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Application of accretive operators theory to evolutive combined conduction, convection and radiation

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Abstract

The accretive operators theory is employed for proving an existence theorem for the evolutive energy equations involving simultaneously conduction, stationary convection (in the sense that the velocity field is assumed to be time independent), and radiation. In doing that we need to use new existence results for elliptic linear problems with mixed boundary conditions and irregular data.

1. Introduction

Radiation heat transfer may have a great importance in processes at high temperatures. In those cases temperatures are affected by variations in the intensity of radiation, which is the solution of a transfer equation like the neutron transport equation, which also depends on the temperature. This means that one has to solve a coupled system for the temperature (T) *and* the intensity (i). Transient solutions for the conduction-radiation problem, i.e., when both conduction and radiation affect the evolution of temperatures, are necessary to examine heat transfer and thermal stress behavior for many practical applications. This includes "heat transfer in ceramic components for high temperature use, thermal protection coatings, glass forming for manufacturing, tempering of glass windows, glass envelopes for high intensity lamps, porous burners and insulation systems, liquid-drop and liquid-sheet space radiators, ceramic insulation for atmospheric reentry of spacecraft, and some high temperature components in advanced aircraft engines" [19].

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Existence results for the steady conduction-radiation problem have been proved [11], [12], while the evolutive problem including only radiation has been object of several studies [10], [16], [14] and references therein. Regarding the evolutive convection-radiation problem, it has been proved to be solvable when the velocity field is regular, steady and incompressible [14]. The result of existence given in this paper is, to the authors' knowledge, the first one for the evolutive problem including combined conduction, convection, and radiation.

The use of accretive operators theory in L^1 spaces makes this work be the natural continuation of Mercier's paper [16]. The inclusion of convection and conduction leads to the difficulty, pointed out in [14], of proving existence and uniqueness in L^1 for $T + \mathbf{v} \cdot \nabla T - \Delta T = \hat{T}$, where $\hat{T}$ is given in L^1. This is an interesting problem by itself. In the case of Dirichlet homogeneous boundary conditions the existence of distributional solutions is proved in [8], [9], [2], [3] and in [4]. In the general case of mixed boundary conditions, the existence of weak solutions was not proved even with regular data (that is $\hat{T} \in \mathrm{L}^2(\Omega)$). Moreover, for what concerns uniqueness results, in [3] (see also [5]) it is also proved that the solution of the homogeneous Dirichlet problem that the authors have constructed satisfies some further regularity conditions (i.e., it is a renormalized solution) and that there is uniqueness in the set of the renormalized solutions. Unfortunately this is insufficient for us because what we need is uniqueness in $\mathrm{W}^{1,1}(\Omega)$. Anyway we have solved this problem in [15] under (also nonhomogeneous) Dirichlet boundary conditions or under suitable boundary mixed conditions (see subsection 4.1.2 below).

The paper is organized as follows. In Section 2 we present the equations, under the name *the convection-conduction-radiation problem.* Section 3 contains the statement of the main theorem (Theorem 3.1), and Section 4 is devoted to its proof. The paper ends with the conclusions in Section 5, and the proofs of some technical lemmas in Section 6 (Appendix).

2. The mathematical model

2.1. Notations

- a: absorption coefficient [units: $1/m$].
- c: speed of light in the medium $[m/s]$.
- c_v: specific heat at constant volume $[J/(kg\ K)]$.
- $d\boldsymbol{\omega}$: surface measure on the unit sphere S^2.
- F: internal heat source $[W/m^3]$.

- i: intensity of radiation $[J/(m^2 sr)]$ (here sr stands for steradian, the unit of solid angle).
- i_b: intensity of radiation emitted from a blackbody $[J/(m^2\ sr)]$. The function i_b obeys Planck's law:

$$i_b(\nu, T) = \frac{2hn^2\nu^3}{c_0^2(\mathrm{e}^{h\nu/kT} - 1)} \tag{2.1}$$

  for a frequency ν and a temperature T, where $h = 6.6262 \times 10^{-34}$ $[J \cdot s]$ is Planck's constant, $k = 1.3806 \times 10^{-23}$ $[J/K]$ is Boltzmann's constant, n is the refractive index of the medium and $c_0 = 2.9979 \times 10^8$ $[m/s]$ is the speed of light in the vacuum.
- $\bar{k}$: thermal conductivity $[W/(m\ K)]$.
- $\mathbf{q_c}$: conductive flux vector $[W/m^2]$.
- $\mathbf{q_r}$: radiant flux vector $[W/m^2]$.
- T: temperature $[K]$.
- $\mathbf{v}$: velocity field $[m/s]$.
- ρ: density $[kg/m^3]$.
- σ_s: scattering coefficient $[1/m]$.
- ϕ: scattering phase function [dimensionless]. This function must satisfy the equality

$$\int_{S^2} \phi(\boldsymbol{\omega}^\star, \boldsymbol{\omega})\ d\boldsymbol{\omega} = 4\pi \quad \forall \boldsymbol{\omega}^\star \in S^2. \tag{2.2}$$

2.2. Mathematical model

Let us consider a bounded, open and connected set $\Omega \subset \mathbb{R}^3$ with regular C^∞-boundary Γ, and a bounded time interval $(0, \tau)$. Let us call $Q = \Omega \times (0, \tau)$ and $\Sigma = (0, \infty) \times \Omega \times S^2 \times (0, \tau)$. For the sake of brevity, we will use the notation $i(\boldsymbol{\omega}) = i(\nu, \mathbf{x}, \boldsymbol{\omega}, t)$.

