Some geometric aspects of Puiseux surfaces

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Abstract

This paper is part of the author's thesis, recently presented, where the following problem is treated: Characterizing the tangent cone and the equimultiple locus of a Puiseux surface (that is, an algebroid embedded surface admitting an equation whose roots are Puiseux power series), using a set of exponents appearing in a root of an equation. The aim is knowing to which extent the well–known results for the quasi–ordinary case can be extended to this much wider family.

1. Introduction. Privileged exponents

Let us consider $\mathbb{C}$ (or any other algebraically closed field of characteristic 0), let X and Y be formally independent variables over $\mathbb{C}$ and let $m \in \mathbb{N}$ be a positive integer, fixed in what follows. We have the following rings and fields:

$$
\begin{aligned}
K &= \mathbb{C}\left((X,Y)\right) \subset L = \mathbb{C}\left(\left(X^{1/m}, Y^{1/m}\right)\right) \\
R &= \mathbb{C}\left[[X,Y]\right] \subset S = \mathbb{C}\left[\left[X^{1/m}, Y^{1/m}\right]\right].
\end{aligned}
$$

The elements of S will be called Puiseux power series.

The field extension is trivially normal and finite, having as Galois group $G \simeq C_m \times C_m$. We will denote the elements of G by

$$
\begin{aligned}
(a,b): \quad L &\longrightarrow L \\
X^{1/m} &\longmapsto \delta^a X^{1/m} \\
Y^{1/m} &\longmapsto \delta^b Y^{1/m}
\end{aligned}
$$

where δ (fixed from now on) is a primitive m–th root of the unity.

Definition 1 *Let $\mathcal{S} = Spec(\mathbb{C}[[X, Y, Z]]/(F)) = Spec(R_{\mathcal{S}})$ be an embedded algebroid surface. Then $\mathcal{S}$ is called a Puiseux surface if the roots of F are Puiseux power series.*

If $\mathcal{S}$ is an irreducible Puiseux surface and F is an equation of $\mathcal{S}$ which is the minimal polynomial of $\zeta \in S$, we will say that $\mathcal{S}$ is defined by ζ.

Definition 2 *Let $\zeta \in S$, written as*

$$\zeta = \sum_{(i,j) \in \Delta(\zeta) \subset \mathbb{N}^2} c_{ij} X^{i/m} Y^{j/m}, 0 \neq c_{ij} \in \mathbb{C}.$$

The set $\Delta(\zeta)$ (Δ, if no confussion is posible) is called the set of exponents of ζ. Note that we consider the exponents (i, j) rather than $(i/m, j/m)$.

A particular case of Puiseux power series (and the surfaces they determine) has been studied quite thoroughly: that of the quasi–ordinary series. A power series $\zeta \in S$ with minimal polynomial $F(Z)$ is called quasi–ordinary if

$$D(F) = X^a Y^b u(X, Y), \text{ with } u(0, 0) \neq 0,$$

where $D(F)$ is the discriminant of F (with respect to Z, of course). These series were already used by Jung ([5]) in his work on the resolution of surface singularities. Later on, the work of Lipman ([7]) and others (see, for instance [4], [3]) has led to a quite well understanding of the geometry and the topology of these surfaces and their resolution.

In particular, it is known that the quasi–ordinary character is preserved by blowing–up equimultiple smooth subvarieties.

Definition 3 *Given a Puiseux power series $\zeta = \sum c_{ij} X^{i/m} Y^{j/m}$, we will say that a set $E = \{(i_1, j_1), \ldots, (i_t, j_t)\} \subset \Delta$ is a set of privileged exponents if*

$$K(\zeta) = K\left(X^{i_1/m} Y^{j_1/m}, \ldots, X^{i_t/m} Y^{j_t/m}\right).$$

Example 1 *Consider the Puiseux power series*

$$\zeta = X^{13/9} Y^{16/9} - 2X^{22/9} Y^{7/9} + 7X^{12/9} Y^{11/9} - 5X^{7/9} Y^{4/9}.$$

