The Singularity of Orbital Measures on Compact Lie Groups

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Abstract

We find the minimal real number k such that the kth power of the Fourier transform of any continuous, orbital measure on a classical, compact Lie group belongs to l^2. This results from an investigation of the pointwise behaviour of characters on these groups. An application is given to the study of L^p-improving measures.

1. Introduction

It is well known that there are many continuous, singular measures on the circle (or any compact abelian group) all of whose convolution powers remain singular to L^1. In contrast, Ragozin in [5] proved the striking fact that if G was a compact, connected, simple Lie group and μ was any central, continuous measure on G then

$$\mu^{\dim G} \in L^1(G).$$

This fact obviously implies that the Fourier transform of any such measure tends to zero, and was used to prove that compact, simple Lie groups admit no infinite, central Sidon sets, again in contrast to the abelian case.

Ragozin's result was first improved in [2] where it was shown that if $k > \dim G/2$ and μ was a continuous orbital measure on G then $\mu^k \in L^2(G)$, while if μ was any central, continuous measure then $\mu^k \in L^1(G)$. In [3] estimates were made on the size of characters (in terms of their degrees) from which one could determine the minimal integer k such that $\mu^k \in L^2$ for all continuous orbital measures μ on the classical, compact Lie groups.

Our interest in this paper is to determine the precise size of the Fourier transform of continuous orbital measures, i.e., to determine the minimal fractional power k for which $\widehat{\mu}^k \in l^2$. The answer depends on the Lie group type and is summarized below.

Main Theorem *Let G be a compact, connected, simple Lie group of type A_n, B_n, C_n or D_n. Then $\widehat{\mu}^k \in l^2$ for all continuous orbital measures μ on G if and only if*

$$k > k_0 \equiv \begin{cases} n + 1/2 & \text{if G is type A_n} \\ 2n - 1/2 & \text{if G is type B_n} \\ n - 1/4 & \text{if G is type C_n, $n \neq 3$} \\ 13/4 & \text{if G is type C_3} \\ n - 3/4 & \text{if G is type D_n.} \end{cases}$$

The necessity of this choice of k_0 follows easily from the earlier work as is shown in Corollary 3.2.

The proof of sufficiency is divided into two parts. In the first part of the proof we continue the study of the pointwise behaviour of characters begun in [3]. This is relevant because if μ_g is the orbital measure associated to $g \in G$, then the Fourier transform of μ_g at the representation λ is given by

$$\widehat{\mu_g}(\lambda) = Tr\lambda(g)/\deg \lambda.$$

Here we show that for many $g \in G$ it is possible to improve the pointwise bounds of characters found in [3]. Using these improved estimates, the orbital measures corresponding to these points are easily seen to have the desired property. The arguments used to establish these statements are similar to those of [3]; the details are sketched in Section 3 and the results are summarized in Corollary 3.4. In the second part of the proof of sufficiency we consider the points $g \in G$ where the pointwise bounds found in [3] are sharp. For the orbital measures corresponding to these points a different, more direct approach to summing the Fourier transform is taken. This approach depends on properties of the particular points g and is the content of Section 4.

Ricci and Stein in [6] proved that surface measures on compact, connected analytic manifolds which generate the group act as convolution operators from L^p to L^2 for some $p < 2$. In Section 5 we use our results to investigate the size of p for the particular case of continuous orbital measures. Previously the minimal p was determined for regular orbital measures in [7].

2. Notation and Structural properties of Subroot Systems

The proof of the theorem relies significantly on the representation theory of simple Lie groups. The main facts we need are recorded below.

2.1. Notation and Basic Facts

Let G be a compact, connected, simple, non-exceptional Lie group of rank n. Let $Z(G)$ denote its centre and W be its Weyl group. Denote by $e_1, \ldots, e_m$ the usual unit vectors in R^m where $m = n + 1$ in type A_n and $m = n$ otherwise. We take a maximal torus T with Φ the set of roots for (G, T) described below.