A gas is called *participating* when it scatters, absorbs, and emits thermal radiation. The following system models the thermal evolution of a participating incompressible gas with given velocity field $\mathbf{v}$ [6], [17], which is being heated by an internal source $F(\mathbf{x}, t)$:

$$\rho c_v \Big(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} T\Big) + \nabla_{\mathbf{x}} \cdot (\mathbf{q_r} + \mathbf{q_c}) = F \quad \text{in} \quad Q. \tag{2.3}$$

$$\frac{1}{c}\frac{\partial i}{\partial t} + \boldsymbol{\omega} \cdot \nabla_{\mathbf{x}} i + a(i - i_b) = L(i) \quad \text{in} \quad \Sigma, \tag{2.4}$$

where

$$L(i) = \frac{\sigma_s}{4\pi} \int_{S^2} [\phi(\boldsymbol{\omega}^\star, \boldsymbol{\omega}) i(\boldsymbol{\omega}^\star) - \phi(\boldsymbol{\omega}, \boldsymbol{\omega}^\star) i(\boldsymbol{\omega})]\ d\boldsymbol{\omega}^\star.$$

The conductive flux vector is given by Fourier's law:

$$\mathbf{q_c} = -\bar{k}\nabla_{\mathbf{x}} T, \tag{2.5}$$

and the radiant flux vector is given by

$$\mathbf{q_r} = \int_0^\infty \int_{S^2} i\boldsymbol{\omega}\ d\boldsymbol{\omega} d\nu. \tag{2.6}$$

If the transient term $(1/c)(\partial i/\partial t)$ is removed, then (2.4) and (2.6) imply, formally,

$$\nabla_{\mathbf{x}} \cdot \mathbf{q_r} = \int_0^\infty \int_{S^2} a(i_b - i)\ d\boldsymbol{\omega} d\nu. \tag{2.7}$$

Thus, we have the following system, which is expected to be a good model in case that $(1/c)(\partial i/\partial t)$ is negligible (this is really the case in many applications [17], not in astrophysics neither in short pulsar lasers):

$$\rho c_v \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} T \right) - \bar{k}\Delta_{\mathbf{x}} T + \int_0^\infty \int_{S^2} a(i_b - i)\ d\boldsymbol{\omega} d\nu = F \quad \text{in } Q. \tag{2.8}$$

$$\frac{1}{c}\frac{\partial i}{\partial t} + \boldsymbol{\omega} \cdot \nabla_{\mathbf{x}} i + a(i - i_b) = L(i) \quad \text{in } \Sigma. \tag{2.9}$$

The system (2.8)-(2.9) has been proved to be solvable in the purely radiant case ($\mathbf{v} \equiv \mathbf{0}$, $\bar{k} = 0$) [16], [13], [14], and in the convective-radiative case ($\bar{k} = 0$) [14]. Here we assume that conduction is always present together with radiation, but the convective term may appear or not; in other words, $\bar{k}$ is strictly positive but $\mathbf{v}$ may be $\mathbf{0}$. We will call this problem, together with the initial and boundary conditions, *the convection-conduction-radiation problem.*

2.2.1. Initial Conditions

We require that

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}) \text{ in } \Omega, \tag{2.10}$$

$$i(\nu, \mathbf{x}, \boldsymbol{\omega}, 0) = i_0(\nu, \mathbf{x}, \boldsymbol{\omega}) \text{ in } (0, \infty) \times \Omega \times S^2. \tag{2.11}$$

2.2.2. Boundary Conditions (from now on, b.c.)

Recall that Γ is the boundary of Ω. We use the notations:

- $D_{\text{in}} = \{(\mathbf{x}, \boldsymbol{\omega}) \in \Gamma \times S^2 :\ \mathbf{n}(\mathbf{x}) \cdot \boldsymbol{\omega} < 0\}$, where $\mathbf{n}(\mathbf{x})$ is the outward unit normal at $\mathbf{x} \in \Gamma$.
- Γ_1 is a measurable subset of Γ.
- $\Gamma_{\mathbf{v}}^- = \{\mathbf{x} \in \Gamma :\ \mathbf{n}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) < 0\}$.

For i the b.c. is

$$i_{|(0,\infty)\times D_{\rm in}\times(0,\tau)}(\nu,\mathbf{x},\boldsymbol{\omega},t)=\bar{h}(\nu,\mathbf{x},\boldsymbol{\omega}), \tag{2.12}$$

which means that the inflow of radiant intensity is known and independent of t.

For T we consider the mixed b.c.

$$T_{|\Gamma_1\times(0,\tau)}(\mathbf{x},t) = \bar{g}(\mathbf{x}), \tag{2.13}$$

$$(\nabla_{\mathbf{x}}T\cdot\mathbf{n})_{|(\Gamma\setminus\Gamma_1)\times(0,\tau)}(\mathbf{x},t) = 0, \tag{2.14}$$

where Γ_1 is such that $\Gamma_{\mathbf{v}}^{-}\subseteq\Gamma_1$.

Remark 2.1 *Note that the Dirichlet b.c.* $T_{|\Gamma\times(0,\tau)}=\bar{g}$ *is obtained for the choice* $\Gamma_1=\Gamma$ *in the mixed b.c. (2.13)-(2.14). Moreover, if* $\mathbf{v}\cdot\mathbf{n}\geq 0$ *on* Γ *(in particular if* $\mathbf{v}\equiv\mathbf{0}$*, which corresponds to the case of a motionless gas),* Γ_1 *may be an arbitrary measurable subset of* Γ*; in that case, the mixed b.c. (2.13)-(2.14) with the choice* $\Gamma_1=\emptyset$ *becomes homogeneous Neumann b.c.*

3. Main result

The density ρ, the specific heat c_v and the conductivity $\bar{k}$ are assumed to be constant and positive. The velocity field $\mathbf{v}$ is assumed to be independent of t, and with regularity

$$\mathbf{v}\in(\mathrm{W}^{1,\infty}(\Omega))^3,\quad \operatorname{div}\mathbf{v}=0. \tag{3.1}$$