Then we have the following minimal sets of privileged exponents:

$$E_1 = \{(7, 4), (12, 11)\}, \ E_2 = \{(12, 11), (13, 16)\}, \ E_3 = \{(12, 11), (22, 7)\}.$$

In fact, these are all the minimal sets of privileged exponents, as

$$X^{12/9} Y^{11/9} \notin K\left(X^{13/9} Y^{16/9}, X^{22/9} Y^{7/9}, X^{7/9} Y^{4/9}\right).$$

It is possible to give a constructive (in an ample sense, taking into account that our objects are power series) process which produces a set of privileged exponents from a given Puiseux power series ([11]). The process relies on a chosen monomial order. If a graded order is taken, the exponents for a quasi–ordinary power series are precisely the so–called characteristic exponents ([7]) of the power series.

Let ζ be a given quasi–ordinary power series ζ with minimal polynomial F, defining a Puiseux surface $\mathcal{S} = Spec(R_{\mathcal{S}})$. The characteristic exponents of ζ have the following properties (see the above references), up to a technical process called normalization:

(a) They are totally ordered by the product ordering

$$(i, j) < (i', j') \iff i < i', \ j < j'.$$

(b) Every exponent $(i, j) \in \Delta$ can be written (mod $\mathbb{Z}m \times \mathbb{Z}m$) as an integral combination of the characteristic exponents.

(c) Their monomials generate the same field extension (of K) than ζ.

(d) They determine the degree of F (that is, $[K(\zeta) : K]$).

(e) They determine the set of equimultiple subvarieties of $\mathcal{S}$.

(f) They determine the tangent cone of $\mathcal{S}$.

(g) They determine (*and* are determined by) the topological type of $\mathcal{S}$ (in the sense of [12]).

(h) They determine the characteristic exponents of any permissible blowing–up of $\mathcal{S}$.

Our abstract aim is studying to which extent the properties of quasi–ordinary singularities can be extended to the general Puiseux singularities. In particular we wanted to know if the tangent cone and the equimultiple locus can be obtained from a finite set of data.

Remark 1 *When working with a given $\zeta \in S$ and its minimal polynomial $F(Z)$, we can always consider that $\Delta \cap (\mathbb{Z}m)^2 = \emptyset$. This can be achieved by performing the Tchirnahusen transformation on F, that is, if*

$$F(Z) = Z^n + a_{n-1}(X, Y)Z^{n-1} + \cdots + a_0(X, Y),$$

then the change of variables

$$Z \longmapsto Z - \frac{1}{n} a_{n-1}(X, Y)$$

will take F to the minimal polynomial of the power series resulting from erasing all the monomials in ζ with exponents in $(\mathbb{Z}m)^2$. So, from now on, we will assume that all the power series we are working with have no exponents in $(\mathbb{Z}m)^2$.

Let $\zeta \in S$ and $\{(i_1, j_1), \ldots, (i_t, j_t)\}$ be a set of privileged exponents of ζ. We have already proved in [11] that this set verify the properties (b), (c) (well, this is obvious) and (d). In particular, with the notations above

$$[K(\zeta) : K] = \frac{m^2}{d(\zeta)},$$

where $d(\zeta)$ is the gcd of the 2×2 minors of the matrix

$$\begin{pmatrix} m & 0 & i_1 & \cdots & i_t \\ 0 & m & j_1 & \cdots & j_t \end{pmatrix}$$

which does not depend on the set of privileged exponents considered.

In this paper we prove that it is not possible, with full generality, to obtain (e) and (f) from a subset of Δ. As the tangent cone and the equimultiple subvarieties are essential for a complete understanding of the resolution process, we conclude that a more complicated set of data is necessary for the general Puiseux case.

2. The tangent cone of Puiseux surfaces

For our purposes, we can take the tangent cone of an embedded algebroid surface to be the affine variety defined by the initial form of an equation of the surface. In fact, the actual definition is the spectrum of the graded ring $\mathrm{gr}_{(X,Y,Z)}(R_{\mathcal{S}})$, but it is easily shown that both versions coincide ([7]).