Type	Root system Φ	Base $\Delta = \{\alpha_j : j = 1, \ldots, n\}$
A_n	$\{e_i - e_j : 1 \leq i \neq j \leq n+1\}$	$\alpha_j = e_j - e_{j+1}$
B_n	$\{\pm e_i, \pm(e_i \pm e_j) : 1 \leq i \neq j \leq n\}$	$\alpha_j = e_j - e_{j+1}$ for $j \neq n$
		$\alpha_n = e_n$
C_n	$\{\pm 2e_i, \pm(e_i \pm e_j) : 1 \leq i \neq j \leq n\}$	$\alpha_j = e_j - e_{j+1}$ for $j \neq n$
		$\alpha_n = 2e_n$
D_n	$\{\pm(e_i \pm e_j) : 1 \leq i \neq j \leq n\}$	$\alpha_j = e_j - e_{j+1}$ for $j \neq n$
		$\alpha_n = e_{n-1} + e_n$

The positive roots associated with the base of simple roots Δ will be denoted by Φ^+, the fundamental dominant weights relative to Δ will be denoted by $\lambda_1, \ldots, \lambda_n$, and Λ^+ will be the set of all dominant weights. The set Λ^+ is in a 1–1 correspondence with $\widehat{G}$; $\sigma_\lambda \in \widehat{G}$ is indexed by its highest weight $\lambda \in \Lambda^+$. The degree of σ_λ will be denoted by d_λ. The weights of $\lambda \in \Lambda^+$ are given by

$$\Pi(\lambda) = \{\mu \in \Lambda : w(\mu) < \lambda \text{ for all } w \in W\}$$

where $\mu < \lambda$ means $\lambda - \mu$ is a non-negative integral sum of positive roots. We set $\rho = \sum_{j=1}^n \lambda_j$. According to the Weyl dimension formula ([9]) the degree of λ is given by

$$(2.1) \qquad \prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha)/(\rho, \alpha).$$

For general facts about root systems we refer the reader to [4].

A measure μ on G is called *central* if μ commutes with all other measures on G under the action of convolution. Central measures are characterized by the fact that their Fourier transforms are scalar multiples of identity matrices:

$$\widehat{\mu}(\lambda) = a_\lambda I_{d_\lambda} \quad \text{where } a_\lambda = \int_G \frac{Tr\lambda(x)}{d_\lambda} d\mu.$$

We will simply write $\widehat{\mu}(\lambda)$ in place of a_λ.

An interesting class of singular, central measures are the orbital measures. The *orbital measure* μ_g, supported on the conjugacy class $C(g)$ containing $g \in G$, is defined by

$$\int_G f d\mu_g = \int_G f(tgt^{-1}) dm_G(t) \quad \text{for } f \in C(G).$$

Orbital measures are continuous if and only if $g \notin Z(G)$. One can easily see that $\widehat{\mu_g}(\lambda) = Tr\lambda(g)/d_\lambda$, hence the behaviour of the Fourier transform of orbital measures is determined by the pointwise behaviour of characters.

Since characters are class functions it suffices to know their values on the torus: For g in the torus the Weyl character formula [9] states

$$Tr\lambda(g) = \frac{e^{i\rho(g)} \sum_{w \in W} \det w \exp i(\rho + \lambda, w(g))}{\prod_{\alpha \in \Phi^+}(e^{i\alpha(g)} - 1)}$$

where this is understood to be a limit if it is an indefinite form. It is important to identify the zeroes of the denominator and so given $g \in T$ we let

$$\Phi(g) = \{\alpha \in \Phi : \alpha(g) \in 2\pi\mathbf{Z}\} \quad \text{and let} \quad \Phi^+(g) = \Phi(g) \bigcap \Phi^+.$$

It is easily seen that $\Phi(g)$ is a subroot system of Φ and that $\Phi^+(g)$ is a complete set of positive roots of this subroot system. It is known that $\Phi(g) = \Phi$ if and only if $g \in Z(G)$ (see [1, p. 189]). When $\Phi(g)$ is empty g is called a regular element of G.

