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L^p Estimates for singular integrals with kernels belonging to certain block spaces

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Abstract

We establish the L^p boundedness of singular integrals with kernels which belong to block spaces and are supported by subvarieties.

1. Introduction

Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma\,(u)$. Let $K\,(\cdot)$ be a singular kernel defined by

$$K\,(y) = \Omega\,(y)\,|y|^{-n}\,h\,(|y|)\ , \tag{1.1}$$

where h is a measurable function on $\mathbb{R}^+$ and Ω is a homogeneous function of degree 0 which satisfies $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega\,(u)\,d\sigma\,(u) = 0. \tag{1.2}$$

For $\gamma > 1$, let $\Delta_\gamma\,(\mathbb{R}^+)$ denote the set of all measurable functions h on $\mathbb{R}^+$ such that

$$\sup_{R>0} \frac{1}{R} \int_0^R |h\,(t)|^\gamma\,dt < \infty. \tag{1.3}$$

It is easy to see that the following inclusions hold and are proper.

$$L^\infty\left(\mathbb{R}^+\right) \subset \Delta_{\gamma_2}\left(\mathbb{R}^+\right) \subset \Delta_{\gamma_1}\left(\mathbb{R}^+\right) \qquad \text{for } \gamma_1 < \gamma_2. \tag{1.4}$$

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Let $\mathcal{P} = (P_1, \ldots, P_m)$ be a mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ with P_j being polynomials on $\mathbb{R}^n$ for $1 \leq j \leq m$. To $\mathcal{P}$ we associate a singular integral operator $T = T_{\mathcal{P},h}$ and a related maximal operator $\mathcal{M}_{\mathcal{P},\Omega}$ defined initially for C_0^∞ functions on $\mathbb{R}^m$ as follows:

$$(1.5) \qquad Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f\left(x - \mathcal{P}\left(u\right)\right) K\left(u\right) du\ ,$$

$$(1.6) \qquad \mathcal{M}_{\mathcal{P},\Omega} f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\leq r} \left|f(x - \mathcal{P}\left(y\right)\right| \left|\Omega\left(y\right)\right| \left|h(|y|)\right| dy.$$

Also, we define the maximal truncated singular integral operator T^* by

$$(1.7) \qquad T^* f\left(x\right) = \sup_{\varepsilon>0} \left| \int_{|u|>\varepsilon} f\left(x - \mathcal{P}\left(u\right)\right) K\left(u\right) du \right|$$

Whenever $m = n$ and $\mathcal{P}\left(y\right) = (y_1, \ldots, y_n)$ we shall denote T by $T_{c,h}$ and T^* by $T^*_{c,h}$. Namely,

$$(1.8) \qquad T_{c,h} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f\left(x - y\right) K\left(y\right) dy\,,$$

and

$$(1.9) \qquad T^*_{c,h} f(x) = \sup_{\varepsilon>0} \left| \int_{|u|>\varepsilon} f\left(x - y\right) K\left(y\right) dy \right|.$$

There has been a considerable amount of research concerning the L^p boundedness of T and T^*. For relevant results one may consult [3], [4], [6], [7], [11], [10], [15], [16], [19], among others. We shall content ourselves here with recalling only the following pertinent results:

Jiang and Lu introduced a special class of block spaces $B_q^{\kappa,\upsilon}\left(\mathbf{S}^{n-1}\right)$ with respect to the study of the L^p mapping properties of the class of singular integral operators $T_{c,h}$ (see [13]). In fact, they obtained the following L^2 boundedness result.

Theorem A ([13]) *Let K, $T_{c,h}$ and $T^*_{c,h}$ be given as in (1.1)-(1.2), and (1.8)-(1.9). Suppose that $h \in L^\infty\left(\mathbb{R}^+\right)$. For $n \geq 2$ we have*

(i) if $\Omega \in B_q^{0,0}\left(\mathbf{S}^{n-1}\right)$, then $T_{c,h}$ is a bounded operator on $L^2\left(\mathbb{R}^n\right)$;

and

(ii) if $\Omega \in B_q^{0,1}\left(\mathbf{S}^{n-1}\right)$, then $T^*_{c,h}$ is a bounded operator on $L^2\left(\mathbb{R}^n\right)$.

One of our main results in this paper is that the L^p boundedness of T and T^* hold for arbitrary polynomial mappings $\mathcal{P}$ and Ω's in $B_q^{0,0}(\mathbf{S}^{n-1})$, $q > 1$. By specializing into the case $\mathcal{P}(y) \equiv y$, one obatins that $T_{c,h}$ and $T^*_{c,h}$ are bounded on L^p for all $p \in (1, \infty)$ and $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$, $q > 1$, which improve Theorem A in both the range of p (in *(i)* and *(ii)*) and Ω (in *(ii)*).

They can also be considered as improvements over the L^p boundedness theorems obtained independently by Duoandikoetxea-Rubio de Francia [7] and Namazi [15] under the stronger condition that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. After the completion of our paper, we learned that these results on the operators $T_{c,h}$ and $T^*_{c,h}$ had also been obtained by Al-Hasan and Fan (see [1] and [2]).

In addition, we shall improve a result in [10] dealing with operators associated to a special class of polynomial mappings. Let us first recall the relevant results in [10].

Theorem B ([10]) *Let T, T^* and K be given as in (1.1)-(1.2), (1.5), and (1.7). Suppose that $\Omega \in H^1(\mathbf{S}^{n-1})$ (the Hardy space on the unit sphere in the sense of Coifman and Weiss [5]).*

(i) If $h \in \Delta_\gamma(\mathbb{R}^+)$ for some and $\gamma > 1$, then for $\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$ there exists a constant $C_p > 0$ such that

$$\|T(f)\|_{L^p(\mathbb{R}^m)} \le C_p \|\Omega\|_{H^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^m)}$$

for any $f \in L^p(\mathbb{R}^m)$.

(ii) If $h \in L^\infty(\mathbb{R}^+)$, then for $1 < p < \infty$, there exists a constant $C_p > 0$ such that

$$\|T^*(f)\|_{L^p(\mathbb{R}^m)} \le C_p \|\Omega\|_{H^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^m)}$$

for any $f \in L^p(\mathbb{R}^m)$.

In both (i) and (ii) the constant C_p may depend on $n, m, h(\cdot)$ and $\deg(P_j)$, *but it is independent of the coefficients of $\{P_j\}$.*

Clearly, the range for p in Theorem B becomes a tiny open interval around 2 as γ approaches 1. Fan and Pan showed that, if $\Omega \in L^q(\mathbf{S}^{n-1})$ and $\mathcal{P}$ lies in a certain class of mappings, then the L^p boundedness of T and T^* can be preserved for the full range $1 < p < \infty$, regardless how close γ is to 1.

Theorem C ([10]) *Let K, T, $\mathcal{M}_{\mathcal{P},\Omega}$, and T^* be given as in (1.1)-(1.2), and (1.5)-(1.8). Suppose that $\mathcal{P} \in \mathcal{F}(n,m)$, $\Omega \in L^q(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $q > 1$ and $\gamma > 1$. Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that*

$$\begin{array}{rrcl} (i) & \|T(f)\|_{L^p(\mathbb{R}^m)} & \leq & C_p \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^m)}; \\ (ii) & \|\mathcal{M}_{\mathcal{P},\Omega}\|_{L^p(\mathbb{R}^m)} & \leq & C_p \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^m)}; \\ (iii) & \|T^*(f)\|_{L^p(\mathbb{R}^m)} & \leq & C_p \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^m)}; \end{array}$$

for any $f \in L^p(\mathbb{R}^m)$. Furthermore, if $\mathcal{P}(-x) = -\mathcal{P}(x)$, $x \in \mathbb{R}^n$, then the constant C_p may depend on n, m, $h(\cdot)$, $\deg(P_j$, but it is independent of the coefficients of $\{P_j\}$.

The class $\mathcal{F}(n,m)$ contains the class of odd polynomial mappings as a proper subset. Its definition will be reviewed in Section 7. A question which arises naturally in light of Theorem C is the following:

Question: *Does the L^p boundedness of the operators in Theorem C still hold under a weaker condition on Ω for $1 < p < \infty$?*

We use the method of block decomposition for functions to obtain an answer to this question. The actual statements of our results will be given in the next section.

We would like to thank the referee for some helpful comments.

2. Statements of results

We shall start with the following result, which gives the L^p boundedness of the operator T whose kernel is allowed to be very rough on the unit sphere as well as in the radial direction. In fact, we have the following:

Theorem 2.1 *Let T and K be given as in (1.1)-(1.2), and (1.5). Suppose:*

(i) $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ for some $q > 1$; and

(ii) $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$.

Then for any p satisfying $\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$ there exists a constant $C_p > 0$ such that

$$\|T(f)\|_{L^p(\mathbb{R}^m)} \leq C_p \|f\|_{L^p(\mathbb{R}^m)} \tag{2.1}$$

for any $f \in L^p(\mathbb{R}^m)$. The constant C_p may depend on n, m, $h(\cdot)$ and $\deg(P_j)$, but it is independent of the coefficients of $\{P_j\}$.

It is worth noting that the range of p given in Theorem 2.1 is the full range $(1, \infty)$ whenever $\gamma \geq 2$.

For T^* we have the following:

Theorem 2.2 *Let T^* and K be given as in (1.1)-(1.2), and (1.7). Suppose that:*

(i) $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ for some $q > 1$; (ii) and $h \in L^\infty(\mathbb{R}^+)$.

Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that

$$\|T^*(f)\|_{L^p(\mathbb{R}^m)} \le C_p \|f\|_{L^p(\mathbb{R}^m)} \tag{2.2}$$

for any $f \in L^p(\mathbb{R}^m)$. The constant C_p may depend on n, m, $h(\cdot)$ and $\deg(P_j)$, but it is independent of the coefficients of $\{P_j\}$.

With regard to the special class of polynomial mappings $\mathcal{F}(n,m)$ we have the following result.

Theorem 2.3 *Let K, T, $\mathcal{M}_{\mathcal{P},\Omega}$ and T^* be given as in (1.1)-(1.2), and (1.5)-(1.7). Suppose that:*

(i) $\mathcal{P} \in \mathcal{F}(n,m)$;

(ii) $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ for some $q > 1$;

(iii) and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$.

Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that

$$\|T(f)\|_{L^p(\mathbb{R}^m)} \le C_p \|f\|_{L^p(\mathbb{R}^m)}, \tag{2.3}$$

$$\|\mathcal{M}_{\mathcal{P},\Omega}(f)\|_{L^p(\mathbb{R}^m)} \le C_p \|f\|_{L^p(\mathbb{R}^m)}, \tag{2.4}$$

and

$$\|T^*(f)\|_{L^p(\mathbb{R}^m)} \le C_p \|f\|_{L^p(\mathbb{R}^m)} \tag{2.5}$$

for any $f \in L^p(\mathbb{R}^m)$. Furthermore, if $\mathcal{P}(-x) = -\mathcal{P}(x)$, $x \in \mathbb{R}^n$, then the constant C_p may depend on n, m, $h(\cdot)$ and $\deg(P_j)$, but it is independent of the coefficients of $\{P_j\}$.