Remark 2 *We can restrict ourselves to irreducible Puiseux surfaces, as the tangent cone of a reducible surface is the union of the tangent cones of its irreducible components.*

Let therefore $\mathcal{S}$ be an irreducible Puiseux surface, defined by ζ. As an equation of $\mathcal{S}$ is

$$F(Z) = \prod_{i=1}^{n} (Z - \zeta_i),$$

where $\{\zeta = \zeta_1, \ldots, \zeta_n\} = \{(a, b)(\zeta) \mid (a, b) \in G\}$, it is plain from [11] that the multiplicity of $\mathcal{S}$ is determined by a set of privileged exponents (for instance,

choosing any graded ordering in $\mathbb{N}^2$ so that the first exponent determines the order of ζ). In fact,

$$mult(\mathcal{S}) = \min\{n, n\nu(\zeta)\} = \min\left\{\frac{m^2}{d(\zeta)}, \frac{m^2\nu(\zeta)}{d(\zeta)}\right\},$$

where ν is the usual order in S. Here we need to make a crucial distinction for the sequel.

Definition 4 *A Puiseux power series $\zeta \in S$ will be called transversal if $\nu(\zeta) \geq 1$. Otherwise it will be called non–transversal.*

In geometrical terms, transversality (respectively, non–transversality) corresponds to the situation in which the generic fiber of the projection

$$(X, Y, Z) \longmapsto (X, Y)$$

has exactly $n = [K(\zeta) : K]$ (respectively strictly less) points.

2.1. The transversal case

This case is the easiest one. Note that, if $\nu(\zeta) \geq 1$, then its minimal polynomial

$$F(Z) = Z^n + \sum_{l=0}^{n-2} a_l(X, Y) Z^l$$

is a Weierstrass polynomial, that is $\nu(a_l) \geq n - l$.

Remark 3 *If $\nu(\zeta) > 1$, the tangent cone of $\mathcal{S}$ is the plane $Z^n = 0$.*

Remark 4 *If $\nu(\zeta) = 1$, let $\overline{\zeta}$ be the initial form of ζ. As Galois conjugation leaves the exponents unchanged, we have*

$$\overline{F} = \prod_{i=1}^{n} \left(Z - \overline{\zeta_i}\right)$$

with $\Delta(\zeta_i) = \Delta(\zeta)$ and $\Delta(\overline{\zeta}) = \Delta(\overline{\zeta_i})$.
Obviously there can be repeated factors in the above decomposition of $\overline{F}$. In particular, it becomes clear that $\overline{F}$ must be a power of the minimal polynomial of $\overline{\zeta}$.
Using the previous expression for the degree of a minimal polynomial from a set of privileged exponents, we can conclude that, in this case, $\overline{F}$ is a $d(\overline{\zeta})/d(\zeta)$ power of an irreducible polynomial (which is the equation of the surface defined by $\overline{\zeta}$).

2.2. The non–transversal case

Let us write $\nu(\zeta) = \lambda = \mu/n < 1$. In this case the situation is less straight-forward. The initial form of F is now

$$\overline{F} = \overline{\zeta_1} \cdots \overline{\zeta_n},$$

which is a form of order $\lambda m^2/d(\zeta)$. Being a homogeneous polynomial in two variables with coefficients in $\mathbb{C}$ it can be written as a product of linear forms

$$\overline{F} = G_1^{a_1} \cdots G_r^{a_r}, \quad \text{with } G_i \neq G_j,$$

that is, the tangent cone is a union of linear planes. It could well be an only plane or a proper product of planes.

Let us begin with a particularly simple case:

Lemma 1 *Let $\zeta = X^{1/m} + \alpha Y^{1/m}$, $0 \neq \alpha \in \mathbb{C}$. Then, the tangent cone of the surface defined by ζ is given by $(X - \alpha^m Y)^m$.*

Proof: Apply Cardano formulae for the cyclotomic case. ∎

In fact, it is easy showing that

$$\prod_{\delta \,|\, \delta^m = 1} \left(X^{1/m} + \delta \alpha Y^{1/m} \right) = X - \alpha^m Y.$$

The factors on the right hand side will be called the monic conjugates of ζ.