It was shown in [2] that one can evaluate the Weyl character formula (by considering suitable directional derivatives if $\Phi^+(g)$ is not empty) to obtain

$$(2.2) \quad \frac{|Tr\lambda(g)|}{d_\lambda} = c(g) \frac{\left| \sum_{w \in W} \det w \prod_{\alpha \in \Phi^+(g)} (\rho + \lambda, w(\alpha)) \exp i(\rho + \lambda, w(g)) \right|}{\prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha)}.$$

From this one can immediately derive the key formula which we use to make estimates:

$$(2.3) \quad \frac{|Tr\lambda(g)|}{d_\lambda} \leq c(g) \frac{\sum_{w \in W} \left| \prod_{\alpha \in \Phi^+(g)} (\rho + \lambda, w(\alpha)) \right|}{\prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha)}.$$

In order to use this formula at $g \notin Z(G)$ it is helpful to understand the structures of the proper subroot systems $\Phi(g)$ and their Weyl conjugates.

Clearly it suffices to analyze those subroot systems which are maximal in the sense that there is no other proper subroot system containing it; we provide a listing of these below for the convenience of the reader. For an explanation on how they are determined see [3].

Type	Maximal subroot systems
A_n	A_{n-1}, $A_k \times A_{n-k}$: $k, n-k \geq 2$
B_n	B_{n-1}, D_n, $A_1 \times D_{n-1}$, $D_k \times B_{n-k}$: $k, n-k \geq 2$
C_n	A_{n-1}, $C_k \times C_{n-k}$: $k, n-k \geq 1$
D_n	D_{n-1}, A_{n-1}, $D_k \times D_{n-k}$: $k, n-k \geq 2$.

Here D_2 is understood to mean $\{e_i \pm e_j\}$, $B_1 = \{e_i\}$, $C_1 = \{2e_i\}$. C_2 and D_3 are the obvious root systems.

3. Pointwise values of characters

The following theorem, which compares the value of a character to its degree, was obtained in [3].

Theorem 3.1 *Let G be a compact, connected, simple Lie group of type A_{n-1}, B_n, C_n or D_n. For every $g \notin Z(G)$ there is a constant $c(g)$ such that*

$$\left| \frac{Tr\lambda(g)}{\deg \lambda} \right| \leq c(g)(\deg \lambda)^{-s}$$

for all $\lambda \in \widehat{G}$ if and only if

$$s \leq s_0 \equiv \begin{cases} 1/(n-1) & \text{if } G \text{ is type } A_{n-1} \text{ or } D_n \\ 1/(2n-1) & \text{if } G \text{ is type } B_n \\ 2/(2n-1) & \text{if } G \text{ is type } C_n, \ n \neq 3 \\ 1/3 & \text{if } G \text{ is type } C_3. \end{cases}$$

Moreover, there is some $g \notin Z(G)$ such that

$$\left| \frac{Tr\lambda(g)}{d_\lambda} \right| \geq c(g) d_\lambda^{-s_0}$$

for the infinite family of representations $\lambda = m\lambda_1$, $m \in 2\mathbb{Z}^+$ ($m\lambda_3$ in type C_3).

From the final statement of this theorem it is easy to prove the necessity of k_0 in the main theorem.

Corollary 3.2 *Let G be a compact, connected, simple Lie group of type A_n, B_n, C_n or D_n. Then there is some $g \notin Z(G)$ such that $\widehat{\mu_g}^{k_0} \notin l^2$.*