We assume also the following dependences: $a=a(\nu,T(\mathbf{x},t))$, $\sigma_s=\sigma_s(\nu,\mathbf{x})$, being the second one different from the first only due to technical reasons. Lastly, let us assume that

$$\begin{aligned}
&F\in\mathrm{L}^\infty(\Omega\times[0,\tau]);\\
&\sigma_s \text{ and } \phi \text{ are measurable, essentially bounded and nonnegative;}\\
&\phi \text{ is symmetric, i.e., } \phi(\boldsymbol{\omega},\boldsymbol{\omega}^\star)=\phi(\boldsymbol{\omega}^\star,\boldsymbol{\omega});\\
&T_0\in\mathrm{L}^1(\Omega);\ i_0\in\mathrm{L}^1((0,\infty)\times\Omega\times S^2);\\
&\bar{h}\in\mathrm{L}^1((0,\infty)\times D_{\rm in},|\boldsymbol{\omega}\cdot\mathbf{n}|d\nu d\gamma d\boldsymbol{\omega});\\
&\text{with } \bar{g} \text{ we denote both } \psi_{|\Gamma_1} \text{ or } \psi_{|\Gamma},\ \text{where } \psi\in\mathcal{D}(\bar{\Omega});
\end{aligned} \tag{3.2}$$

and that, for certain real numbers m and M with $0<m<M$,

$$a_{|(0,\infty)\times[m,M]} \text{ is continuous} \tag{3.3}$$

and there exist positive constants C_1 and C_2 such that

$$\begin{aligned}
0\leq a(\nu,T) &\leq C_1,\ \forall(\nu,T)\in(0,\infty)\times[m,M];\\
|a(\nu,T_1)-a(\nu,T_2)| &\leq C_2|T_1-T_2|,\ \forall(\nu,T_i)\in(0,\infty)\times[m,M].
\end{aligned} \tag{3.4}$$

Let us set

$$\begin{aligned} X &= \mathrm{L}^1(\Omega) \times \mathrm{L}^1((0,\infty) \times \Omega \times S^2), \\ \|(T,i)\|_X &= \|T\|_{\mathrm{L}^1(\Omega)} + \|i\|_{\mathrm{L}^1((0,\infty)\times\Omega\times S^2)}, \\ f &= \frac{F}{\rho c_v}, \text{ and} \\ \|f\|_\infty &= \|f\|_{\mathrm{L}^\infty(\Omega\times[0,\tau])}. \end{aligned}$$

Then we have the following theorem, where the concept of solution is weak in the sense of the nonlinear semigroup theory (*mild solution* in [1] and [7], C^0-*solution* in [18]).

Theorem 3.1 *Under the previous hypotheses, the convection-conduction-radiation problem has a unique solution* $\{T,i\} \in \mathrm{C}([0,\tau];X)$ *satisfying*

$$m \le T \le M \quad \textit{and} \quad i_b(\nu,m) \le i \le i_b(\nu,M),$$

provided that

- $m + \tau\|f\|_\infty \le T_0 \le M - \tau\|f\|_\infty$ *a.e. in* Ω,
- $m + \tau\|f\|_\infty \le \bar{g} \le M - \tau\|f\|_\infty$ *a.e. on* Γ_1,
- $i_b(\nu, m + \tau\|f\|_\infty) \le i_0 \le i_b(\nu, M - \tau\|f\|_\infty)$ *a.e. in* $(0,\infty) \times \Omega \times S^2$,
- $i_b(\nu, m + \tau\|f\|_\infty) \le \bar{h} \le i_b(\nu, M - \tau\|f\|_\infty)$ *a.e. in* $(0,\infty) \times D_{\mathrm{in}}$.

Remark 3.1 *The assumption* $\operatorname{div}\mathbf{v} = 0$ *in Theorem 3.1 can be weakened in the hypothesis* $\operatorname{div}\mathbf{v} \le 0$, *although it is not clear the physical meaning of* $\operatorname{div}\mathbf{v} < 0$ *in the energy equation (2.3).*

Remark 3.2 *Theorem 3.1 is valid in the particular case of the conduction-radiation problem, which corresponds to the choice* $\mathbf{v} \equiv \mathbf{0}$, *for the problem with no internal heating* ($f \equiv 0$), *and for the combination of both of them.*

Remark 3.3 *When* $f \equiv 0$, *the solution exists in the time interval* $[0,\infty)$.

Remark 3.4 *When the internal heat source* F *is nonnegative* (f *nonnegative) then the same result holds if we replace* $m + \tau\|f\|_\infty$ *by* m. *This fact has the physical explanation that the temperature can only increase with time, and not decrease, due to the effect of a nonnegative heat source.*

4. Application of the accretive operators theory

The basic result that we use is a perturbed version of Crandall-Liggett generation theorem [1], [7], [18], which states that, if X is a Banach space, the initial value problem $(du/dt)+Au+Bu=\bar{f},\ u(0)=u_0$, has a unique (mild) solution provided that $A:D(A)\subset X\to X$ is m-accretive, $B:X\to X$ is Lipschitz, $\bar{f}\in \mathrm{L}^1(0,\tau;X)$, and $u_0\in\overline{D(A)}$.

We can write the problem (2.8)-(2.11) in the abstract form

$$\frac{du}{dt}+Au+Bu=\bar{f},\ u(0)=\{T_0,i_0\}, \tag{4.1}$$

where $u=\{T,i\}$, $\bar{f}=\{f,0\}$, and the operators A and B are defined as follows:

$$Au=\{\mathbf{v}\cdot\nabla T-\frac{\bar{k}}{\rho c_v}\Delta T, c\ \boldsymbol{\omega}\cdot\nabla_{\mathbf{x}} i-c\ L(i)\}. \tag{4.2}$$

$$Bu=\{-\frac{1}{\rho c_v}\int_0^\infty\int_{S^2} q(u)\ d\nu d\boldsymbol{\omega}, c\ q(u)\}, \tag{4.3}$$

with

$$q(u)(\nu,\mathbf{x},\boldsymbol{\omega})=a(\nu,T(\mathbf{x}))[i(\nu,\mathbf{x},\boldsymbol{\omega})-i_b(\nu,T(\mathbf{x}))] \tag{4.4}$$

and

$$X=\mathrm{L}^1(\Omega)\times\mathrm{L}^1((0,\infty)\times\Omega\times S^2). \tag{4.5}$$

The domain of A is

$$\begin{aligned} D(A)=\{u=\{T,i\}\in X:\ &\boldsymbol{\omega}\cdot\nabla_{\mathbf{x}} i\in \mathrm{L}^1((0,\infty)\times\Omega\times S^2),\\ &T\in \mathrm{W}^{1,1}(\Omega),\ \Delta T\in \mathrm{L}^1(\Omega),\quad i \text{ and } T \text{ satisfy the b.c.}\}. \end{aligned} \tag{4.6}$$

The proof of the following lemma is given in the appendix.

Lemma 4.1 *$D(A)$ is dense in X.*

Lemma 4.1 guarantees that the initial condition $\{T_0,i_0\}$ can be chosen an arbitrary element of X.

