}) \le n\} \quad \text{and} \quad \lambda_n = \lambda_{|E_n}$$

for every positive integer n. Then $\lambda_n \le n\mu$, so that we have

$$\{|\psi_{\lambda_n}| < 1\} \subset \{|\psi_{n\mu}| < 1\} = \{|\psi_\mu| < 1\} \subset Z(\psi_\nu).$$

Since $\lambda = \sum_{n=1}^{\infty} \lambda_n$, by Theorem 2.1 we have our assertion. ■

Corollary 2.4 *Let $\{\mu_n\}_n$ be a sequence in M_s^+ such that μ_n has an outer vanishing measure for every n. If $\| \vee_{n=1}^{\infty} \mu_n\| < \infty$, then $\vee_{n=1}^{\infty}\mu_n$ has an outer vanishing measure.*

Proof. There exists a sequence of positive numbers $\{a_n\}_n$ such that $\mu = \sum_{n=1}^{\infty} a_n\mu_n \in M_s^+$. Since $a_n\mu_n$ has an outer vanishing measure, by Corollary 2.1 μ has an outer vanishing measure. Since $\mu \ll \vee_{n=1}^{\infty}\mu_n \ll \mu$, by Corollary 2.3 $\vee_{n=1}^{\infty}\mu_n$ has an outer vanishing measure. ■

For a closed subset E of ∂D, put

$$M_E(H^\infty + C) = \{x \in M(H^\infty + C); z(x) \in E\},$$

where z is the identity function on D. Let $\mu \in M_s^+$. If $S(\mu) \subset E$, then $\{|\psi_\mu| < 1\} \subset M_E(H^\infty + C)$ and $|\psi_\mu| = 1$ on $M(H^\infty + C) \setminus M_E(H^\infty + C)$, see Hoffman's book [8].

Theorem 2.2 *Let $\mu \in M_s^+$. Then the following conditions are equivalent.*

(i) μ has an outer vanishing measure.

(ii) For each open subset U of ∂D such that $U \cap S(\mu) \neq \emptyset$, $\mu_{|U}$ has an outer vanishing measure.

(iii) For every $\zeta \in S(\mu)$, there exists an open neighborhood V_ζ of ζ in ∂D such that $\mu_{|V_\zeta}$ has an outer vanishing measure.

Proof. (i) ⇒ (ii) Suppose that μ has an outer vanishing measure. Then there exists $\nu \in L^1_+(\mu)$ such that

$$\{|\psi_\mu| < 1\} \subset Z(\psi_\nu). \tag{2.7}$$

Let U be an open subset of ∂D such that $U \cap S(\mu) \neq \emptyset$. Then $\nu_{|U} \in L^1_+(\mu_{|U})$. Take a sequence of increasing closed subsets $\{E_n\}_n$ of ∂D such that $\bigcup_{n=1}^\infty E_n = U$. Put $\mu_1 = \mu_{|E_1}$ and $\mu_n = \mu_{|(E_n \setminus E_{n-1})}$ for $n \geq 2$.

Then $\mu_{|U} = \sum_{n=1}^\infty \mu_n$ and

$$\{|\psi_{\mu_n}| < 1\} \subset \{|\psi_\mu| < 1\} \cap M_{E_n}(H^\infty + C). \tag{2.8}$$

Since $U^c \cap E_n = \emptyset$, $|\psi_{\nu_{|U^c}}| = 1$ on $M_{E_n}(H^\infty + C)$. Since $\psi_\nu = \psi_{\nu_{|U}}\psi_{\nu_{|U^c}}$, we have

$$|\psi_{\nu_{|U}}| = |\psi_\nu| \quad \text{on} \quad M_{E_n}(H^\infty + C). \tag{2.9}$$

Then

$$\begin{aligned} \{|\psi_{\mu_n}| < 1\} &\subset \{|\psi_\mu| < 1\} \cap M_{E_n}(H^\infty + C) && \text{by (2.8)} \\ &\subset Z(\psi_\nu) \cap M_{E_n}(H^\infty + C) && \text{by (2.7)} \\ &= Z(\psi_{\nu|U}) \cap M_{E_n}(H^\infty + C) && \text{by (2.9)} \\ &\subset Z(\psi_{\nu|U}). \end{aligned}$$

Hence by Theorem 2.1, we have our assertion.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i) By (iii), there exist $\zeta_1, \zeta_2, \ldots, \zeta_n \in S(\mu)$ and open subsets $V_{\zeta_1}, V_{\zeta_2}, \ldots, V_{\zeta_n}$ of ∂D such that $S(\mu) \subset \bigcup_{j=1}^n V_{\zeta_j}$, $\zeta_j \in V_{\zeta_j}$, and $\mu_{|V_{\zeta_j}}$ has an outer vanishing measure for every j. Put $\sigma = \sum_{j=1}^n \mu_{|V_{\zeta_j}}$. Then $\sigma \ll \mu \ll \sigma$. By Corollaries 2.1 and 2.3, we have our assertion. ■

Theorem 2.3 *Let $\mu \in M_s^+$. Then μ has a unique decomposition $\mu = \mu_a + \mu_b$, where μ_a and μ_b satisfy the following conditions.*

(i) $\mu_a, \mu_b \in M_s^+$ *and* $\mu_a \perp \mu_b$.

(ii) μ_a *has an outer vanishing measure.*

(iii) Let $\nu \in M_s^+$ such that $0 \leq \nu \leq \mu$. If ν has an outer vanishing measure, then $\nu \leq \mu_a$.

(iv) There are no nonzero measures $\lambda \in L^1_+(\mu_b)$ which have outer vanishing measures.

Proof. Let Ω be the set of measures ν such that $0 \leq \nu \leq \mu$ and ν has an outer vanishing measure.

First, suppose that $\Omega = \emptyset$. Put $\mu_a = 0$ and $\mu_b = \mu$. If there is a measure $\lambda \in L^1_+(\mu), \lambda \neq 0$, which has an outer vanishing measure, by Corollary 2.3 we have $\Omega \neq \emptyset$. Hence μ_a and μ_b satisfies our conditions.

Next, suppose that $\Omega \neq \emptyset$. Put

$$\alpha = \sup\{\|\nu\|; \nu \in \Omega\}. \tag{2.10}$$

Then $0 < \alpha \leq \|\mu\|$, and there is a sequence $\{\nu_n\}_n$ in Ω such that $\|\nu_n\| \to \alpha$ as $n \to \infty$. Put $\mu_a = \vee_{n=1}^{\infty} \nu_n$. Then we have $\|\mu_a\| = \alpha$ and $\mu_a \leq \mu$. Since ν_n has an outer vanishing measure, by Corollary 2.4 μ_a has an outer vanishing measure. Put $\mu_b = \mu - \mu_a$.

To prove (i), suppose not. Then $\mu_a \wedge \mu_b \neq 0$. By Corollary 2.3, $\mu_a + \mu_a \wedge \mu_b \in \Omega$ and $\|\mu_a + \mu_a \wedge \mu_b\| > \|\mu_a\| = \alpha$. This contradicts (2.10).

(iii) By Corollary 2.4, $\mu_a \vee \nu$ has an outer vanishing measure. If $\nu \not\leq \mu_a$, then $\|\mu_a \vee \nu\| > \|\mu_a\| = \alpha$. Since $\mu_a \vee \nu \leq \mu$, this contradicts (2.10).

(iv) follows from (iii). ∎

3. Codivisibility of singular inner functions

Up to now, Guillory and Sarason's problem [7] is still open, that is, it is not known the existence of measures $\mu, \nu \in M_s^+$ such that $\psi_\mu/\psi_\nu, \psi_\nu/\psi_\mu \in H^\infty + C$, and $\mu \neq \nu$.

Suppose, for a while, that there exist $\mu, \nu \in M_s^+$ such that ψ_μ and ψ_ν are codivisible in $H^\infty + C$ and $\mu \neq \nu$. Then $\psi_{\mu - \mu\wedge\nu}$ and $\psi_{\nu - \mu\wedge\nu}$ are codivisible in $H^\infty + C$. Hence moreover we may assume that $\mu \perp \nu$. By the codivisibility, we have $|\psi_\mu| = |\psi_\nu|$ on $M(H^\infty + C)$, so that $\{|\psi_\mu| < 1\} = \{|\psi_\nu| < 1\}$ and $S(\mu) = S(\nu)$.

It is also not known the existence of $\mu, \nu \in M_s^+$ such that $\{|\psi_\mu| < 1\} = \{|\psi_\nu| < 1\}$ and $\mu \perp \nu$. But we have the following.

Proposition 3.1 *There exist $\mu, \nu \in M_s^+$ such that $\{|\psi_\mu| < 1\} \subset \{|\psi_\nu| < 1\}$, $\mu \perp \nu$, and $S(\mu) = S(\nu)$.*

Proof. Let $\mu, \lambda \in M_s^+$ such that $\mu \perp \lambda$ and $S(\mu) = S(\lambda) = \partial D$. By Lemma 2.1, there is an interpolating Blaschke product b such that $\{|b| < 1\} = \{|\psi_\mu| < 1\}$. By [17, Theorem 2.1], there exists $\nu \in L^1_+(\lambda)$ such that $\{|b| < 1\} \subset Z(\psi_\nu)$. It is not difficult to see that μ and ν satisfy our conditions. ∎

The following is the main theorem is this section, and we prove this as an application of Theorem 2.1.

Theorem 3.1 *Let $\mu, \lambda \in M_s^+$ such that $\{|\psi_\mu| < 1\} \subset \{|\psi_\lambda| < 1\}$, $\mu \perp \lambda$, and $S(\mu) = S(\lambda)$. Then μ has an outer vanishing measure.*

Proof. Since $\mu \perp \lambda$, by the regularity of measures there is a sequence of closed subsets $\{E_n\}_n$ of ∂D such that $\{E_n\}_n$ is mutually disjoint, $\mu(E_n) > 0$ for every n, $\mu = \sum_{n=1}^{\infty} \mu_{|E_n}$, and $\lambda(E_n) = 0$ for every n.

