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Some questions on quasinilpotent groups and related classes

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Abstract

In this paper we will prove that if G is a finite group, X a subnormal subgroup of $X\,\mathrm{F}^*(G)$ such that $X\,\mathrm{F}^*(G)$ is quasinilpotent and Y is a quasinilpotent subgroup of $\mathrm{N}_G(X)$, then $Y\,\mathrm{F}^*(\mathrm{N}_G(X))$ is quasinilpotent if and only if $Y\,\mathrm{F}^*(G)$ is quasinilpotent. Also we will obtain that $\mathrm{F}^*(G)$ controls its own fusion in G if and only if $G = \mathrm{F}^*(G)$.

The generalized Fitting subgroup $\mathrm{F}^*(G)$ of a finite group G is the product of the Fitting subgroup and the semisimple radical of G.

This generalized Fitting subgroup satisfies $\mathrm{C}_G(\mathrm{F}^*(G)) \leq \mathrm{F}^*(G)$, for every finite group G. This property is similar to the corresponding one for the Fitting subgroup of a soluble group: $\mathrm{C}_G(\mathrm{F}(G)) \leq \mathrm{F}(G)$. Quasinilpotent groups are those groups which coincide with their generalized Fitting subgroup. A group G such that $\mathrm{F}^*(G) = \mathrm{F}(G)$ is a nilpotent-constrained group.

H. Bender stated that if G is a nilpotent-constrained group, X a subgroup of G such that $X\,\mathrm{F}(G)$ is nilpotent and $Y \leq \mathrm{N}_G(X)$, then $Y\,\mathrm{F}(\mathrm{N}_G(X))$ is nilpotent if and only if $Y\,\mathrm{F}(G)$ is nilpotent.

A well known theorem of Frobenius states that if a p-Sylow subgroup of G controls its own fusion in G, then G has a normal p-complement.

In this paper we will prove that if G is a finite group, X a subnormal subgroup of $X\,\mathrm{F}^*(G)$ such that $X\,\mathrm{F}^*(G)$ is quasinilpotent and Y is a quasinilpotent subgroup of $\mathrm{N}_G(X)$, then $Y\,\mathrm{F}^*(\mathrm{N}_G(X))$ is quasinilpotent if and only if $Y\,\mathrm{F}^*(G)$ is quasinilpotent. Also we characterize when a nilpotent

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injector controls its own fusion in a nilpotent-constrained group or when a quasinilpotent injector controls its own fusion in a finite group.

Notations. All groups considered in this paper are assumed to be finite. The non-explicit notations are standard, see for instance [3]. We quote nevertheless the following:

$\mathfrak{N}$: class of nilpotent groups,

$\mathfrak{S}$: class of soluble groups,

$\mathfrak{N}^*$: class of quasinilpotent groups,

$\mathrm{F}(G)$ is the Fitting subgroup of G, i.e., the largest nilpotent normal subgroup of G.

If $\mathfrak{F}$ is a class of groups,

$$\begin{aligned} \mathrm{s}_n\mathfrak{F} &= \{G;\ G \trianglelefteq X \text{ for some } X \in \mathfrak{F}\}, \\ \mathrm{N}_0\mathfrak{F} &= \{G;\ G = \langle X_1, \dots, X_n \rangle \text{ for some } X_i \trianglelefteq\trianglelefteq G,\ X_i \in \mathfrak{F},\ 1 \leq i \leq n\}, \end{aligned}$$

A *Fitting class* $\mathfrak{F}$ is an s_n- and N_0-closed class, that is, a class such that $\mathfrak{F} = \mathrm{s}_n\mathfrak{F} = \mathrm{N}_0\mathfrak{F}$.

If $\mathfrak{F}$ is a Fitting class, a subgroup H of G is an $\mathfrak{F}$-*injector* of G whenever $H \cap N$ is an $\mathfrak{F}$-maximal subgroup of N, for every subnormal subgroup N of G. We denote by $\mathrm{Inj}_{\mathfrak{F}}(G)$ the set of all $\mathfrak{F}$-injectors of G. The quasinilpotent injectors of a group G are characterized as the maximal quasinilpotent subgroups containing the generalized Fitting subgroup of G ([3]).

A group G is said to be *quasisimple* if G is perfect and $G/\mathrm{Z}(G)$ is simple.

The quasisimple subnormal subgroups of a (finite) group G are called the *components* of G. The *semisimple radical* $\mathrm{E}(G)$ of G is the join of its components.

We will need the description of some properties about the semisimple radical of a group. As we did not find any complete reference to it in the literature, for the sake of being selfcontained, we include the following:

Lemma 1 *Let G be a group. Then:*

(1) If $\mathrm{F}^*(G) \leq H \leq G$ it follows that $\mathrm{E}(H) = \mathrm{E}(G)$.

(2) If H is a subnormal subgroup of G then $\mathrm{E}(H)$ is the product of all components Q of G such that $[Q, H] \neq 1$. In particular $\mathrm{E}(G) \leq \mathrm{N}_G(H)$.

(3) If $H \trianglelefteq\trianglelefteq H\,\mathrm{F}^*(G)$, then $\mathrm{E}(\mathrm{N}_G(H)) = \mathrm{E}(G)$.