When $\Gamma\backslash\Gamma_1\neq\emptyset$, we have to give sense to the b.c.

$$\nabla T\cdot\mathbf{n}=0\quad\text{on } \Gamma\backslash\Gamma_1. \tag{4.7}$$

This is done in the following definition.

Definition 4.1 *Given $T \in \mathrm{W}^{1,1}(\Omega)$ such that $\Delta T \in \mathrm{L}^1(\Omega)$, we say that T satisfies the b.c. (4.7) if*

$$\int_\Omega \Delta T \varphi \, d\mathbf{x} = -\int_\Omega \nabla T \cdot \nabla \varphi \, d\mathbf{x} \tag{4.8}$$

for all $\varphi \in \mathrm{H}^1(\Omega) \cap \mathrm{W}^{1,\infty}(\Omega)$ such that $\varphi_{|\Gamma_1} = 0$.

Remark 4.1 *Note that, if $T \in \mathrm{W}^{2,1}(\Omega)$, (4.8) implies*

$$\int_{\Gamma \setminus \Gamma_1} (\nabla T \cdot \mathbf{n}) \varphi \, d\gamma = 0$$

for all $\varphi \in \mathrm{H}^1(\Omega) \cap \mathrm{W}^{1,\infty}(\Omega)$ such that $\varphi_{|\Gamma_1} = 0$.

The operator B is not well defined from X into X; instead, we consider a "truncation" B_m^M which is well defined and Lipschitz, if the hypotheses described in Section 3 are fulfilled. Before defining B_m^M, we need some other definitions. Recall that m and M are two real numbers such that $0 < m < M$. In the following definition ζ may be a or i_b:

$$\Theta_m^M(\zeta)(\nu, T) = \begin{cases} 0 & \text{if} \quad T < 0 \\ (\zeta(\nu, m)/m)T & \text{if} \quad 0 \le T < m \\ \zeta(\nu, T) & \text{if} \quad m \le T \le M \\ \zeta(\nu, M) & \text{if} \quad T > M. \end{cases} \tag{4.9}$$

Remark 4.2 *Since $m > 0$, $\Theta_m^M(a)$ maintains us away from the singularity of the absorption coefficient a at $T = 0$ [10].*

Let us define also

$$\varphi^M(i)(\nu, \mathbf{x}, \boldsymbol{\omega}) = \begin{cases} 0 & \text{if} \quad i(\nu, \mathbf{x}, \boldsymbol{\omega}) < 0 \\ i(\nu, \mathbf{x}, \boldsymbol{\omega}) & \text{if} \quad 0 \le i(\nu, \mathbf{x}, \boldsymbol{\omega}) \le i_b(\nu, M) \\ i_b(\nu, M) & \text{if} \quad i(\nu, \mathbf{x}, \boldsymbol{\omega}) > i_b(\nu, M). \end{cases} \tag{4.10}$$

Now B_m^M is given, for all $u \in X$, by

$$B_m^M u = \{-\frac{1}{\rho c_v} \int_0^\infty \int_{S^2} q_m^M(u) \, d\boldsymbol{\omega} d\nu, c \, q_m^M(u)\}, \tag{4.11}$$

where

$$q_m^M(u)(\nu, \mathbf{x}, \boldsymbol{\omega}) = \Theta_m^M(a)(\nu, T(\mathbf{x}))[\varphi^M(i)(\nu, \mathbf{x}, \boldsymbol{\omega}) - \Theta_m^M(i_b)(\nu, T(\mathbf{x}))]. \tag{4.12}$$

Clearly, it results $B_m^M u = Bu$ whenever

$$m \le T \le M \quad \text{and} \quad 0 \le i \le i_b(\nu, M).$$

Let us call *the truncated problem* the one obtained by replacing in (4.1) B by B_m^M. The plan of the paper is the following:

(1) Prove that A is m-accretive (that is, accretive and $R(I+\lambda A)=X$ for $\lambda>0$, where $R(I+\lambda A)$ is the range of $I+\lambda A$).

(2) Prove that B_m^M is Lipschitz.

(3) Give conditions on the initial and boundary data in order that the unique solution of the truncated problem satisfies $m \le T \le M$ and $i_b(\nu, m) \le i \le i_b(\nu, M)$, which implies that it is also solution of (4.1).

Once we know the results in [16], basically we have to deal only with the convective and conductive terms (see also [14]).

In what concerns the proof of accretivity, some integral estimates are needed, the proofs of which are given in this paper; the key for proving that $R(I+\lambda A)=X$ for $\lambda>0$ is the existence (uniqueness is always guaranteed by the accretivity) for the elliptic linear problem

$$T + \lambda \mathbf{v}\cdot\nabla T - \lambda C \Delta T = \hat{T}, \tag{4.13}$$

with mixed ($T=\bar{g}$ on Γ_1, $\nabla T\cdot\mathbf{n}=0$ on $\Gamma\backslash\Gamma_1$) b.c., where C is a positive constant and $\hat{T}\in \mathrm{L}^1(\Omega)$; we have proved in [15] that this result holds. Item (2) is known from [14]. To solve item (3) we adapt the techniques of [16] and use analogous arguments to those used to prove accretivity.

4.1. m-accretivity of A

Recall the definition of the sign function:

$$\operatorname{sgn}(t) = \begin{cases} -1 & \text{if} \quad t<0 \\ 0 & \text{if} \quad t=0 \\ 1 & \text{if} \quad t>0. \end{cases} \tag{4.14}$$

