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Periodic Quasiregular Mappings of Finite Order

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Abstract

The authors construct a periodic quasiregular function of any finite order ρ, $1 \leq \rho < \infty$. This completes earlier work of O. Martio and U. Srebro.

1. Introduction

Let f be a (sense-preserving) quasiregular map on $\mathbb{R}^m$ $(m \geq 2)$. Thus f is ACL^m and there is a $K < \infty$ with

$$|f'(x)|^m \leq K J_f(x) \quad \text{a.e.},$$

where the left side is the norm of the induced operator on the tangent space at x, and the right side is the Jacobian determinant. The now-standard reference is Rickman's monograph [4]. These mappings carry much of the geometric theory of analytic and meromorphic functions to higher dimensions. Suppose in addition that f is entire. We then set

$$M(r, f) = \max_{|x| \leq r} |f(x)|,$$

and define the order ρ of f by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

Perhaps the most important function in the theory is V. Zoric's analogue of the exponential function, $Z(x)$ (cf. [4, p. 15]). It it is not a local homeomorphism, has order one, and is periodic in $m - 1$ of the variables. Using the Zoric function, O. Martio and U. Srebro [3] observed that there exist $(m-1)$–periodic mappings of order 1 and ∞, and (Theorem 8.7) that 1 is a lower bound for the orders of such functions.

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They raise a question [3, p. 38] which is answered by our

Theorem 1.1 *Let ρ, $1 \le \rho \le \infty$ be given. Then there exists an $(m-1)$–periodic $K(m)$-quasiregular map g of exact order ρ.*

In view of [3], this theorem has significance only when $\rho \in (1, \infty)$. The main step in our construction is Theorem 2.1, in which we associate an entire K–qr map f to any of a class of slowly increasing functions $\nu(r)$ which satisfy (2.2) below; K will be independent of the specific choice of ν and depend only on the dimension m. For example, let $\nu(r) = \rho(\log r)^{\rho-1}$ for any fixed $\rho > 1$. Not only will we have $\log M(r,f) \sim (\log r)^\rho$, but for most large x,

$$\log|f(x)| \sim (\log|x|)^\rho, \tag{1.2}$$

where the symbol $\sim$ means that the ratio of the two sides is bounded above and below by positive constants. From this it is routine to see that

$$g(x) = f \circ Z(x) \tag{1.3}$$

is entire, $(m-1)$–periodic, K_1–qr and of exact order ρ. In the special case $m = 2$ and $K = 1$ (analytic functions), the functions of Theorem 2.1 exhaust the class of entire functions of very slow completely regular growth. These functions are discussed, for example, in [1, §6.7].

In [3, p. 38] Martio and Srebro raise another question, for which Theorem 1.1 yields a negative answer. So long as $\rho > 1$, the function f will have infinitely many zeros in $\mathbb{R}^m$. Then (1.3) guarantees that g also has infinitely many zeros in each fundamental region Ω of the function Z in $\mathbb{R}^m$. Martio and Srebro had asked if ρ must always be infinite whenever g is quasiregular, $(m-1)$–periodic and some equation $g(x) = a$ has infinitely many solutions in a fundamental region. They show in Theorem 8.7 that when $\rho = 1$ each $a \in \mathbb{R}^m$ has only finitely many preimages in each Ω. Our Theorem 1.1 implies that their theorem is sharp: when f is chosen as in (1.2) and (1.3), then g assumes all values infinitely often in each Ω.

2. A generalization of the power mapping

Theorem 2.1 *Let $\nu(r)$ be a positive increasing function such that $\nu \to \infty$,*

$$r\nu'(r) < \frac{\nu(r)}{2}, \quad r\nu'(r) = o(\nu(r)) \qquad (r \to \infty), \tag{2.2}$$

and set

$$A(r) = \exp\int_1^r \nu(t)t^{-1}dt. \tag{2.3}$$

Then there exists an entire $K = K(m) - qr$ map f on $\mathbb{R}^m$ with

$$M(r,f) \sim A(r) \qquad (r \to \infty). \tag{2.4}$$

Moreover, on $S(r) = \{x; |x| = r\}$, we have (h_{m-1} is $(m-1)$-Hausdorff measure)

$$|f(x)| > \big(1 + o(1)\big)A(r) \qquad (|x| \to \infty,\ x \in S(r) \setminus E(r)),$$

where $h_{m-1}(E(r)) = o(r^{m-1}) = o(h_{m-1}(S(r)))$.