Proof. (1) Clearly $\mathrm{E}(G) \leq \mathrm{E}(H)$. As $\mathrm{F}^*(G) \leq \mathrm{N}_G(\mathrm{E}(H))$ then $\mathrm{E}(H) \leq \mathrm{E}(G)$ ([2] 4.25), thus $\mathrm{E}(G) = \mathrm{E}(H)$.

(2) If Q is a component of G and $H \leq G$, then either $[Q, H] = 1$ or $Q \leq [Q, H]$ ([7], X 13.18). When $H \trianglelefteq\trianglelefteq G$ the second alternative implies that $Q \leq H$. Therefore $\mathrm{E}(H)$ is the product of all components Q of G such that $[Q, H] \neq 1$. In consequence $\mathrm{E}(G) \leq \mathrm{N}_G(H)$.

(3) By ([2], 4.26) $\mathrm{E}(\mathrm{N}_G(H)) \leq \mathrm{E}(G)$. On the other hand, by (1) and (2) we have that $\mathrm{E}(H\,\mathrm{F}^*(G)) = \mathrm{E}(G) \leq \mathrm{N}_G(H)$, thus $\mathrm{E}(G) \leq \mathrm{E}(\mathrm{N}_G(H))$. Therefore $\mathrm{E}(G) = \mathrm{E}(\mathrm{N}_G(H))$. ■

In [8] we proved the following result:

> *Suppose that N is a nilpotent normal subgroup of G and let X be a nilpotent subgroup of G satisfying $\mathrm{C}_G(N \cap X) \leq X$. Then NX is nilpotent.*

As a consequence of this result, it is easy to obtain:

> *If X is a subgroup of $\mathrm{F}(G)$ and Y is a nilpotent subgroup of $\mathrm{N}_G(X)$ containing $\mathrm{F}(\mathrm{N}_G(X))$, then $Y\,\mathrm{F}(G)$ is nilpotent.*

A generalization of this result would be:

> *If $X\,\mathrm{F}(G)$ is a nilpotent subgroup of G and Y a subgroup of $\mathrm{N}_G(X)$ satisfying $Y\,\mathrm{F}(\mathrm{N}_G(X))$ is nilpotent, then $Y\,\mathrm{F}(G)$ is nilpotent.*

In [1] H. Bender had given an affirmative answer when G is a nilpotent-constrained group. Next we will prove that this result is true without any restriction:

Proposition 2 *Let $X \leq G$ with $X\,\mathrm{F}(G)$ nilpotent and let $Y \leq \mathrm{N}_G(X)$ with $Y\,\mathrm{F}(\mathrm{N}_G(X))$ nilpotent, then $Y\,\mathrm{F}(G)$ is nilpotent.*

Proof. Work by induction on the order of G.

If $R = \mathrm{F}(G)\,\mathrm{N}_G(X) < G$ then $X\,\mathrm{F}(G) \trianglelefteq R$ thus $X\,\mathrm{F}(G) \leq \mathrm{F}(R)$ and $X\,\mathrm{F}(R) = \mathrm{F}(R)$. Therefore, since $\mathrm{N}_G(X) = \mathrm{N}_R(X)$, by the inductive hypothesis, it follows that $Y\,\mathrm{F}(R)$ is nilpotent, so $Y\,\mathrm{F}(G)$ is nilpotent.

Thus we can suppose that $G = \mathrm{F}(G)\,\mathrm{N}_G(X)$, so $X\,\mathrm{F}(G) \trianglelefteq G$, then $X \leq \mathrm{F}(G)$ and by the consequence of ([8], 2.2), it follows that $Y\,\mathrm{F}(\mathrm{N}_G(X))\,\mathrm{F}(G)$ is nilpotent, so $Y\,\mathrm{F}(G)$ is nilpotent. ■

Proposition 3 *Let X be a quasinilpotent subgroup of G satisfying $X \cap \mathrm{F}^*(G) \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$. If $\mathrm{C}_{\mathrm{F}^*(G)}(X \cap \mathrm{F}^*(G)) \leq X$ then $X\,\mathrm{F}^*(G)$ is quasinilpotent.*

Proof. Since $U = X \cap \mathrm{F}^*(G) \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$ and $\mathrm{C}_{\mathrm{F}^*(G)}(U) \leq U$, by ([7], X 15.1) it follows that $U = \mathrm{E}(G)(U \cap \mathrm{F}(G))$ and $\mathrm{C}_{\mathrm{F}(G)}(U \cap \mathrm{F}(G)) \leq U$.

Then

$$\begin{aligned}\mathrm{C}_{\mathrm{F}(G)}(\mathrm{F}(X) \cap \mathrm{F}(G)) &= \mathrm{C}_{\mathrm{F}(G)}(X \cap \mathrm{F}(G)) = \mathrm{C}_{\mathrm{F}(G)}(U \cap \mathrm{F}(G)) \leq U \cap \mathrm{F}(G)\\ &= \mathrm{F}(X) \cap \mathrm{F}(G)\,.\end{aligned}$$