Let fix n. Then there is a decreasing sequence of open subsets $\{U_j\}_j$ of ∂D such that

$$E_n = \bigcap_{j=1}^{\infty} U_j, \tag{3.1}$$

$$\lambda(\overline{U}_j \setminus U_j) = \mu(\overline{U}_j \setminus U_j) = 0 \quad \text{for every } j, \tag{3.2}$$

and $\sum_{j=1}^{\infty} \|\lambda_{|U_j}\| < \infty$. Put

$$\sigma = \sum_{j=1}^{\infty} \lambda_{|U_j}. \tag{3.3}$$

Then $\sigma \in M_s^+$ and $S(\sigma) \subset S(\lambda) = S(\mu)$. Moreover we have

$$\{|\psi_{\mu_{|E_n}}| < 1\} \subset Z(\psi_\sigma). \tag{3.4}$$

Let $x \in \{|\psi_{\mu_{|E_n}}| < 1\}$. Then $x \in M_{E_n}(H^\infty + C)$. By (3.1), $|\psi_\lambda| = |\psi_{\lambda_{|U_j}}|$ on $M_{E_n}(H^\infty + C)$. Hence we have

$$|\psi_\lambda(x)| = |\psi_{\lambda_{|U_j}}(x)| \quad \text{for every } j. \tag{3.5}$$

Since $|\psi_\mu(x)| \le |\psi_{\mu_{|E_n}}(x)| < 1$, by our assumption we have $|\psi_\lambda(x)| < 1$.

Hence by (3.3) and (3.5), we get

$$|\psi_\sigma(x)| \le \prod_{j=1}^{t} |\psi_{\lambda_{|U_j}}(x)| = |\psi_\lambda(x)|^t \quad \text{for every } t.$$

Since $|\psi_\lambda(x)| < 1$, by the above we obtain (3.4).

Next, we prove the existence of $\tau_n \in L^1_+(\mu)$ such that

$$\{|\psi_{\mu_{|E_n}}| < 1\} \subset Z(\psi_{\tau_n}). \tag{3.6}$$

By Lemma 2.1, there exists an interpolating Blaschke product b such that

$$\{|b| < 1\} = \{|\psi_{\mu_{|E_n}}| < 1\}. \tag{3.7}$$

Let $\{z_i\}_i$ be the zeros of b in D. Then

$$\overline{\{z_i\}_i} \setminus \{z_i\}_i = S(\mu_{|E_n}) \subset E_n. \tag{3.8}$$

To show (3.6), by (3.7) and Lemma 2.3 it is sufficient to prove

$$\psi_{\tau_n}(z_i) \to 0 \quad \text{as } i \to \infty. \tag{3.9}$$

To prove the existence of $\tau_n \in L^1_+(\mu)$ satisfying (3.9), put

$$\lambda_j = j\lambda_{|U_j \setminus U_{j+1}} \quad \text{and} \quad \mu_j = \mu_{|U_j \setminus U_{j+1}} \quad \text{for every } j. \tag{3.10}$$

Since $\{U_j\}_j$ is decreasing, by (3.3) we have

$$\sigma = \sum_{j=1}^{\infty} \lambda_j. \tag{3.11}$$

Since $S(\lambda) = S(\mu)$, by (3.2) we have $S(\lambda_j) = S(\mu_j)$. Then for each j, there is a sequence of measures

$$\{\mu_{j,k}\}_k \subset L^1_+(\mu_j) \tag{3.12}$$

such that

$$\|\mu_{j,k}\| \leq \|\lambda_j\| \quad \text{for every } k \tag{3.13}$$

and $\mu_{j,k} \to \lambda_j$ as $k \to \infty$ in the weak*-topology of $M(\partial D)$ as the dual space of $C(\partial D)$. Then

$$\psi_{\mu_{j,k}} \to \psi_{\lambda_j} \quad \text{uniformly on each compact subset of } \overline{D} \setminus S(\lambda_j) \text{ as } k \to \infty. \tag{3.14}$$

By (3.1), (3.8), and (3.10), we have $|\psi_{\lambda_j}(z_i)| \to 1$ as $i \to \infty$ for each j, so that by (3.14),

$$|\psi_{\mu_{j,k}}|/|\psi_{\lambda_j}| \to 1 \quad \text{uniformly on } \{z_i\}_i \text{ as } k \to \infty. \tag{3.15}$$

Take a sequence of positive numbers $\{r_j\}_j$ such that $r_j > 1$ for every j and

$$\prod_{j=1}^{\infty} r_j < 2. \tag{3.16}$$

By (3.15), for every j there exists a positive integer k_j such that

$$|\psi_{\mu_{j,k_j}}(z_i)| < r_j|\psi_{\lambda_j}(z_i)| \quad \text{for every } i. \tag{3.17}$$

We put $\tau_n = \sum_{j=1}^{\infty} \mu_{j,k_j}$. Since $\|\sigma\| < \infty$, by (3.11) $\sum_{j=1}^{\infty} \|\lambda_j\| < \infty$. Hence by (3.13), we have $\|\tau_n\| < \infty$ and $\tau_n \in M_s^+$. By (3.10) and (3.12), we have $\tau_n \in L^1_+(\mu)$. For every i, we have

$$\begin{aligned} |\psi_{\tau_n}(z_i)| &= \prod_{j=1}^{\infty} |\psi_{\mu_{j,k_j}}(z_i)| \\ &< \Big(\prod_{j=1}^{\infty} r_j\Big)\Big(\prod_{j=1}^{\infty} |\psi_{\lambda_j}(z_i)|\Big) \quad \text{by (3.17)} \\ &< 2|\psi_\sigma(z_i)| \quad \text{by (3.11) and (3.16).} \end{aligned}$$

By (3.6) and (3.7), $\psi_\sigma(z_i) \to 0$ as $i \to \infty$. Hence by the above, $\psi_{\tau_n}(z_i) \to 0$ as $i \to \infty$. Thus we get (3.9), so that we have (3.6).

Hence we can apply Theorem 2.1 and we get our assertion. ■

In [5], Gorkin proved that for every $\mu \in M_s^+$ there exists $\lambda \in M_s^+$ such that $\{|\psi_\mu| < 1\} \subset Z(\psi_\lambda)$. In this case, we have $S(\mu) \subset S(\lambda)$. Moreover, if μ does not have an outer vanishing measure, then by Theorem 3.1 we have $S(\mu) \neq S(\lambda)$.

Corollary 3.1 *Let $\mu \in M_s^+$. If there exists $\nu \in M_s^+$ such that $\mu \perp \nu$, and ψ_μ and ψ_ν are codivisible in $H^\infty + C$, then μ has an outer vanishing measure.*

4. Examples of measures

The following answers the problem posed in [17, p. 809] negatively.

Theorem 4.1 *There exists $\mu \in M_{s,c}^+$ satisfying the following.*

(i) $\{|\psi_\mu| < 1\} \not\subset Z(\psi_\nu)$ for every $\nu \in M_s^+$ with $S(\nu) \subset S(\mu)$.

(ii) For every $\nu \in L^1_+(\mu)$, ν does not have outer vanishing measures.

(iii) For every $\nu \in L^1_+(\mu)$, $\{|\psi_\nu| < 1\} \not\subset Z(\psi_\mu)$.

(iv) For every $\nu \in M_{s,d}^+$ with $S(\nu) \subset S(\mu)$, ν does not have outer vanishing measures.

To prove the existence of such μ, we need some preparation. The following lemma follows from elementary properties of Poisson kernels.

Lemma 4.1 *Let $J = \{e^{i\theta}; a \le \theta \le b\}, 0 \le a < b \le 1$, and $K \subset \partial D$ be a closed subset such that $J \cap K = \emptyset$. For each $R > 0$ and $\varepsilon > 0$, there exist a positive number $r, 0 < r < 1$, and disjoint closed subarcs $J_1 = \{e^{i\theta}; a \le \theta \le a'\}$, $J_2 = \{e^{i\theta}; b' \le \theta \le b\}, a < a' < b' < b$, of J satisfying the following conditions.*

(i) $P_{\zeta_1}(e^{ia}) = P_{\zeta_1}(e^{ia'}) = R, \quad$ *where* $\zeta_1 = re^{i(a+a')/2}$.

(ii) $P_{\zeta_1}(e^{i\theta}) < \varepsilon \quad$ *for* $e^{i\theta} \in K \cup J_2$.

(iii) $P_{\zeta_2}(e^{ib}) = P_{\zeta_2}(e^{ib'}) = R, \quad$ *where* $\zeta_2 = re^{i(b+b')/2}$.

(iv) $P_{\zeta_2}(e^{i\theta}) < \varepsilon \quad$ *for* $e^{i\theta} \in K \cup J_1$.

(v) $|J_1| = |J_2| \le |J|/4$.

Moreover, we may take a' and b' such that both $a' - a$ and $b - b'$ are sufficiently small.

Put

$$\Lambda_n = \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n); \varepsilon_i = 0 \text{ or } 1 \text{ for every } i\} \quad \text{and} \quad \Lambda = \bigcup_{n=1}^{\infty} \Lambda_n.$$

Lemma 4.2 *Let $J = \{e^{i\theta}; a \le \theta \le b\}, 0 \le a < b \le 1$. Then there exist a family of points $\{z_\alpha\}_{\alpha \in \Lambda}$ in D and closed arcs $\{J_\alpha\}_{\alpha \in \Lambda}$, say $J_\alpha = \{e^{i\theta}; a_\alpha \le \theta \le b_\alpha\}, a_\alpha < b_\alpha$, satisfying the following conditions for every n.*

(i) $J_\alpha \subset J \quad$ *for* $\alpha \in \Lambda_n$.