Let us move on to the general case and let $\zeta \in S$ with

$$\overline{\zeta} = L_1^{(1)} \cdots L_{r_1}^{(1)} L_1^{(2)} \cdots L_{r_2}^{(2)} \cdots L_1^{(l)} \cdots L_{r_l}^{(l)},$$

where

$$L_b^{(a)} = X^{1/m} - \alpha_b^{(a)} Y^{1/m},$$

and they are written in such a way that two linear forms $L_{b_1}^{(a)}$ and $L_{b_2}^{(a)}$ share the same superindex if and only there exists δ, an m–th root of the unity, with

$$\delta \alpha_{b_1}^{(a)} = \alpha_{b_2}^{(a)}.$$

Clearly $r_1 + \cdots + r_l = \mu$.

Moreover, when conjugating ζ by $(a, b) \in G$, $L_1^{(1)}$ must be taken (up to product by a constant) into one of its monic conjugates, and so does all $L_j^{(1)}$, $j = 2, \ldots, r_1$. In fact, when we have computed all the $m^2/d(\zeta)$ posible conjugates of ζ appearing in the decomposition of F, the m monic conjugates of $L_1^{(1)}$ must appear among all the conjugates of the $L_j^{(1)}$'s.

Further, all of these monic conjugates must appear the same number of times. If this were not so, after gathering together the remaining forms, we would get a product

$$\prod_{i=1}^{p}\left(X^{1/m} - \alpha_i Y^{1/m}\right) = s(X,Y) \in R.$$

Now, as this product must remain invariant by any $(a,b) \in G$, the unique factorization in S implies that all the monic conjugates of every factor must lie in the product.

Therefore, the total number of monic conjugates of the $L_j^{(1)}$'s appearing in the constant term of F is $m^2 r_1/d(\zeta)$, and all of them appear the same number of times. So, together with the previous lemma, this proves that the decomposition of the tangent cone is

$$(X - \alpha_1^m Y)^{mr_1/d} \cdots (X - \alpha_l^m Y)^{mr_l/d}.$$

Example 2 *If we consider*

$$\zeta = X^{3/4} + 2X^{2/4}Y^{1/4} - X^{1/4}Y^{2/4} - 2Y^{3/4} + X^{6/4},$$

we have the following decomposition for $\overline{\zeta}$:

$$\begin{aligned}\overline{\zeta} &= X^{3/4} + 2X^{2/4}Y^{1/4} - X^{1/4}Y^{2/4} - 2Y^{3/4} \\ &= \left(X^{1/4} - Y^{1/4}\right)\left(X^{1/4} + Y^{1/4}\right)\left(X^{1/4} + 2Y^{1/4}\right).\end{aligned}$$

In the previous notation, this means

$$\begin{aligned}L_1^{(1)} &= X^{1/4} - Y^{1/4}, \quad L_2^{(1)} = X^{1/4} + Y^{1/4}, \quad r_1 = 2 \\ L_1^{(2)} &= X^{1/4} + 2Y^{1/4}, \qquad\qquad\qquad\qquad\quad r_2 = 1\end{aligned}$$

and so the decomposition of the tangent cone is

$$\left[(X - Y)^4\right]^2\left[(X - 2^4 Y)^4\right]^1 = (X - Y)^8 (X - 16Y)^4.$$

Remark 5 *In the quasi–ordinary case, Lipman showed ([7]) that the tangent cone could consist in, at most, two planes. In our case, there is no bound for the amount of planes appearing, as we will show now.*

Obviously, for m fixed, at most $m - 1$ planes can occur to join in the tangent cone, but there can be in fact $m - 1$ of them. A simple example is obtained by taking

$$\zeta = \overline{\zeta} = \prod_{k=1}^{\mu}\left(X^{1/m} + m^k Y^{1/m}\right),$$

whose tangent cone is the union of μ differents planes. By varying μ from 1 to $m - 1$ we obtain all the possible structures.