4.1.1. Accretivity

The operator A is accretive if, and only if, the following inequality is satisfied for all $\{T,i\}$ and $\{\hat{T},\hat{\imath}\}$ in $D(A)$:

$$\begin{aligned} &\int_\Omega [\mathbf{v}\cdot\nabla(T-\hat{T}) - \frac{\bar{k}}{\rho c_v}\Delta(T-\hat{T})]\ \operatorname{sgn}(T-\hat{T})\ d\mathbf{x} + \\ &\qquad \int_{\{T=\hat{T}\}} |\mathbf{v}\cdot\nabla(T-\hat{T}) - \frac{\bar{k}}{\rho c_v}\Delta(T-\hat{T})|\ d\mathbf{x} + \\ &\int_0^\infty \int_\Omega \int_{S^2} [c\ \boldsymbol{\omega}\cdot\nabla_{\mathbf{x}}(i-\hat{\imath}) - c\ L(i-\hat{\imath})]\ \operatorname{sgn}(i-\hat{\imath})\ d\boldsymbol{\omega} d\mathbf{x} d\nu + \\ &\qquad \int_{\{i=\hat{\imath}\}} |c\ \boldsymbol{\omega}\cdot\nabla_{\mathbf{x}}(i-\hat{\imath}) - c\ L(i-\hat{\imath})|\ d\boldsymbol{\omega} d\mathbf{x} d\nu \ge 0. \end{aligned} \tag{4.15}$$

In fact, we show that every term in the sum in nonnegative. Obviously, the second and fourth integrals are nonnegative. Moreover, we know from [16] that the third one is nonnegative. To prove that the first one is nonnegative note that, since $\operatorname{div}\mathbf{v} = 0$,

$$\begin{aligned}\int_\Omega \mathbf{v}\cdot\nabla(T-\hat{T})\ \operatorname{sgn}(T-\hat{T})\ d\mathbf{x} &= \int_\Omega \mathbf{v}\cdot\nabla(|T-\hat{T}|)\ d\mathbf{x}\\ &= \int_\Gamma |T-\hat{T}|\ \mathbf{v}\cdot\mathbf{n}\ d\gamma = \int_{\Gamma\setminus\Gamma_1} |T-\hat{T}|\ \mathbf{v}\cdot\mathbf{n}\ d\gamma \geq 0,\end{aligned}$$

where we have used the facts that $T = \hat{T} = \bar{g}$ on Γ_1 and that $\Gamma_{\mathbf{v}}^- \subseteq \Gamma_1$.

The proof of the accretivity of A ends by proving that

$$-\int_\Omega \Delta(T-\hat{T})\ \operatorname{sgn}(T-\hat{T})\ d\mathbf{x} \geq 0, \tag{4.16}$$

which is a consequence of the following lemma, the proof of which is given in the appendix.

Lemma 4.2 *Let us take $\beta \in \mathrm{C}^1(\mathbb{R})\cap\mathrm{L}^\infty(\mathbb{R})$ such that $\beta(0) = 0$ and $\beta' \geq 0$, and let Γ_1 be an arbitrary measurable subset of Γ. Then, for all $T \in \mathrm{W}^{1,1}(\Omega)$ such that $\Delta T \in \mathrm{L}^1(\Omega)$, $T = 0$ on Γ_1 and $\nabla T\cdot\mathbf{n} = 0$ on $\Gamma\setminus\Gamma_1$ (in the sense of Definition 4.1),*

$$-\int_\Omega \Delta T\beta(T)\ d\mathbf{x} \geq 0. \tag{4.17}$$

Now (4.16) holds in virtue of the following corollary.

Corollary 4.1 *Let Γ_1 and T be as in the Lemma 4.2. Then*

$$-\int_\Omega \Delta T \operatorname{sgn}(T)\ d\mathbf{x} \geq 0. \tag{4.18}$$

Proof. Let us consider $\{\beta_n\} \subset \mathrm{C}^1(\mathbb{R})\cap\mathrm{L}^\infty(\mathbb{R})$ such that $\{\beta_n\} \to \operatorname{sgn}$ pointwise, $\beta_n(0) = 0$, $\beta_n' \geq 0$ and $\|\beta_n\|_{\mathrm{L}^\infty(\mathbb{R})} \leq C$ for all n, with C independent of n. According to Lemma 4.2,

$$-\int_\Omega \Delta T\ \beta_n(T)\ d\mathbf{x} \geq 0, \tag{4.19}$$

and the result follows by taking the limit as $n \to \infty$. ■

4.1.2. m-accretivity

The m-accretivity means, first, accretivity, and, second, that if

$$\hat{u} = \{\hat{T}, \hat{\imath}\} \in X \quad \text{and} \quad \lambda > 0, \tag{4.20}$$

then the problem $u + \lambda A u = \hat{u}$ has some solution $u \in D(A)$ (this solution is necessarily unique in virtue of accretivity). Since the operator A is m-accretive when $\mathbf{v} \equiv \mathbf{0}$ and $\bar{k} = 0$ [16], it suffices to prove that, for $\hat{T}$ given in $\mathrm{L}^1(\Omega)$ and $\lambda > 0$, the problem

$$T + \lambda \mathbf{v} \cdot \nabla T - \lambda \frac{\bar{k}}{\rho c_v} \Delta T = \hat{T} \tag{4.21}$$

has a unique distributional solution $T \in \mathrm{W}^{1,1}(\Omega)$ with the mixed b.c.

$$T = \bar{g} \quad \text{on } \Gamma_1, \tag{4.22}$$

$$\nabla T \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \backslash \Gamma_1, \tag{4.23}$$

where Γ_1 is a measurable subset of Γ. This is indeed the case, and the reader can find the proof in [15, Theorem 2.3].

4.2. Boundedness of the solution of the truncated problem

The functions $\mathrm{s}_\pm : \mathbb{R} \to \mathbb{R}$ are defined by:

$$\mathrm{s}_+(t) = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \le 0, \end{cases} \qquad \mathrm{s}_-(t) = \begin{cases} -1 & \text{if} \quad t < 0 \\ 0 & \text{if} \quad t \ge 0. \end{cases} \tag{4.24}$$

If H is a real function, set $H_\pm = H \cdot \mathrm{s}_\pm(H)$.

We assume that the hypotheses described in Section 3 are satisfied. Recall that $f = F/(\rho c_v)$ and $\bar{f} = \{f, 0\}$. Let $u = \{T, i\} \in \mathrm{C}^0([0,\tau]; X)$ be the unique (mild) solution of the truncated problem.