One observes that Theorem 2.3 represents an improvement over Theorem C because Ω is allowed to be in the space $B_q^{0,0}(\mathbf{S}^{n-1})$ and bearing in mind the following relation

$$L^q\left(\mathbf{S}^{n-1}\right) \subset B_q^{0,0}\left(\mathbf{S}^{n-1}\right).$$

Also, notice that for the class of odd polynomials, Theorem 2.3 gives the L^p boundedness of the operator T for the full range $1 < p < \infty$ which is much better than the range $\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$ as $\gamma \to 1$ if we apply Theorem 2.1. Furthermore, Theorem 2.3 gives that T^* is bounded on L^p for $1 < p < \infty$ even if $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ which is considerably better than the condition $h \in L^\infty(\mathbb{R}^+)$ as assumed in both Theorem A (for the special class of operators $T^*_{c,h}$) and Theorem 2.2.

3. Some technical lemmas

All the results are obtained on the basis of the following general lemmas. These lemmas are similar in spirit to the general results established in [7] and [10].

By following exactly the proofs of Theorem B in [7] and Lemma 5.2 in [8] and keeping track of the constants we obtain the following:

Lemma 3.1 *Let* $n,\ N \in \mathbb{N}$, $\{a_l : 1 \le l \le N\} \subseteq \mathbb{R}^+ \setminus \{1\}$, $\{\alpha_l : 1 \le l \le N\} \subseteq \mathbb{R}^+$, $\{m_l : 1 \le l \le N\} \subseteq \mathbb{N}$, *and let* $L_l : \mathbb{R}^n \to \mathbb{R}^{m_l}$ *linear transformations for* $1 \le l \le N$. *Let* $\{\sigma_k^{(l)} : 0 \le l \le N, k \in \mathbb{Z}\}$ *be a family of Borel measures which satisfies the following: For all* $k \in \mathbb{Z}$, $1 \le l \le N$, $\xi \in \mathbb{R}^n$ *with* $\xi \ne 0$ *and for some constant* $A > 1$, $p_0 \in (2, \infty)$,

$$(i) \qquad \sigma_k^{(0)} = 0\,;$$

$$(ii) \qquad \left\|\sigma_k^{(l)}\right\| \le 1;$$

$$(iii) \qquad \left|\hat{\sigma}_k^{(l)}(\xi)\right| \le C\left|a_l^k L_l(\xi)\right|^{-\alpha_l/A}\ ;$$

$$(iv) \qquad \left|\hat{\sigma}_k^{(l)}(\xi) - \hat{\sigma}_k^{(l-1)}(\xi)\right| \le C\left|a_l^k L_l(\xi)\right|^{\alpha_l/A}\ ;$$

$$(3.1) \qquad (iv) \qquad \left\|\Big(\sum_{k\in\mathbb{Z}}\left|\sigma_k^{(l)} * g_k\right|^2\Big)^{\frac{1}{2}}\right\|_{p_0} \le C\left\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{\frac{1}{2}}\right\|_{p_0}$$

for arbitrary functions $\{g_k\}$ *on* $\mathbb{R}^n$.

Then for $p_0' < p < p_0$ *there exists a constant* C_p *such that*

$$(3.2) \qquad \left\|\sum_{k\in\mathbb{Z}}\sigma_k^{(N)} * f\right\|_{L^p(\mathbb{R}^n)} \le C_p A\,\|f\|_{L^p(\mathbb{R}^n)}$$

and

$$(3.3) \qquad \left\|\Big(\sum_{k\in\mathbb{Z}}\left|\sigma_k^{(N)} * f\right|^2\Big)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^n)} \le C_p A\,\|f\|_{L^p(\mathbb{R}^n)}$$

for all f *in* $L^p(\mathbb{R}^n)$. *The constant* C_p *is independent of* A *and the linear transformations* $\{L_l\}_{l=1}^N$.

The above lemma will be used in the proof of Theorem 2.1. To prove Theorem 2.2 and Theorem 2.3 we need to take a somewhat different approach. We first need a little more notation. Let η be a fixed positive integer. For $1 \le s \le \eta$ we define the projection operator $\pi_s^\eta : \mathbb{R}^\eta \to \mathbb{R}^s$ by $\pi_s^\eta(\xi) = (\xi_1, \dots, \xi_s)$. Also, let $t^{\pm\alpha} = \inf(t^\alpha,\ t^{-\alpha})$.

Lemma 3.2 *Let $\{\sigma_k : k \in \mathbb{Z}\}$ be a sequence of Borel measures on $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that for all $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$, for some $a \geq 2$, $\alpha, C > 0$, $A > 1$ and $p_0 \in (2, \infty)$ we have*

$$(i) \qquad |\hat{\sigma}_k(\xi)| \leq CA(a^{kA}\,|L(\xi)|\,)^{\pm\frac{\alpha}{A}}$$

$$(3.4) \qquad (ii) \qquad \Big\|(\sum_{k\in\mathbb{Z}} |\sigma_k * g_k|^2)^{\frac{1}{2}}\Big\|_{p_0} \leq CA\Big\|(\sum_{k\in\mathbb{Z}} |g_k|^2)^{\frac{1}{2}}\Big\|_{p_0}$$

for arbitrary functions $\{g_k\}$ on $\mathbb{R}^n$.

Then for $p_0' < p < p_0$ there exists a positive constant C_p such that

$$(3.5) \qquad \Big\|\sum_{k\in\mathbb{Z}} \sigma_k * f\Big\|_{L^p(\mathbb{R}^n)} \leq C_p A\,\|f\|_{L^p(\mathbb{R}^n)}$$

and

$$(3.6) \qquad \Big\|(\sum_{k\in\mathbb{Z}} |\sigma_k * f|^2)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \leq C_p A\,\|f\|_{L^p(\mathbb{R}^n)}$$

hold for all f in $L^p(\mathbb{R}^n)$. The constant C_p is independent of A and the linear transformation L.

Proof. We shall use a variation of the methods in [7] and [10]. Without loss of generality, we may assume that $0 < \alpha \leq 1$. By the arguments in the proof of Lemma 6.2 in [10], we may assume without loss of generality that $m \leq n$ and $L = \pi_m^n$. Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the intervals $[a^{-(j+1)A}, a^{-(j-1)A}]$. More precisely, we require the following:

$$\Phi_j \in C^\infty,\ 0 \leq \Phi_j \leq 1,\ \sum_j [\Phi_j(t)]^2 = 1,$$

$$\text{supp } \Phi_j \subseteq \left\{t : a^{-(j+1)A} < t < a^{-(j-1)A}\right\}$$

and

$$\left|\frac{d^s\Phi_j(t)}{dt^s}\right| \leq \frac{C}{t^s}$$

where C can be chosen to be independent of the lacunary sequence $\{a^{jA}\}$ (We would like to thank Ahmad Al-Salman for a useful discussion about the construction of such partition of unity).

Let $F(f) = \sum_{k\in\mathbb{Z}} \sigma_k * f$ and let S_k be the multiplier operators in $\mathbb{R}^n$ given by

$$(\widehat{S_k f})(\xi) = \Phi_k(|\pi_m^n\,\xi|)\,\hat{f}(\xi).$$

Define $F_j(f) = \sum_{k\in\mathbb{Z}} S_{k+j}(\sigma_k * S_{k+j}\ f)$. Then it is easy to see that the following identity

$$F(f) = \sum_{j\in\mathbb{Z}} F_j(f)$$

holds for $f \in \mathcal{S}(\mathbb{R}^n)$. Now

$$\begin{aligned} \|F_j(f)\|_{p_0} &\le C\Big\|\Big(\sum_{k\in\mathbb{Z}} |\sigma_k * S_{k+j} f|^2\Big)^{\frac{1}{2}}\Big\|_{p_0} \\ &\le CA\Big\|\Big(\sum_{k\in\mathbb{Z}} |S_{k+j} f|^2\Big)^{\frac{1}{2}}\Big\|_{p_0} \le CA\|f\|_{p_0} \qquad (3.7) \end{aligned}$$

for all $j \in \mathbb{Z}$ and $f \in L^{p_0}(\mathbb{R}^n)$ with C independent of the essential variables. The middle inequality is a consequence of (3.4) whereas the first and the last inequalities follow from both Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [18], p. 96.

On the other hand, by Plancherel's theorem we have

$$\|F_j(f)\|_{L^2}^2 \le \sum_{k\in\mathbb{Z}} \int_{\Delta_{k,j}} \left|\hat{f}\,(\xi)\right|^2 |\hat{\sigma}_k\,(\xi)|^2\, d\xi$$

where

$$\Delta_{k,j} = \{\xi \in \mathbb{R}^n : (a^A)^{-k-j-1} \le |\pi_m^n \xi| < (a^A)^{-k-j+1}\}.$$

Then by (i) we get easily

$$\|F_j(f)\|_{L^2} \le CA\ a^{-\alpha|j|}\, \|f\|_{L^2}\,. \qquad (3.8)$$

Now, if $p \in (p_0', p_0)$, we have

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0} \quad \text{for some } \theta,\ 0 < \theta \le 1,$$

and by interpolating between estimates (3.7) and (3.8) we get

$$\|F(f)\|_p \le \sum_j \|F_j(f)\|_p \le CA \sum_j a^{-\alpha|j|}\, \|f\|_p = C_p A\, \|f\|_p\,.$$

The proof of (3.6) is similar. Alternatively, we can deduce it easily from the above proof by observing that for every sequence $\varepsilon = \{\varepsilon_k\}$, $\varepsilon_k = +1$ or -1, the linear operator

$$F_\varepsilon(f) = \sum_{k\in\mathbb{Z}} \varepsilon_k \sigma_k * f$$

has the same bound in L^p as that of $F(f)$ and this bound is independent of the sequence of signs $\{\varepsilon_k\}$. Then, the inequality (3.6) can be obtained by the usual argument using Rademacher functions. This completes the proof of our lemma. ■

Lemma 3.3 *Let $N \in \mathbb{N}$ and $\{\sigma_k^{(l)} : k \in \mathbb{Z}, 0 \le l \le N\}$ be a family of Borel measures on $\mathbb{R}^n$ with $\sigma_k^{(0)} = 0$ for every $k \in \mathbb{Z}$. Let $\{a_l : 1 \le l \le N\} \subseteq [2,\infty)$, $\{m_l : 1 \le l \le N\} \subseteq \mathbb{N}$, $\{\alpha_l : 1 \le l \le N\} \subseteq \mathbb{R}^+$, and let $L_l : \mathbb{R}^n \to \mathbb{R}^{m_l}$ be linear transformations for $1 \le l \le N$. Suppose that for all $k \in \mathbb{Z}$, $1 \le l \le N$, for all $\xi \in \mathbb{R}^n$ and for some $C > 0$, $A > 1$, $p_0 \in (2,\infty)$, we have the following:*

$$(i) \qquad \left\|\sigma_k^{(l)}\right\| \le CA\,;$$

$$(ii) \qquad \left|\hat{\sigma}_k^{(l)}(\xi)\right| \le CA\left|a_l^{kA}L_l(\xi)\right|^{-\alpha_l/A}\,;$$

$$(iii) \qquad \left|\hat{\sigma}_k^{(l)}(\xi) - \hat{\sigma}_k^{(l-1)}(\xi)\right| \le CA\left|a_l^{kA}L_l(\xi)\right|^{\alpha_l/A}\,;$$

$$(3.9) \qquad (iv) \qquad \Big\|\Big(\sum_{k\in\mathbb{Z}}\left|\sigma_k^{(l)} * g_k\right|^2\Big)^{\frac{1}{2}}\Big\|_{p_0} \le CA\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{\frac{1}{2}}\Big\|_{p_0}$$

holds for all functions $\{g_k\}$ on $\mathbb{R}^n$.