We summarize the results on this section.

Proposition 1 *Let $\mathcal{S}$ be a Puiseux surface defined by $\zeta \in S$ and $F(Z)$ the minimal polynomial of ζ; n and $d(\zeta)$ defined as above. Then*

(a) *If $\nu(\zeta) > 1$ the equation of the tangent cone is the $Z^n = 0$.*

(b) *If $\nu(\zeta) = 1$ the equation of the tangent cone is the $d(\overline{\zeta})/d(\zeta)$–th power of an irreducible polynomial (the minimal polynomial of $\overline{\zeta}$).*

(c) *If $\nu(\zeta) < 1$ the equation of the tangent cone is a product of linear forms.*

Remark 6 *Note that, while $\nu(\zeta)$ can be known from a set of privileged exponents, the number of different linear forms occuring in case (c) cannot be found out from such a set. A simple example is given by*

$$\zeta_1 = X^{2/4} + (1+i)X^{1/4}Y^{1/4} + iY^{2/4}, \quad \zeta_2 = X^{2/4} + 2X^{1/4}Y^{1/4} + 2Y^{2/4}$$

which have the same Δ (hence the same set of privileged exponents for any choice of a total ordering in $\mathbb{N}^2$). However, the first power series defines a surface whose tangent cone is given by $(X - Y)^4$, while the tangent cone of the surface defined by the second is $(X - Y)^2(X - 4Y)^2$.

3. The equimultiple locus of Puiseux surfaces

The equimultiple locus of an algebroid surface is the set of primes $P \subset R[[Z]]$ such that $F \in P^{(\nu)}$, where ν is the usual order of F. We will be interested only on primes of height 1 as the maximal ideal of $R[[Z]]$ lies always in the equimultiple locus.

Remark 7 *Analogously as in the previous section, we will reduce ourselves to the irreducible case, as a given prime is equimultiple in an algebroid surface if and only if it is equimultiple in all of its irreducible components.*

3.1. The transversal case

Again this is the easiest part. Firstly note that, due to the fact that the minimal polynomial has the form

$$F(Z) = Z^n + a_{n-2}(X,Y)Z^{n-2} + \cdots + a_0(X,Y),$$

with $\nu(F) = n$, a prime ideal belonging to the equimultiple locus can be taken to be $(Z, G(X,Y))$, with $G(X,Y) \in R$, $\nu(G) > 0$.

Let us begin by considering the simplest posible question: is (X, Z) in the equimultiple locus? If $\mathcal{S}$ is defined by ζ and the corresponding equation is

$$F(Z) = \prod_{i=1}^{n} (Z - \zeta_i) = Z^n + \sum_{l=0}^{n-2} a_l(X, Y) Z^l,$$

with $\nu(a_l) \geq n - l$, it becomes obvious that (X, Z) lies in the singular locus if and only if $X^{n-l} | a_l$ for all $l = 0, \dots, n - 2$.

But every monomial of a_l is of the form

$$\alpha X^{(i_1 + \cdots + i_l)/m} \, Y^{(j_1 + \cdots + j_l)/m},$$

for some choice $\{(i_1, j_1), \dots, (i_l, j_l)\} \subset \Delta$. In particular, if we choose $(i_0, j_0) \in \Delta$ and take $i_1 = \cdots = i_l = i_0$ and $j_1 = \cdots = j_l = j_0$ we will get the monomial

$$\alpha X^{li_0/m} \, Y^{lj_0/m}.$$

Hence, it must be $li_0/m > l$, that is, $i_0 > m$ for all $(i_0, j_0) \in \Delta$. This condition clearly suffices, but it is also necessary. For, if we had an exponent $(i_0, j_0) \in \Delta$ with $i_0 < m$, the n–th power of this exponent will appear as an exponent in $a_0(X, Y)$ (at least, if we take (i_0, j_0) with minimal degree in X for avoiding cancellations) and its degree in X will be strictly lower than n, hence (X, Z) cannot be in the equimultiple locus.