Next, we will prove that $\mathrm{F}(X)\,\mathrm{F}(G)$ is nilpotent. It suffices to show that $\mathrm{F}(X)\,\mathrm{O}_p(G)$ is nilpotent for every prime p in order of $\mathrm{F}(G)$. Consider the action of $(\mathrm{O}_p(G) \cap \mathrm{O}_p(X)) \times \mathrm{O}_{p'}(\mathrm{F}(X))$ on $\mathrm{O}_p(G)$. Since

$$\mathrm{C}_{\mathrm{O}_p(G)}(\mathrm{O}_p(G) \cap \mathrm{O}_p(X)) \leq \mathrm{C}_{\mathrm{F}(G)}(\mathrm{F}(G) \cap \mathrm{F}(X)) \leq \mathrm{F}(X),$$

we have $\mathrm{C}_{\mathrm{O}_p(G)}(\mathrm{O}_p(G) \cap \mathrm{O}_p(X)) \leq \mathrm{O}_p(X)$ and $\mathrm{O}_{p'}(\mathrm{F}(X))$ acts trivially on $\mathrm{C}_{\mathrm{O}_p(G)}(\mathrm{O}_p(G) \cap \mathrm{O}_p(X))$. The Thompson's $P \times Q$-lemma implies that $\mathrm{O}_{p'}(\mathrm{F}(X))$ also acts trivially on $\mathrm{O}_p(G)$. Then $\mathrm{F}(X)\,\mathrm{O}_p(G)$ is nilpotent.

On the other hand, since $\mathrm{E}(X)$ is a quasinilpotent perfect U-invariant subgroup, by ([7], X 15.2), it follows that $\mathrm{E}(X) \trianglelefteq \mathrm{E}(G)$ so $\mathrm{E}(X) = \mathrm{E}(G)$, then $X\,\mathrm{F}^*(G) = \mathrm{E}(X)(\mathrm{F}(X)\,\mathrm{F}(G))$ that is quasinilpotent. ■

Remarks.

1. Notice that, as the following example shows, the condition of subnormality in the above result is necessary.

   Let $G = \mathrm{GL}(2,5)$ and $Z = \mathrm{Z}(G)$. By ([6], II 7.3) there exists $X \leq G$, $X \cong \mathrm{C}_{24}$ satisfying $\mathrm{C}_G(X) = X$. If $D = X \cap \mathrm{SL}(2,5)$ then $|D| = 6$ and if $\langle x \rangle \leq D$ such that $o(x) \not| 4$ then by ([10], page 163) $\mathrm{C}_{\mathrm{SL}(2,5)}(\langle x \rangle) = D$. Since $\mathrm{F}^*(G) = \mathrm{SL}(2,5)Z$, then:

   $$\begin{aligned}\mathrm{C}_{\mathrm{F}^*(G)}(\mathrm{F}^*(G) \cap X) &= Z\,\mathrm{C}_{\mathrm{SL}(2,5)}(\mathrm{SL}(2,5)Z \cap X)\\ &= Z\,\mathrm{C}_{\mathrm{SL}(2,5)}(\mathrm{SL}(2,5) \cap X) \leq Z\,\mathrm{C}_{\mathrm{SL}(2,5)}(\langle x \rangle) = ZD \leq X.\end{aligned}$$

   As $|X\,\mathrm{SL}(2,5)| = |G|$, it follows that $G = X\,\mathrm{SL}(2,5) = X\,\mathrm{F}^*(G)$, that is not quasinilpotent.

2. It is easy to prove that Proposition 3 is equivalent to the following:

   > *Let $H \leq G$ such that $\mathrm{C}_{\mathrm{F}^*(G)}(H \cap \mathrm{F}^*(G)) \leq H$ and $H \cap \mathrm{F}^*(G) \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$. If X is a quasinilpotent subgroup of G, such that $\mathrm{F}^*(H) \leq X \leq H$, then $X\,\mathrm{F}^*(G)$ is quasinilpotent.*

Next we will obtain a version for quasinilpotent groups of ([12], 2.1).

Recall that if N is a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{I}_G(\theta) = \{g \in G | \theta^g = \theta\}$ is the stabilizer of θ in G.

Corollary 4 *Let N be a quasinilpotent normal subgroup of G. Let $\theta \in \mathrm{Irr}(N)$ and let $T = \mathrm{I}_G(\theta)$ the stabilizer of θ in G. If $T \cap \mathrm{F}^*(G) \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$ and X is a quasinilpotent subgroup of G satisfying $\mathrm{F}^*(T) \leq X \leq T$ then $X\,\mathrm{F}^*(G)$ is quasinilpotent.*

Proof. Since $N\,\mathrm{C}_G(N) \leq T$ we have

$$\mathrm{C}_{\mathrm{F}^*(G)}(\mathrm{F}^*(G) \cap T) \leq \mathrm{C}_{\mathrm{F}^*(G)}(N \cap T) = \mathrm{C}_{\mathrm{F}^*(G)}(N) \leq T.$$

Now, by Remark 2, we obtain that $X\,\mathrm{F}^*(G)$ is quasinilpotent. ■

Corollary 5 *If $X \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$ and Y is a quasinilpotent subgroup satisfying $\mathrm{F}^*(\mathrm{N}_G(X)) \leq Y \leq \mathrm{N}_G(X)$, then $Y\,\mathrm{F}^*(G)$ is quasinilpotent.*