Then for $p_0' < p < p_0$ there exists a positive constant C_p such that

$$(3.10) \qquad \Big\|\sum_{k\in\mathbb{Z}}\sigma_k^{(N)} * f\Big\|_{L^p(\mathbb{R}^n)} \le C_pA\,\|f\|_{L^p(\mathbb{R}^n)}$$

and

$$(3.11) \qquad \Big\|\Big(\sum_{k\in\mathbb{Z}}\left|\sigma_k^{(N)} * f\right|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \le C_pA\,\|f\|_{L^p(\mathbb{R}^n)}$$

hold for all f in $L^p(\mathbb{R}^n)$. The constant C_p is independent of the linear transformations $\{L_l\}_{l=1}^N$.

Proof. The idea of the proof will be very similar to the one appearing in the proof of Theorem 7.6 in [10]. Without loss of generality, we may assume that $0 < \alpha_l \le 1$, $m_l \le n$ and $L_l = \pi_{m_l}^n$ for $1 \le l \le N$.

Define the sequence of measures $\{\lambda_k^{(l)} : 1 \le l \le N, k \in \mathbb{Z}\}$ as follows: choose and fix a function $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta(t) = 1$ for $|t| \le \frac{1}{2}$ and $\theta(t) = 0$ for $|t| \ge 1$. Let $\zeta(t) = \theta(t^2)$ and for $k \in \mathbb{Z}$, let

$$(3.12) \quad \hat{\lambda}_k^{(l)}(\xi) = \hat{\sigma}_k^{(l)}(\xi)\prod_{l<i\le N}\zeta(a_i^{kA}\left|\pi_{m_i}^n\xi\right|) - \hat{\sigma}_k^{(l-1)}(\xi)\prod_{l-1<i\le N}\zeta(a_i^{kA}\left|\pi_{m_i}^n\xi\right|)$$

when $1 \le l \le N-1$ and

$$(3.13) \qquad \hat{\lambda}_k^{(N)}(\xi) = \hat{\sigma}_k^{(N)}(\xi) - \hat{\sigma}_k^{(N-1)}(\xi)\;\zeta(a_N^{kA}\left|\pi_{m_N}^n\,\xi\right|).$$

By the assumptions of the lemma one obtains that

$$\left|\hat{\lambda}_k^{(l)}(\xi)\right| \leq \; CA(a_l^{kA}\left|\pi_{m_l}^n(\xi)\right|)^{\pm\alpha_l/A} \tag{3.14}$$

for all $1 \leq l \leq N$. By condition (iv), it is easy to see that

$$\Big\|(\sum_{k\in\mathbb{Z}}\left|\lambda_k^{(l)} * g_k\right|^2)^{\frac{1}{2}}\Big\|_{p_0} \leq C_pA\Big\|(\sum_{k\in\mathbb{Z}}|g_k|^2)^{\frac{1}{2}}\Big\|_{p_0} \tag{3.15}$$

for $1 \leq l \leq N$ and for arbitrary functions $\{g_k\}$ on $\mathbb{R}^n$. By (3.14)-(3.15) and Lemma 3.2, for $p_0' < p < p_0$, there exists a positive constant C_p such that

$$\Big\|\sum_{k\in\mathbb{Z}}\lambda_k^{(l)} * f\Big\|_{L^p(\mathbb{R}^n)} \leq C_pA\,\|f\|_{L^p(\mathbb{R}^n)} \tag{3.16}$$

and

$$\Big\|(\sum_{k\in\mathbb{Z}}\left|\lambda_k^{(l)} * f\right|^2)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \leq C_pA\,\|f\|_{L^p(\mathbb{R}^n)} \tag{3.17}$$

hold for all f in $L^p(\mathbb{R}^n)$. By (3.16)-(3.17) and observing that

$$\sigma_k^{(N)} = \sum_{l=1}^{N}\lambda_k^{(l)} \tag{3.18}$$

we get (3.10)-(3.11). This completes the proof of the lemma. ■

By a quick examination of the proof given in [7], page 544, it is easy to see that the following result holds.

Lemma 3.4 *Let $\{\Upsilon_k\}$ be a sequence of Borel measures in $\mathbb{R}^n$ and let Υ^* be the maximal operator given by $\Upsilon^*(f) = \sup_{k\in\mathbb{Z}}||\Upsilon_k| * f|$. Assume that*

$$\|\Upsilon^*(f)\|_q \leq C\,\|f\|_q \text{ for some } q > 1 \text{ and } C > 0. \tag{3.19}$$

Then the following vector valued inequality

$$\Big\|(\sum_{k\in\mathbb{Z}}|\Upsilon_k * g_k|^2)^{\frac{1}{2}}\Big\|_{p_0} \leq \sqrt{C\sup_{k\in\mathbb{Z}}\|\Upsilon_k\|}\;\;\Big\|(\sum_{k\in\mathbb{Z}}|g_k|^2)^{\frac{1}{2}}\Big\|_{p_0} \tag{3.20}$$

holds for $\left|\frac{1}{p_0} - \frac{1}{2}\right| = \frac{1}{2q}$ and for arbitrary functions $\{g_k\}$ on $\mathbb{R}^n$.

Remark: It is worth pointing out that the constant on the right hand side of the original version of (3.20) in [7] didn't appear explicitly in the form of $\left(C \sup_{k\in\mathbb{Z}} \|\Upsilon_k|\right)^{1/2}$, since it is not significant for the applications given there. However, this newly introduced constant will play a major and indispensable role in the proofs of both Theorems 2.2 and 2.3.

For a given sequence of measures $\{\sigma_k^{(l)} : k \in \mathbb{Z},\ 0 \le l \le N\}$, define S_k and S_* by

$$S_k f(x) = \sum_{j=k}^{\infty} \sigma_j^{(N)} * f(x)$$

and

$$S_* f(x) = \sup_{k\in\mathbb{Z}} |S_k f(x)| .$$

To study the L^p boundedness of the maximal truncated singular integral we need to establish the following result.

Lemma 3.5 *Let $N \in \mathbb{N}$ and $\{\sigma_k^{(l)} : k \in \mathbb{Z}, 0 \le l \le N\}$ be a family of Borel measures on $\mathbb{R}^n$ with $\sigma_k^{(0)} = 0$ for every $k \in \mathbb{Z}$. Let $\{a_l : 1 \le l \le N\} \subseteq [2,\infty)$, $\{m_l : 1 \le l \le N\} \subseteq \mathbb{N}$, $\{\alpha_l : 1 \le l \le N\} \subseteq \mathbb{R}^+$, and let $L_l : \mathbb{R}^n \to \mathbb{R}^{m_l}$ be linear transformations for $1 \le l \le N$. Suppose that for all $k \in \mathbb{Z}$, $1 \le l \le N$, for all $\xi \in \mathbb{R}^n$ and for some $C > 0$, $A > 1$, we have*

$$(i) \qquad \left\|\sigma_k^{(l)}\right\| \le CA ;$$

$$(ii) \qquad \left|\hat{\sigma}_k^{(l)}(\xi)\right| \le CA \left|a_l^{kA} L_l(\xi)\right|^{-\alpha_l/A} ;$$

$$(iii) \qquad \left|\hat{\sigma}_k^{(l)}(\xi) - \hat{\sigma}_k^{(l-1)}(\xi)\right| \le CA \left|a_l^{kA} L_l(\xi)\right|^{\alpha_l/A} .$$

Assume that

$$\left\|\sigma^{*(l)} f\right\|_p \le C_p A \|f\|_p \tag{3.21}$$

for $1 < p < \infty$ and every $f \in L^p(\mathbb{R}^n)$, where $\sigma^{*(l)}(f) = \sup_{k\in\mathbb{Z}} \left|\left|\sigma_k^{(l)}\right| * f\right|$, $1 \le l \le N$.

Then for every $1 < p < \infty$ there exists a constant $C_p > 0$ which is independent of the linear transformations $\{L_l\}$ such that

$$\|S_* f\|_p \le C_p A \|f\|_p \tag{3.22}$$

for every $f \in L^p(\mathbb{R}^n)$.

Proof. As above, there is no loss of generality in assuming that $0<\alpha_l\leq 1$, $m_l\leq n$ and $L_l=\pi^n_{m_l}$. Let $\{\lambda_k^{(l)}: 1\leq l\leq N,\ k\in\mathbb{Z}\}$ be defined by (3.12)-(3.13). For each $1\leq l\leq N$, let

$$\begin{aligned} S^{(l)}f(x) &= \sum_{k=-\infty}^{\infty}\lambda_k^{(l)}*f(x), \quad \lambda^{*(l)}(f)=\sup_{k\in\mathbb{Z}}\left|\left|\lambda_k^{(l)}\right|*f\right|, \\ S_k^{(l)}f(x) &= \sum_{j=k}^{\infty}\lambda_j^{(l)}*f(x) \quad \text{and} \quad S_*^{(l)}f(x)=\sup_{k\in\mathbb{Z}}\left|S_k^{(l)}f(x)\right|. \end{aligned}$$

By the definition of $\lambda_k^{(l)}$ and (3.21) we have

$$\left\|\lambda^{*(l)}f\right\|_p\leq C_pA\left\|f\right\|_p \tag{3.23}$$

for $1<p<\infty$, $f\in L^p(\mathbb{R}^n)$ and $1\leq l\leq N$. Then by (3.23) and Lemma 3.4 in conjunction with $\|\lambda_k^{(l)}\|\leq CA$ we get

$$\left\|\Big(\sum_{k\in\mathbb{Z}}\left|\lambda_k^{(l)}*g_k\right|^2\Big)^{\frac{1}{2}}\right\|_p\leq C_pA\left\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{\frac{1}{2}}\right\|_p \tag{3.24}$$

for $1<p<\infty$, $f\in L^p(\mathbb{R}^n)$, $1\leq l\leq N$ and for arbitrary functions $\{g_k\}$ on $\mathbb{R}^n$. By (3.14), (3.24) together with Lemma 3.2 we have

$$\left\|S^{(l)}f\right\|_p\leq C_pA\left\|f\right\|_p \tag{3.25}$$

for some constant $C_p>0$, $1<p<\infty$, $f\in L^p(\mathbb{R}^n)$ and $1\leq l\leq N$.

