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Two–microlocal Besov spaces and wavelets

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Abstract

We give a characterization of the two–microlocal Besov spaces in terms of the local Besov type conditions. As an easy consequence, we obtain the inclusions between the two–microlocal Besov spaces and the local Besov spaces. These results are natural extensions of those obtained by Jaffard and Meyer, who treated the pointwise Hölder regularity in terms of two–microlocal estimates. The Daubechies wavelets play a key role throughout the paper.

1. Introduction

Our aim in this note is to characterize the two–microlocal Besov spaces in terms of the local Besov type conditions.

This characterization is a natural extension of Theorem 1.2 in [3]. As an easy consequence, we obtain the inclusions between the two–microlocal Besov spaces and the local Besov spaces, which are a natural extension of Proposition 1.3 in [3], too.

In Section 2, we begin with the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ from [5]. After introducing an orthonormal wavelet basis composed of compactly supported smooth wavelets from [1], we define the two–microlocal Besov spaces $B^{s,s'}_{p,q}(U)$, where U is an open subset in $\mathbb{R}^n$. However, we treat only the case where $p = q$ in our theorems, which are stated in Section 3. The proof is carried out in Section 4. The main point is the dyadic decomposition of the domain under consideration.

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2. Notations and definitions

Let $\mathbb{R}^n$ be n–dimensional real Euclidean space and $\mathbb{Z}^n$ be the lattice of all points $k = (k_1, \dots, k_n) \in \mathbb{R}^n$, where the components $k_1, \dots, k_n$ are integers. Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ be the set of all tempered distributions on $\mathbb{R}^n$. If f belongs to the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, then

$$\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x,\, \xi\rangle} f(x)dx, \qquad \xi \in \mathbb{R}^n,$$

denotes the Fourier transform of f. Here $\langle x,\, \xi\rangle = \sum_{j=1}^n x_j\xi_j$ is the scalar product of $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$. The inverse Fourier transform $\mathcal{F}^{-1}f$ is given by

$$\mathcal{F}^{-1}f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\, \xi\rangle} f(\xi)d\xi, \qquad x \in \mathbb{R}^n.$$

The transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended in the usual way from $\mathcal{S}$ to $\mathcal{S}'$.

Let $\{\varphi_j(x)\}_{j=-\infty}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ satisfy

1) $\operatorname{supp} \varphi_j \subset \{x \in \mathbb{R}^n;\ 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}, \qquad j \in \mathbb{Z},$

2) for every multi–index α there exists a positive number C_α such that

$$2^{j|\alpha|}|D^\alpha \varphi_j(x)| \leqslant C_\alpha, \qquad j \in \mathbb{Z},\ x \in \mathbb{R}^n,$$

and

3) $\displaystyle\sum_{j=-\infty}^{\infty} \varphi_j(x) \equiv 1, \qquad x \in \mathbb{R}^n \setminus \{0\}.$

Here D^α in 2) above are classical derivatives. Let $s > 0$ and $1 \leqslant p,\ q \leqslant \infty$. Then the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined as the set of all tempered distributions f (modulo polynomials) satisfying

$$\|f|\dot{B}^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)|L_p(\mathbb{R}^n)\|^q \right)^{1/q} < \infty$$

(usual modification if $q = \infty$). Here $\|\cdot|L_p(\mathbb{R}^n)\|$ stands for the usual L_p–norm. See Definition 2 of Section 5.1.3 in [5]. The definition of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is independent of the choice $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$. See Theorem 5.1.5 in [5].

Let us now consider an orthonormal wavelet basis on $\mathbb{R}^n$. Such a basis is composed by translations and dilations of $(2^n - 1)$ functions $\psi^{(i)}(i \in \{0,1\}^n - (0, \dots, 0))$. We assume in the following that these wavelets are compactly supported smooth wavelets, whose supports are included in a ball centered at the origin. See [1]. Let $\psi^{(i)}_{j,k}(x) = 2^{nj/2}\psi^{(i)}(2^j x - k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. Then the wavelet decomposition of $f \in \mathcal{S}'$ will be written

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} C_{j,k}\psi_{j,k}(x), \quad C_{j,k} = \langle f,\, \psi_{j,k}\rangle,$$

where we can forget the index i.