Proof. Since $X \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$, by Lemma 1 (3), it follows that $\mathrm{E}(G) = \mathrm{E}(\mathrm{N}_G(X))$, thus

$$\begin{aligned}\mathrm{N}_G(X) \cap \mathrm{F}^*(G) &= \mathrm{E}(G)(\mathrm{N}_G(X) \cap \mathrm{F}(G)) \leq \mathrm{E}(G)\,\mathrm{F}(\mathrm{N}_G(X)) \\ &= \mathrm{F}^*(\mathrm{N}_G(X)) \leq Y.\end{aligned}$$

Hence,

$$Y \cap \mathrm{F}^*(G) = \mathrm{N}_G(X) \cap \mathrm{F}^*(G) \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$$

Then

$$\begin{aligned}\mathrm{C}_{\mathrm{F}^*(G)}(Y \cap \mathrm{F}^*(G)) &= \mathrm{C}_{\mathrm{F}^*(G)}(\mathrm{N}_G(X) \cap \mathrm{F}^*(G)) \leq \mathrm{C}_{\mathrm{F}^*(G)}(X) \leq \mathrm{N}_{\mathrm{F}^*(G)}(X) \\ &= \mathrm{F}^*(G) \cap \mathrm{N}_G(X) \leq Y.\end{aligned}$$

Therefore, by Proposition 3, it follows that $Y\,\mathrm{F}^*(G)$ is quasinilpotent. ■

The following example shows that, in the above result, the subnormality condition is necessary:

Example. Let $\Sigma_7 = \mathrm{A}_7\langle(6,7)\rangle$ and $\mathrm{A}_5 \leq \mathrm{A}_7 = \mathrm{E}(\Sigma_7) = \mathrm{F}^*(\Sigma_7)$ (where A_5 is considered as the group of all even permutations of the set $\{1,2,3,4,5\}$).

Clearly $\mathrm{N}_{\Sigma_7}(\mathrm{A}_5) = \Sigma_5\langle(6,7)\rangle$ and $\mathrm{F}^*(\mathrm{N}_{\Sigma_7}(\mathrm{A}_5)) = \mathrm{A}_5\langle(6,7)\rangle$ however $\mathrm{F}^*(\mathrm{N}_{\Sigma_7}(\mathrm{A}_5))\,\mathrm{F}^*(\Sigma_7)$ coincides with Σ_7, that is not quasinilpotent.

Theorem 6 *Let $X \trianglelefteq\trianglelefteq X\,\mathrm{F}^*(G)$ where $X\,\mathrm{F}^*(G)$ is quasinilpotent and let Y be a quasinilpotent subgroup of $\mathrm{N}_G(X)$. Then $Y\,\mathrm{F}^*(\mathrm{N}_G(X))$ is quasinilpotent if and only if $Y\,\mathrm{F}^*(G)$ is quasinilpotent.*

Proof. Suppose that $Y\,\mathrm{F}^*(\mathrm{N}_G(X))$ is quasinilpotent. We argue by induction on $|G|$.

If $R = \mathrm{N}_G(X)\,\mathrm{F}^*(G) < G$, then $X\,\mathrm{F}^*(G) \trianglelefteq R$ and $X\,\mathrm{F}^*(G) \leq \mathrm{F}^*(R)$, thus $X \trianglelefteq\trianglelefteq X\,\mathrm{F}^*(G) \trianglelefteq \mathrm{F}^*(R)$. Since $\mathrm{N}_G(X) = \mathrm{N}_R(X)$, by induction we obtain that $Y\,\mathrm{F}^*(R)$ is quasinilpotent. Since $\mathrm{F}^*(G) \leq Y\,\mathrm{F}^*(R)$, by Lemma 1 (1) we have $\mathrm{E}(Y\,\mathrm{F}^*(R)) = \mathrm{E}(G)$, thus $Y\,\mathrm{F}^*(G)/\mathrm{E}(G) \leq Y\,\mathrm{F}^*(R)/\mathrm{E}(G)$ that is nilpotent so $Y\,\mathrm{F}^*(G) \trianglelefteq\trianglelefteq Y\,\mathrm{F}^*(R)$, then $Y\,\mathrm{F}^*(G)$ is quasinilpotent.

Thus we can suppose that $G = \mathrm{N}_G(X)\,\mathrm{F}^*(G)$. Then $X\,\mathrm{F}^*(G) \trianglelefteq G$ and $X \trianglelefteq\trianglelefteq \mathrm{F}^*(G)$. Using Corollary 5 it follows that $Y\,\mathrm{F}^*(\mathrm{N}_G(X))\,\mathrm{F}^*(G)$ is quasinilpotent. Since $\mathrm{E}(Y\,\mathrm{F}^*(\mathrm{N}_G(X))\,\mathrm{F}^*(G)) = \mathrm{E}(G)$ we have $Y\,\mathrm{F}^*(G)$ is a subnormal subgroup of $Y\,\mathrm{F}^*(\mathrm{N}_G(X))\,\mathrm{F}^*(G)$ and $Y\,\mathrm{F}^*(G)$ is quasinilpotent as desired.