(ii) $J_\alpha \cap J_\beta = \emptyset \quad$ *for* $\alpha, \beta \in \Lambda_n, \alpha \ne \beta$.

(iii) $P_{z_\alpha}(e^{ia_\alpha}) = P_{z_\alpha}(e^{ib_\alpha}) = 2^n \quad$ *for* $\alpha \in \Lambda_n$.

(iv) $J_{(\alpha,0)} \cup J_{(\alpha,1)} \subset \{e^{i\theta} \in J_\alpha; 2^n \le P_{z_\alpha}(e^{i\theta}) \le 2^{n+1}\} \quad$ *for* $\alpha \in \Lambda_n$.

(v) $P_{z_\alpha}(e^{i\theta}) < 1/2^n \quad$ *for* $e^{i\theta} \in \bigcup_{\beta \in \Lambda_n, \beta \ne \alpha} J_\beta$.

(vi) $\sum_{\alpha \in \Lambda_{n+1}} |J_\alpha| \le \sum_{\alpha \in \Lambda_n} |J_\alpha|/2$.

(vii) $|J_{(\alpha,0)}| = |J_{(\alpha,1)}| \le |J_\alpha|/4 \quad$ *for* $\alpha \in \Lambda$.

Proof. By induction, we shall prove our assertion. First, take $R = 2$ and $\varepsilon = 1/2$ in Lemma 4.1. Then there exist $\zeta_1, \zeta_2 \in D$ and disjoint closed subarcs J_1, J_2 satisfying the conditions in Lemma 4.1. Put

$$z_{(0)} = \zeta_1, \; z_{(1)} = \zeta_2, \; J_{(0)} = J_1, \text{ and } J_{(1)} = J_2.$$

Next, suppose that z_α and J_α, $\alpha \in \bigcup_{n=1}^k \Lambda_n$, are already chosen satisfying (i)–(vii). Let $\alpha \in \Lambda_k$. Apply Lemma 4.1 for $J = J_\alpha, R = 2^{k+1}$, and $\varepsilon = (1/2)^{k+1}$. Then there exist $z_{(\alpha,0)}, z_{(\alpha,1)} \in D$ and disjoint closed subarcs $J_{(\alpha,0)} = \{e^{i\theta}; a_{(\alpha,0)} \leq \theta \leq b_{(\alpha,0)}\}, J_{(\alpha,1)} = \{e^{i\theta}; a_{(\alpha,1)} \leq \theta \leq b_{(\alpha,1)}\}$, $a_\alpha = a_{(\alpha,0)} < b_{(\alpha,0)} < a_{(\alpha,1)} < b_{(\alpha,1)} = b_\alpha$, of J_α such that

$$P_{z_{(\alpha,0)}}(e^{ia_{(\alpha,0)}}) = P_{z_{(\alpha,0)}}(e^{ib_{(\alpha,0)}}) = 2^{k+1},$$

$$P_{z_{(\alpha,1)}}(e^{ia_{(\alpha,1)}}) = P_{z_{(\alpha,1)}}(e^{ib_{(\alpha,1)}}) = 2^{k+1},$$

$$P_{z_{(\alpha,0)}}(e^{i\theta}) < (1/2)^{k+1} \quad \text{for } e^{i\theta} \in \Big(\bigcup_{\beta\in\Lambda_k, \beta\neq\alpha} J_\beta \Big) \cup J_{(\alpha,1)},$$

$$P_{z_{(\alpha,1)}}(e^{i\theta}) < (1/2)^{k+1} \quad \text{for } e^{i\theta} \in \Big(\bigcup_{\beta\in\Lambda_k, \beta\neq\alpha} J_\beta \Big) \cup J_{(\alpha,0)},$$

and

$$|J_{(\alpha,0)}| = |J_{(\alpha,1)}| \leq |J_\alpha|/4.$$

Since $P_{z_\alpha}(e^{ia_\alpha}) = P_{z_\alpha}(e^{ib_\alpha}) = 2^n$, we may further assume that

$$J_{(\alpha,0)} \cup J_{(\alpha,1)} \subset \{e^{i\theta} \in J_\alpha; 2^n \leq P_{z_\alpha}(e^{i\theta}) \leq 2^{n+1}\}.$$

This completes the induction. ■

Lemma 4.3 ([17], Theorem 5.1) . *Let $\mu = \sum_{n=1}^\infty a_n \delta_{\zeta_n} \in M_{s,d}^+$, where $\zeta_n \in \partial D$ and $a_n > 0$ for every n. Then μ has an outer vanishing measure if and only if for each n there exists $\lambda_n \in M_s^+$ such that $S(\lambda_n) \subset S(\mu)$ and $\{|\psi_{\delta_{\zeta_n}}| < 1\} \subset Z(\psi_{\lambda_n})$.*

Proof of Theorem 4.1. First, we construct $\mu \in M_{s,c}^+$. Take a closed subarc J of ∂D such that $|J| > 0$ and $J \subset \{e^{i\theta}; 0 \leq \theta \leq 1\}$ and apply Lemma 4.2. Then there exist a family of points $\{z_\alpha\}_{\alpha\in\Lambda}$ in D and a family of open subarcs $\{J_\alpha\}_{\alpha\in\Lambda}$ of J, say $J_\alpha = \{e^{i\theta}; a_\alpha \leq \theta \leq b_\alpha\}, a_\alpha < b_\alpha$, satisfying

$$J_\alpha \cap J_\beta = \emptyset \quad \text{for } \alpha, \beta \in \Lambda_n, \alpha \neq \beta, \tag{4.1}$$

$$J_{(\alpha,0)} \cup J_{(\alpha,1)} \subset \{e^{i\theta} \in J_\alpha; 2^n \leq P_{z_\alpha}(e^{i\theta}) \leq 2^{n+1}\} \quad \text{for } \alpha \in \Lambda_n, \tag{4.2}$$

$$P_{z_\alpha}(e^{i\theta}) < 1/2^n \quad \text{for } e^{i\theta} \in \bigcup_{\beta\in\Lambda_n, \beta\neq\alpha} J_\beta, \tag{4.3}$$

$$P_{z_\alpha}(e^{ia_\alpha}) = P_{z_\alpha}(e^{ib_\alpha}) = 2^n \quad \text{for } \alpha \in \Lambda_n, \tag{4.4}$$

$$\sum_{\alpha\in\Lambda_{n+1}} |J_\alpha| \le \sum_{\alpha\in\Lambda_n} |J_\alpha|/2, \tag{4.5}$$

and

$$|J_{(\alpha,0)}| = |J_{(\alpha,1)}| \le |J_\alpha|/4 \quad \text{for } \alpha \in \Lambda. \tag{4.6}$$

For each $\alpha \in \Lambda_n$, put

$$\lambda_\alpha = \frac{1}{2^{n+1}} (\delta_{e^{ia_\alpha}} + \delta_{e^{ib_\alpha}}). \tag{4.7}$$

For each positive integer n, put

$$\lambda_n = \sum_{\alpha\in\Lambda_n} \lambda_\alpha.$$

Then $\|\lambda_\alpha\| = 1/2^n$ and $S(\lambda_\alpha) \subset J_\alpha$. Since number of elements of Λ_n is 2^n,

$$\|\lambda_n\| = 1 \quad \text{and} \quad S(\lambda_n) \subset \bigcup_{\alpha\in\Lambda_n} J_\alpha.$$

It is not difficult to see that $\lambda_n \to \mu$ in the weak*-topology as $n \to \infty$ for some positive continuous measure μ on ∂D such that

$$\|\mu\| = 1 \tag{4.8}$$

and

$$S(\mu) = \bigcap_{n=1}^{\infty} \Big(\bigcup_{\alpha\in\Lambda_n} J_\alpha \Big). \tag{4.9}$$

By (4.5),

$$|S(\mu)| \le \sum_{\alpha\in\Lambda_n} |J_\alpha| \le (1/2)^{n-1}(|J_{(0)}| + |J_{(1)}|)$$

for every n, so that we have $|S(\mu)| = 0$.

For $\alpha \in \Lambda_n$, let

$$\mu_\alpha = \mu_{|J_\alpha}. \tag{4.10}$$

For each positive integer k, let $\Lambda_{k,\alpha} = \{\beta \in \Lambda_k; \lambda_\beta(J_\alpha) \neq 0\}$. Then by (4.1), (4.2), and (4.7), we have $\sum_{\beta\in\Lambda_{k,\alpha}} \lambda_\beta \to \mu_\alpha$ as $k \to \infty$ and

$$\Big\| \sum_{\beta\in\Lambda_{k,\alpha}} \lambda_\beta \Big\| = \lambda_\alpha(J_\alpha) = 1/2^n \quad \text{for } k \ge n.$$

Hence

$$\|\mu_\alpha\| = 1/2^n, \tag{4.11}$$

$$\mu = \sum_{\alpha\in\Lambda_n} \mu_\alpha, \tag{4.12}$$

and

$$\mu_\alpha = \mu_{(\alpha,0)} + \mu_{(\alpha,1)}. \tag{4.13}$$

By (4.11) and (4.12), $\mu \in M^+_{s,c}$. Now we have

$$\begin{aligned}
\int_{\partial D} P_{z_\alpha}\, d\mu &= \sum_{\beta\in\Lambda_n} \int_{\partial D} P_{z_\alpha}\, d\mu_\beta \quad \text{by (4.12)} \\
&= \int_{\partial D} P_{z_\alpha}\, d\mu_{(\alpha,0)} + \int_{\partial D} P_{z_\alpha}\, d\mu_{(\alpha,1)} + \sum_{\beta\in\Lambda_n,\beta\neq\alpha} \int_{\partial D} P_{z_\alpha}\, d\mu_\beta \\
&\qquad \text{by (4.13)} \\
&\leq 2^{n+1}\, \|\mu_{(\alpha,0)} + \mu_{(\alpha,1)}\| + \frac{1}{2^n} \sum_{\beta\in\Lambda_n,\beta\neq\alpha} \|\mu_\beta\| \\
&\qquad \text{by (4.2) and (4.3)} \\
&\leq 2^{n+1}\, \|\mu_\alpha\| + \frac{1}{2^n} \quad \text{by (4.8) and (4.13)} \\
&\leq 3 \quad \text{by (4.11)}.
\end{aligned}$$