Let us recall the fact that $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ if and only if

$$\sum_{j\in\mathbb{Z}} 2^{jq(s+n/2-n/p)}\left(\sum_{k\in\mathbb{Z}^n} |C_{j,k}|^p\right)^{q/p} < \infty.$$

See Chapter VI, (10.5) in [4].

After these preliminaries we can define the local Besov spaces $B^s_{p,q}(U)$ and the two–microlocal Besov spaces $B^{s,s'}_{p,q}(U)$, where U is an open subset in $\mathbb{R}^n$.

Definition 2.1 *Let $s > 0$ and $1 \leqslant p,\ q \leqslant \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to the local Besov space $B^s_{p,q}(U)$ if there exists an $F \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ such that $f|_U = F|_U$, where $f|_U$ denotes the restriction of f to U. The norm $\|f|B^s_{p,q}(U)\|$ of f is then the infimum of all possible norms of F in $\dot{B}^s_{p,q}(\mathbb{R}^n)$.*

Definition 2.2 *Let $s > 0$, $s' \in \mathbb{R}$ and $1 \leqslant p,\ q \leqslant \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to the two–microlocal Besov space $B^{s,s'}_{p,q}(U)$ if the following two–microlocal estimate holds:*

$$\|f|B^{s,s'}_{p,q}(U)\| = \left[\sum_{j\in\mathbb{Z}} 2^{j\,q\,(s+\frac{n}{2}-\frac{n}{p})}\left\{\sum_{k\in\mathbb{Z}^n}\left|\left(1+2^j\, d(k2^{-j},U)\right)^{s'} C_{j,k}\right|^p\right\}^{\frac{q}{p}}\right]^{\frac{1}{q}} < \infty,$$

where $d(k2^{-j},U)$ denotes the distance from $k2^{-j}$ to U (usual modification if $p = \infty$ or $q = \infty$).

The two–microlocal estimate in Definition 2.2 above can be described equivalently by using the Littlewood–Paley decompositions. See Definition 1.1 and Proposition 1.4 in [3].

Let $x_0 \in \mathbb{R}^n$. Then by taking the inductive limit with respect to $x_0 \in U$ of the function spaces in Definitions 2.1 and 2.2, we can define the pointwise function spaces as follows:

Definition 2.3 *Let $s > 0$, $s' \in \mathbb{R}$ and $1 \leqslant p,\ q \leqslant \infty$. Then*

$$1)\ B^s_{p,q}(x_0) = \varinjlim_{x_0\in U} B^s_{p,q}(U), \quad \textit{and} \quad 2)\ B^{s,s'}_{p,q}(x_0) = \varinjlim_{x_0\in U} B^{s,s'}_{p,q}(U).$$

It is easy to see that $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the two–microlocal Besov space $B^{s,s'}_{p,q}(x_0)$ if and only if the following two–microlocal estimate holds:

$$\|f|\, B^{s,s'}_{p,q}(x_0)\| = \\ = \left[\sum_{j\in\mathbb{Z}} 2^{jq(s+n/2-n/p)} \left\{ \sum_{k\in\mathbb{Z}^n} \left| \left(1+2^j|k2^{-j}-x_0|\right)^{s'} C_{j,k}\right|^p \right\}^{q/p} \right]^{1/q} < \infty.$$

In order to state the local Besov type conditions in Theorem 3.1 below, we shall use the following notation: If $\rho(\varepsilon)$ is a function of the real variable ε, defined for all positive ε, we write $\rho(\varepsilon) = \mathcal{O}^{(p)}(\varepsilon^{-s})$ if and only if

$$\int_0^\infty (\rho(\varepsilon)\varepsilon^s)^p \frac{d\varepsilon}{\varepsilon} = \int_0^\infty \rho(\varepsilon)^p \varepsilon^{sp-1} d\varepsilon < \infty.$$

We can say that the symbol $\mathcal{O}^{(p)}$ is a homogeneous L_p–version of the Hörmander symbol $\mathcal{O}^{(2)}$ used in Theorem 7.1 in [2].