Now, by (3.18) we have

$$\left\|S_*f\right\|_p\leq\sum_{l=1}^{N}\left\|S_*^{(l)}f\right\|_p$$

and hence we need to prove only that

$$\left\|S_*^{(l)}f\right\|_p\leq C_pA\left\|f\right\|_p \tag{3.26}$$

for some constant C_p, $1<p<\infty$, $f\in L^p(\mathbb{R}^n)$ and $1\leq l\leq N$.

The proof of (3.26) follows now by using (3.14), (3.23), (3.25) and the same line of arguments as in the proof of Lemma 6.3 in [10]. We omit the details. The lemma is proved. ■

4. Block spaces on $\mathbf{S}^{n-1}$

The method of block decomposition for functions was invented by M. H. Taibleson and G. Weiss in their study of the convergence of the Fourier series (see [20]). Later on, many applications of the block decomposition to harmonic analysis were discovered (see [14], [17], etc.). For further background and information about the theory of spaces generated by blocks and its applications to harmonic analysis one can consult the book [13]. Let us first recall the definition of a block function on $\mathbf{S}^{n-1}$.

Definition 4.1 *For $1 < q \leq \infty$ we say that a measurable function $b(x)$ on $\mathbf{S}^{n-1}$ is a $q-$block if it satisfies the following:*

(i) $\operatorname{supp}(b) \subseteq I$ *where I is an interval on $\mathbf{S}^{n-1}$; i.e.,*

$$I = \left\{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \alpha\right\} \text{ with } x_0' \in \mathbf{S}^{n-1} \text{ and } \alpha > 0;$$

(ii) $\|b\|_{L^q} \leq |I|^{-1/q'}$ *where* $\frac{1}{q} + \frac{1}{q'} = 1$.

The class of block spaces $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$ for $\kappa \geq 0$ and $\upsilon \in \mathbb{R}$ is defined as follows.

Definition 4.2. $B_q^{\kappa,\upsilon} = B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1})$: $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ *where each c_μ is a complex number; each b_μ is a $q-$block supported in an interval I_μ; and $M_q^{\kappa,\upsilon}(\{c_\mu\}) < \infty\}$ where*

$$M_q^{\kappa,\upsilon}(\{c_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu| \left(1 + \phi_{\kappa,\upsilon}(|I_\mu|)\right)$$

and

$$\phi_{\kappa,\upsilon}(t) = \begin{cases} \displaystyle\int_t^1 u^{-1-\kappa} \log^{\upsilon}\left(u^{-1}\right) du, & \text{if } 0 < t < 1; \\ 0, & \text{if } t \geq 1. \end{cases} \tag{4.1}$$

One observes that $\phi_{\kappa,\upsilon}(t) \sim t^{-\kappa} \log^{\upsilon}(t^{-1})$ as $t \to 0$ for $\kappa > 0$, $\upsilon \in \mathbb{R}$, and $\phi_{0,\upsilon}(t) \sim \log^{\upsilon+1}(t^{-1})$ as $t \to 0$ for $\upsilon > -1$.

The following properties of $B_q^{\kappa,\upsilon}$ can be found in [13] and [12]: for $1 < q \leq \infty$ we have

$$B_q^{\kappa,\upsilon_2} \subset B_q^{\kappa,\upsilon_1} \quad (\upsilon_2 > \upsilon_1 > -1 \text{ and } \kappa \geq 0), \tag{4.2}$$

$$B_q^{\kappa_2,\upsilon_2} \subset B_q^{\kappa_1,\upsilon_1} \quad (\upsilon_i > -1, i = 1, 2, \text{ and } 0 \leq \kappa_1 < \kappa_2) \tag{4.3}$$

$$B_{q_2}^{\kappa,\upsilon} \subset B_{q_1}^{\kappa,\upsilon} \quad (1 < q_1 < q_2) \tag{4.4}$$

and

$$L^{q}\left(\mathbf{S}^{n-1}\right)\subseteq B_{q}^{\kappa,\upsilon}\left(\mathbf{S}^{n-1}\right)\quad(\text{for }\upsilon>-1,\ \text{and }\kappa\geq 0). \tag{4.5}$$

Also, Keitoku and Sato in [12] proved the following interesting results which give a clear relation between the spaces $B_{q}^{\kappa,\upsilon}$ and the L^{q}-space on the unit sphere:

Theorem 4.3 *(i) If $1<p\leq q\leq\infty$, then for $\kappa>\frac{1}{p'}$ we have*

$$B_{q}^{\kappa,\upsilon}\left(\mathbf{S}^{n-1}\right)\subseteq L^{p}\left(\mathbf{S}^{n-1}\right)\ \text{ for any }\upsilon>-1; \tag{4.6}$$

$$(ii)\ B_{q}^{\kappa,\upsilon}\left(\mathbf{S}^{n-1}\right)=L^{q}\left(\mathbf{S}^{n-1}\right)\ \text{ if and only if }\kappa\geq\frac{1}{q'}\ \text{ and }\upsilon\geq 0; \tag{4.7}$$

and

(iii) for any $\upsilon>-1$, we have

$$B_{q}^{0,\upsilon}\left(\mathbf{S}^{n-1}\right)\not\subseteq\bigcup_{p>1}L^{p}\left(\mathbf{S}^{n-1}\right). \tag{4.8}$$

5. Certain maximal functions

Let $E_k=\{x\in\mathbb{R}^n:2^k\leq|x|<2^{k+1}\}$. For suitable mappings $\Gamma:\mathbb{R}^n\to\mathbb{R}^m$ and $\vartheta:\mathbf{S}^{n-1}\to\mathbb{R}$, we define the measures $\{\sigma_{\Gamma,\vartheta,k}:k\in\mathbb{Z}\}$ and $\{|\sigma_{\Gamma,\vartheta,k}|:k\in\mathbb{Z}\}$ on $\mathbb{R}^m$ by

$$\int_{\mathbb{R}^m}f\,d\sigma_{\Gamma,\vartheta,k}=\int_{E_k}f\left(\Gamma\left(u\right)\right)h\left(\left|u\right|\right)\frac{\vartheta\left(u'\right)}{\left|u\right|^{n}}du \tag{5.1}$$

and

$$\int_{\mathbb{R}^m}f\ d\left|\sigma_{\Gamma,\vartheta,k}\right|=\int_{E_k}f\left(\Gamma\left(u\right)\right)\left|h\left(\left|u\right|\right)\right|\frac{\left|\vartheta\left(u'\right)\right|}{\left|u\right|^{n}}du. \tag{5.2}$$

Also, define the maximal operator $\sigma^{*}_{\Gamma,\vartheta}$ on $\mathbb{R}^m$ by

$$\sigma^{*}_{\Gamma,\vartheta}\left(f\right)=\sup_{k\in\mathbb{Z}}\left|\,\left|\sigma_{\Gamma,\vartheta,k}\right|*f\right|. \tag{5.3}$$

For $l\in\mathbb{N}$, let $\mathcal{A}_l$ denote the class of polynomials of l variables with real coefficients. Let $\mathcal{Q}\left(t\right)=\left(\mathcal{Q}_1\left(t\right),\ldots,\mathcal{Q}_m\left(t\right)\right)$ be a mapping defined on $\mathbb{R}$ with $\mathcal{Q}_i\in\mathcal{A}_1$ for $1\leq i\leq m$. Let

$$\mathcal{M}_{\mathcal{Q}}f\left(x\right)=\sup_{r>0}\frac{1}{r}\int_{|t|<r}\left|f\left(x-\mathcal{Q}\left(t\right)\right)\right|dt.$$

We need the following L^p boundedness result which can be found in [19], pp. 476-478.

Lemma 5.1 *For every $1 < p \leq \infty$, there exists a positive constant C_p such that*

$$\left\| \mathcal{M}_{\mathcal{Q}} f \right\|_p \leq \ C_p \left\| f \right\|_p$$

for $f \in L^p(\mathbb{R}^m)$. The constant C_p may depend on the degrees of the polynomials $\{\mathcal{Q}_i\}$, but it is independent of the coefficients of $\{\mathcal{Q}_i\}$.

Also, we shall need the following two results from [10], pp. 823-824.

Theorem 5.2 *Let $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let $\mathcal{Q}_x(t) = \Gamma(tx)$ for $t \in \mathbb{R}$, $x \in \mathbf{S}^{n-1}$. Suppose that $\vartheta \in L^1(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. If $\mathcal{Q}_x(\cdot) \in \mathcal{A}_1$ for every $x \in \mathbf{S}^{n-1}$, $\Lambda(\Gamma) = \sup_{x \in \mathbf{S}^{n-1}} \{\deg(\mathcal{Q}_x)\} < \infty$, then for any p, $\gamma' < p \leq \infty$, there exists a constant C_p such that*

$$\left\| \sigma^*_{\Gamma,\vartheta}(f) \right\|_p \leq C_p \left\| \vartheta \right\|_{L^1(\mathbf{S}^{n-1})} \left\| f \right\|_p$$

for $f \in L^p(\mathbb{R}^m)$. The constant C_p depends on p, n, m, $h(\cdot)$ and $\Lambda(\Gamma)$.

Theorem 5.3 *Let $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let $\mathcal{Q}_x(t) = \Gamma(tx)$ for $t \in \mathbb{R}$, $x \in \mathbf{S}^{n-1}$. Suppose that*

$$K(x) = \frac{\vartheta(x')}{|x|^n} h(|x|)$$

where $\vartheta \in L^1(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $1 < \gamma \leq 2$. If $\mathcal{Q}_x(\cdot) \in \mathcal{A}_1$ for every $x \in \mathbf{S}^{n-1}$ and

$$\Lambda(\Gamma) = \sup_{x \in \mathbf{S}^{n-1}} \{\deg(\mathcal{Q}_x)\} < \infty\,,$$

then for any $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{\gamma'}$, there exists a constant C_p such that

$$\left\| \Big(\sum_{k \in \mathbb{Z}} |\sigma_{\Gamma,\vartheta,k} * g_k|^2 \Big)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \vartheta \right\|_{L^1(\mathbf{S}^{n-1})} \left\| \Big(\sum_{k \in \mathbb{Z}} |g_k|^2 \Big)^{\frac{1}{2}} \right\|_p$$

holds for arbitrary functions $\{g_k\}$ on $\mathbb{R}^m$. The constant C_p depends on p, n, m, $h(\cdot)$ and $\Lambda(\Gamma)$.