On the other hand, by (4.4) we have $P_{z_\alpha} \geq 2^n$ on J_α. By (4.10) and (4.11), we get $1 \leq \int_{\partial D} P_{z_\alpha}\, d\mu_\alpha$. Hence $1 \leq \int_{\partial D} P_{z_\alpha}\, d\mu \leq 3$ for every $\alpha \in \Lambda$. Since $-\log|\psi_\mu(z_\alpha)| = \int_{\partial D} P_{z_\alpha}\, d\mu$, we have

$$e^{-3} \leq |\psi_\mu(z_\alpha)| \leq e^{-1} \quad \text{for every } \alpha \in \Lambda. \tag{4.14}$$

To prove (i), let $\nu \in M^+_s$ such that $S(\nu) \subset S(\mu)$. Put $\nu_\alpha = \nu_{|J_\alpha}$ for $\alpha \in \Lambda$. Then in the same way as the above paragraph, we have

$$\int_{\partial D} P_{z_\alpha}\, d\nu \leq 2^{n+1}\|\nu_\alpha\| + \frac{\|\nu\|}{2^n} \quad \text{for } \alpha \in \Lambda_n. \tag{4.15}$$

Since $S(\nu) \subset S(\mu)$, by (4.9) we have $\sum_{\alpha\in\Lambda_n} \|\nu_\alpha\| = \|\nu\| < \infty$. Since the number of elements of Λ_n is 2^n, there exists α_n in Λ_n such that $\|\nu_{\alpha_n}\| \leq \|\nu\|/2^n$. Hence by (4.15), $\int_{\partial D} P_{z_{\alpha_n}}\, d\nu \leq 3\|\nu\|$ for every n. Thus we get $0 < e^{-3\|\nu\|} \leq |\psi_\nu(z_{\alpha_n})|$ for every n. Therefore, by (4.14) we have $\{|\psi_\mu| < 1\} \not\subset Z(\psi_\nu)$.

Next, we prove (ii) and (iii). Let $\nu, \lambda \in L^1_+(\mu)$. It is sufficient to prove that $\{|\psi_\nu| < 1\} \not\subset Z(\psi_\lambda)$. We may assume that $\|\nu\| = \|\lambda\| = 1$. Then there exist $K > 0$ and a sequence $\{\alpha_n\}_n, \alpha_n \in \Lambda_n$, such that

$$\lambda(J_{\alpha_n})/K \le \mu(J_{\alpha_n}) \le K\nu(J_{\alpha_n}) \quad \text{for every } n. \tag{4.16}$$

In the similar way as above, we have

$$\begin{aligned}\int_{\partial D} P_{z_{\alpha_n}}\, d\lambda &\le \int_{J_{\alpha_n}} P_{z_{\alpha_n}}\, d\lambda + \frac{1}{2^n} \\ &\le 2^{n+1}\lambda(J_{\alpha_n}) + \frac{1}{2^n} \\ &\le 2^{n+1}K\mu(J_{\alpha_n}) + \frac{1}{2^n} && \text{by (4.16)} \\ &\le 2K + 1 && \text{by (4.11)}\end{aligned}$$

and

$$\begin{aligned}\int_{\partial D} P_{z_{\alpha_n}}\, d\nu &\ge \int_{J_{\alpha_n}} P_{z_{\alpha_n}}\, d\nu \\ &\ge 2^n\nu(J_{\alpha_n}) && \text{by (4.2)} \\ &\ge 2^n\mu(J_{\alpha_n})/K && \text{by (4.16)} \\ &= 1/K && \text{by (4.11).}\end{aligned}$$

Hence we get $0 < e^{-(2K+1)} \le |\psi_\lambda(z_{\alpha_n})|$ and $|\psi_\nu(z_{\alpha_n})| \le e^{-1/K} < 1$ for every n. Thus we obtain $\{|\psi_\nu| < 1\} \not\subset Z(\psi_\lambda)$.

(iv) Let $\zeta = e^{i\theta_0} \in S(\mu)$. By Lemma 4.3, it is sufficient to prove that

$$\{|\psi_{\delta_\zeta}| < 1\} \not\subset Z(\psi_\sigma). \tag{4.17}$$

for every measure $\sigma \in M_s^+$ such that

$$S(\sigma) \subset S(\mu) \tag{4.18}$$

To prove (4.17), let $\sigma \in M_s^+$ satisfying (4.18). We may assume that $\|\sigma\| = 1$. By (4.9), there exists a sequence $\{\alpha_n\}_n$ in $\Lambda, \alpha_n \in \Lambda_n$, such that $\zeta \in J_{\alpha_n}$ for every n. Then $\zeta \in J_{(\alpha_n,0)} \cup J_{(\alpha_n,1)}$. Here we may assume that $\zeta \in J_{(\alpha_n,0)}$, that is, $\alpha_{n+1} = (\alpha_n, 0)$. Then $a_{(\alpha_n,0)} \le \arg\zeta = \theta_0 \le b_{(\alpha_n,0)} < a_{(\alpha_n,1)}$. Put

$$\xi_n = \exp\Big(\frac{i}{2}(\theta_0 + a_{(\alpha_n,1)})\Big) \quad \text{and} \quad \zeta_n = \exp\Big(\frac{i}{4}(\theta_0 + 3\,a_{(\alpha_n,1)})\Big). \tag{4.19}$$

Then by (4.6),

$$\arg\zeta \le b_{(\alpha_n,0)} < \arg\xi_n < \arg\zeta_n < a_{(\alpha_n,1)}$$

and ζ_n is the center of the arc jointing ξ_n and $e^{ia_{(\alpha_n,1)}}$. Hence

$$|\xi_n - \zeta_n| = |e^{ia_{(\alpha_n,1)}} - \zeta_n| = d(\zeta_n, S(\mu)) < |\zeta - \zeta_n|. \tag{4.20}$$

Let $w_n \in D$ such that

$$|\psi_{\delta_\zeta}(w_n)| = e^{-1} \quad \text{and} \quad w_n/|w_n| = \zeta_n. \tag{4.21}$$

Put

$$\theta_n = \arg \zeta_n \overline{\xi}_n. \tag{4.22}$$

Then $\theta_n \to 0$ as $n \to \infty$. Since $\zeta = e^{i\theta_0}$, by (4.19) we have $3\theta_n = \arg \zeta_n \overline{\zeta}$. Hence by (4.21), we get

$$|w_n| = \cos(3\theta_n). \tag{4.23}$$

Now we have

$$\begin{aligned} -\log |\psi_\sigma(w_n)| &= \int_{\partial D} P_{w_n}\, d\sigma \\ &\le \int_{\partial D} P_{w_n}\, d\delta_{\xi_n} \qquad \text{by (4.18), (4.20), and (4.21)} \\ &= \int_{\partial D} P_{|w_n|}\, d\delta_{\xi_n \overline{\zeta}_n} \qquad \text{by (4.21)} \\ &= \frac{1 - \cos^2(3\theta_n)}{1 - 2\cos(3\theta_n)\cos\theta_n + \cos^2(3\theta_n)}, \end{aligned}$$

where the last equality follows from (4.22) and (4.23). It is not difficult to see that

$$\lim_{\theta \to 0} \frac{1 - \cos^2(3\theta)}{1 - 2\cos(3\theta)\cos\theta + \cos^2(3\theta)} = 9.$$

Hence we obtain

$$\liminf_{n \to \infty} |\psi_\sigma(w_n)| \ge e^{-9}.$$

By (4.21), we get (4.17). ∎

The following follows from Theorem 4.1 and Corollary 3.1.

Corollary 4.1 *Let μ be the measure given in Theorem 4.1 and $\nu \in L^1_+(\mu)$. Then there are not singular inner functions which are codivisible with ψ_ν.*

Relating to Theorem 4.1 (iv), we have the following problem.

Problem 4.1. Let μ be the measure given in Theorem 4.1. Does there exist $\nu \in M_s^+$ such that $S(\nu) \subset S(\mu)$ and ν has an outer vanishing measure?

Theorem 4.2 *There exists a measure $\mu \in M_{s,c}^+$ which has an outer vanishing measure and $|S(\mu)| = 0$.*

To prove this, we need the following lemma.

Lemma 4.4 *Let $\mu \in M_s^+$ and let U be an open subset of ∂D such that $S(\mu) \subset U$. Then for every $\varepsilon > 0$, there exists $\lambda \in M_{s,c}^+$ satisfying the following conditions.*

(i) $S(\mu) \subset S(\lambda) \subset U$.

(ii) $|S(\lambda)| = |S(\mu)|$.

(iii) $\|\lambda\| < \varepsilon$.

(iv) $\{|\psi_\mu| < 1\} \subset Z(\psi_\lambda)$.