When $\nu(r) \equiv n \in \mathbb{Z}^+$, the construction is a more complicated version of the power mapping as described in [4, Ch.1, §3.2]. The theorem can be reformulated to allow ν to tend to a finite limit, but since $\nu \to \infty$ in cases of interest, we impose this additional hypothesis.

The map f depends on a sequence $\{r_n\}$ with

$$\nu(r_n) = n, \tag{2.5}$$

and will be defined on the boundary of each m–cube Q_r,

$$Q_r = \{x;\ \|x\|_\infty \le r\}.$$

Every ∂Q_r has $2m$ faces $\{F_j\}$, on each of which $x_j \equiv \pm r$ for some $1 \le j \le m$.

Note from (2.2) and (2.5) that

$$n \log \frac{r_{n+1}}{r_n} \to \infty, \tag{2.6}$$

since $1 = \int_{r_n}^{r_{n+1}} t\nu'(t)dt/t = o(1) n \log(r_{n+1}/r_n)$. We choose $\varepsilon_0 = \varepsilon_0(m)$ with

$$0 < \varepsilon_0 < \frac{1}{2}, \quad \sin^{-1}\varepsilon_0 < \frac{1}{2}\sin^{-1} m^{-1/2}. \tag{2.7}$$

Then (2.6) yields r_0 and $n_0 = n_0(\varepsilon_0, \nu) \ge 4$ so that

$$(m+1)r\nu'(r)/\nu(r) \le \varepsilon_0 \qquad (r > r_0), \qquad \nu(r_0) = n_0 \in \mathbb{Z}, \tag{2.8}$$

$$n \log \frac{r_{n+1}}{r_n} > (m+1)\varepsilon_0^{-1} \qquad (n \ge n_0). \tag{2.9}$$

In this and the next two sections we construct f on $\cup \partial Q_r$ $(r \ge r_0)$, leaving the simpler range $0 \le r \le r_0$ to §5.

With the $\{r_n\}$ as in (2.5), let J_n $(n \ge n_0) = [r_n, r_{n+1}]$. We partition J_n into $m+1$ intervals $J_n^\ell = [r'_{n,\ell}, r''_{n,\ell}]$ $(0 \le \ell \le m)$, subject to $r'_{n,0} = r_n$, $r''_{n,\ell} = r'_{n,\ell+1}$, $r''_{n,m} = r_{n+1}$; (2.9) shows that we may suppose

$$\varepsilon_0 \log\left(\frac{r''_{n,\ell}}{r'_{n,\ell}}\right) = \log\left(\frac{n+1}{n}\right), \ (1 \le \ell \le m,\ n \ge n_0). \tag{2.10}$$

Thus for each $1 \le \ell \le m$, $r''_{n,\ell} = (1+o(1))r'_{n,\ell}$ $(n \to \infty)$, while $r'_{n,1}/r_n \to \infty$. Since $n \ge n_0$ is usually fixed in §§2-4, we often ignore it in our notations.

In §3 we construct f on

$$\bigcup_{n \ge n_0} \bigcup_{r \in J_n^0} Q_r,$$

where we set $J^0 = J_n^0 = [r'_{n,0}, r''_{n,0}] \equiv [r'_0, r''_0]$ $n \ge n_0$. The situation is simpler here since the combinatorics on each ∂Q_r does not change with r, while in §4 we modify this approach on the $\{J_n^k\}$, $n \ge n_0$, $k \ge 1$.

The map f has to evolve in $J = J_n$ subject to:

(A) on ∂Q_{r_n} f is (a constant multiple of) a power-type map of 'degree' n (cf. [4, p. 14]). Thus each of the $2m$ faces of ∂Q_{r_n} is first divided into $(2n)^{m-1}$ congruent $(m-1)$-'boxes' $\mathcal{K}$, where a box is the product of m closed intervals: $\mathcal{K} = I_1 \times \ldots \times I_m$, with one $I_j = \{+r\}$ or $\{-r\}$ and $|I_i| = r/n$ when $i \ne j$. With $S_{m-1} = 2^{m-1}(m-1)!$ as determined below (3.1), we then divide each $\mathcal{K}$ into S_{m-1} $(m-1)$-simplices Λ_r. The map f is defined on each Λ_r by (3.6), so that f is K–qc on Λ_r, K–qr on Q_r, with $|f(x)| \sim A(r_n)$ for $x \in \partial Q_{r_n}$;

(B) situation (A) holds on $\partial Q_{r_{n+1}}$, with $n+1$ in place of n;

(C) the process is such that f is K–qr and $|f(x)| \sim A(|x|)$ for most x on every ∂Q_r, $r \ge r_0$.