Similarly, (Y, Z) lies in the equimultiple locus if and only if $Y | \zeta$. Note that this condition can be observed in a set of privileged exponents if we choose, for instance, the inverse lexicographic ordering as total ordering in $\mathbb{N}^2$ (see [11] for how this implies $Y | \zeta$).

The general case follows the same lines: a prime $(Z, c(X, Y))$ lies in the equimultiple locus if and only if $c(X, Y)^{n-l} | a_l$, for all $l = 0, \dots, n - 2$. So, it obviously suffices that $c(X, Y) | \zeta$.

Let us see that this is also necessary. First $c(X, Y) | \zeta$ is equivalent to $c(X, Y) | \zeta_i$ for all $i = 1, \dots, n$, as $c(X, Y) \in R$. So if $c(X, Y)$ lies in the equimultiple locus but does not divide ζ, it does not divide any ζ_i. Write

$$c(X, Y) = c_1 \left(X^{1/m}, Y^{1/m} \right) \cdots c_s \left(X^{1/m}, Y^{1/m} \right),$$

the factorization of $c(X, Y)$ in S. There must be some c_i not dividing ζ, so write $\zeta_1, \dots, \zeta_j$ the conjugates of ζ not divided by c_i. Then

$$a_{n-j}(X, Y) = \sum_{l_1, \dots, l_j \in \{1, \dots, n\}} \zeta_{l_1} \cdots \zeta_{l_j},$$

so c_i divides all terms of the right hand side except $\zeta_1 \cdots \zeta_j$, which is impossible as $c(X, Y) | a_{n-j}$.

The only remaining possibility is that c_i does not divide any conjugate of ζ. This is clearly absurd again, as c_i must divide $a_0 = \zeta_1 \cdots \zeta_n$.

3.2. The non–transversal case

Write the minimal polynomial of $\zeta \in S$ as

$$F(Z) = Z^n + \cdots + a_0(X, Y),$$

with $\nu(F) = l < n$. We will look up first for equimultiple curves P, not lying in $Z = 0$. If the initial form of ζ contains both $X^{a/m}$ and $Y^{b/m}$ it comes out that both X^a and Y^b appear in $\overline{F}$. Then, as $D(F) \in P$, for any derivation D of order $l - 1$, it is plain that $P = (X, Y)$, which is impossible since F is a monic polynomial in Z.

Hence ζ can be assumed to be

$$\zeta = X^{a/m} + \zeta',$$

with $a < m$, $\nu(\zeta') > \nu(\zeta)$. But if ζ' contains a monomial of the form $X^{b/m}Y^{c/m}$, with $b < a$, and we take one of these monomials with minimal b, it follows (for the same reason as above) that $P = (X, Y)$.

So, if there are equimultiple curves not lying in $Z = 0$, the series has the form $\zeta = X^\lambda u(X^{1/m}, Y^{1/m})$, and, therefore, (X, Z) is an equimutiple curve as well. This situation features a curious phenomenon, which is shown in the following lemma, whose proof follows the same lines as Lipman's normalization for quasi–ordinary surfaces ([7]).

Lemma 2 *Let $\zeta \in S$, $\nu(\zeta) < 1$ and (X, Z) lying in the equimultiple locus of the surface $\mathcal{S}$, defined by ζ. Then there is a transversal Puiseux power series $\eta \in S$ defining $\mathcal{S}$.*

Proof: Let us write, as above, $\zeta = X^\lambda u(X^{1/m}, Y^{1/m})$. Let us call F the minimal polynomial of ζ; μ/m the order of ζ. F is regular with respect to Z and X (with order $\lambda m^2/d$). So, let us call $\widehat{F}$ the Weierstrass polynomial associated to F with respect to X.