6. Oscillatory integrals

For a positive integer l, we let $\mathcal{V}_l$ denotes the space of real-valued homogeneous polynomials of degree l on $\mathbb{R}^n$ and for $P(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$, we let $\|P\| = \sum_{|\alpha|=l} |a_\alpha|$. Let $Z_l^n : \mathcal{V}_l \to \mathcal{V}_l$ be the linear transformation defined as in [10], p. 807. The following result can be found in [10], p. 810.

Proposition 6.1 *Let $h \in \Delta_\gamma(\mathbb{R}^+)$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $\gamma > 1$, $q > 1$ and let $\omega = \min\{2, \gamma, q\}$. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is a function given by*

$$F(x) = \sum_{j=0}^{l} P_j(x) + W(|x|)$$

where $P_j(\cdot)$ is a homogeneous polynomial of degree j for $0 \le j \le l$ and $W(\cdot)$ is an arbitrary function. Let

$$\mathcal{J}_k(\Omega) = \int_{2^k}^{2^{k+1}} \int_{\mathbf{S}^{n-1}} e^{iF(tx)} \Omega(x) h(t) \; d\sigma(x) \frac{dt}{t}$$

for $k \in \mathbb{Z}$. Then we have

$$|\mathcal{J}_k(\Omega)| \le C \|\Omega\|_{L^q(\mathbf{S}^{n-1})} (2^{kl} \|Z_l^n(P_l)\|)^{-1/(4l\omega')}. \tag{6.1}$$

The constant C is independent of $\Omega(\cdot)$, k, $W(\cdot)$ and the coefficients of $\{P_j(\cdot)\}$.

If G is a subspace of $\mathcal{V}_l$ with $|x|^l \notin G$, then there exists a constant C' such that

$$|\mathcal{J}_k(\Omega)| \le C' \|\Omega\|_{L^q(\mathbf{S}^{n-1})} (2^{kl} \|P_l\|)^{-1/(4l\omega')} \tag{6.2}$$

holds for all $k \in \mathbb{Z}$ and F with $P_l(\cdot) \in G$. The constant C' may depend on the subspace G if l is even, but it is independent of G if l is odd.

One thing which makes working with block functions difficult is the lack of mean zero property. In order to remove this obstacle and elaborate on the proofs of certain known results on block spaces (see, for example, [9]), we find it is useful to introduce the following notion:

Definition 6.2 A function $\tilde{b}(\cdot)$ on $\mathbf{S}^{n-1}$ is called a $q-$*blocklike* function associated with an interval I on $\mathbf{S}^{n-1}$, $1 < q \le \infty$, if it satisfies the following conditions:

$$(i) \qquad \int_{\mathbf{S}^{n-1}} \tilde{b}(u)\, d\sigma(u) = 0\,;$$

$$(ii) \qquad \|\tilde{b}\|_{L^q(\mathbf{S}^{n-1})} \le |I|^{-1/q'}\,;$$

$$(iii) \qquad \|\tilde{b}\|_{L^1(\mathbf{S}^{n-1})} \le 1.$$

Let b be a q-block function on $\mathbf{S}^{n-1}$ supported in an interval I with $q>1$ and $\|b\|_{L^q(\mathbf{S}^{n-1})} \leq |I|^{-1/q'}$, $\frac{1}{q}+\frac{1}{q'}=1$. To each function b we associate a function $\tilde{b}$ defined by

$$\tilde{b}(x) = b(x) - \int_{\mathbf{S}^{n-1}} b(u)d\sigma(u). \tag{6.3}$$

Then the function $\tilde{b}$ enjoys the following properties:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}(u)\, d\sigma(u) \;=\; 0 \tag{6.4}$$

$$\|\tilde{b}\|_{L^q(\mathbf{S}^{n-1})} \;\leq\; 2\,|I|^{-1/q'} \tag{6.5}$$

$$\|\tilde{b}\|_{L^1(\mathbf{S}^{n-1})} \;\leq\; 2. \tag{6.6}$$

We notice that, with the exception of a constant factor, the function $\tilde{b}$ is a $q-$blocklike function associated with the interval I on $\mathbf{S}^{n-1}$. *We call the function $\tilde{b}$ the blocklike function corresponding to the block function b.*

Our aim now is to establish the necessary Fourier transform estimates related to blocklike functions $\tilde{b}$.

Proposition 6.3 *Let $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma>1$, $\tilde{b}$ be a q-blocklike function associated with an interval I on $\mathbf{S}^{n-1}$ and $\omega = \min\{2,\gamma,q\}$. Suppose that $F:\mathbb{R}^n \to \mathbb{R}$ is a function given by $F(x) = \sum_{j=0}^{l} P_j(x) + W(|x|)$ where $P_j(\cdot)$ is a homogeneous polynomial of degree j for $0 \leq j \leq l$ and $W(\cdot)$ is an arbitrary function. Let $\mathcal{J}_k = \mathcal{J}_k(\tilde{b})$ for $k \in \mathbb{Z}$. Then*

$$|\mathcal{J}_k| \leq C\left(2^{kl}\,\|Z_l^n(P_l)\|\right)^{1/(4\,l\,\omega'\,\log|I|)} \quad \text{if } |I| < e^{-2} \tag{6.7}$$

and

$$|\mathcal{J}_k| \leq C\left(2^{kl}\,\|Z_l^n(P_l)\|\right)^{-1/(4\,l\,\omega')} \quad \text{if } |I| \geq e^{-2}. \tag{6.8}$$

The constant C is independent of $\tilde{b}(\cdot)$, k, $W(\cdot)$ and the coefficients of the polynomials $\{P_j(\cdot) : 0 \leq j \leq l\}$.

Proof. By Proposition 6.1 and the definition of $\tilde{b}$, property (ii) we have

$$|\mathcal{J}_k| \leq C\,|I|^{-1/q'}\left(2^{kl}\left\|Z_l^{(n,1)}(P_l)\right\|\right)^{-1/(8\,l\,\omega')}. \tag{6.9}$$

Also, by the definition of $\tilde{b}$, property (iii), we get

$$|\mathcal{J}_k| \leq C\|\tilde{b}\|_{L^1(\mathbf{S}^{n-1})}\left[\sup_{R>0}\frac{1}{R}\int_0^R |h(t)|\,dt\right] \leq C.$$

If $|I| < e^{-2}$, then by interpolating between the preceding estimates of $|\mathcal{J}_k|$ we get (6.7). On the other hand, if $|I| \geq e^{-2}$, (6.8) follows easily from (6.9).
■

The oscillatory estimates in Proposition 6.3 will be used in the proof of Theorem 2.1. To prove Theorems 2.2 and 2.3 we need to use somewhat different measures for decomposing our operators at hand. For this purpose, we define the following class of measures related to blocklike functions $\tilde{b}$.

Definition 6.4 *For a suitable mapping* $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ *we define the measures* $\{\Upsilon_{\Gamma,\tilde{b},k} : k \in \mathbb{Z}\}$ *and the maximal operators* $\Upsilon^*_{\Gamma,\tilde{b}}$ *on* $\mathbb{R}^m$ *by*

$$\int_{\mathbb{R}^m} f \, d\Upsilon_{\Gamma,\tilde{b},k} = \int_{\beta^k \le |u| < \beta^{k+1}} f\left(\Gamma\left(u\right)\right) \frac{\tilde{b}\left(u'\right)}{\left|u\right|^n} \, h\left(|u|\right) \, du \tag{6.10}$$

and

$$\Upsilon^*_{\Gamma,\tilde{b}} f\left(x\right) = \sup_{k \in \mathbb{Z}} \left| \left|\Upsilon_{\Gamma,\tilde{b},k}\right| * f\left(x\right) \right| \tag{6.11}$$

where $\beta = 2^{\,\log(1/|I|)}$ *and* $|I| < e^{-2}$.

We would like to thank Ahmad Al-salman for a very fruitful discussion concerning the usefulness of decomposing our operator T using the measures $\{\Upsilon_{\Gamma,\tilde{b},k}\}$.

Now let us establish the following proposition which will provide us with the necessary Fourier transform estimates related to $\tilde{b}$ whenever $|I| < e^{-2}$.

Proposition 6.5. *Let* $h \in \Delta_\gamma\left(\mathbb{R}^+\right)$ *for some* γ, $1 < \gamma \le 2$ *and* $\tilde{b}$ *be a* q*-blocklike associated with an interval* I. *Let* $F : \mathbb{R}^n \to \mathbb{R}$ *be a function given by* $F\left(x\right) = \sum_{j=0}^{l} P_j\left(x\right) + W\left(|x|\right)$ *where* $P_j(\cdot)$ *is a homogeneous polynomial of degree* j *for* $0 \le j \le l$ *and* $W\left(\cdot\right)$ *is an arbitrary function. Let*

$$J_k(\tilde{b}) = \int_{\beta^k}^{\beta^{k+1}} \int_{\mathbf{S}^{n-1}} e^{iF(tx)} \tilde{b}(x) h(t) d\sigma\left(x\right) \frac{dt}{t}$$

for $k \in \mathbb{Z}$. *If* $|I| < e^{-2}$, *then there exists a constant* C *such that*

$$\left|J_k(\tilde{b})\right| \le C \log\left(\frac{1}{|I|}\right) \left(2^{lk(\log \frac{1}{|I|})} \ \|Z_l(P_l)\|\right)^{1/(4\,l\,\gamma'\,q'\,\log|I|)} \tag{6.12}$$

holds for all $k \in \mathbb{Z}$. *The constant* C *is independent of* k, $\tilde{b}$, $W\left(\cdot\right)$ *and the coefficients of* $P_j(\cdot)$.