We conclude this section with a PL version of the sphere S^m. While Rickman's map is based on the manifold S^m being in the range (and is a so-called Alexander map) our construction in §4 seems to require the polyhedron P of Proposition 2.12. Let $S' = \{|x'| = 1\} \cap \{x_m = 0\}$ be the unit $(m-2)$–sphere. Depending on the context, we may view $\alpha \in S'$ as a vector in $\mathbb{R}^{m-1}$ or one in $\mathbb{R}^m$ whose final coordinate is zero. Choose m points $\alpha^0, \ldots, \alpha^{m-1} \in S'$ so that the vectors $\alpha^j - \alpha^0$ $(1 \le j \le m-1)$ form a basis of $\mathbb{R}^{m-1}$ which is $L(m)$–bilipschitz equivalent to the standard basis, the origin is in the convex hull of the $\{\alpha^i\}$, and the map $(\alpha^j - \alpha^0) \to e^j$ is sense-preserving; the $\{e^j\}$ are the standard basis of $\mathbb{R}^{m-1}$. Let Δ be the convex hull of the $\{\alpha^i\}$, and $s\Delta = \{sp\,; p \in \Delta\}$. For $s > 0$ and $q = s\sum \lambda_i \alpha^i \in \Delta_s$, consider the function

$$\lambda(q) = \lambda_s(q) = ms \inf_i \lambda_i \qquad (q \in \Delta_s). \tag{2.11}$$

(The factor m ensures that $\max_{\Delta_s} \lambda(q) = s$).

Proposition 2.12 *For each $s > 0$, the graph of the function $\lambda_s(q)$, $q \in \Delta_s$, is a polyhedron $P^+ = P_s^+ \subset \{x_m \geq 0\}$. If we define P^- as the graph of $-\lambda_s(q)$, then*

$$P = P^+ \cup P^-$$

is a polyhedron composed of subsets of a finite number of hyperplanes with 0 in its interior. If $q \in \partial\Delta_s$, then $\lambda(q) = 0$.

The ray from 0 to the point $(q, \pm\lambda(q)) \in P$ makes an angle Φ with P such that

$$|\sin\Phi| > 3\tau > 0, \tag{2.13}$$

where τ depends only on the specific choice of the $\{\alpha^i\}$.

Proof. It suffices to consider $s = 1$. Then P determined by $2m$ hyperplanes each of which contains $m - 1$ of the $\{\alpha^i\}$ and one of the points $(\alpha, \pm 1)$, where $\alpha = \sum \alpha^i/m$ is the barycenter of Δ, so it is clear that 0 is interior to P. The normal to each of these hyperplanes has a nonzero component orthogonal to the hyperplane $\{x_m = 0\}$, so the result follows by elementary linear algebra. ■

3. The first stage

Recall the $\{J_n\} = \{\cup_{0\leq\ell\leq m} J_n^\ell\}$, $n \geq n_0$, from the discussion of (2.10). Let $r \in J_n^0$, and consider a face $F \subset \partial Q_r$ on which $x_j = \epsilon r$, for $\epsilon = \pm 1$. Then for $1 \leq i \leq n$, $i \neq j$, the planes

$$\Pi_p^i(n) = \{x_i = pr/n\}, \qquad |p| \leq n, \tag{3.1}$$

divide F into $(2n)^{m-1}$ $(m-1)$-boxes $\mathcal{K}$, and barycentric subdivision of each box in turn partitions F into a union of $(m-1)$-simplices Λ_r, which are positively or negatively oriented with respect to the standard orientation ∂Q_r inherits from $\mathbb{R}^m$. As $r \in \cup_{n\geq n_0} J_n^0$ and $1 \leq j \leq m$ vary, note that each vertex $b(r)$ of Λ_r may be associated to a vector $p \in \mathbb{Z}^m$:

$$b(r) = \Big(\frac{p_1}{2n}, \frac{p_2}{2n}, \ldots, \frac{p_m}{2n}\Big) r, \tag{3.2}$$

with $|p_i| \leq 2n$; on F, $p_j \equiv 2\epsilon n$. Each Λ_r is L–bilipschitz equivalent to the standard $(m-1)$–simplex, up to the scaling factor (cf. (2.3))

$$\frac{r}{\nu(r)} = \frac{A(r)}{A'(r)},$$

with $L = L(m)$. Thus

$$L^{-1}\frac{r}{\nu(r)} \le |\,b^i(r) - b^j(r)| \le L\frac{r}{\nu(r)} \qquad (i \ne j). \tag{3.3}$$