3. Theorems

Our first theorem is now stated as follows:

Theorem 3.1 *Let* $s > 0$, $s' < 0$ *and* $1 \leqslant p \leqslant \infty$. *Let* U *be an open subset in* $\mathbb{R}^n$ *and* $A_\rho = \{x \in \mathbb{R}^n;\ d(x,U) < \rho,\ x \notin U\}$. *Then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^{s,s'}_{p,p}(U)$ *if and only if there exists a decomposition* $f = f_1 + f_2$ *such that*

$$f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n),$$

and

$$\|f_2|B^{s+s'}_{p,p}(A_\rho)\| = \mathcal{O}^{(p)}(\rho^{-s'}) \qquad \text{for every } \rho > 0.$$

The following theorem is an easy consequence of Theorem 3.1 above and the characterizations of the homogeneous Besov spaces by using differences. See Section 5.2.3, Theorem 2 and Section 2.5.12, Remark 3 in [5].

Theorem 3.2 *Let* $s > s' > 0$ *and* $1 \leqslant p \leqslant \infty$. *Let* U *be an open subset in* $\mathbb{R}^n$ *and* x_0 *a point in* $\mathbb{R}^n$. *Then we have the following inclusions:*

$$B^{s,-s'}_{p,p}(U) \subset B^s_{p,p}(U) \subset B^{s,-s}_{p,p}(U),$$

and

$$B^{s,-s'}_{p,p}(x_0) \subset B^s_{p,p}(x_0) \subset B^{s,-s}_{p,p}(x_0).$$

Remark 3.3 Theorem 1.2 and Proposition 1.3 in [3] treat the case where $p = \infty$ of our theorems, as mentioned in the introduction.

4. Proof of Theorem 3.1

We denote by C' the diameter of the support of the wavelet ψ. Let $f \in B^{s,s'}_{p,p}(U)$. Then its wavelet coefficients satisfy

$$\sum_{j\in\mathbb{Z}} 2^{jp(s+n/2-n/p)} \sum_{k\in\mathbb{Z}^n} \left| \left(1+2^j d(k2^{-j},U)\right)^{s'} C_{j,k}\right|^p < \infty. \tag{4.1}$$

We write f as

$$f = \sum_{\operatorname{supp}\psi_{j,k}\cap U\neq\phi} C_{j,k}\psi_{j,k} + \sum_{\operatorname{supp}\psi_{j,k}\cap U=\phi} C_{j,k}\psi_{j,k} = f_1+f_2.$$

If $\operatorname{supp}\psi_{j,k}\cap U \neq \phi$, then $2^j d(k2^{-j},U)$ is estimated from above by some constant comparable to C'. Therefore $f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n)$. Next we split the wavelet decomposition of f_2 into three sums $f_2 = \sum_1 + \sum_2 + \sum_3$:

The first, $\sum_1$, corresponds to the wavelets whose supports do not intersect A_ρ, and we can forget this sum because of Definition 2.1.

Next we consider the sum $\sum_2$ whose coefficients satisfy $2^j\rho \leqslant 10\,C'$; in that case, because $2^j d(k2^{-j},U)$ can be estimated from above by some constant comparable to $10\,C'$, we have that $\sum_2 \in \dot{B}^s_{p,p}(\mathbb{R}^n)$.

Finally we consider the remaining sum $\sum_3$ whose coefficients satisfy $2^j\rho \geqslant 10\,C'$. We decompose A_ρ into the "curved annuli" as follows:

$$A_\rho = \bigcup_{m\in\mathbb{Z};\,2^{-m}\leqslant\rho} \left\{x\in\mathbb{R}^n;\ 2^{-m-1}\leqslant d(x,U)\leqslant 2^{-m}\right\} = \bigcup_{m;\,2^m\rho\geqslant 1} D_m. \tag{4.2}$$

By using this decomposition (4.2), we can write (4.1) as follows:

$$\sum_{j;\,2^j\rho\geqslant 10C'} 2^{jp(s+n/2-n/p)} \sum_{m;\,2^m\rho\geqslant 1} \left(1+2^{(j-m)}\right)^{s'p} \sum_{k;\,k2^{-j}\in D_m} |C_{j,k}|^p < \infty. \tag{4.3}$$

The case where $m > j+L(C')$, $L(C')$ being an integer dependent only on C', is negligible because $\operatorname{supp}\psi_{j,k}\cap U=\phi$. Therefore we obtain from (4.3) that

$$\begin{aligned} &\sum_{j;\,2^j\rho\geqslant 10C'} \sum_{\substack{m;\,2^m\rho\geqslant 1,\\ m\leqslant j+L(C')}} 2^{jp(s+n/2-n/p)} 2^{(j-m)s'p} \sum_{k;\,k2^{-j}\in D_m} |C_{j,k}|^p = \\ &= \sum_{m;\,2^m\rho\geqslant 1} 2^{-ms'p} \sum_{\substack{j;\,2^j\rho\geqslant 10C',\\ j\geqslant m-L(C')}} 2^{jp(s+n/2-n/p+s')} \sum_{k;\,k2^{-j}\in D_m} |C_{j,k}|^p < \infty. \end{aligned} \tag{4.4}$$