Assume now that $Y\,\mathrm{F}^*(G)$ is quasinilpotent. As $Y\,\mathrm{F}^*(G)/\mathrm{E}(G)$ is nilpotent, then $Y\,\mathrm{E}(G)$ is a subnormal quasinilpotent subgroup of $Y\,\mathrm{F}^*(G)$. Write $Y_1 = Y\,\mathrm{E}(G)$. Then $Y_1 \leq \mathrm{N}_G(X)$ by Lemma 1 (3). Notice that $\mathrm{F}(X)$, $\mathrm{F}(Y_1)$, $\mathrm{F}(\mathrm{N}_G(X))$ are subgroups of $C = \mathrm{C}_G(\mathrm{E}(G))$, that is the nilpotent-constrained radical of G. As $\mathrm{F}(X)\,\mathrm{F}(G)$, $\mathrm{F}(Y_1)\,\mathrm{F}(G)$ are nilpotent subgroups of C and $\mathrm{F}(Y_1) \leq \mathrm{N}_C(\mathrm{F}(X))$ it follows from ([1]) that $\mathrm{F}(Y_1)\,\mathrm{F}(\mathrm{N}_C(\mathrm{F}(X)))$ is nilpotent.

On the other hand, as $X = \mathrm{F}(X)\,\mathrm{E}(X)$ and $\mathrm{E}(X) \leq \mathrm{E}(X\,\mathrm{F}^*(G)) = \mathrm{E}(G)$ it follows that $\mathrm{N}_C(\mathrm{F}(X)) = \mathrm{N}_C(X)$. Moreover, $\mathrm{N}_C(X) = C \cap \mathrm{N}_G(X) \trianglelefteq \mathrm{N}_G(X)$, thus $\mathrm{F}(\mathrm{N}_C(X)) \leq \mathrm{F}(\mathrm{N}_G(X)) \leq C$, hence $\mathrm{F}(\mathrm{N}_C(X)) = \mathrm{F}(\mathrm{N}_G(X))$ and $\mathrm{F}(Y_1)\,\mathrm{F}(\mathrm{N}_G(X))$ is nilpotent. As $Y_1\,\mathrm{F}^*(G)/\mathrm{E}(G)$ is nilpotent, it follows that $\mathrm{E}(Y_1) \leq \mathrm{E}(G)$. Therefore $Y\,\mathrm{F}^*(\mathrm{N}_G(X)) = Y_1\,\mathrm{F}^*(\mathrm{N}_G(X)) = \mathrm{F}(Y_1)\,\mathrm{F}(\mathrm{N}_G(X))\,\mathrm{E}(G)$ is quasinilpotent. ∎

Corollary 7 *Let $X \trianglelefteq\trianglelefteq X\,\mathrm{F}^*(G)$, where $X\,\mathrm{F}^*(G)$ is a quasinilpotent subgroup of G and let Y be a quasinilpotent injector of $\mathrm{N}_G(X)$. Then there exists a quasinilpotent injector K of G satisfying $K \cap \mathrm{N}_G(X) = Y$.*

Proof. By Theorem 6, $Y\,\mathrm{F}^*(G)$ is quasinilpotent. Let K be a maximal quasinilpotent subgroup of G containing $Y\,\mathrm{F}^*(G)$, then K is a quasinilpotent injector of G. Thus $K = \mathrm{E}(G)I$, where I is a nilpotent injector of $\mathrm{C}_G(\mathrm{E}(G))$; hence $Y \leq K \cap \mathrm{N}_G(X) = \mathrm{E}(G)(I \cap \mathrm{N}_G(X))$, that is quasinilpotent. Therefore $Y = K \cap \mathrm{N}_G(X)$. ∎

Recall that, if $H \leq G$, it is said that H controls its own G-fusion (briefly H is c-closed in G), if any two elements of H, that are G-conjugate, are already H-conjugate. It is well known the Frobenius theorem , that states that in a finite group G, a Sylow p-subgroup of G is c-closed in G if and only if G has a normal p-complement. Also, C. Sah proved, in π-separable groups, an analogous result for Hall π-subgroups. We will prove corresponding results for nilpotent injectors in nilpotent-constrained groups and for quasinilpotent injectors in finite groups.

Lemma 8 *Let H be c-closed in G. Then:*

(i) $H \leq K \leq G$ implies that H is c-closed in K.

(ii) If $K \leq H \leq G$ and K is c-closed in H then K is c-closed in G.

(iii) If $K \leq H$ and $K \trianglelefteq G$ then H/K is c-closed in G/K.

(iv) If $N \trianglelefteq G$ and $(|N|, |H|) = 1$ then HN/N is c-closed G/N.

Proof. See ([13], 2.2) ∎

Theorem 9 *Let G be a nilpotent-constrained group and let I be a nilpotent injector of G. The following conditions are equivalent:*

(i) G is nilpotent.

(ii) I is c-closed in G.

(iii) $\mathrm{F}(G)$ is c-closed in G.

Proof. Clearly (i) implies (ii).

(ii) $\Rightarrow$ (iii) Let $p \in \pi(|I|)$. As I is c-closed in G it follows that I_p is c-closed in G. Since $I_p \in \mathrm{Syl}_p(\mathrm{C}_G(\mathrm{O}_{p'}(\mathrm{F}(G))))$ by ([11], 1), then I_p is c-closed in $C_p = \mathrm{C}_G(\mathrm{O}_{p'}(\mathrm{F}(G)))$, thus C_p is p-nilpotent $C_p = I_p \mathrm{O}_{p'}(C_p) = I_p \mathrm{Z}(\mathrm{O}_{p'}(\mathrm{F}(G)))$, therefore $I_p \trianglelefteq C_p$ and then $I_p = \mathrm{O}_p(C_p) = \mathrm{O}_p(G)$.