The vertices of $\cup_{\partial Q_r}\Lambda_r$ are put into m classes b^i, $0 \le i \le m-1$, using the standard model Δ of Proposition 2.12. On some face $F \subset \partial Q_r$ choose a positively oriented simplex Λ_r^0, and label its vertices $b^i(r)$, $0 \le i \le m-1$, the ordering taken so that the map

$$\sum \lambda_i b^i(r) \to \sum \lambda_i \alpha^i \qquad (\lambda_1 \ge 0,\ \sum \lambda_i = 1) \tag{3.4}$$

from Λ_r^0 to Δ has positive Jacobian. We may then consistently assign clases b^i to any of the vertices of all $\Lambda_r \subset \partial Q_r$, so that if Λ_r and Λ_r' share a lower dimensional subsimplex, the vertices common to both simplexes belong to the same class. Note that the mapping (3.4) when defined on each simplex Λ_r is sense preserving if Λ_r is positively oriented, and sense reversing otherwise.

With $s = A(r)\,(r \in J_n^0)$ from (2.3), let $p = \sum \lambda_i b^i(r) \in \Lambda_r \subset \partial Q_r$, set

$$p' = s(\sum \lambda_i \alpha^i) \qquad (s = A(r)), \tag{3.5}$$

and, recalling the function $\lambda(p')$ of (2.11), define

$$f(p) = (p', \pm\lambda(p')) = \big(s\sum \lambda_i \alpha^i, \pm\lambda(p')\big) \qquad (s = A(r)). \tag{3.6}$$

The first entry on the right side of (3.6) is an $(m-1)$-vector, and the second is a scalar, and the $\pm$ sign is taken according to whether (3.4) preserves or reverses orientation. Thus (3.6) is always sense preserving.

Lemma 3.7 *Let $\mathcal{B} : e^1, \dots, e^m$ be the standard basis of $\mathbb{R}^m$. Then there is a $K_1 < \infty$ such that at almost each point p and $f(p)$ exist bases $\mathcal{V} = \{v^i\}$ and $\mathcal{W} = \{w^i\}$ of the tangent spaces T_p and $T_{f(p)}$ such that the linear maps determined by*

$$e^i \leftrightarrow v^i, \qquad e^i \leftrightarrow w^i$$

are K_1-quasiconformal. Moreover, if $\mathcal{J}_f$ is the Jacobian matrix relative to the bases $\mathcal{V}$ and $\mathcal{W}$, then

$$\mathcal{J}_f = A'(r)I.$$

Hence, if K_2 is the dilatation of the map (3.4), then f is $K = K_1^2K_2$-quasiregular.

Proof. Given $p = \sum \lambda_i b^i(r) \in \Lambda_r \subset \partial Q_r$, define p' by (3.5). Assume there is a + sign in (3.6), and $\lambda_k = \min_i \lambda_i$ in a neighborhood of p. The basis for T_p consists of $\mathcal{V} = \{v^1, \dots, v^m\}$ such that $v^m = \sum \lambda_i (b^i)'(r)$, and for $1 \le t \le m-1$, the $\{v^t\}$ are the vectors $(\nu(r)/r)(b^{\sigma(t)} - b^k)$, where the $\{\sigma(t)\}_{i=1}^{m-1}$ exhaust the range $1 \le t \le m$, $\sigma \ne k$, ordered so that $\mathcal{V}$ is positively oriented with respect to $\mathcal{B}$. At $f(p) = (p', \lambda(p))$ the basis of $T_{f(p)}$ will be normalized Df-images of $\mathcal{V}$, so that when $t < m$, $w^t = (\alpha^{h(t)} - \alpha^k, -m)$. When $r \in J_n^0$ ($n \ge n_0$) the final basis vector w^m in $\mathcal{W}$ is $w^m = \left(\sum \lambda_i \alpha^i, m\lambda_k\right)$, but this will be modified in Lemma 4.7 for the situation $r \in \cup_{\ell \ge 1} J_n^\ell$, $n \ge n_0$.