We can choose [1], [13, Lemma 2.2.2] a family of partitions of $[0, \tau]$

$$0 = t_0^n < \cdots < t_{P(n)}^n = \tau \quad (n \in \mathbb{N}) \tag{4.25}$$

and a family of finite sequences in X

$$\bar{f}_1^n = \{f_1^n, 0\}, \ldots, \bar{f}_{P(n)}^n = \{f_{P(n)}^n, 0\} \tag{4.26}$$

such that

$$\lim_{n\to\infty} \max_{1 \le j \le P(n)} \lambda_j^n = \lim_{n\to\infty} \|\bar{f} - \bar{f}_n\|_{\mathrm{L}^1(0,\tau;X)} = 0, \tag{4.27}$$

where $\lambda_j^n = t_j^n - t_{j-1}^n$ and $\bar{f}_n(t) = \bar{f}_j^n \ \forall t \in (t_{j-1}^n, t_j^n]$, for $j = 1, \ldots, P(n)$.

Set $u_0^n = \{T_0^n, i_0^n\} = u_0 = \{T_0, i_0\}$ $\forall n \in \mathbb{N}$, and let $u_j^n = \{T_j^n, i_j^n\}$ be the unique solution of

$$\frac{u_j^n - u_{j-1}^n}{\lambda_j^n} + Au_j^n + B_m^M u_j^n = \bar{f}_j^n, \tag{4.28}$$

for $j = 1, \dots, P(n)$.

Then u is the uniform limit on $[0, \tau]$ of the sequence $\{u_n\}$, piecewise defined as follows:

$$u_n(0) = u_0, \tag{4.29}$$

$$u_n(t) = u_j^n \quad \text{if} \quad t \in (t_{j-1}^n, t_j^n]. \tag{4.30}$$

Remark 4.3 *Since A is m-accretive and B_m^M is Lipschitz with domain X, the problem (4.28) has a unique solution in $D(A)$ for small values of λ_j^n, to be precise for $0 < \lambda_j^n < 1/M^\star$, where $M^\star$ is a Lipschitz constant for B_m^M. See [14, Theorem 2.3]; a proof can be done by using the Banach fixed point theorem.*

At this point it must be clear that it results

$$m \le T \le M \quad \text{and} \quad i_b(\nu, m) \le i \le i_b(\nu, M) \tag{4.31}$$

if, for every $n \in \mathbb{N}$,

$$m \le T_j^n(\mathbf{x}) \le M \quad \text{and} \quad i_b(\nu, m) \le i_j^n(\nu, \mathbf{x}, \boldsymbol{\omega}) \le i_b(\nu, M) \tag{4.32}$$

$\forall j \in \{0, 1, \dots, P(n)\}$. We need the following lemma to obtain bounds for the stationary problem (4.28).

Lemma 4.3 *Let us take: $f^\star \in \mathrm{L}^1(\Omega)$, $M^\star$ a Lipschitz constant for B_m^M, $\lambda \in \mathbb{R}$ such that $0 < \lambda < 1/M^\star$, and $\hat{u} = \{\hat{T}, \hat{\imath}\} \in X$. Let $u = \{T, i\} \in D(A)$ be the unique solution of*

$$u + \lambda(Au + B_m^M u) = \hat{u} + \{\lambda f^\star, 0\}.$$

If N' and N are constants such that $m \le N' \le N \le M$, then

$$N' \le T \le N \quad \text{a.e. in } \Omega, \tag{4.33}$$

$$i_b(\nu, N') \le i \le i_b(\nu, N) \quad \text{a.e. in } (0, \infty) \times \Omega \times S^2 \tag{4.34}$$

provided that

- $N' \le \hat{T} + \lambda f^\star \le N$ *a.e. in* Ω,
- $N' \le \bar{g} \le N$ *a.e. on* Γ_1,
- $i_b(\nu, N') \le \hat{\imath} \le i_b(\nu, N)$ *a.e. in* $(0, \infty) \times \Omega \times S^2$, *and*
- $i_b(\nu, N') \le \bar{h} \le i_b(\nu, N)$ *a.e. in* $(0, \infty) \times D_{\mathrm{in}}$.

Proof. We prove $T \leq N$ and $i \leq i_b(N)$, and what is left can be proved analogously. In this proof we use the shortened notation $i_b(N) = i_b(\nu, N)$. From the equation $u + \lambda(Au + B_m^M u) = \hat{u} + \{\lambda f^\star, 0\}$ it follows

$$\rho c_v \|(T-N)_+\|_{\mathrm{L}^1(\Omega)} + \frac{1}{c}\|(i - i_b(N))_+\|_{\mathrm{L}^1((0,\infty)\times\Omega\times S^2)} + \tag{4.35}$$

$$+\lambda \int_\Omega \rho c_v \mathbf{v}\cdot\nabla(T-N)\ \mathrm{s}_+(T-N)\ d\mathbf{x}$$

$$+\lambda \int_0^\infty \int_\Omega \int_{S^2} \boldsymbol{\omega}\cdot\nabla_{\mathbf{x}} i\ \mathrm{s}_+(i - i_b(N))\ d\boldsymbol{\omega} d\mathbf{x} d\nu$$

$$+\lambda \int_0^\infty \int_\Omega \int_{S^2} \Theta_m^M(a)[\Theta_m^M(i_b) - i_b(N)][\mathrm{s}_+(T-N) - \mathrm{s}_+(i - i_b(N))]\ d\boldsymbol{\omega} d\mathbf{x} d\nu$$

$$+\lambda \int_0^\infty \int_\Omega \int_{S^2} \Theta_m^M(a)[\varphi^M(i) - i_b(N)][\mathrm{s}_+(i - i_b(N)) - \mathrm{s}_+(T-N)]\ d\boldsymbol{\omega} d\mathbf{x} d\nu$$

$$= \int_\Omega \rho c_v(\hat{T} + \lambda f^* - N)\ \mathrm{s}_+(T-M)\ d\mathbf{x}$$

$$+ \frac{1}{c}\int_0^\infty \int_\Omega \int_{S^2} (\hat{\imath} - i_b(N))\ \mathrm{s}_+(i - i_b(N))\ d\boldsymbol{\omega} d\mathbf{x} d\nu$$

$$+ \lambda \int_\Omega \bar{k}\Delta T\ \mathrm{s}_+(T-N)\ d\mathbf{x} + \lambda \int_0^\infty \int_\Omega \int_{S^2} L(i)\ \mathrm{s}_+(i - i_b(N))\ d\boldsymbol{\omega} d\mathbf{x} d\nu.$$