Proof. By Lemma 2.1, there is an interpolating Blaschke product b with zeros $\{z_n\}_n$ such that

$$\{|b| < 1\} = \{|\psi_\mu| < 1\}. \tag{4.24}$$

Since we may discard finitely many zeros from $\{z_n\}_n$, we may assume that $z_n \neq 0$ for every n and

$$\sum_{n=1}^{\infty} 1 - |z_n| < \varepsilon. \tag{4.25}$$

By (4.24), we have $\overline{\{z_n\}_n} \setminus \{z_n\}_n = S(\mu)$. Put $e^{i\theta_n} = z_n/|z_n|$. Then

$$\overline{\{e^{i\theta_n}\}_n} \setminus \{e^{i\theta_n}\}_n \subset S(\mu) \subset \overline{\{e^{i\theta_n}\}_n}, \tag{4.26}$$

so that we may assume that $e^{i\theta_n} \in U$ for every n. Put

$$E_n = \{e^{i\theta}; |\theta - \theta_n| \leq 1 - |z_n|\}. \tag{4.27}$$

Then

$$P_{z_n} \geq 1/(1 - |z_n|) \quad \text{on } E_n \text{ for every } n. \tag{4.28}$$

By (4.25), there exists a sequence of positive numbers $\{p_n\}_n$ such that

$$\sum_{n=1}^{\infty} p_n(1 - |z_n|) < \varepsilon \quad \text{and} \quad p_n \to \infty \quad \text{as } n \to \infty. \tag{4.29}$$

There is a measure λ_n in $M^+_{s,c}$ such that

$$\|\lambda_n\| = p_n(1 - |z_n|), \tag{4.30}$$

$$e^{i\theta_n} \in S(\lambda_n) \subset E_n \cap U, \tag{4.31}$$

and

$$|S(\lambda_n)| = 0. \tag{4.32}$$

Put $\lambda = \sum_{n=1}^{\infty} \lambda_n$. Then by (4.29) and (4.30), $\|\lambda\| < \varepsilon$. Since $\lambda_n \in M^+_{s,c}$, $\lambda \in M^+_{s,c}$. By (4.27), $|E_n| \to 0$ as $n \to \infty$, so that by (4.26) and (4.31) we have

$$S(\lambda) = S(\mu) \cup \big(\bigcup_{n=1}^{\infty} S(\lambda_n)\big) \subset U.$$

Hence by (4.32), $|S(\lambda)| = |S(\mu)|$.

Now we have

$$\begin{aligned} -\log|\psi_\lambda(z_n)| &= \int_{\partial D} P_{z_n}\, d\lambda \\ &\geq \int_{E_n} P_{z_n}\, d\lambda_n \\ &\geq p_n \qquad \text{by (4.28) and (4.30)} \\ &\to \infty \qquad \text{by (4.29)}. \end{aligned}$$

Therefore $\psi_\lambda(z_n) \to 0$ as $n \to \infty$. Hence by Lemma 2.3, $\{|b| < 1\} \subset Z(\psi_\lambda)$, so that by (4.24) we have $\{|\psi_\mu| < 1\} \subset Z(\psi_\lambda)$. This completes the proof. ∎

Proof of Theorem 4.2. By induction, we shall prove the existence of $\{\mu_n\}_n$ in $M^+_{s,c}$ and open subsets $\{U_n\}_n$ of ∂D satisfying

$$S(\mu_{n-1}) \subset S(\mu_n) \subset U_n \subset U_{n-1}, \tag{4.33}$$

$$|S(\mu_n)| = 0, \tag{4.34}$$

$$\|\mu_n\| < 1/2^n, \tag{4.35}$$

$$\{|\psi_{\mu_{n-1}}| < 1\} \subset Z(\psi_{\mu_n}), \tag{4.36}$$

and

$$|\overline{U}_n| < 1/n. \tag{4.37}$$

First, take a measure $\mu_1 \in M^+_{s,c}$ such that $\|\mu_1\| < 1/2$ and $|S(\mu_1)| = 0$. Then take an open subset U_1 of ∂D such that $S(\mu_1) \subset U_1$ and $|\overline{U}_1| < 1$. We use Lemma 4.4 for $\mu = \mu_1, U = U_1$, and $\varepsilon = (1/2)^2$. Then there exists $\mu_2 \in M^+_{s,c}$ such that $S(\mu_1) \subset S(\mu_2) \subset U_1$, $|S(\mu_2)| = |S(\mu_1)| = 0$, $\|\mu_2\| < (1/2)^2$, and $\{|\psi_{\mu_1}| < 1\} \subset Z(\psi_{\mu_2})$.

Assume that $\mu_1, \ldots, \mu_n$ and $U_1, \ldots, U_{n-1}$ are chosen satisfying the above conditions. Since $S(\mu_n) \subset U_{n-1}$, by (4.34) there is an open subset U_n such that $S(\mu_n) \subset U_n \subset U_{n-1}$ and $|\overline{U}_n| < 1/n$. By Lemma 4.4, there exists $\mu_{n+1} \in M^+_{s,c}$ such that $S(\mu_n) \subset S(\mu_{n+1}) \subset U_n$, $|S(\mu_{n+1})| = |S(\mu_n)| = 0$, $\|\mu_{n+1}\| < 1/2^{n+1}$, and $\{|\psi_{\mu_n}| < 1\} \subset Z(\psi_{\mu_{n+1}})$. This completes our induction.

Put $\mu = \sum_{n=1}^\infty \mu_n$. Then by (4.35), $\|\mu\| < \infty$ so that $\mu \in M^+_{s,c}$. By (4.33), $S(\mu_n) \subset U_k$ for every pair of positive integers n and k. Hence $S(\mu) \subset \overline{U}_k$ for every k. Then by (4.37), we have $|S(\mu)| = 0$. By (4.36) and Theorem 2.1, μ has an outer vanishing measure. ■

We have the following problem:

PROBLEM 4.2. Let E be a closed subset of ∂D such that $|E| > 0$. Does there exist $\mu \in M^+_s$ such that $S(\mu) \subset E$ and μ has an outer vanishing measure?

5. Inner vanishing measures and factorization in $H^\infty + C$

In this section, we characterize $\mu \in M^+_s$ satisfying conditions (A) and (B).

Lemma 5.1 ([21]) . *Let B be a Douglas algebra and let ψ be an inner function. Then $\psi B \subset H^\infty + C$ if and only if $\psi = 0$ on $M(H^\infty + C) \setminus M(B)$.*

In the same way as [11, Lemma 4.2], we have

Lemma 5.2 *Let ψ be an inner function. If $f \in H^\infty + C$ and $|f| \le |\psi|$ on $M(H^\infty + C)$, then $\psi = 0$ on $M(H^\infty + C) \setminus M(H^\infty[f/\psi])$.*

Hence the proof of [11, Theorem 4.2] actually proved the following.

Proposition 5.1 *Let $f \in H^\infty + C$ and let ψ be an inner function such that $|f| \le |\psi|$ on $M(H^\infty + C)$. Then there is a Blaschke product b such that $b = 0$ on $M(H^\infty + C) \setminus M([H^\infty[f/\psi])$ and $\{|b| < 1\} \subset Z(\psi)$.*

Corollary 5.1 *Let $f \in H^\infty + C$ and let ψ be an inner function such that $|f| \le |\psi|$ on $M(H^\infty + C)$. Then there is a Blaschke product b satisfying the following conditions.*

(i) $(bf)/\psi \in H^\infty + C$ and $|bf| = |f|$ on $M(H^\infty + C)$.

(ii) $\psi/b \in H^\infty + C$, $|\psi/b| = |\psi|$ on $M(H^\infty + C)$, and $f/(\psi/b) \in H^\infty + C$.

Proof. By Proposition 5.1, there is a Blaschke product b such that

$$b = 0 \quad \text{on } M(H^\infty + C) \setminus M([H^\infty[f/\psi]) \tag{5.1}$$

and $\{|b| < 1\} \subset Z(\psi)$. By our assumption, $Z(\psi) \subset Z(f)$, so that we have $|bf| = |f|$ on $M(H^\infty + C)$. By (5.1) and Lemma 5.1, $(bf)/\psi = b(f/\psi) \in H^\infty + C$.

By Lemma 2.2, $\psi/b \in H^\infty + C$ and $|\psi/b| = |\psi|$ on $M(H^\infty + C)$. And we have $f/(\psi/b) = b(f/\psi) \in H^\infty + C$. ■

Proposition 5.2 *Let $\zeta \in \partial D$. Then δ_ζ does not satisfy condition (A).*

Proof. There exists an interpolating sequence $\{z_n\}_n$ in D such that

$$\psi_{\delta_\zeta}(z_n) \to 0 \quad \text{as } n \to \infty.$$

Let b be the interpolating Blaschke product with zeros $\{z_n\}_n$. Then by Lemma 2.3, $f = \psi_{\delta_\zeta}/b \in H^\infty + C$ and $|f| = |\psi_{\delta_\zeta}|$ on $M(H^\infty + C)$. We also have $f/\psi_{\delta_\zeta} = 1/b \notin H^\infty + C$. Let $\nu \in M_s^+$ such that $\nu \le \delta_\zeta$ and $|\psi_\nu| = |\psi_{\delta_\zeta}|$ on $M(H^\infty + C)$. Then $\nu = \delta_\zeta$. Hence δ_ζ does not satisfy condition (A). ■

Let $\mu \in M_s^+$. Recall that a measure ν with $0 \le \nu \le \mu$ is called an inner vanishing measure for μ if $\{|\psi_\nu| < 1\} \subset Z(\psi_\mu)$. Let μ_α be the upper band of inner vanishing measures for μ. We put $\mu_\beta = \mu - \mu_\alpha$. Generally μ_α is not an inner vanishing measure for μ. The measure μ_α is called the inner vanishing part of μ.

The following is the main theorem in this section.

Theorem 5.1 *Let $\mu \in M_s^+$ and let μ_α be the inner vanishing part of μ. Then the following conditions are equivalent.*

(i) $Z(\psi_{\mu_\alpha}) = Z(\psi_\mu)$.

(ii) μ satisfies condition (A).

(iii) μ satisfies condition (B).

To prove our theorem, we need some lemmas.

Lemma 5.3 *Let $\mu \in M_s^+$. Then there exists a sequence of measures $\{\lambda_n\}_n$ in M_s^+ satisfying the following conditions.*

(i) λ_n *is an inner vanishing measure for* μ *for every* n.

(ii) $0 \le \lambda_n \le \lambda_{n+1} \le \mu$ *for every* n.

(iii) $\mu_\alpha = \vee_{n=1}^\infty \lambda_n$.