Since $\lambda(p')$ is also determined by the coefficient λ_k of b^k for p' near p, (3.6) shows that f is linear near p. Hence if $t < m$ and h is small,

$$p + hv^t = b^k + \sum_{i \ne \sigma(t), k} \lambda_i b^i + (\lambda_{\sigma(t)} + h(\nu(r)/r))(b^{\sigma(t)} - b^k),$$

and (2.3), (2.11), (3.5) and (3.6) yield for $1 \le t \le m-1$ that

$$(3.8)\quad Df(v^t) = \frac{f(p + hv^{\sigma(t)}) - f(p)}{h} = \frac{\nu(r)}{r} A(r)(\alpha^{\sigma(t)} - \alpha^k, -m) \equiv A'(r) w^t.$$

Next, consider $Df(v^m)$. Let $r' = r + h$ and consider the image of $p + hv^m = \sum \lambda_i (b^i + h(b^i)')$. By (3.1),

$$p + hv^m = \sum \lambda_i (b^i(r) + h(b^i)'(r)) = \sum \lambda_i b^i(r') \qquad (r' = r + h),$$

so that $f(p + hv^m) - f(p) = (A(r') - A(r))(\sum \lambda_i \alpha^i, m\lambda_k)$, and

$$(3.9)\qquad Df(v^m) = A(r') w^m.$$

We check that the bases $\mathcal{V}$ and $\mathcal{W}$ satisfy the assertions of Lemma 3.7. First consider $p \in \Lambda_r$. The explicit form of the simplices Λ_r and the arrangement of the $\{\sigma(t)\}$ show that the first $m-1$ vectors v^i form part of such a basis at T_p and lie parallel to that face F of ∂Q_r which contains p, while (3.3) implies $|v^i| \sim 1$. In addition, we deduce from (3.1) that $|v^m| \sim 1$, and that (the vector from 0 to) p makes an angle Θ with F such that $|\sin \Theta| > m^{-1/2}$, so Θ is uniformly bounded away from 0. Thus $\mathcal{V}$ is related to $\mathcal{B}$ as claimed in the Lemma.

Now consider $\mathcal{W}$. That $|w^i| = |(\alpha^i - \alpha^k, -m)| \sim 1$ for $i < m$ follows from properties of the $\{\alpha^i\}$. In addition, we have that $|w^m| = |(\sum \lambda_i \alpha^i, m\lambda_k)| \sim 1$. This follows from (2.11) and (3.6) when $\lambda_k (= \min \lambda_i) > \eta > 0$, but when λ_k is small, then $\sum \lambda_i \alpha^i$ lies near $\partial \Delta$, and so $\sum \lambda_i$ already has magnitude at least h for some fixed $h > 0$. To check that the $\{w^i\}$ span $\mathbb{R}^m$ appropriately, note that the $\{w^j\}$ ($j < m$) span the tangent plane at $f(p) \in A(r)P$. Hence (2.13) ensures that w^m has a uniformly nontrivial normal component to $A(r)P$ at $f(p)$. ∎

4. Interpolation

In order to define f on ∂Q_r for $r \in J_n^k (k \geq 1,\ n \geq n_0)$ we follow the scheme of §3, but need to arrange new simplices (or partial simplices) so that (B) in §2 holds when $r = r_{n+1}$. We do this by working with the $(m-1)$ free coordinates on a given face F one at a time, and when $r \in J_n^\ell$, this will be x_ℓ.

Consider, for example, the face $F \subset \partial Q_r$ on which $x_j \equiv r$. For each $1 \leq i \leq m,\ i \neq j$, F again is partitioned by $(m-1)$-planes orthogonal to the x_i-axis. This has already been described when $r \in J^0$, so consider a fixed $\ell \geq 1$. Then for each $i < \ell,\ i \neq j$, the planes

$$\Pi_p^i(n+1) = \{x_i = pr/(n+1)\}, \quad |p| \leq n+1 \tag{4.1}$$

divide F into $2(n+1)$ congruent slices, and when $i > \ell, i \neq j$, the $\{\Pi_p^i(n)\}$, $|p| \leq n$ of (3.1) divide F into $2n$ congruent slices.