The idea is to prove that all the terms in the left-hand side are nonnegative and all the terms in the right-hand side are nonpositive, since this implies

$$\rho c_v \|(T-N)_+\| + (1/c)\|(i - i_b(N))_+\| \leq 0, \tag{4.36}$$

as desired. Taking into account that the case with $\mathbf{v} \equiv \mathbf{0}$ and $\bar{k} = 0$ has already be studied [16], [13] and [14], it suffices to prove

$$\int_\Omega \mathbf{v}\cdot\nabla(T-N)\ \mathrm{s}_+(T-N)\ d\mathbf{x} \geq 0 \tag{4.37}$$

and

$$-\int_\Omega \Delta T\ \mathrm{s}_+(T-N)\ d\mathbf{x} \geq 0. \tag{4.38}$$

The proof becomes now similar to that of accretivity: since $\operatorname{div}\mathbf{v} = 0$,

$$\begin{aligned} \int_\Omega \mathbf{v}\cdot\nabla(T-N)\ \mathrm{s}_+(T-N)\ d\mathbf{x} &= \int_\Gamma (T-N)_+ \mathbf{v}\cdot\mathbf{n}\ d\gamma \\ &= \int_{\Gamma\setminus\Gamma_1} (T-N)_+ \mathbf{v}\cdot\mathbf{n}\ d\gamma \geq 0, \end{aligned} \tag{4.39}$$

where we have taken into account that $T = \bar{g} \leq N$ on Γ_1 and that $\Gamma_{\mathbf{v}}^- \subseteq \Gamma_1$.

The inequality (4.38) is a consequence of the following lemma, the proof of which is given in the appendix.

Lemma 4.4 *Let Γ_1 be an arbitrary measurable subset of Γ, and let us suppose that g is such that $g = \psi_{|\Gamma_1}$ for certain $\psi \in \mathcal{D}(\bar{\Omega})$ and that $g \le 0$ on Γ_1. Let us take $\beta \in \mathrm{C}^1(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ such that $\beta(s) = 0$ $\forall s \le 0$ and $\beta' \ge 0$. Then, for all $T \in \mathrm{W}^{1,1}(\Omega)$ such that $\Delta T \in \mathrm{L}^1(\Omega)$, $T = g$ on Γ_1 and $\nabla T \cdot \mathbf{n} = 0$ on $\Gamma \backslash \Gamma_1$ (in the sense of Definition 4.1),*

$$-\int_\Omega \Delta T \beta(T)\, d\mathbf{x} \ge 0. \tag{4.40}$$

Now (4.38) holds in virtue of the following corollary, since

$$\Delta T = \Delta(T - N), \tag{4.41}$$

and $T - N = \bar{g} - N \le 0$ on Γ_1.

Corollary 4.2 *Let Γ_1, g and T be as in the Lemma 4.4. Then*

$$-\int_\Omega \Delta T \, \mathrm{s}_+(T)\, d\mathbf{x} \ge 0. \tag{4.42}$$

The proof of the Corollary 4.2 follows closely the arguments used to prove Corollary 4.1. This ends the proof of Lemma 4.3. ■

Finally, the following theorem gives Theorem 3.1 simply by taking $N' = m$ and $N = M$. Note that a new restriction arises for m and M, since they must satisfy $m + 2\tau\|f\|_\infty \le M$. Recall that $\|f\|_\infty = \|f\|_{\mathrm{L}^\infty(\Omega\times[0,\tau])}$.

Theorem 4.1 *Let N' and N be constants such that*

$$m \le N', \quad N' + \tau\|f\|_\infty \le N - \tau\|f\|_\infty, \quad \text{and} \quad N \le M. \tag{4.43}$$

Let $u = \{T, i\}$ be the solution of the truncated problem. Then u satisfies

$$N' \le T \le N \quad \text{a.e. in } \Omega, \tag{4.44}$$

$$i_b(\nu, N') \le i \le i_b(\nu, N) \quad \text{a.e. in } (0,\infty) \times \Omega \times S^2 \tag{4.45}$$

for all $t \in [0, \tau]$, provided that

- $N' + \tau\|f\|_\infty \le T_0 \le N - \tau\|f\|_\infty$ *a.e. in* Ω,
- $N' + \tau\|f\|_\infty \le \bar{g} \le N - \tau\|f\|_\infty$ *a.e. on* Γ_1,
- $i_b(\nu, N' + \tau\|f\|_\infty) \le i_0 \le i_b(\nu, N - \tau\|f\|_\infty)$ *a.e. in* $(0,\infty) \times \Omega \times S^2$,
- $i_b(\nu, N' + \tau\|f\|_\infty) \le \bar{h} \le i_b(\nu, N - \tau\|f\|_\infty)$ *a.e. in* $(0,\infty) \times D_{\mathrm{in}}$.

Proof. In this proof $\|f\|$ is $\|f\|_\infty$. By previous discussion it suffices to prove (4.32) replacing m and M by N' and N, respectively. Note that

$$N' + \tau\|f\| \le T_0^n = T_0 \le N - \tau\|f\| \tag{4.46}$$

and

$$i_b(\nu, N' + \tau\|f\|) \le i_0^n = i_0 \le i_b(\nu, N - \tau\|f\|). \tag{4.47}$$

By succesive application of Lemma 4.3 for $j = 1, \dots, P(n)$ we obtain

$$N' + (\tau - \sum_{\alpha=1}^{j} \lambda_\alpha^n)\|f\| \le T_j^n \le N - (\tau - \sum_{\alpha=1}^{j} \lambda_\alpha^n)\|f\|, \tag{4.48}$$

$$i_b(\nu, N' + (\tau - \sum_{\alpha=1}^{j} \lambda_\alpha^n)\|f\|) \le i_j^n \le i_b(\nu, N - (\tau - \sum_{\alpha=1}^{j} \lambda_\alpha^n)\|f\|), \tag{4.49}$$

which proves (4.32), since $i_b(\nu, \cdot)$ is increasing and $\sum_{\alpha=1}^{P(n)} \lambda_\alpha^n = \tau$. ■

5. Conclusions

We have proved solvability of the transient radiative transfer equations for a participating and incompressible gas in the presence of conduction and the possible presence of convection. There are some clear limits in this work, perhaps the more important one is that the velocity field and the boundary data must be stationary, due (in principle) only to technical reasons, since otherwise our operator A would depend on time, making more difficult the treatment of the problem.