Hence $\mathrm{F}(G) = I$ and $\mathrm{F}(G)$ is c-closed in G.

(iii) $\Rightarrow$ (i) Suppose that there exists $p \in \pi(|G|) \setminus \pi(|\mathrm{F}(G)|)$. Let $P \in \mathrm{Syl}_p(G)$, then $\mathrm{F}(G)$ is a Hall p'-subgroup of $\mathrm{F}(G)P$ and $\mathrm{F}(G)$ is c-closed in $\mathrm{F}(G)P$. Then, by ([13], 1) , we obtain that $P \trianglelefteq \mathrm{F}(G)P$ so $P \leq \mathrm{C}_G(\mathrm{F}(G)) \leq \mathrm{F}(G)$, that is a contradiction. Consequently, $\pi(|\mathrm{F}(G)|) = \pi(|G|)$.

As $\mathrm{F}(G)$ is c-closed in G, it follows that $\mathrm{O}_{p'}(\mathrm{F}(G))$ is c-closed in G, for every $p \in \pi(|G|)$. Take $P \in \mathrm{Syl}_p(G)$, then $\mathrm{O}_{p'}(\mathrm{F}(G))$ is c-closed in $P\,\mathrm{O}_{p'}(\mathrm{F}(G))$, thus $P \trianglelefteq P\,\mathrm{O}_{p'}(\mathrm{F}(G))$ by ([13],1). Then $P \leq \mathrm{C}_G(\mathrm{O}_{p'}(\mathrm{F}(G)))$ and by ([11], 1), we conclude that G is nilpotent. ∎

Theorem 10 *Let I be a quasinilpotent injector of G. The following conditions are equivalent:*

(i) *G is quasinilpotent.*

(ii) *I is c-closed in G.*

(iii) $\mathrm{F}^*(G)$ *is c-closed in G.*

Proof. Clearly (i) implies (ii).

(ii) $\Rightarrow$ (iii) We know that $I = \mathrm{E}(G)V$ where V is a nilpotent injector of $\mathrm{C}_G(\mathrm{E}(G))$.

Since V is c-closed in $\mathrm{C}_G(\mathrm{E}(G))$, by Theorem 9, it follows that $\mathrm{C}_G(\mathrm{E}(G))$ is nilpotent. Therefore $\mathrm{C}_G(\mathrm{E}(G)) = \mathrm{F}(G)$, and $I = \mathrm{E}(G)\,\mathrm{F}(G) = \mathrm{F}^*(G)$.

(iii) $\Rightarrow$ (i) By induction on order of G. Suppose that $Z = \mathrm{Z}(G) \neq 1$. Then, by Lemma 8 (iii) and the inductive hypothesis, we obtain that $G/Z = \mathrm{F}^*(G/Z) = \mathrm{F}^*(G)/Z$ so $G = \mathrm{F}^*(G)$. Therefore, we can suppose that $Z = 1$. Since $G = \mathrm{F}^*(G)\,\mathrm{C}_G(x)$, for every $x \in \mathrm{F}^*(G)$, we can conclude that $\mathrm{Z}(\mathrm{F}(G)) \leq \mathrm{Z}(G) = 1$, thus $\mathrm{F}(G) = 1$ and $\mathrm{F}^*(G) = \mathrm{E}(G)$.

Suppose that $\mathrm{E}(G) \leq L$, where L is a maximal subgroup of G. By Lemma 1 (1) it follows that $\mathrm{E}(G) = \mathrm{E}(L)$, and as $\mathrm{F}(L) \leq \mathrm{C}_G(\mathrm{E}(G)) = \mathrm{Z}(\mathrm{E}(G)) = 1$ we conclude , by induction, that $\mathrm{E}(G) = L$. Then $\mathrm{E}(G)$ is a maximal subgroup of G, so there exists a prime p such that $|G/\mathrm{E}(G)| = p$.

Let Q be a component of G. Since $G = \mathrm{E}(G)\,\mathrm{C}_G(x)$ for all $x \in \mathrm{E}(G)$, it follows that $\mathrm{Z}(Q) = 1$. Therefore $\mathrm{E}(G) = Q_1 \times \ldots \times Q_r$, where $Q_1 \ldots, Q_r$ are the components of G which are nonabelian simple groups. Also they are c-closed in G.

Let $i \in \{1, \ldots, r\}$ $g \in G$ and let α_g be the inner automorphism of G determined by g. Since $Q_i \trianglelefteq G$, one has that the restriction $\alpha_{g|Q_i}$ is an automorphism of Q_i. Note that $\alpha_g(C) = C$ for any conjugacy class C of Q_i; hence, by ([4], Theorem C), there exists $z_i \in Q_i$ such that $\alpha_g(x) = x^{z_i}$ for every $x \in Q_i$. If $x \in E(G)$, then $x = x_1 ... x_r,\ x_i \in Q_i,\ 1 \leq i \leq r$. Thus, $x^g = {x_1}^g ... {x_r}^g = {x_1}^{z_1} ... {x_r}^{z_r} = {x_1}^z ... {x_r}^z = x^z$, where $z = z_1 ... z_r \in E(G)$. Therefore $\alpha_g\,|_{\mathrm{E}(G)}$ is the inner automorphism of $\mathrm{E}(G)$ of G determined by $z = z_1 . \ldots . z_r$. It follows from ([7] , X 13.1) that G is quasinilpotent. ■