We next consider $i = \ell$, and recall ε_0 in (2.7) and that $J_n^\ell = [r'_\ell, r''_\ell]$. Then use (2.10) to define $\nu_\ell(r)$ with

$$\begin{gathered} \nu_\ell(r'_\ell) = n,\ \nu_\ell(r''_\ell) = n+1, \\ \frac{d(\log \nu_\ell(r))}{d(\log r)} \equiv \frac{r\nu'_\ell(r)}{\nu_\ell(r)} = \frac{1}{\log(r''_\ell/r'_\ell)} \equiv \varepsilon_0 \qquad (r'_\ell \leq r \leq r''_\ell), \end{gathered} \tag{4.2}$$

and partition F by planes $\Pi_p^\ell(\nu_\ell) \equiv \{x_\ell = pr/\nu_\ell(r),\, p \in \mathbb{Z},\, 0 \leq |p| \leq n\}$. As r increases in J_n^ℓ, each $\Pi_{\pm p}^\ell(\nu_\ell)$ recedes from $\{x_\ell = \pm r\}$ and so for the appropriate choice of $n^* \in \{n, n+1\}$, the $\{\Pi_p^i(n^*)\}$ $(i \neq j,\ \ell$, and $|p| \leq n^*)$, $\{\Pi_p^\ell(\nu_\ell)\}$ and $\{x_\ell = \pm r\}$ create new boxes $\mathcal{K} \subset F$, which when $r = r''_\ell$ are all congruent. Boxes whose boundary is disjoint from $\{x_\ell = \pm r\}$ are called interior boxes, and the others are boundary boxes.

As in §3, these boxes must be divided into simplices, and f defined simplex by simplex. If $\mathcal{K}_0$ is an interior box, its barycentric subdivision leads at once to oriented simplies Λ_r as in §3, with vertices $b(r)$ having coordinates $b_i(r)$, such that for $i \neq j,\ i < \ell$, we have $b_i = (2p_i)r/2(n+1)$ $(|p_i| \leq n+1)$, while $b_\ell = (2p_\ell)r/(2\nu_\ell(r))$ $(|p_\ell| \leq n)$ and $b_i = (2p_i)r/(2n),\ |p_i| \leq n$ when $i > \ell,\ i \neq j$. On F we have $b_j \equiv r$. This again allows the simplex structure and orientation to be transferred to the interior boxes. The only new feature is that the coordinate b_ℓ of each vertex satisfies

$$rb'_\ell = b_\ell\Big(1 - \frac{r\nu'_l}{\nu_\ell}\Big) \equiv b_\ell(1-\varepsilon_0), \tag{4.3}$$

instead of what appears in (3.2). Since $n \leq \nu_\ell(r) \leq n+1$, these simplices Λ_r are $(1+o(1)$-bilipschitz equivalent to those Λ_r for $r \in J_n^0$, and so the mappings (3.4) are uniformly $(1+o(1))K_2$-qc (perhaps sense reversing).

We next consider the boundary boxes, and partition them into what we call partial simplices Λ_r^*. It suffices to work in $\{x_\ell \geq 0\} \cap Q_r$. The x_i-coordinates ($i \neq \ell$) of these boxes are the same as those corresponding to vertices of interior boxes, while the x_ℓ-coordinate, b_ℓ, is either $(n/\nu_\ell(r))r$ or r. Let

$$r^* = \frac{1}{2}\Big(1 + \frac{n}{\nu_\ell(r)}\Big)r = \Big(\frac{n + \nu_\ell(r)}{2\nu_\ell(r)}\Big)r,$$

and $H : \{x_\ell = r^*\}$. Then H lies midway between $\Pi_n^\ell(\nu_\ell)$ and $\{x_\ell = r\}$, and each boundary box $\mathcal{K}$ is divided by H into two congruent subboxes $\mathcal{K}_\pm$. Let $\mathcal{K}_- = \mathcal{K} \cap \{(nr/\nu_\ell) \leq x_\ell \leq r^*\}$ and $\mathcal{K}_+$ the reflection of $\mathcal{K}_-$ in H. In an obvious sense $\mathcal{K}_-$ may be considered as a subset of a (phantom) box $\mathcal{K}'$ which is bounded by the hyperplanes $\Pi_n^\ell(\nu_\ell)$ and $\Pi_{n+1}^\ell(\nu_\ell) \equiv \{x_\ell = r(n+1)/\nu_\ell(r)\}$, as well as the various hyperplanes $\Pi_p^i(n^*)$ $(i \neq j, \ell,\ n^* \in \{n, n+1\})$ which meet $\partial\mathcal{K}$. In particular, $\mathcal{K}'_-$ may be divided into oriented simplices Λ_r generated by vertices in the classes $b^i(r)$ exactly as with the interior boxes $\mathcal{K}$. The vertices Λ_r^* of $\mathcal{K}_-$ are of the form $\Lambda_r^* = \Lambda_r \cap \mathcal{K}'$, with inherited orientation. In the same way, we obtain simplices $(\Lambda_r')^* \subset \mathcal{K}_+$; these are reflections of the $\{\Lambda_r^*\}$ across H.