(iv) $\mu_\alpha \perp \mu_\beta$.

(v) If $0 \ne \nu \le \mu_\beta$, *then* ν *is not an inner vanishing measure for* μ.

Proof. Let $\mathcal{M}$ be the set of inner vanishing measures for μ and $A = \sup\{\|\lambda\|; \lambda \in \mathcal{M}\}$. If $\lambda \in \mathcal{M}$ and $0 \le \sigma \le \lambda$, then $\{|\psi_\sigma| < 1\} \subset \{|\psi_\lambda| < 1\}$, so that $\sigma \in \mathcal{M}$. If $\nu_1, \nu_2 \in \mathcal{M}$ and $\nu_1 + \nu_2 \le \mu$, then we have $\nu_1 + \nu_2 \in \mathcal{M}$ and $\nu_1 \vee \nu_2 \in \mathcal{M}$. Then it is not difficult to find a sequence $\{\lambda_n\}_n$ in $\mathcal{M}$ satisfying conditions (i) and (ii), and $\|\lambda_n\| \to A$ as $n \to \infty$. Put $\mu'_\alpha = \vee_{n=1}^\infty \lambda_n$. Then $\|\mu'_\alpha\| = A$ and $\mu'_\alpha \le \mu_\alpha$.

To prove (iii), suppose not. Then there exists $\nu \in \mathcal{M}$ such that $\nu \not\le \mu'_\alpha$. Then $\|\mu'_\alpha \vee \nu\| > \|\mu'_\alpha\| = A$. Since $\lambda_n \vee \nu \in \mathcal{M}$, $\|\lambda_n \vee \nu\| \le A$. Since $\|\lambda_n \vee \nu\| \to \|\mu'_\alpha \vee \nu\|$, we have a contradiction.

To prove (iv), suppose not. Then $\mu_\alpha \wedge \mu_\beta \ne 0$. By (iii), $\lambda_{n_0} \wedge \mu_\beta \ne 0$ for some n_0. Since $\lambda_{n_0} \wedge \mu_\beta \le \lambda_{n_0} \in \mathcal{M}$, $\lambda_{n_0} \wedge \mu_\beta \in \mathcal{M}$. Since $\lambda_n + (\lambda_{n_0} \wedge \mu_\beta) \le \mu_\alpha + \mu_\beta = \mu$, we have $\lambda_n + (\lambda_{n_0} \wedge \mu_\beta) \in \mathcal{M}$ and $\|\lambda_n + (\lambda_{n_0} \wedge \mu_\beta)\| \to A + \|\lambda_{n_0} \wedge \mu_\beta\| > A$. This is a contradiction.

(v) follows from (iv). ∎

Lemma 5.4 *Let $\mu \in M_s^+$ and let μ_α be the inner vanishing part of μ. Then there is a sequence of inner vanishing measures $\{\mu_n\}_n$ for μ such that $\mu_\alpha = \sum_{n=1}^\infty \mu_n$.*

Proof. Let $\{\lambda_n\}_n$ be a sequence of measures given in Lemma 5.3. Put $\mu_1 = \lambda_1$ and $\mu_n = \lambda_n - \lambda_{n-1}$ for $n \ge 2$. Then μ_n is an inner vanishing measure for μ and $\sum_{n=1}^\infty \mu_n = \vee_{n=1}^\infty \lambda_n$. ∎

Lemma 5.5 *Let $\mu \in M_s^+$ and let μ_α be the inner vanishing part of μ. Suppose that $Z(\psi_{\mu_\alpha}) \ne Z(\psi_\mu)$. Then there is an interpolating Blaschke product b such that $\{|b| < 1\} \subset Z(\psi_\mu)$, $\psi_\mu / b \in H^\infty + C$, $|\psi_\mu / b| = |\psi_\mu|$ on $M(H^\infty + C)$, and $\psi_\nu / b \notin H^\infty + C$ for every measure $\nu \in M_s^+$ satisfying $Z(\psi_\nu) \subset Z(\psi_{\mu_\alpha})$.*

Proof. By our assumption, $Z(\psi_\mu) \not\subset Z(\psi_{\mu_\alpha})$. Take $x \in M(H^\infty + C)$ such that $\psi_\mu(x) = 0$ and $\psi_{\mu_\alpha}(x) \neq 0$. Then there exists an interpolating Blaschke product b with zeros $\{z_n\}_n$ such that $\psi_\mu(z_n) \to 0$ and $\psi_{\mu_\alpha}(z_n) \to \psi_{\mu_\alpha}(x)$ as $n \to \infty$. By Lemma 2.3, $\psi_\mu/b \in H^\infty + C$ and $|\psi_\mu/b| = |\psi_\mu|$ on $M(H^\infty + C)$. Since $Z(b) = cl\,\{z_n\}_n \backslash \{z_n\}_n$, we have $\psi_{\mu_\alpha} = \psi_{\mu_\alpha}(x)$ on $Z(b)$. Take $y \in Z(b)$. Then $b(y) = 0$ and $\psi_{\mu_\alpha}(y) \neq 0$. Let $\nu \in M_s^+$ satisfying $Z(\psi_\nu) \subset Z(\psi_{\mu_\alpha})$. Then $\psi_\nu(y) \neq 0$. If $\psi_\nu/b \in H^\infty + C$, $\psi_\nu = bh$ for some $h \in H^\infty + C$. Then we have $0 \neq \psi_\nu(y) = (bh)(y) = b(y)h(y) = 0$. This is a contradiction. ■

The following is a key to prove Theorem 5.1.

Lemma 5.6 *Let $\mu \in M_s^+$ and let $\{\mu_n\}_n$ be a sequence of inner vanishing measures for μ such that $\mu = \sum_{n=1}^\infty \mu_n$. Let $\{b_n\}_n$ be a sequence of interpolating Blaschke products such that $\bigcup_{n=1}^\infty \{|b_n| < 1\} \subset Z(\psi_\mu)$. Then there exists a measure λ such that $0 \le \lambda \le \mu$ and*

$$\bigcup_{n=1}^\infty \{|b_n| < 1\} \subset Z(\psi_\lambda) \subset \{|\psi_\lambda| < 1\} \subset Z(\psi_\mu)\,.$$

Proof. Let $\{c_j\}_j$ be a sequence of positive numbers such that

(5.2) $$c_j \to 0 \text{ as } j \to \infty \quad \text{and} \quad 0 < c_{j+1} \le c_j \le 1 \text{ for every } j.$$

Put

(5.3) $$\lambda = \sum_{j=1}^\infty c_j \mu_j.$$

Then $0 \le \lambda \le \mu$. Moreover we have

(5.4) $$\{|\psi_\lambda| < 1\} \subset Z(\psi_\mu).$$

For, let $x \in M(H^\infty + C)$ such that $|\psi_\lambda(x)| < 1$. Since $\lambda \le \mu$, $Z(\psi_\lambda) \subset Z(\psi_\mu)$. Hence to show (5.4), we may assume that

(5.5) $$0 < |\psi_\lambda(x)| < 1.$$

Since $\{|\psi_{\mu_n}| < 1\} \subset Z(\psi_\mu)$, we may further assume that

(5.6) $$|\psi_{\mu_j}(x)| = 1 \quad \text{for every } j.$$

For each positive integer n, put

(5.7) $$\lambda_n = \sum_{j=n}^\infty c_j \mu_j \quad \text{and} \quad \mu_n = \sum_{j=n}^\infty \mu_j.$$

Then by (5.2),

(5.8) $$c_n \mu_n \ge \lambda_n \quad \text{for every } n.$$

Now we have

$$\begin{aligned} |\psi_\mu(x)| &= |\psi_{\mu_n}(x)| && \text{by (5.6) and (5.7)} \\ &\le |\psi_{\lambda_n}(x)|^{1/c_n} && \text{by (5.8)} \\ &= |\psi_\lambda(x)|^{1/c_n} && \text{by (5.3), (5.6), and (5.7).} \end{aligned}$$

Then by (5.2) and (5.5), we have $\psi_\mu(x) = 0$. Thus we get (5.4). Next, we shall find a sequence $\{c_j\}_j$ satisfying (5.2) and λ defined by (5.3) satisfies

$$\bigcup_{n=1}^{\infty} \{|b_n| < 1\} \subset Z(\psi_\lambda). \tag{5.9}$$

Let $\{z_{n,k}\}_k$ be the zeros of b_n in D. Since $\bigcup_{n=1}^{\infty}\{|b_n| < 1\} \subset Z(\psi_\mu)$, for each n we have

$$\psi_\mu(z_{n,k}) \to 0 \quad \text{as } k \to \infty. \tag{5.10}$$

By induction, we shall find a strictly increasing sequence of positive integers $\{k_i\}_i$ and a family of sequence of positive integers $\{n_{i,j}\}_{1\le i\le j}$ satisfying $n_{i,j} < n_{i,j+1}$ for $j \ge i$,

$$|\psi_{\nu_i'}(z_{t,s})| \le 2|\psi_{\nu_{i-1}}(z_{t,s})| \quad \text{for } 1 \le t \le i, 1 \le s \le n_{t,i}, \tag{5.11}$$

and

$$|\psi_{\nu_{i-1}}(z_{t,s})| \le (1/2)^i \quad \text{for } 1 \le t \le i, s \ge n_{t,i}, \tag{5.12}$$

where we put $\nu_0 = \mu$, and for each positive integer i we set

$$\nu_i' = \sum_{j=1}^{k_1} \mu_j + \sum_{j=k_1+1}^{k_2} (1/2)\mu_j + \ldots + \sum_{j=k_{i-1}+1}^{k_i} (1/2)^{i-1}\mu_j \tag{5.13}$$

and

$$\nu_i = \nu_i' + \sum_{j=k_i+1}^{\infty} (1/2)^i \mu_j. \tag{5.14}$$

Since $\nu_0 = \mu$, by (5.10) there is a positive integer $n_{1,1}$ such that

$$|\psi_{\nu_0}(z_{1,s})| \le \frac{1}{2} \quad \text{for } s \ge n_{1,1}.$$

Hence (5.12) holds for $i = 1$. Then there exists a positive integer k_1, so we get ν_1' by (5.13), such that $|\psi_{\nu_1'}(z_{1,s})| \le 2|\psi_{\nu_0}(z_{1,s})|$ for $1 \le s \le n_{1,1}$. Hence (5.11) holds for $i = 1$.