If G *is a subspace of* $\mathcal{V}_l$ *satisfying* $|x|^l \notin G$ *for some* $l \in \mathbb{N}$ *then for* $|I| < e^{-2}$ *there exists a constant* C' *such that*

$$\left|J_k(\tilde{b})\right| \le C' \log\left(\frac{1}{|I|}\right) \left(2^{lk(\log \frac{1}{|I|})} \ \|P_l\|\right)^{1/(4\,l\,\gamma'\,q'\,\log|I|)} \tag{6.13}$$

holds for all $k \in \mathbb{Z}$ *and* $P_l \in G$. *The constant* C' *may depend on the subspace* G *if* l *is even, but it is independent of* G *if* l *is odd.*

Proof. By Hölder's inequality we have

$$\left|J_k(\tilde{b})\right| \leq \left(\int_{\beta^k}^{\beta^{k+1}} |h(t)|^{\gamma} \frac{dt}{t}\right)^{1/\gamma} \left(\int_1^{\beta} |S_k(t)|^{\gamma'} \frac{dt}{t}\right)^{1/\gamma'},$$

where

$$S_k(t) = \int_{\mathbf{S}^{n-1}} e^{iF\left(\beta^k tx\right)}\, \tilde{b}(x) d\sigma(x).$$

Since

$$\int_{\beta^k}^{\beta^{k+1}} |h(t)|^{\gamma} \frac{dt}{t} \leq \sum_{s=1}^{[\log \frac{1}{|I|}]+1} \int_{\beta^k 2^{s-1}}^{\beta^k 2^s} |h(t)|^{\gamma} \frac{dt}{t} \text{ and } |S_k(t)| \leq 2,$$

we obtain

$$\left|J_k(\tilde{b})\right| \leq C\Big(\log \frac{1}{|I|}\Big)^{1/\gamma} \left(\int_1^{\beta} |S_k(t)|^2 \frac{dt}{t}\right)^{1/\gamma'}.$$

where $[\cdot]$ denotes the usual greatest integer function. By writing

$$|S_k(t)|^2 = \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \tilde{b}(x)\, \overline{\tilde{b}(y)} e^{i\left(F\left(\beta^k tx\right) - F\left(\beta^k ty\right)\right)}\, d\sigma(x)\, d\sigma(y)$$

and using Van der Corput's lemma we obtain

$$\left|\int_1^{\beta} e^{i\left(F\left(\beta^k tx\right) - F\left(\beta^k ty\right)\right)} \frac{dt}{t}\right| \leq C \min\left\{\log\left(\frac{1}{|I|}\right), \left|\beta^{kl}(P_l(x) - P_l(y))\right|^{-1/l}\right\}$$

$$\leq C \log\left(\frac{1}{|I|}\right) \left|\beta^{kl}(P_l(x) - P_l(y))\right|^{-1/(4\,l\,q')}.$$

Therefore, by Hölder's inequality, Lemma 3.3 and Lemma 3.4 in [10] we get

$$\left|J_k(\tilde{b})\right| \leq C \log\left(\frac{1}{|I|}\right) \|\,\tilde{b}\,\|_{L^q(\mathbf{S}^{n-1})}^{2/\gamma'} \left(\beta^{kl} \|Z_l(P_l)\|\right)^{-1/(4\,l\,\gamma'\,q')}.$$

Then by the definition of $\tilde{b}$, property (ii), we get

$$\left|J_k(\tilde{b})\right| \leq C \log\left(\frac{1}{|I|}\right) |I|^{-2/(q'\,\gamma')} \left(\beta^{kl} \|Z_l(P_l)\|\right)^{-1/(4\,l\,\gamma'\,q')}.$$

By interpolating between the preceding estimate and the trivial estimate

$$\left|J_k(\tilde{b})\right| \leq C \log\left(\frac{1}{|I|}\right)$$

we obtain (6.12). (6.13) can be established by following essentially the same argument as in the proof of (6.12) and using Lemma 3.7 in [10]. This completes the proof of our proposition. ∎

7. Proofs of Theorems 2.1 and 2.2

Let $\deg(\mathcal{P}) = \max\{\deg(P_j) : 1 \le j \le m\}$, $0 < n_1 < n_2 < \ldots, n_N = \deg(\mathcal{P})$ be non-negative integers and polynomials $\{P_j^l : 1 \le j \le m,\, 1 \le l \le N\}$ such that

$$\mathcal{P}(x) = \sum_{l=1}^{N} \mathcal{P}^l(x) + \mathcal{R}(|x|)$$

where $\mathcal{P}^l(x) = (P_1^l(x), \ldots, P_m^l)$, $x \in \mathbb{R}^n$, $\mathcal{W}(t) = (\mathcal{W}_1(t), \ldots, \mathcal{W}_m(t))$, $t \in \mathbb{R}$, $Z_{n_l}(P_j^l) = P_j^l$, with $P_j^l \in V_{n_l} \subseteq \mathcal{A}_n$ and $\mathcal{W}_j \in \mathcal{A}_1$ for $1 \le j \le m, 1 \le l \le N$.

For $1 \le l \le N$, let ρ_l denote the number of elements of $\{\beta \in (\mathbb{N} \cup \{0\})^n$: $|\beta| = n_l\}$ and write $\{\beta \in (\mathbb{N} \cup \{0\})^n : |\beta| = n_l\} = \{\beta(1), \ldots, \beta(\rho_l)\}$. Write

$$P_j^l(x) = \sum_{k=1}^{\rho_l} \eta_{k,j}\ x^{\beta(k)}$$

and define the linear mappings $L_l : \mathbb{R}^m \to \mathbb{R}^{\rho_l}$ by

$$L_l(\xi) = \Big(\sum_{j=1}^{m} \eta_{1,j}^l\, \xi_j, \ldots, \sum_{j=1}^{m} \eta_{\rho_l,j}^l\, \xi_j\Big) \text{ for } 1 \le j \le m, 1 \le l \le N. \tag{7.1}$$

Let

$$\Gamma_l(x) = \sum_{j=1}^{l} \mathcal{P}^j(x) + \mathcal{W}(|x|) \text{ for } 1 \le l \le N \text{ and } \Gamma_0(x) = \mathcal{W}(|x|). \tag{7.2}$$

Also, let

$$\sigma_{\vartheta,k}^{(l)} = \sigma_{\Gamma_l,\vartheta,k} \text{ for } \vartheta \in L^1(\mathbf{S}^{n-1}) \text{ and for each } 1 \le l \le N. \tag{7.3}$$

Since $\Delta_\gamma(\mathbb{R}^+) \subseteq \Delta_2(\mathbb{R}^+)$ when $\gamma \ge 2$, we may assume that $1 < \gamma \le 2$ and p satisfies $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{\gamma'}$. By assumption Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbb{C}$, b_μ is a q-block with support on an interval I_μ on $\mathbf{S}^{n-1}$ and

$$M_q^{0,0}(\{c_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu| \left(1 + \left(\log^+ \frac{1}{|I_\mu|}\right)\right) < \infty. \tag{7.4}$$

For each $\mu \ge 1$ let $\tilde{b}_\mu$ be the blocklike function corresponding to the block function b_μ. Then by the mean zero property of Ω, condition (1.2), we have

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu. \tag{7.5}$$

Thus, the operator T in (1.5) can be decomposed as

$$Tf(x) = \sum_{\mu=1}^{\infty} c_\mu T_{\tilde{b}_\mu} f(x) \tag{7.6}$$

where

$$T_{\tilde{b}_\mu} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f\left(x - \mathcal{P}(u)\right) \frac{\tilde{b}_\mu\left(u'\right)}{|u|^n} h\left(|u|\right) du. \tag{7.7}$$

It is easy to see that, by (6.6),

$$\left\|\sigma^{(l)}_{\tilde{b}_\mu,k}\right\| \leq C \tag{7.8}$$

holds uniformly in l, μ and k. By Proposition 6.3 and (6.5)-(6.6) we have

$$\left|\hat{\sigma}^{(l)}_{\tilde{b}_\mu,k}(\xi)\right| \leq C(2^{kn_l} \left|L_l(\xi)\right|^{1/(4\,n_l\,\log|I_\mu|)} \quad \text{if } |I_\mu| < e^{-2} \tag{7.9}$$

and

$$\left|\hat{\sigma}^{(l)}_{\tilde{b}_\mu,k}(\xi)\right| \leq C(2^{kn_l} \left|L_l(\xi)\right|)^{-1/(4\,n_l\,q')} \quad \text{if } |I_\mu| \geq e^{-2}, \tag{7.10}$$

where C is independent of $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$ and the blocklike function $\tilde{b}_\mu(\cdot)$. We also observe that

$$\left|\sigma^{(l)}_{\tilde{b}_\mu,k}(\xi) - \sigma^{(l-1)}_{\tilde{b}_\mu,k}(\xi)\right| \leq C\|\tilde{b}_\mu\|_{L^1(\mathbf{S}^{n-1})}\Big(\sup_{R>0} \frac{1}{R}\int_0^R |h(t)|\,dt)(2^{kn_l}\left|L_l(\xi)\right|\Big)$$

which, together with (6.6), implies that

$$\left|\hat{\sigma}^{(l)}_{\tilde{b}_\mu,k}(\xi) - \hat{\sigma}^{(l-1)}_{\tilde{b}_\mu,k}(\xi)\right| \leq C(2^{kn_l}\left|L_l(\xi)\right|) \tag{7.11}$$

with a positive constant C independent of $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$ and the blocklike $\tilde{b}_\mu(\cdot)$.

By Theorem 5.3 and (6.6) we have

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\left|\sigma^{(l)}_{\tilde{b}_\mu,k} * g_k\right|^2\Big)^{\frac{1}{2}}\Big\|_p \leq C_p\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{\frac{1}{2}}\Big\|_p \tag{7.12}$$

for p satisfying $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{\gamma'}$ and for arbitrary functions $\{g_k\}$ on $\mathbb{R}^n$ with a C_p independent of the blocklike function $\tilde{b}_\mu$, $\mu = 1, 2, \ldots$.

Now by (7.8)-(7.12) and invoking Lemma 3.1 we get

$$\left\|T_{\tilde{b}_\mu} f\right\|_p = \Big\|\sum_{k\in\mathbb{Z}} \sigma^{(N)}_{\tilde{b}_\mu,k} * f\Big\|_p \leq C_p \log\left(\frac{1}{|I_\mu|}\right)\|f\|_p \;\text{ if } |I_\mu| < e^{-2}, \tag{7.13}$$

and

$$\left\|T_{\tilde{b}_\mu} f\right\|_p = \left\|\sum_{k\in\mathbb{Z}} \sigma^{(N)}_{\tilde{b}_\mu,k} * f\right\|_p \leq C_p \left\|f\right\|_p \text{ if } |I_\mu| \geq e^{-2} \tag{7.14}$$

for p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{\gamma'}$ and for any $f \in L^p(\mathbb{R}^n)$. Then by (7.4), (7.6)-(7.7), (7.13)-(7.14) we get (2.1). This completes the proof of Theorem 2.1. ■

Proof of Theorem 2.2. By (7.5) we have

$$T^*(f) \leq \sum_{\mu=1} |c_\mu| \, T^*_{\tilde{b}_\mu}(f) \tag{7.15}$$

where $T^*_{\tilde{b}_\mu}$ is the maximal truncated singular integral corresponding to the operator $T_{\tilde{b}_\mu}$. Thus, it suffices to establish appropriate L^p bounds for $T^*_{\tilde{b}_\mu}$, $\mu = 1, 2, \ldots$. For the sake of simplicity, we shall work with an arbitrarily fixed μ and write $I = I_\mu$ and $\tilde{b} = \tilde{b}_\mu$.

By Theorem C (iii) and (6.6) we have

$$\left\|T^*_{\tilde{b}} f\right\|_p \leq C_p \left\|f\right\|_p \tag{7.16}$$

for $|I| \geq e^{-2}$ and every $p \in (1,\infty)$ with C_p independent of $\tilde{b}$.