6. Appendix

In this appendix we give the proofs of Lemmas 4.1, 4.2 and 4.4. Let us recall that Ω is a bounded, open and connected subset of $\mathbb{R}^3$, with C^∞-boundary Γ.

Proof of Lemma 4.1. Here Γ_1 is an arbitrary measurable subset of Γ. Note that, once we know [14, Corollary 1], it suffices to prove that, given $f \in \mathrm{L}^1(\Omega)$, there exists a sequence $\{T_n\} \subset \mathrm{W}^{1,1}(\Omega)$ converging to f in $\mathrm{L}^1(\Omega)$ such that $\Delta T_n \in \mathrm{L}^1(\Omega)$, $T_n = \bar{g}$ on Γ_1, and $\nabla T_n \cdot \mathbf{n} = 0$ on $\Gamma \backslash \Gamma_1$.

Let u be the unique solution of the following elliptic problem:

$$\begin{cases} u - \Delta u = 0, \\ u = \bar{g} \quad \text{on } \Gamma_1, \\ \nabla u \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \backslash \Gamma_1. \end{cases} \tag{6.1}$$

Since $f - u \in \mathrm{L}^1(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $\mathrm{L}^1(\Omega)$, there exists a sequence

$$\{\varphi_n\} \subset \mathcal{D}(\Omega) \quad \text{such that} \quad \{\varphi_n\} \to f - u \quad \text{in} \quad \mathrm{L}^1(\Omega). \tag{6.2}$$

Now

$$T_n = \varphi_n + u \tag{6.3}$$

defines the desired sequence converging to f. ■

To prove Lemma 4.2 and Lemma 4.4 we use the following result, which is a direct consequence of [15, Theorem 6.1, Remark 6.1].

Lemma 6.1 *Let Γ_1 be an arbitrary measurable subset of Γ. Given $\hat{T} \in \mathrm{L}^1(\Omega)$, the problem*

$$\begin{cases} T - \Delta T &= \hat{T}, \\ T &= 0 \quad \text{on } \Gamma_1, \\ \nabla T \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma \backslash \Gamma_1, \end{cases} \tag{6.4}$$

has a unique distributional solution, i.e., there exists a unique $T \in \mathrm{W}^{1,1}(\Omega)$ such that $T_{|\Gamma_1} = 0$ and

$$\int_\Omega T\varphi \, d\mathbf{x} + \int_\Omega \nabla T \cdot \nabla \varphi \, d\mathbf{x} = \int_\Omega \hat{T}\varphi \, d\mathbf{x} \tag{6.5}$$

for every test function $\varphi \in \mathrm{W}^{1,\infty}(\Omega)$ satisfying $\varphi_{|\Gamma_1} = 0$. Moreover, this solution satisfies

$$\|T\|_{\mathrm{L}^1(\Omega)} \le c\|\hat{T}\|_{\mathrm{L}^1(\Omega)}, \tag{6.6}$$

where c is a constant which depends only on $|\Omega|$.

The following result contains as particular cases both Lemma 4.2 and Lemma 4.4.

Lemma 6.2 *Let Γ_1 be an arbitrary measurable subset of Γ, and let us suppose that g is such that $g = \psi_{|\Gamma_1}$ for certain $\psi \in \mathcal{D}(\bar{\Omega})$. Let us take $\beta \in \mathrm{C}^1(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ such that $\beta \circ g = 0$ on Γ_1 and $\beta' \ge 0$. Then, for all $T \in \mathrm{W}^{1,1}(\Omega)$ such that $\Delta T \in \mathrm{L}^1(\Omega)$, $T = g$ on Γ_1 and $\nabla T \cdot \mathbf{n} = 0$ on $\Gamma \backslash \Gamma_1$ (in the sense of Definition 4.1),*

$$-\int_\Omega \Delta T \beta(T) \, d\mathbf{x} \ge 0. \tag{6.7}$$

Proof. Set $f = T - \Delta T$. Note that $f \in \mathrm{L}^1(\Omega)$, and thus there exists a sequence

$$\{f_n\} \subset \mathcal{D}(\Omega) \quad \text{such that} \quad \{f_n\} \to f \quad \text{in } \mathrm{L}^1(\Omega). \tag{6.8}$$

Denote by T_n the unique solution of the following problem:

$$\begin{cases} T_n - \Delta T_n = f_n, \\ T_n = g \quad \text{on } \Gamma_1, \\ \nabla T_n \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \backslash \Gamma_1. \end{cases} \tag{6.9}$$

Then T_n is regular, and we can integrate by parts to obtain

$$-\int_\Omega \Delta T_n \beta(T_n) d\mathbf{x} = \int_\Omega |\nabla T_n|^2 \beta'(T_n)\, d\mathbf{x} - \int_\Gamma \beta(T_n) \nabla T_n \cdot \mathbf{n}\, d\gamma, \tag{6.10}$$

which proves that

$$-\int_\Omega \Delta T_n \beta(T_n) d\mathbf{x} \geq 0, \tag{6.11}$$

since $\beta(T_n) = \beta(g) = 0$ on Γ_1 and $\nabla T_n \cdot \mathbf{n} = 0$ on $\Gamma \backslash \Gamma_1$. The proof ends by noting that, in virtue of Lemma 6.1,

$$\|T - T_n\|_{\mathrm{L}^1(\Omega)} \leq c \|f - f_n\|_{\mathrm{L}^1(\Omega)}, \tag{6.12}$$

which implies

$$\{T_n\} \to T \quad \text{in } \mathrm{L}^1(\Omega), \tag{6.13}$$

and consequently, at least for a certain subsequence $\{T_{n_k}\}$ of $\{T_n\}$ (which converges also a.e. to T),

$$-\int_\Omega \Delta T \beta(T)\, d\mathbf{x} = -\lim_{k \to \infty} \int_\Omega \Delta T_{n_k} \beta(T_{n_k}) d\mathbf{x} \geq 0, \tag{6.14}$$

as desired. ■

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