Next, suppose that $\{k_1, k_2, \ldots, k_N\}$ and $\{n_{i,j}; 1 \le i \le j \le N\}$ are chosen satisfying our conditions. We get ν_N' and ν_N by (5.13) and (5.14), respectively. By (5.13) and (5.14), $(1/2)^N\mu \le \nu_N \le \mu$. Hence by (5.10), $\psi_{\nu_N}(z_{n,k}) \to 0$ as $k \to \infty$ for every n.

Therefore for each $t, 1 \le t \le N+1$, there exists a positive integer $n_{t,N+1}$ such that $n_{t,N} < n_{t,N+1}$ and $|\psi_{\nu_N}(z_{t,s})| \le (1/2)^{N+1}$ for $s \ge n_{t,N+1}$. Then there exists a positive integer k_{N+1} such that $k_N < k_{N+1}$ and $|\psi_{\nu'_{N+1}}(z_{t,s})| \le 2|\psi_{\nu_N}(z_{t,s})|$ for $1 \le t \le N+1, 1 \le s \le n_{t,N+1}$. This completes the induction.

Now we define λ as

$$\lambda = \sum_{i=0}^{\infty} \Big(\sum_{j=k_i+1}^{k_{i+1}} (1/2)^i \mu_j \Big), \quad \text{where } k_0 = 0. \tag{5.15}$$

Put $c_j = (1/2)^i$ for $k_i + 1 \le j \le k_{i+1}$. Then (5.2) is satisfied.

To show (5.9) for this λ, by Lemma 2.3 it is sufficient to prove

$$\psi_\lambda(z_{t,k}) \to 0 \quad \text{as } k \to \infty \text{ for every } t. \tag{5.16}$$

Fix a positive integer t. Suppose that k is a sufficiently large integer. Then there exists a positive integer N, depends on k, such that $t < N$ and

$$n_{t,N} \le k < n_{t,N+1}. \tag{5.17}$$

By (5.13) and (5.15), $\nu'_{N+1} \le \lambda$. Hence by (5.11) we have

$$|\psi_\lambda(z_{t,k})| \le |\psi_{\nu'_{N+1}}(z_{t,k})| \le 2|\psi_{\nu_N}(z_{t,k})|. \tag{5.18}$$

By (5.13) and (5.14), we have $\nu_{N-1} \le 2\nu_N$, so that $|\psi_{\nu_N}|^2 \le |\psi_{\nu_{N-1}}|$ on $M(H^\infty)$. Hence by (5.18), we have $|\psi_\lambda(z_{t,k})| \le 2|\psi_{\nu_{N-1}}(z_{t,k})|^{1/2}$. Therefore by (5.12) and (5.17), $|\psi_\lambda(z_{t,k})| \le 2(1/2)^{N/2}$. When $k \to \infty$, we have $N \to \infty$. Hence $\psi_\lambda(z_{t,k}) \to 0$ as $k \to \infty$. Thus we get (5.16). This completes the proof. ∎

Proof of Theorem 5.1. (i) ⇒ (ii) Let $f \in H^\infty + C$ satisfying $|f| \le |\psi_\mu|$ on $M(H^\infty + C)$. By Lemma 5.2,

$$\psi_\mu = 0 \quad \text{on } M(H^\infty + C) \setminus M(H^\infty[f/\psi_\mu]). \tag{5.19}$$

By [10], there is a sequence of interpolating Blaschke products $\{b_n\}_n$ such that

$$\bigcup_{n=1}^{\infty} \{|b_n| < 1\} = M(H^\infty + C) \setminus M(H^\infty[f/\psi_\mu]). \tag{5.20}$$

By Lemma 5.4, there is a sequence of measures $\{\mu_n\}_n$ in $L^1_+(\mu)$ such that $\mu_\alpha = \sum_{n=1}^\infty \mu_n$ and $\{|\psi_{\mu_n}| < 1\} \subset Z(\psi_\mu)$ for every n. Since $Z(\psi_{\mu_\alpha}) = Z(\psi_\mu)$, we have $\{|\psi_{\mu_n}| < 1\} \subset Z(\psi_{\mu_\alpha})$ for every n. By (5.19) and (5.20), we have $\bigcup_{n=1}^\infty \{|b_n| < 1\} \subset Z(\psi_{\mu_\alpha})$. Then by Lemma 5.6, there exists a measure λ such that $0 \le \lambda \le \mu_\alpha$ and

$$\bigcup_{n=1}^{\infty} \{|b_n| < 1\} \subset Z(\psi_\lambda) \subset \{|\psi_\lambda| < 1\} \subset Z(\psi_{\mu_\alpha}). \tag{5.21}$$

Since $\lambda/2$ also satisfies (5.21) instead of λ, we may assume that

$$0 \le \lambda \le \mu_\alpha/2. \tag{5.22}$$

Put $\nu = \mu - \lambda$. Then $\nu \in M_s^+$. By (5.20), (5.21), and Lemma 5.1, $\psi_\lambda(f/\psi_\mu) \in H^\infty + C$. Thus we get $f/\psi_\nu \in H^\infty + C$. By (5.22), $\mu/2 \le \nu \le \mu$. Hence $Z(\psi_\nu) = Z(\psi_\mu) = Z(\mu_\alpha)$. By (5.21), we have $\{|\psi_\lambda| < 1\} \subset Z(\psi_{\mu_\alpha}) = Z(\psi_\nu)$. Therefore we obtain $|\psi_\mu| = |\psi_\nu||\psi_\lambda| = |\psi_\nu|$ on $M(H^\infty + C)$. Thus we get (ii).

(i) $\Rightarrow$ (iii) Let ψ be an inner function such that $|\psi_\mu| \le |\psi|$ on $M(H^\infty + C)$. By Lemma 5.2, $\psi = 0$ on $M(H^\infty + C) \setminus M(H^\infty[\psi_\mu/\psi])$. Since $Z(\psi) \subset Z(\psi_\mu)$, we have $\psi_\mu = 0$ on $M(H^\infty + C) \setminus M(H^\infty[\psi_\mu/\psi])$. So that by the proof of (i) $\Rightarrow$ (ii), there exists a measure λ such that $0 \le \lambda \le \mu_\alpha/2$ and

$$\begin{aligned} M(H^\infty + C) \setminus M(H^\infty[\psi_\mu/\psi]) &\subset Z(\psi_\lambda) \subset \{|\psi_\lambda| < 1\} \\ &\subset Z(\psi_{\mu_\alpha}) = Z(\psi_\mu). \end{aligned} \tag{5.23}$$

Put $\nu = \lambda + \mu$. Then by (5.23) and Lemma 5.1, $\psi_\nu/\psi = \psi_\lambda(\psi_\mu/\psi) \in H^\infty + C$. We have $\mu \le \nu \le \mu + (\mu_\alpha/2) \le 2\mu$. By (5.23) again, we have $|\psi_\nu| = |\psi_\lambda||\psi_\mu| = |\psi_\mu|$. Thus we get (iii).

Suppose that (i) does not hold. We shall prove that both (ii) and (iii) do not hold. By Lemma 5.5, there is an interpolating Blaschke product b such that $\psi_\mu/b \in H^\infty + C$, $\{|b| < 1\} \subset Z(\psi_\mu)$, $|\psi_\mu/b| = |\psi_\mu|$ on $M(H^\infty + C)$, and

$$\psi_\nu/b \notin H^\infty + C \quad \text{for every } \nu \in M_s^+ \text{ satisfying } Z(\psi_\nu) \subset Z(\psi_{\mu_\alpha}). \tag{5.24}$$

First, to prove that (ii) does not hold, suppose that (ii) holds. Put $f = \psi_\mu/b$. Then $f \in H^\infty + C$ and $|f| = |\psi_\mu|$. Since (ii) holds,

$$f/\psi_\nu \in H^\infty + C \tag{5.25}$$

for some $\nu \in M_s^+$ such that $0 \le \nu \le \mu$ and

$$|\psi_\nu| = |\psi_\mu| \quad \text{on } M(H^\infty + C). \tag{5.26}$$

Put $\lambda = \mu - \nu \ge 0$. Then by (5.26), $|\psi_\mu| = |\psi_\nu||\psi_\lambda| = |\psi_\lambda||\psi_\mu|$ on $M(H^\infty + C)$. Hence λ is an inner vanishing measure, so that $\lambda \le \mu_\alpha$. Since $Z(\psi_\lambda) \subset Z(\psi_{\mu_\alpha})$, by (5.24) we have $f/\psi_\nu = (\psi_\mu/b)/\psi_\nu = \psi_\lambda/b \notin H^\infty + C$. This contradicts (5.25). Thus we get (ii) $\Rightarrow$ (i).