Corollary 11 *If G is a group then*

$$\mathrm{F}^*(G) = \bigcap_{\theta \in \mathrm{Irr}(\mathrm{F}^*(G))} \mathrm{I}_G(\theta).$$

Proof. Work by induction on $|G|$. We know that

$$\mathrm{F}^*(G) \leq \bigcap_{\theta \in \mathrm{Irr}(\mathrm{F}^*(G))} \mathrm{I}_G(\theta) = N \trianglelefteq G.$$

Suppose that $N < G$; since $\mathrm{F}^*(G) = \mathrm{F}^*(N)$, by induction , it follows that

$$\mathrm{F}^*(G) = \mathrm{F}^*(N) = \bigcap_{\theta \in \mathrm{Irr}(\mathrm{F}^*(N))} \mathrm{I}_G(\theta) = N.$$

Therefore, we can suppose that $N = G$. Then $\mathrm{I}_G(\theta) = G$, for all $\theta \in \mathrm{Irr}(\mathrm{F}^*(G))$. Let $\mathrm{Irr}(\mathrm{F}^*(G)) = \{\theta_1, \theta_2, .., \theta_m\}$. Suppose that $x, y \in \mathrm{F}^*(G)$ with $x^g = y$, for some $g \in G$, then

$$\theta_i(x) = \theta_i^{g^{-1}}(x) = \theta_i(x^g) = \theta_i(y),\ 1 \leq i \leq m.$$

Thus $\sum \theta_i(x)\theta_i(y^{-1}) = \sum \theta_i(y)\theta_i(y^{-1}) \neq 0$. Then x and y are conjugate in $\mathrm{F}^*(G)$ and, in consequence, $\mathrm{F}^*(G)$ is c-closed in G. Then, using Theorem 10, it follows that $G = \mathrm{F}^*(G)$ as desired. ■

Corollary 12 *If G is a nilpotent-constrained group then*

$$\mathrm{F}(G) = \bigcap_{\theta \in \mathrm{Irr}(\mathrm{F}(G))} \mathrm{I}_G(\theta).$$

Proof. Since G is a nilpotent-constrained group, we have $\mathrm{F}^*(G) = \mathrm{F}(G)$. Now apply the above result. ■

Corollary 13 *Let $\mathfrak{F}$ be a Fitting class such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{N}^*$ and let G be an $\mathfrak{F}$-constrained group (i.e. $\mathrm{C}_G(G_{\mathfrak{F}}) \leq G_{\mathfrak{F}}$). If $I \in \mathrm{Inj}_{\mathfrak{F}}(G)$, the following statements are equivalent:*

(i) $G \in \mathfrak{F}$.

(ii) *I is c-closed in G.*

(iii) *$G_{\mathfrak{F}}$ is c-closed in G.*

Proof. Note that, as G is an $\mathfrak{F}$-constrained group, by ([9], 2), we have $\mathrm{F}^*(G) = G_{\mathfrak{F}}$ and, by ([9], 8), $\mathrm{Inj}_{\mathfrak{F}}(G) = \mathrm{Inj}_{\mathfrak{N}^*}(G)$. Now the result follows from Theorem 10. ■

Remarks.

1. The last results suggest that, perhaps, one can obtain a general result for any Fitting class, but there exist Fitting classes of full characteristic and finite groups, do not belong to the corresponding Fitting class, but whose injectors are c-closed:

Consider $G = \mathrm{A}_5$ and $\mathfrak{F} = \mathfrak{S}$. If $S \in \mathrm{Syl}_5(G)$ then $\mathrm{N}_G(S) \cong \mathrm{D}_{10}$ is an $\mathfrak{F}$-injector of G. Moreover $\mathrm{N}_G(S)$ is c-closed in A_5. Indeed, let $x \in \mathrm{N}_G(S) \setminus \{1\}$ and $g \in G$ such that $x^g \in \mathrm{N}_G(S)$.

If $\mathrm{o}(x) = 5$, then $\langle x \rangle = S$ and we obtain that $g \in \mathrm{N}_G(S)$.

If $\mathrm{o}(x) = 2$, then $\langle x \rangle$, $\langle x^g \rangle$ are Sylow 2-subgroups of $\mathrm{N}_G(S)$, thus there exists $h \in \mathrm{N}_G(S)$ such that $\{1, x^g\} = \langle x^g \rangle = \langle x \rangle^h = \{1, x^h\}$ and so $x^g = x^h$.