On the other hand, the $\mathcal{O}^{(p)}$–condition that for every $\varepsilon > 0$,

$$\int_0^{\varepsilon} \left(\rho^{s'} \|f_2 | B^{s+s'}_{p,p}(A_\rho)\|\right)^p \frac{d\rho}{\rho} < \infty$$

follows from the condition that

$$\sum_{u \in \mathbb{Z};\, 2^{-u} \leqslant \varepsilon} 2^{-us'p} \sum_{j;\, 2^j \rho \geqslant 10C'} 2^{jp(s+s'+n/2-n/p)} \sum_{v \in \mathbb{Z};\, v \geqslant u} \sum_{k;\, k2^{-j} \in D_v} |C_{j,k}|^p < \infty. \tag{4.5}$$

Because $\operatorname{supp} \psi_{j,k} \cap U = \phi$, and the geometric series $\sum_{u;\, u \leqslant v} 2^{-us'p}$ is estimated from above by some constant comparable to $2^{-vs'p}$ (note that $s' < 0$), this last condition (4.5) follows from that

$$\sum_{v;\, 2^v \varepsilon \geqslant 1} 2^{-vs'p} \sum_{\substack{j;\, 2^j \rho \geqslant 10C', \\ j \geqslant v - L(C')}} 2^{jp(s+s'+n/2-n/p)} \sum_{k;\, k2^{-j} \in D_v} |C_{j,k}|^p < \infty. \tag{4.6}$$

It follows from (4.4) and (4.6) that the remaining sum $\sum_3$ satisfies the local Besov $\mathcal{O}^{(p)}$–condition, as desired.

Conversely let us assume that $f = f_1 + f_2$ satisfies the following conditions:

$$f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n), \tag{4.7}$$

and

$$\|f_2 | B^{s+s'}_{p,p}(A_\rho)\| = \mathcal{O}^{(p)}(\rho^{-s'}) \qquad \text{for every } \rho > 0. \tag{4.8}$$

We note that if the support of the wavelet $\psi_{j,k}$ is completely included in A_ρ, then any function extending f_2 outside A_ρ has the same wavelet coefficient $C_{j,k}$. From this remark, Definition 2.1 and the assumption (4.8), we have that for any $\rho > 0$,

$$\sum_{u;\, 2^u \rho \geqslant 1} 2^{-us'p} \sum_{j \in \mathbb{Z}} 2^{jp(s+s'+n/2-n/p)} \sum_{k;\, k2^{-j} \in A_{2^{-u}}} |C_{j,k}|^p < \infty. \tag{4.9}$$

The condition (4.9) is equivalent to that

$$\sum_{j \in \mathbb{Z}} 2^{jp(s+s'+n/2-n/p)} \sum_{k \in \mathbb{Z}^n} |C_{j,k}|^p \sum_{\substack{u;\, 2^u \rho \geqslant 1, \\ 2^u d(k2^{-j}, U) \leqslant 1}} 2^{-us'p} < \infty.$$

After the calculation of the geometric sum, we arrive at the following:

$$\sum_{j \in \mathbb{Z}} 2^{jp(s+s'+n/2-n/p)} \sum_{k \in \mathbb{Z}^n} |C_{j,k}|^p \left(d(k2^{-j}, U)^{s'p} - \rho^{s'p}\right) < \infty. \tag{4.10}$$

Note that $s' < 0$. Then as $\rho \to \infty$ in (4.10), we obtain that

$$\sum_{j \in \mathbb{Z}} 2^{jp(s+n/2-n/p)} \sum_{k \in \mathbb{Z}^n} \left| \left(1 + 2^j d(k2^{-j}, U)\right)^{s'} C_{j,k} \right|^p < \infty,$$

that is, $f_2 \in B^{s,s'}_{p,p}(U)$. Taking into account the assumption (4.7) that $f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n)$, we conclude that $f = f_1 + f_2 \in B^{s,s'}_{p,p}(U)$.

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