Next, to prove that (iii) does not hold, suppose that (iii) holds. We put $\psi = b\psi_\mu$. Since $\{|b| < 1\} \subset Z(\psi_\mu)$, we have $|\psi| = |\psi_\mu|$ on $M(H^\infty + C)$. Since (iii) holds, there exists $\nu \in L^1_+(\mu)$ such that $\mu \le \nu \le 2\mu$, $|\psi_\nu| = |\psi_\mu|$ on $M(H^\infty + C)$, and $\psi_\nu/\psi \in H^\infty + C$. Then we have

$$\psi_{(\nu-\mu)}/b = \psi_\nu/\psi \in H^\infty + C. \tag{5.27}$$

Put $\lambda = \nu - \mu$. Then $\lambda \le \mu$ and $|\psi_\mu| = |\psi_\nu| = |\psi_\lambda||\psi_\mu| \le |\psi_\mu|$. Thus λ is an inner vanishing measure for μ. Hence $\lambda \le \mu_\alpha$. Therefore $Z(\psi_\lambda) \subset Z(\psi_{\mu_\alpha})$. Hence by (5.24), we get $\psi_\lambda/b \notin H^\infty + C$. This contradicts (5.27). ∎

Corollary 5.2 *Let $\mu \in M_s^+$ be an outer vanishing measure for some $\nu \in M_s^+$. Then $\mu_\alpha = \mu$ and μ satisfies conditions (A) and (B).*

Proof. We have $\mu \in L_+^1(\nu)$ and $\{|\psi_\nu| < 1\} \subset Z(\psi_\mu)$. Suppose that $\mu_\beta \ne 0$. Then $\nu \wedge \mu_\beta \ne 0$. We have $\{|\psi_{\nu\wedge\mu_\beta}| < 1\} \subset \{|\psi_\nu| < 1\} \subset Z(\psi_\mu)$. This contradicts Lemma 5.3 (v). Hence $\mu_\beta = 0$ and $\mu_\alpha = \mu$. By Theorem 5.1, μ satisfies (A) and (B). ∎

By [17, Section 5], there exists a measure $\nu \in M_{s,d}^+$ which has an outer vanishing measure. Applying Corollary 5.2, we get a measure $\mu \in M_{s,d}^+$ satisfying conditions (A) and (B). In the same way, by Theorem 4.2 there exists $\mu \in M_{s,c}^+$ satisfying conditions (A), (B), and $|S(\mu)| = 0$.

By Theorem 4.1 (iii), we show an existence of a measure $\mu \in M_{s,c}^+$ such that $\{|\psi_\nu| < 1\} \not\subset Z(\psi_\mu)$ for every $\nu \in L_+^1(\mu)$. For this μ, we have $\mu_\alpha = 0$. Hence μ does not satisfy conditions (A) and (B).

Relating to Theorem 5.1, we have the following problem.

Problem 5.1. Does there exist $\mu \in M_s^+$ such that $Z(\psi_{\mu_\alpha}) = Z(\psi_\mu)$ and $\mu_\beta \ne 0$?

In this paper, we have two decompositions $\mu = \mu_a + \mu_b = \mu_\alpha + \mu_\beta$ for $\mu \in M_s^+$. We shall give an example of $\mu \in M_{s,d}^+$ such that $\mu_a = \mu$ and $\mu_\alpha = 0$.

Example 5.1. Let $\{\zeta_j\}_j$ be a distinct sequence in ∂D such that $\{\zeta_j\}_j$ is dense in ∂D. Let $\{a_j\}_j$ be a sequence of positive numbers such that $\sum_{j=1}^\infty a_j < \infty$. Let $\mu = \sum_{j=1}^\infty a_j \delta_{\zeta_j}$. Then $\mu \in M_{s,d}^+$, and by Corollary 2.2 we have $\mu_a = \mu$.

We show the existence of $\{a_n\}_n$ such that $\mu_\alpha = 0$. Let $\nu_1 = \sum_{j=1}^\infty (1/2)^j \delta_{\zeta_j}$. Put $a_1 = 1/2$. By [14, Theorem 4.3], there exists a sequence of positive numbers $\{c_{2,j}\}_{j\ge 2}$ such that $\{|\psi_{\delta_{\zeta_1}}| < 1\} \cap Z(\psi_{\nu_2}) = \emptyset$ and $\nu_2 \le \nu_1$, where $\nu_2 = \sum_{j=2}^\infty c_{2,j}\delta_{\zeta_j}$. Put $a_2 = c_{2,2}$. Then there exists a sequence of positive numbers $\{c_{3,j}\}_{j\ge 3}$ such that $\{|\psi_{\delta_{\zeta_2}}| < 1\} \cap Z(\psi_{\nu_3}) = \emptyset$ and $\nu_3 \le \nu_2$, where $\nu_3 = \sum_{j=3}^\infty c_{3,j}\delta_{\zeta_j}$. Put $a_3 = c_{3,3}$. Repeat the above argument. Then for each positive integer n, there exists a sequence of positive numbers $\{c_{n,j}\}_{j\ge n}$ such that

$$\{|\psi_{\delta_{\zeta_{n-1}}}| < 1\} \cap Z(\psi_{\nu_n}) = \emptyset \tag{5.28}$$

and $\nu_n \le \nu_{n-1}$, where $\nu_n = \sum_{j=n}^\infty c_{n,j}\delta_{\zeta_j}$.

Put $a_n = c_{n,n}$. As a consequence, we have a sequence $\{a_n\}_n$. Put

$$\mu = \sum_{j=1}^{\infty} a_j \delta_{\zeta_j} \quad \text{and} \quad \mu_n = \Big(\sum_{j=1}^{n} a_j \delta_{\zeta_j}\Big) + \nu_{n+1}. \tag{5.29}$$

Since $\nu_n \le \nu_{n-1}$, we have $\mu \le \mu_n$ for every n. To prove $\mu_\alpha = 0$, suppose not. Then there exists $\nu \in M_s^+$ such that $0 \ne \nu \le \mu$ and $\{|\psi_\nu| < 1\} \subset Z(\psi_\mu)$. Then there exists a positive integer n_0 such that $\nu(\{\zeta_{n_0}\}) > 0$. Hence, $\{|\psi_{\delta_{\zeta_{n_0}}}| < 1\} \subset \{|\psi_\nu| < 1\} \subset Z(\psi_\mu) \subset Z(\psi_{\mu_{n_0}})$. Therefore we have

$$\begin{aligned}\{|\psi_{\delta_{\zeta_{n_0}}}| < 1\} &= \{|\psi_{\delta_{\zeta_{n_0}}}| < 1\} \cap Z(\psi_{\mu_{n_0}}) \\ &= \{|\psi_{\delta_{\zeta_{n_0}}}| < 1\} \cap \Big(\Big(\bigcup_{j=1}^{n_0} Z(\psi_{\delta_{\zeta_j}})\Big) \cup Z(\psi_{\nu_{n_0+1}})\Big) \quad \text{by (5.29)} \\ &= \{|\psi_{\delta_{\zeta_{n_0}}}| < 1\} \cap Z(\psi_{\delta_{\zeta_{n_0}}}) \quad \text{by (5.28)} \\ &= Z(\psi_{\delta_{\zeta_{n_0}}}).\end{aligned}$$

Thus we get $\{|\psi_{\delta_{\zeta_{n_0}}}| < 1\} = Z(\psi_{\delta_{\zeta_{n_0}}})$. This is a contradiction.

We have the following problem.

Problem 5.2. Does there exist $\mu \in M_{s,c}^+$ such that $\mu_a = \mu$ and $\mu_\alpha = 0$?

References

[1] Axler, S., Gorkin, P.: Divisibility in Douglas algebras. *Michigan Math. J.* **31** (1984), 89–94.

[2] Carleson, L.: Interpolation by bounded analytic functions and the corona problem. *Ann. of Math.* **76** (1962), 547–559.

[3] Chang, S.-Y.: A characterization of Douglas subalgebras. *Acta Math.* **137** (1976), 81–89.

[4] Garnett, J.: *Bounded Analytic Functions*. Academic Press, 1981.

[5] Gorkin, P.: Singular functions and division in $H^\infty + C$. *Proc. Amer. Math. Soc.* **92** (1984), 268–270.

[6] Guillory, C., Izuchi, K., Sarason, D.: Interpolating Blaschke products and division in Douglas algebras. *Proc. Roy. Irish Acad. Sect. A* **84** (1984), 1–7.

[7] Guillory, C., Sarason, D.: Division in $H^\infty + C$. *Michigan Math. J.* **28** (1981), 173–181.

[8] Hoffman, K.: *Banach Spaces of Analytic Functions*. Prentice Hall, Englewood Cliffs, 1962.

[9] Hoffman, K.: Bounded analytic functions and Gleason parts. *Ann. of Math. (2)* **86** (1967), 74–111.

[10] Izuchi, K.: Countably generated Douglas algebras. *Trans. Amer. Math. Soc.* **299** (1987), 171–192.

[11] Izuchi, K.: Factorization of Blaschke products. *Michigan Math. J.* **40** (1993), 53–75.

[12] Izuchi, K.: Interpolating Blaschke products and factorization theorems. *J. London Math. Soc. (2)* **50** (1994), 547–567.

[13] Izuchi, K.: Weak infinite powers of Blaschke products. *J. Anal. Math.* **75** (1998), 135–154.

[14] Izuchi, K.: Singular inner functions of L^1-type II. *J. Math. Soc. Japan* **53** (2001), 285–305.

[15] Izuchi, K.: Common zero sets of equivalent singular measures, preprint.

[16] Izuchi, K., Izuchi, Y.: Inner functions and division in Douglas algebras. *Michigan Math. J.* **33** (1986), 435–443.

[17] Izuchi, K., Niwa, N.: Singular inner functions of L^1-type. *J. Korean Math. Soc.* **36** (1999), 787–811.

[18] Marshall, D.: *Approximation and interpolation by inner functions*. Thesis, University of California, Los Angeles, 1976.

[19] Marshall, D.: Subalgebras od L^∞ containing H^∞. *Acta Math.* **137** (1976), 91–98.

[20] Sarason, D.: Algebras of functions on the unit circle. *Bull. Amer. Math. Soc.* **79** (1973), 286–299.

[21] Younis, R.: Division in Douglas algebras and some applications. *Arch. Math.* **45** (1985), 550–560.

[22] Ziskind, S.: Interpolating sequences and Shilov boundary of $H^\infty(\Delta)$. *J. Funct. Anal.* **21** (1976), 380–388.

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