We place $\Lambda_r^* \subset \mathcal{K}'$ in groups according to how many vertices $\Lambda_r \supset \Lambda_r^*$ does *not* have on $\Pi_n^\ell(\nu_\ell)$. This number, $t(\Lambda_r^*)$, is at least 1 and at most $m-1$. If $(\Lambda_r')^* \subset \mathcal{K}_+$ is the reflection of Λ_r^* across H, set $t(\Lambda_r')^* = t(\Lambda_r^*)$, and note that the vertices of Λ_r and Λ_r' which contribute to the appropriate t are of the same classes $\{b^i\}$, while orientations of the simplices are reversed. Let $\mathcal{T} = \mathcal{T}(\Lambda_r^*)$ be the vertices of Λ_r which contribute to $t(\Lambda_r^*)$: we call these the phantom vertices.

The mapping f of (3.7) must be modified so that

$$f \text{ is } L\text{–bilipschitz and } K\text{-}qc \text{ in each } \Lambda_r^*,$$
$$(f(x))_m \geq 0 \text{ on } \Lambda_r^*,\ (f(x))_m = 0 \text{ on } \partial\Lambda_r^*,$$

where $(\cdot)_m$ is the m-th coordinate. The important requirement is that $(f(x))_m$ vanish in $\partial\Lambda_r^*$; otherwise reflection across the boundary (compare with (3.6)) will not be possible. Note that (3.6) cannot be used, since $(f(x))_m$ is usually nonzero when $x \in \mathcal{K}_+ \cap \mathcal{K}_- = H \cap \mathcal{K}$. To avoid this we use $\mathcal{T}$ to modify the function λ of (2.11). According to the definition of $t(\Lambda)$, if $p = \sum \lambda_i b^i(r) \subset \Lambda_r^*$, then

$$0 \leq \sum_{\mathcal{T}} \lambda_i \leq L(r) \equiv \frac{\nu_\ell(r) - n}{2}, \tag{4.4}$$

where the left equality holds when $p \in \Pi_n^\ell(\nu_\ell)$ and the right when $p \in H$.

Thus if K_s is the image of $\Lambda_r^* \cap H$, we have

$$p' = s \sum \lambda_i \alpha^i \in K_s \Longleftrightarrow \sum_{\mathcal{T}} \lambda_i = \frac{\nu_\ell(r) - n}{2} = L(r).$$

Now with p' and $\lambda(p')$ as in (3.5) and (2.11), we define λ_s^* to have the same effect relative to Λ_r^*: if

$$p' = s\Big(\sum \lambda_i \alpha^i\Big) \in \Delta_{A(r)}$$

and L is from (4.4), set

$$\lambda^*(p') = s \min\Big(\lambda(p'),\ (L(r) - \sum_{\mathcal{T}} \lambda_i)\Big), \tag{4.5}$$

so that now $\lambda^* \equiv 0$ on $K_{A(r)}$. Then when $r \in J_n^\ell$ and $p \in \Lambda_r^*$ $(1 \le \ell \le m)$, we modify (3.6) to

$$f(p) = (p', \pm\lambda^*(p')) = \big(s \sum \lambda_i \alpha^i, \pm\lambda^*(p')\big) \qquad (s = A(r)), \tag{4.6}$$

signs chosen so that f is sense preserving. If $p \in \partial\Lambda_r^*$ and $L(r) - \sum_{\mathcal{T}} \lambda_i = 0$, then $p \in H$, and the extension to the symmetric $(\Lambda_r')^*$ is by reflection across H and K.