2. Even more, there exist Fitting classes of soluble groups with full characteristic and soluble groups, do not belong to the corresponding Fitting class, but whose injectors are c-closed:

If G is a soluble group, we define an homomorphism $\mathrm{d}_G : G \longrightarrow \mathrm{GF}(5)^*$ as follows: let $M_1, M_2, \ldots, M_r$ the 5-chief factors of a prefixed chief series of G. If $g \in G$ and $\mathrm{d}_i(g)$ denotes the determinant of the linear map which g induces on M_i, then

$$\mathrm{d}_G(g) = \prod_{i=1}^{r} \mathrm{d}_i(g)$$

The class $\mathfrak{F} = \{G \in \mathfrak{S} \mid \mathrm{d}_G(G) = 1\}$ is a normal Fitting class ([3], IX 2.14). Let

$$A = \langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \rangle \le GL(2,5)\,.$$

Consider A acting in the natural way on $\mathrm{C}_5 \times \mathrm{C}_5$. Let G be the semidirect product of $\mathrm{C}_5 \times \mathrm{C}_5$ by A:

$$G = [\mathrm{C}_5 \times \mathrm{C}_5]\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \rangle$$

and let

$$S = [\mathrm{C}_5 \times \mathrm{C}_5]\langle \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \rangle$$

Observe that $|G| = 2^4 \cdot 5^2 = 400$ and $|S| = 2^3 \cdot 5^2$.

We will see that S is c-closed in G.

We have that $G = S\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rangle$ and if $S_2 = \langle \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \rangle \in \mathrm{Syl}_2(S)$ then $\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rangle \le \mathrm{C}_G(S_2)$.

Hence, if $x \in S_2$ y $g \in G$, we have $g = cs$, where $c \in \langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rangle$, so $x^g = x^{cs} = x^s$. Therefore $G = \mathrm{C}_G(x)S$.

Let $x \in S$. Since S does not have composed order elements, x is a 2-element or a 5-element.

If x is a 2-element then $x = y^s$ where $y \in S_2$ and $s \in S$.

Hence $\mathrm{C}_G(x)S = \mathrm{C}_G(y^s)S = (\mathrm{C}_G(y))^s S = (\mathrm{C}_G(y)S)^s = G$. Thus, if $g \in G$, it follows that $g = ls$, where $l \in \mathrm{C}_G(x)$, $s \in S$. Then $x^g = x^{ls} = x^s$.

If x is a 5-element, then $x \in \mathrm{C}_5 \times \mathrm{C}_5$. We will see that $G = \mathrm{C}_G(x)S$. It is enough to prove that there exists $g \in G \setminus S$ such that $g \in \mathrm{C}_G(x)$.

If $H \leq G$ write $H^* = H \setminus \{1\}$. Then

$$(\mathrm{C}_5 \times \mathrm{C}_5)^* = \langle h_1\rangle^* \cup \langle h_2\rangle^* \cup \langle h_3\rangle^* \cup \langle h_4\rangle^* \cup \langle h_5\rangle^* \cup \langle h_6\rangle^*$$

where $h_1 = (1, 0)$, $h_2 = (0, 1)$, $h_3 = (1, 1)$, $h_4 = (2, 1)$, $h_5 = (1, 2)$, $h_6 = (4, 1)$.

Notice that if $h \in \langle h_i\rangle$, then $\mathrm{C}_G(h) = \mathrm{C}_G(h_i)$ $1 \leq i \leq 6$ and it is enough to show that $G = \mathrm{C}_G(h_i)S$, $1 \leq i \leq 6$.

We have,

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \in \mathrm{C}_G(h_1) \setminus S, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{C}_G(h_2) \setminus S, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{C}_G(h_3) \setminus S,$$
$$\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \in \mathrm{C}_G(h_4) \setminus S, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \in \mathrm{C}_G(h_5) \setminus S, \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \in \mathrm{C}_G(h_6) \setminus S,$$

Hence $G = \mathrm{C}_G(x)S$. Therefore, if $g \in G$, $g = cs$, where $c \in \mathrm{C}_G(x)$ and $s \in S$. Then $x^g = x^{cs} = x^s$.Thus, S is c-closed in G.

Now consider the chief series of G:

$$1 \trianglelefteq \mathrm{C}_5 \times \mathrm{C}_5 \trianglelefteq [\mathrm{C}_5 \times \mathrm{C}_5]\langle \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \rangle$$
$$\trianglelefteq [\mathrm{C}_5 \times \mathrm{C}_5]\langle \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \rangle \trianglelefteq S \trianglelefteq G.$$

The only 5-chief factor of this series is $\mathrm{C}_5 \times \mathrm{C}_5$.

Notice that $G \notin \mathfrak{F}$ since $\det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 \neq 1$.

The part of the above series from 1 to S is a chief series of S and the only 5-chief factor of this series is $\mathrm{C}_5 \times \mathrm{C}_5$.

Since $\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 1$ and $\det\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 1$, it follows that $S \in \mathfrak{F}$, then $S \in \mathrm{Iny}_{\mathfrak{F}}(G)$ is c-closed in G, but $G \notin \mathfrak{F}$.

3. It is said that a subgroup H in a group G has property CR (Character Restriction) if every ordinary irreducible character $\theta \in \mathrm{Irr}(H)$ is the restriction χ_H of some $\chi \in \mathrm{Irr}(G)$. It is well known that if H satisfies CR property in G then H is c-closed in G.

A number of authors have shown that property CR, together with suitable additional hypothesis on H and G, does imply the existence of a normal complement for H. For instance Hawkes and Humphreys ([5]) prove that CR yields a normal complement if G is solvable and H is an $\mathfrak{F}$-projector for G, where $\mathfrak{F}$ is any saturated formation. The last example shows that the corresponding result for Fitting classes and injectors satisfying property CR, does not work.

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