On the other hand, by Theorem 5.2 and (6.6) we obtain easily that

$$\left\|\Upsilon^*_{\Gamma_l,\tilde{b}} f\right\|_p \leq C_p(\log\frac{1}{|I|}) \left\|f\right\|_p \tag{7.17}$$

holds for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, $|I| < e^{-2}$ and $1 \leq l \leq N$. For $1 \leq l \leq N$, let $\Upsilon^{(l)}_{\tilde{b},k} = \Upsilon_{\Gamma_l,\tilde{b},k}$. Then by (6.5)-(6.6) and Proposition 6.5 we get for $|I| < e^{-2}$ the following estimates:

$$\left\|\Upsilon^{(l)}_{\tilde{b},k}\right\| \leq C \log\left(\frac{1}{|I|}\right) \tag{7.18}$$

and

$$\left|\hat{\Upsilon}^{(l)}_{\tilde{b},k}(\xi)\right| \leq C \log\left(\frac{1}{|I|}\right) \left|2^{kn_l(\log\frac{1}{|I|})} L_l(\xi)\right|^{1/(4\, n_l\, \gamma'\, q' \, \log|I|)}. \tag{7.19}$$

In addition, by interpolating between

$$\left|\hat{\Upsilon}^{(l)}_{\tilde{b},k}(\xi) - \hat{\Upsilon}^{(l-1)}_{\tilde{b},k}(\xi)\right| \leq C \log\left(\frac{1}{|I|}\right) \left|2^{kn_l(\log\frac{1}{|I|})} L_l(\xi)\right|$$

and the trivial estimate

$$\left|\hat{\Upsilon}^{(l)}_{\tilde{b},k}(\xi) - \hat{\Upsilon}^{(l-1)}_{\tilde{b},k}(\xi)\right| \leq C \log\left(\frac{1}{|I|}\right)$$

we get

(7.20)
$$\left|\hat{\Upsilon}^{(l)}_{\tilde{b},k}(\xi) - \hat{\Upsilon}^{(l-1)}_{\tilde{b},k}(\xi)\right| \leq C \log\left(\frac{1}{|I|}\right) \left|2^{kn_l(\log \frac{1}{|I|})} L_l(\xi)\right|^{-1/(4\, n_l\, \gamma'\, q'\, \log |I|)}$$

for all $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^m$ and $1 \leq l \leq N$. By (7.17)-(7.20) and Lemma 3.5 we get

$$\left\|\sup_{k\in\mathbb{Z}}\left|\sum_{j=k}^{\infty} \Upsilon^{(N)}_{\tilde{b},j} * f\right|\right\|_p \leq C_p(\log \frac{1}{|I|}) \left\|f\right\|_p \tag{7.21}$$

for every $f \in L^p(\mathbb{R}^n)$ and $1 < p < \infty$. Since

$$\left|T^*_{\tilde{b}} f(x)\right| \leq \sup_{k\in\mathbb{Z}}\left|\sum_{j=k}^{\infty} \Upsilon^{(N)}_{\tilde{b},j} * f(x)\right| + \Upsilon^*_{\Gamma_N,\tilde{b}} f(x),$$

by (7.17) and (7.21) we obtain

$$\left\|T^*_{\tilde{b}} f\right\|_p \leq C_p(\log \frac{1}{|I|}) \left\|f\right\|_p \tag{7.22}$$

for every $p \in (1, \infty)$. Hence by (7.4), (7.15), (7.16), and (7.22) we get (2.2). This concludes the proof of Theorem 2.2. ■

8. Classes of maximal functions and singular integrals associated to special polynomial mappings

Let us start with the definition of the special class of polynomial mappings $\mathcal{F}(n,m)$. This class was introduced by Fan and Pan in their study of singular integral operators in [10], p. 833. It is defined as follows: for $n, m, l \in \mathbb{N}$ let $\mathcal{F}_{n,m,0} = \mathbb{R}^m$,

$$\mathcal{F}_{n,m,l} = \left\{(P_1, \ldots, P_m) \in (\mathcal{V}_l)^m : |x|^l \notin \text{span}\{P_1, \ldots, P_m\}\right\}$$

and

$$\mathcal{F}(n,m) = \{\sum_{l=0}^{m} P^l_j : m \geq 0, \mathcal{P}^l \in \mathcal{F}_{n,m,l} \text{ for } 0 \leq l \leq m\}$$

where $\mathcal{V}_l$ represents the linear space of real-valued homogeneous polynomials of degree l on $\mathbb{R}^n$. It is clear that $\mathcal{F}_{n,m,l} = (\mathcal{V}_l)^m$ if l is odd. Also, notice that if $\mathcal{P} = (P_1, \ldots, P_m)$ with $P_j \in \mathcal{A}_n$ and $\mathcal{P}(-x) = -\mathcal{P}(x)$, then $\mathcal{P} \in \mathcal{F}(n,m)$.

Our purpose in this section is to study singular integrals and maximal functions associated to polynomial mappings which belong to the special class $\mathcal{F}(n,m)$. The main thrust in the proof of Theorem 2.3 will be in establishing the following theorem.

Theorem 8.1. *Let* $h \in \Delta_\gamma(\mathbb{R}^+)$ for some γ, $1<\gamma\le 2$ *and* $\tilde{b}$ *be a* q-*blocklike function associated with an interval* I *with* $|I|<e^{-2}$. *Suppose that* $\mathcal{P}\in\mathcal{F}(n,m)$. *Then for* $1<p\le\infty$ *and* $f\in L^p(\mathbb{R}^m)$ *there exists a positive constant* $C_p>0$ *such that*

$$\left\|\Upsilon^*_{\mathcal{P},\tilde{b}}(f)\right\|_{L^p(\mathbb{R}^m)} \le C_p \log\left(\frac{1}{|I|}\right)\|f\|_{L^p(\mathbb{R}^m)}. \tag{8.1}$$

Furthermore, if $\mathcal{P}(-x)=-\mathcal{P}(x)$, *then the constant* C_p *depends only on* p, n, m, h, $\deg(\mathcal{P})$ *and neither on the function* $\tilde{b}$ *nor on the coefficients of the polynomial components of the mapping* $\mathcal{P}$.

Proof. Without loss of generality we may assume that $\tilde{b}\ge 0$ and $h\ge 0$. We shall prove (8.1) by induction on $\deg(\mathcal{P})$. First, if $\deg(\mathcal{P})=0$, then by the definition of $\tilde{b}$, property (iii),

$$\Upsilon^*_{\mathcal{P},\tilde{b}}(f)(x)\le C\log\left(\frac{1}{|I|}\right)|f(x-\mathcal{P}(0))|$$

and hence (8.1) holds trivially. Next, assume that (8.1) holds for all $\mathcal{P}\in\mathcal{F}(n,m)$ with $\deg(\mathcal{P})\le d-1$.

Now suppose that $\deg(\mathcal{P})=d$. Then $\mathcal{P}=\mathcal{H}(x)+\mathcal{R}(x)$ for some non zero $\mathcal{H}\in\mathcal{F}_{n,m,d}$, $\mathcal{R}\in\mathcal{F}(n,m)$ and with $\deg(\mathcal{R})\le d-1$. By the inductive hypothesis we have

$$\left\|\Upsilon^*_{\mathcal{R},\tilde{b}}(f)\right\|_{L^p(\mathbb{R}^m)} \le C_p \log\left(\frac{1}{|I|}\right)\|f\|_{L^p(\mathbb{R}^m)} \tag{8.2}$$

for $1<p<\infty$ and $f\in L^p(\mathbb{R}^m)$.

Let $m_d=\dim(\mathcal{V}_d)$ and $L:\mathbb{R}^m\to\mathbb{R}^{m_d}$ be a linear transformation such that $\|(\xi\cdot\mathcal{H})(\cdot)\|=|L(\xi)|$ for $\xi\in\mathbb{R}^m$. Then by Proposition 6.5 and the proof of (7.20) we obtain

$$\left\|\Upsilon_{\mathcal{P},\tilde{b},k}\right\|\le C\log\left(\frac{1}{|I|}\right),\ \left\|\Upsilon_{\mathcal{R},\tilde{b},k}\right\|\le C\log\left(\frac{1}{|I|}\right) \tag{8.3}$$

$$\left|\hat{\Upsilon}_{\mathcal{P},\tilde{b},k}(\xi)\right|\le C\log\left(\frac{1}{|I|}\right)\left(2^{kd(\log\frac{1}{|I|})}|L(\xi)|\right)^{1/(4\,d\,\gamma'\,q'\,\log|I|)} \tag{8.4}$$

and

$$\left|\hat{\Upsilon}_{\mathcal{P},\tilde{b},k}(\xi)-\hat{\Upsilon}_{\mathcal{R},\tilde{b},k}(\xi)\right|\leq C\log\left(\frac{1}{|I|}\right)\left(2^{kd(\log\frac{1}{|I|})}\left|L(\xi)\right|\right)^{-1/(\log|I|\,d)}. \tag{8.5}$$

Without loss of generality we may assume that $L=\pi^m_{m_d}$ for some $m_d\leq m$. Let $\phi\in\mathcal{S}(\mathbb{R}^{m_d})$ be a Schwartz function such that $\hat{\phi}(x)=1$ for $|x|\leq\frac{1}{2}$ and $\hat{\phi}(x)=0$ for $|x|\geq 1$. Define the functions $\{\phi_k\}$ and measures $\{\nu_k\}$ by

$$(\phi_k\hat{)}(x)=\hat{\phi}\left(2^{kd(\log\frac{1}{|I|})}x\right),$$

and

$$\hat{\nu}_k(\xi)=\hat{\Upsilon}_{\mathcal{P},\tilde{b},k}(\xi)-(\phi_k\hat{)}\left(\pi^m_{m_d}\xi\right)\hat{\Upsilon}_{\mathcal{R},\tilde{b},k}(\xi). \tag{8.6}$$

We observe that

$$|\hat{\nu}_k(\xi)|\leq C\log\left(\frac{1}{|I|}\right)\left|2^{kd(\log\frac{1}{|I|})}\pi^m_{m_d}\xi\right|^{\pm 1/(4\,d\,q'\,\gamma'\,\log|I|)} \tag{8.7}$$

Let

$$g(f)=\Big(\sum_{k\in\mathbb{Z}}|\nu_k*f|^2\Big)^{\frac{1}{2}}\text{ and }\nu^*(f)=\sup_{k\in\mathbb{Z}}||\nu_k|*f|.$$

Then by (8.6) we have

$$\Upsilon^*_{\mathcal{P},\tilde{b}}f(x)\leq g(f)(x)+\sup_{k\in\mathbb{Z}}\left|(\phi_k\otimes\delta_{\mathbb{R}^{m-m_d}})*\Upsilon_{\mathcal{R},\tilde{b},k}*f(x)\right|. \tag{8.8}$$