Lemma 4.7 *Let $p \in \partial Q_r$, $r \in J_n^\ell$ $\ell \ge 1, n \ge n_0$. Then at almost every point p there are bases $\mathcal{V}$ and $\mathcal{W}$ of T_p and $T_{f(p)}$ so that Lemma 3.7 holds.*

Proof. Let p and $p' = f(p)$ be as in Lemma 3.7, with λ_k the minimum λ near p. Take $\mathcal{V}$ and $\{w^1, \dots, w^{m-1}\}$ exactly as in Lemma 3.7, but with the final basis vector, w^m, replaced by a certain $\hat{w}^m$. The first $(m-1)$ components of $\hat{w}^m$ are those of w^m, but $(\hat{w}^m)_m$ is modified to the bracketed term in (4.9) below (so that the factor $A'(r)$ in (4.9) does not appear in $\hat{w}^m$).

When $\lambda^*(p') = \lambda(p')$, the lemma reduces to Lemma 3.7, so we compute J_f when in a neighborhood Ω of p

$$\lambda^*(p') = s\Big(L(r) - \sum_{\mathcal{T}} \lambda_i\Big) < \lambda(p'), \tag{4.8}$$

so that the same set $\mathcal{T}$ is common to all $p' \in \Omega$. The first $(m-1)$ rows of J_f are unchanged, as are all but the diagonal entry of the bottom row. If $p = \sum \lambda_i b^i(r)$, then $p + hv^m = \sum \lambda_i b^i(r')$, $r' = r + h$, so that once again $\sum_{\mathcal{T}} \lambda_i$ is invariant. Hence when (4.8) holds, (4.5) and (4.6) show that if $p \in \Omega$ and h is small,

$$\big(f(p+hv^m)-f(p)\big)_m = \big(A(r')-A(r)\big)\big(L(r')-\sum_{\mathcal{T}} \lambda_i\big)+A(r)\big(L(r')-L(r)\big),$$

and hence (2.3), (4.2), (4.4) and (4.6) give that

$$\begin{aligned}(Df(v^m))_m &= A'(r)\Big(L(r)-\sum_{\mathcal{T}} \lambda_i\Big)+A(r)\frac{\nu_k'}{2}\\ &= A'(r)\Big(L(r)-\sum_{\mathcal{T}} \lambda_i\Big)+\frac{1}{2}\Big(\frac{\nu(r)}{r}\Big)A(r)\Big(\frac{r\nu_\ell'}{\nu_\ell}\Big)\Big(\frac{\nu_\ell}{\nu}\Big)\\ &= A'(r)\Big[\Big(L(r)-\sum \lambda_i\Big)+\frac{1}{2}\varepsilon_0\Big(\frac{\nu_\ell}{\nu}\Big)\Big]. \qquad (4.9)\end{aligned}$$

Thus if $Df(v^m) = \hat{w}^m$, the mth component, $(\hat{w})_m$, satisfies

$$(\hat{w})_m = \max\Big((w^m)_m, \Big(L(r)-\sum_{\mathcal{T}} \lambda_i\Big)+\frac{1}{2}\varepsilon_0\frac{\nu_\ell}{\nu}\Big)$$

(recall w^m from (3.9)). But $(1/2 \geq (L-\sum \lambda_i) \geq 0$ and $2\nu \geq \nu_\ell \geq (\nu/2)$ when $r \in J_n^\ell$. This implies that $1 \geq (\hat{w})_m \geq \varepsilon_0/4$.

We check that these bases satisfy the assertions of Lemma 3.7, and so only need consider $\hat{w}^m$ in the situation that (4.8) holds near p. Now $\varepsilon_0/4 \leq (\hat{w})_m \leq |w^m|$, while for $j < m$, $(w^j)_m \equiv -m$. Hence $\hat{w}^m$ makes an angle with $\mathrm{span}[w^1, \dots, w^{m-1}]$ whose sine is uniformly bounded below. This proves the Lemma. ■

5. Completion of proof

To extend f to Q_{r_0}, recall from §3 that

$$f(x) = A(r_0)\Psi(x) \qquad (x \in \partial Q_{r_0}),$$

where $\Psi : \partial Q_{r_0} \to P_{A(r_0)}$, the polyhedron P of Proposition 3.5. Then exactly as in [2, p. 14] f is extended to the rest of $\mathbb{R}^m$:

$$f(x) = \Big(\frac{r}{r_0}\Big)^{n_0} A(r_0)\Psi\Big(\frac{r_0}{r}x\Big) \qquad (x \in \partial Q_r,\ r \leq r_0).$$

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