Let now T be a new variable. As the units of R have all of their μ–th roots inside R, the power series

$$X^\mu u(X, Y) - T^\mu \in \mathbb{C}[[X, Y, T]]$$

has a factor $Xu'(X, Y) - T$, where u' is an μ–th root of u. By the Weierstrass Preparation Theorem, there exists a unit $u''(X, Y, T)$ such that

$$u''(X, Y, T)(Xu'(X, Y) - T) = X - R(Y, T) \qquad (\star)$$

Doing $X = 0$ we can easily find $R(Y, T) = Tu''(0, Y, T)$. Define then $G(Y, T) = u''(0, Y, T)$, which is a unit in $\mathbb{C}[[Y, T]]$. Writing $X = R(Y, T)$ in the above formula we get

$$0 = u''(R(Y, T), Y, T)\left[R(Y, T)u'(R(Y, T), Y) - T\right].$$

Hence, as $R(Y,T) = TG(Y,T)$,

$$TG(Y,T)u'(TG(Y,T),Y) - T = 0.$$

This means $X = TG(Y,T)$ annihilates a factor of $(\star)$ and therefore

$$(TG(Y,T))^\mu \, u(TG(Y,T),Y) - T^\mu = 0.$$

If we write now $X^{1/m}$ instead of X, $Y^{1/m}$ instead of Y and $Z^{1/\mu}$ instead of T, we can assure that there is a unit $G(Y^{1/m}, Z^{1/\mu})$ such that, if we write $X^{1/m} = Z^{1/\mu}G(Y^{1/m}, Z^{1/\mu})$, we get

$$\zeta\left(Z^{1/\mu}G\left(Y^{1/m}, Z^{1/\mu}\right), Y^{1/m}\right) = Z.$$

As we have

$$F\left(X, Y, \zeta\left(X^{1/m}, Y^{1/m}\right)\right) = 0,$$

doing $X^{1/m} = Z^{1/\mu}G(Z^{1/\mu}, Y^{1/m})$ we obtain

$$F\left(Z^{m/\mu}G^m\left(Z^{1/\mu}, Y^{1/m}\right), Y, Z\right) = 0.$$

Hence $\eta = Z^{m/\mu}G^m\left(Z^{1/\mu}, Y^{1/m}\right)$ is a transversal Puiseux power series which is a root of $\widehat{F}$. ∎

Remark 8 *As for other kind of primes in the equimultiple locus of $\mathcal{S}$, note that the arguments for the transversal case do not use transversality at all. That means $(Z, c(X,Y))$ lies in the equimultiple locus if and only if $c(X,Y)|\zeta$, which is impossible when $\nu(\zeta) < 1$.*

This finishes the considerations for the non–transversal case.

We summarize the results on this section.

Proposition 2 *Let $\mathcal{S}$ be a Puiseux surface defined by $\zeta \in S$ and $F(Z)$ the minimal polynomial of ζ. Then*

(a) *If $\nu(\zeta) \geq 1$, a prime ideal $(Z, c(X,Y))$ lies in the equimultiple locus if and only if $c(X,Y)|\zeta$.*

(b) *If $\nu(\zeta) < 1$ either there are no primes of height 1 in the equimultiple locus or there exists a change of variables which takes the surface into the transversal case.*

Remark 9 *Again, it is not posible to express the equimultiple locus from a set of exponents of ζ, as it is shown in the following example:*

$$\zeta_1 = X^{3/2} + Y^{1/2}X - YX^{1/2} - Y^{3/2}, \quad \zeta_2 = X^{3/2} + 3Y^{1/2}X + 3YX^{1/2} + Y^{3/2}.$$

The first Puiseux power series defines a surface with equation

$$F_1 = Z^4 - 2(X - Y)^2(X + Y)Z^2 + (X - Y)^6$$

where the prime $P = (Z, X - Y)$ lies in the equimultiple locus, while an equation for the second surface is

$$F_2 = Z^4 - 2\left(X^3 + 15X^2Y + 15XY^2 + Y^3\right)Z^2 + (X - Y)^6,$$

hence the equimultiple locus of this surface is simply the closed point.