If we let $\mathcal{M}_s$ denote the Hardy-Littlewood maximal function on $\mathbb{R}^s$, then

$$\begin{aligned}&\sup_{k\in\mathbb{Z}}|[(\phi_k\otimes\delta_{\mathbb{R}^{m-m_d}})]*\Upsilon_{\mathcal{R},\tilde{b},k}*f(x)|\\&\qquad\leq C(\mathcal{M}_{m_d}\otimes id_{\mathbb{R}^{m-m_d}})(\Upsilon^*_{\mathcal{R},\tilde{b}}f(x)).\end{aligned} \tag{8.9}$$

By (8.6) and (8.8)-(8.9) we get

$$\nu^*f(x)\leq g(f)(x)+2C[(\mathcal{M}_{m_d}\otimes id_{\mathbb{R}^{m-m_d}})](\Upsilon^*_{\mathcal{R},\tilde{b}}f(x)). \tag{8.10}$$

It follows from (8.7) and Plancherel's theorem that

$$\|g(f)\|_{L^2}\leq C\log\left(\frac{1}{|I|}\right)\|f\|_{L^2}. \tag{8.11}$$

By the L^p boundedness of the Hardy-Littlewood maximal function, (8.2), (8.10)-(8.11) we get

$$\|\nu^*(f)\|_{L^2}\leq C\log\left(\frac{1}{|I|}\right)\|f\|_{L^2} \tag{8.12}$$

with a C independent of $\tilde{b}$. By using the fact $\|\nu_k\| \leq C\log\left(\frac{1}{|I|}\right)$ together with Lemma 3.4 (for $q = 2$) we get

$$\left\|(\sum_{k\in\mathbb{Z}}(|\nu_k * g_k|^2)^{\frac{1}{2}}\right\|_{p_0} \leq C_{p_0}\log\left(\frac{1}{|I|}\right)\left\|(\sum_{k\in\mathbb{Z}}|g_k|^2)^{\frac{1}{2}}\right\|_{p_0} \tag{8.13}$$

if $\frac{1}{4} = \left|\frac{1}{p_0} - \frac{1}{2}\right|$.

Now, by (8.7), (8.13) and invoking Lemma 3.2 we get

$$\|g(f)\|_{L^p} \leq C_p\log\left(\frac{1}{|I|}\right)\ \|f\|_{L^p} \quad \text{ for } p \in \left(\frac{4}{3}, 4\right). \tag{8.14}$$

By the L^p boundedness of the Hardy-Littlewood maximal function, (8.2) and (8.10), we get

$$\|\nu^*(f)\|_{L^p} \leq C\log\left(\frac{1}{|I|}\right)\|f\|_{L^p} \quad \text{ for } p \in \left(\frac{4}{3}, 4\right). \tag{8.15}$$

Reasoning as above, (8.7), (8.12), Lemma 3.2 and Lemma 3.4 provide

$$\|g(f)\|_{L^p} \leq C_p\log\left(\frac{1}{|I|}\right)\|f\|_{L^p} \quad \text{ for } p \in \left(\frac{8}{7}, 8\right). \tag{8.16}$$

By using this argument repeatedly we ultimately obtain that

$$\|g(f)\|_{L^p} \leq C_p\log\left(\frac{1}{|I|}\right)\|f\|_{L^p} \quad \text{ for } p \in (1,\infty). \tag{8.17}$$

Therefore, by (8.2) and (8.8)-(8.9) we conclude that

$$\left\|\Upsilon^*_{\mathcal{P},\tilde{b}}(f)\right\|_{L^p} \leq C_p\log\left(\frac{1}{|I|}\right)\|f\|_{L^p} \quad \text{ for } p \in (1,\infty). \tag{8.18}$$

Since

$$\left\|\Upsilon^*_{\mathcal{P},\tilde{b}}(f)\right\|_{L^\infty} \leq C\log\left(\frac{1}{|I|}\right)\|f\|_{L^\infty}$$

holds trivially, the proof of (8.1) is complete.

Finally, if $\mathcal{P}(-x) = -\mathcal{P}(x)$, then at each step of our inductive argument d is always an odd number. Therefore, by Proposition 6.5 and the above argument, the constant C_p in (8.1) depends only p, n, m, h, $\deg(\mathcal{P})$ and neither on the function $\tilde{b}$ nor on the coefficients of the polynomial components of the mapping $\mathcal{P}$. This concludes the proof of our theorem. ■

Sketch of the proof of Theorem 2.3. Since $\Delta_\gamma(\mathbb{R}^+) \subseteq \Delta_2(\mathbb{R}^+)$ when $\gamma \geq 2$, we may assume that $1 < \gamma \leq 2$. Let $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ be a block function, where $c_\mu \in \mathbf{C}$; each b_μ is a q-block supported in an interval I_μ; $\|b_\mu\|_{L^q(\mathbf{S}^{n-1})} \leq |I|^{-1/q'}$ and $M_q^{0,0}(\{c_\mu\})$ satisfies (7.5). For each $\mu = 1, 2, \dots$ let $\tilde{b}_\mu$ be the blocklike function corresponding to the q-block b_μ.

Since $\mathcal{P} \in \mathcal{F}(n,m)$, there are integers $0 < n_1 < n_2 < \cdots < n_N = \deg(\mathcal{P})$, and nonzero $\mathcal{P}^d \in \mathcal{F}_{n,m,n_d}$ for $1 \leq d \leq N$ such that $\mathcal{P}(x) = \mathcal{P}(0) + \sum_{d=1}^{N} \mathcal{P}^d(x)$. Let

$$\Gamma_0(x) = \mathcal{P}(0) \text{ and } \Gamma_d(x) = \mathcal{P}(0) + \sum_{j=1}^{d} \mathcal{P}^j(x) \text{ for } 1 \leq d \leq N\,. \tag{8.19}$$

Then by Proposition 6.5 and using an argument similar to the one in the proof of (7.20) we get for suitable linear transformations $G_d : \mathbb{R}^m \to \mathbb{R}^{\rho_d}$ and $|I_\mu| < e^{-2}$ the following:

$$\left\|\Upsilon^{(d)}_{\tilde{b}_\mu,k}\right\| \leq C \log\left(\frac{1}{|I_\mu|}\right) \tag{8.20}$$

$$\left|\hat{\Upsilon}^{(d)}_{\tilde{b}_\mu,k}(\xi)\right| \leq C \log\left(\frac{1}{|I_\mu|}\right) \left|2^{kn_d \log(\frac{1}{|I_\mu|})} G_d(\xi)\right|^{1/(4\, n_d\, \gamma'\, q'\, \log|I_\mu|)} \tag{8.21}$$

$$\left|\hat{\Upsilon}^{(d)}_{\tilde{b}_\mu,k}(\xi) - \hat{\Upsilon}^{(d-1)}_{\tilde{b}_\mu,k}(\xi)\right| \leq C \log\left(\frac{1}{|I_\mu|}\right) \left|2^{kn_d \log(\frac{1}{|I_\mu|})} G_d(\xi)\right|^{-1/(\log|I_\mu|)} \tag{8.22}$$

for all $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^m$, $1 \leq d \leq N$.

By Theorem 8.1, (8.21) and Lemma 3.4 we have

$$\left\|\Upsilon^*_{\Gamma_d,\tilde{b}_\mu}(f)\right\|_{L^p(\mathbb{R}^m)} \leq C_p \log\left(\frac{1}{|I_\mu|}\right) \|f\|_{L^p(\mathbb{R}^m)} \tag{8.23}$$

and

$$\left\|\Big(\sum_{k\in\mathbb{Z}} \left|\Upsilon^{(d)}_{\tilde{b}_\mu,k} * g_k\right|^2\Big)^{\frac{1}{2}}\right\|_p \leq C \log\left(\frac{1}{|I_\mu|}\right) \left\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^2\Big)^{\frac{1}{2}}\right\|_p \tag{8.24}$$

for $1 < p < \infty$, $1 \leq d \leq N$ and for arbitrary functions $\{g_k\}$ on $\mathbb{R}^m$.

By (8.20)-(8.24), Lemma 3.3 and Lemma 3.5 we obtain

$$\left\|\tilde{T}_{\tilde{b}_\mu} f\right\|_p = \left\|\sum_{k\in\mathbb{Z}} \Upsilon^{(N)}_{\tilde{b}_\mu,k} * f\right\|_p \leq C_p \log\left(\frac{1}{|I_\mu|}\right) \|f\|_p \tag{8.25}$$

and

$$\left\|\tilde{T}^*_{\tilde{b}_\mu} f\right\|_p \leq C_p \log\left(\frac{1}{|I_\mu|}\right) \|f\|_p \tag{8.26}$$

for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^m)$.

On the other hand, for $|I_\mu| \geq e^{-2}$ we obtain by Theorem C and (6.5)

$$\left\| \tilde{T}_{\tilde{b}_\mu} f \right\|_p \leq C_p \left\| f \right\|_p \tag{8.27}$$

and

$$\left\| \tilde{T}^*_{\tilde{b}_\mu} f \right\|_p \leq C_p \left\| f \right\|_p . \tag{8.28}$$

Now (2.3) and (2.5) follow by combining (7.4), (7.6), (7.15) and (8.25)-(8.28).

Finally, by (7.5), it is easy to see that

$$\mathcal{M}_{\mathcal{P},\Omega} f(x) \leq 2 \sum_{\mu=1}^{\infty} |c_\mu| \, \sigma^*_{\mathcal{P},\tilde{b}_\mu}(|f|)(x)$$

$$\leq 2 \sum_{\mu=1, |I_\mu| \geq e^{-2}}^{\infty} |c_\mu| \, \mathcal{M}_{\mathcal{P},\tilde{b}_\mu}(|f|)(x) + 4 \sum_{\mu=1, |I_\mu| < e^{-2}}^{\infty} |c_\mu| \, \tau^*_{\mathcal{P},\tilde{b}_\mu}(|f|)(x). \tag{8.29}$$

Therefore, by Theorem C, (6.5), (7.4), (8.23) and (8.29) we obtain (2.4). This concludes the proof of our theorem. ■

9. Oscillatory singular integrals

By a well-known method we can obtain the L^p boundedness of the following oscillatory singular integral operator

$$Sf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} K(x-y) f(y) dy$$

where the phase P is a polynomial. In fact, we have the following.

Theorem 8.1 *Let* $K(x) = \dfrac{\Omega(x)}{|x|^n} h(|x|)$ *where* Ω *satisfies (1.2) and* $h \in \Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$. *Then if* $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ *for some* $q > 1$ *we have*

(i) the operator S *is bounded from* $L^p(\mathbb{R}^n)$ *to itself for* p *satisfying*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \min\left\{ \frac{1}{2}, \frac{1}{\gamma'} \right\}.$$

(ii) If $P(-x) = -P(x)$, $x \in \mathbb{R}^n$, *then for* $1 < p < \infty$ *the operator* $S(f)$ *is bounded from* $L^p(\mathbb{R}^n)$ *to itself.*

Moreover, the bound for the operator norm in (i) and (ii) is independent of the coefficients of the polynomial P.

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