4. Final comments

Puiseux singularities are still far from being well understood. In particular, two goals seem highly interesting to attain:

(a) A simple (as simple as possible, at least) criterion for deciding whenever a given surface is Puiseux. In the quasi–ordinary case, this is the celebrated Jung–Abhyankar theorem ([5], [1]) but, for the more general case, we cannot know whether a given polynomial has Puiseux roots unless we actually compute them. This is now posible, thanks to the work of McDonald ([9]) and Aroca–Cano ([2]); but, while useful for working with examples, this method has not been adopted so far for its use on abstract argumentations.

(b) A numerical (say arithmetical, say combinatorial) control of the behaviour of the singularity in the resolution process. In the quasi–ordinary case, this is the main result in Lipman's thesis ([7]). For the general case, only partial results are known (see [10]), which we summarize now:

   (i) It is possible to describe a resolution process for Puiseux surfaces which, in each step, preserves the Puiseux character. This resolution process is, basically, the Levi resolution ([6]) which consists in blowing–up maximal subvarieties of the equimultiple locus; up to some previous monoidal transformations which take our surface into another one whose roots are ν–quasi–ordinary (this is a mild generalization of the quasi–ordinary concept, see, for instance, [8]).

(ii) For some cases (transversal and ν–quasi–ordinary) it is possible to prove that the blowing–up of a Puiseux surface is still a Puiseux surface, and also we are able to know a set of privileged exponents of the transformed surface from a set of the original one. Still, we do not know what happens in the general case.

Example 3 *Let us end with an example showing the difficulties of the behaviour under blowing–up of a Puisuex surface. The computations below have been posible thanks to the software developed by F. Aroca and J. Cano ([2]).*

Let us consider

$$\zeta = X^{2/5} + Y^{2/5},$$

whose minimal polynomial is

$$F(Z) = Z^{25} + a_{20}(X,Y)Z^{20} + a_{15}(X,Y)Z^{15} + a_{10}(X,Y)Z^{10}$$
$$+ a_5(X,Y)Z^5 + a_0(X,Y),$$

where

$$
\begin{aligned}
a_{20}(X,Y) &= -5X^2 - 5Y^2 \\
a_{15}(X,Y) &= 10X^4 + 10Y^4 - 605X^2Y^2 \\
a_{10}(X,Y) &= -10X^6 - 10Y^6 - 1905X^4Y^2 - 1905X^2Y^4 \\
a_5(X,Y) &= 5X^8 + 5Y^8 - 605X^6Y^2 - 605X^2Y^6 + 1905X^4Y^4 \\
a_0(X,Y) &= -X^{10} - Y^{10} - 5X^8Y^2 - 5X^2Y^8 - 10X^4Y^6 - 10X^6Y^4
\end{aligned}
$$

A quadratic transform in the point $(0 : 0 : 1)$ of the tangent cone gives the equation

$$Z^{15} + a_{20}(X,Y)Z^{12} + a_{15}(X,Y)Z^9 + a_{10}(X,Y)Z^6 + a_5(X,Y)Z^3 + a_0(X,Y).$$

The roots of this equation (which is a Weierstrass equation w.r.t. X, Y and Z) as a polynomial in Z are

$$\eta = Y^{2/3} + \frac{5}{3}X^{2/5}Y^{4/15} + \frac{5}{9}\frac{X^{4/5}}{Y^{2/15}} - \frac{5}{81}\frac{X^{6/5}}{Y^{8/15}};$$

and all of its conjugates. Analogously, its roots as a polynomial in X are

$$\varsigma = Z^{3/2} - \frac{5}{2}Z^{9/10}Y^{2/5} + \frac{15}{8}Z^{3/10}Y^{4/5} - \frac{5}{16}\frac{Y^{6/5}}{Z^{3/10}},$$

and all of its conjugates. As the polynomial is symmetric in X and Y, we cannot find an easy parametrization which gives us an equation with Puiseux roots. The existence of such an equation is a problem which relates with the two open questions set above.

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