Proof. By definition,

$$\left\|\widehat{\mu_g}^k\right\|_2^2 = \sum_{\lambda \in \widehat{G}} d_\lambda^2 \left|\frac{Tr\lambda(g)}{d_\lambda}\right|^{2k},$$

thus for the choices of g for which Theorem 3.1 is optimal we obtain

$$\left\|\widehat{\mu_g}^k\right\|_2^2 \geq \sum_{m \text{ even}} d_{m\lambda_i}^2 \left|\frac{Tr\ m\lambda_i(g)}{d_{m\lambda_i}}\right|^{2k} \geq c(g) \sum_{m \text{ even}} d_{m\lambda_i}^{2-2ks_0}$$

for $i = 3$ in type C_3 and $i = 1$ otherwise. The degrees of the representations $m\lambda_1$ are easily seen to be $O(m^n)$ in type A_n, $O(m^{2n-1})$ in type B_n and C_n, and $O(m^{2n-2})$ in type D_n; the degree of $m\lambda_3$ in C_3 is $O(m^6)$. It is a routine exercise to check that

$$d_{m\lambda_i}^{2-2k_0 s_0} = m^{-1},$$

and thus $\widehat{\mu_g}^{k_0} \notin l^2$. ∎

Now we turn to the problem of proving the sufficiency of this choice of k_0. Our first step is to show that the pointwise bounds of Theorem 3.1 can be sharpened for many points of G.

Proposition 3.3 *Suppose G is a classical, compact, connected, simple Lie group. Then*

$$\left|\frac{Tr\lambda(g)}{\deg \lambda}\right| \leq c(g)(\deg \lambda)^{-s}$$

for all $\lambda \in \widehat{G}$ provided $G, \Phi^+(g)$ and s are as described below:

Type	Type of Subroot System $\Phi^+(g)$	s
A_n	$A_k \times A_{n-k-1}$	$\frac{1}{n-1}$
B_n	B_{n-1}	$\frac{1}{2n-2}$
	$A_1 \times D_{n-1}$	$\frac{1}{n}$
	$D_k \times B_{n-k};\ k, n-k \geq 2$	$\frac{1}{2n-4}$
C_n	A_{n-1}	$\frac{2}{n+1}$
	$C_k \times C_{n-k};\ k, n-k \geq 2$	$\frac{2}{2n-3}\ \left(\frac{1}{3}; n=4\right)$
D_n	$A_{n-1};\ n \geq 5$	$\frac{1}{n-2}\ \left(\frac{2}{7}; n=5\right)$
	$D_k \times D_{n-k};\ k, n-k \geq 2$	$\frac{1}{n-2}.$

Proof.   The arguments used to show this are similar to those used in [3], but more delicate. We first note that inequality (2.3), together with the Weyl dimension formula (2.1), show that it is sufficient to prove that there is some constant $c = c(g)$ such that for all representations λ,

$$(3.1) \quad \left| \frac{\prod_{\alpha \in \Phi'^+} (\rho + \lambda, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha)^{1-s}} \right| = \left| \prod_{\alpha \in \Phi'^+} (\rho + \lambda, \alpha)^s \prod_{\alpha \in \Phi^+ \backslash \Phi'^+} (\rho + \lambda, \alpha)^{s-1} \right| \leq c,$$

whenever Φ'^+ is the set of positive roots of a maximal subroot system Weyl conjugate to $\Phi^+(g)$. One should observe that the action of the Weyl group preserves the type and basic structure of the subroot system.

Throughout the proof we will assume $\rho + \lambda$ can be expressed in terms of the fundamental dominant weights as $\sum_{i=1}^{n} m_i \lambda_i$. We will also assume $m_M = \max_{i=1,\dots,n} m_i$. The letter c will denote a constant which may vary from one line to another.

The proof proceeds by considering each Lie type separately. We give the details here for type C_n and maximal proper subroot system $C_k \times C_{n-k}$, $k, n-k \geq 2$ to illustrate the ideas. This subroot system has positive roots

$$\Phi'^+ = \{2e_l, e_i \pm e_j : i < j, l \in J_1\} \bigcup \{2e_l, e_i \pm e_j : i < j, l \in J_2\}$$

where J_1, J_2 are disjoint subsets of $\{1, \dots, n\}$, $|J_1| = k$, $|J_2| = n - k$. By symmetry we may assume $1 \in J_1$. Let Ψ^+ be the set of roots in Φ^+ on the letters $\{2, \dots, n\}$ and $\Psi'^+ = \Psi^+ \bigcap \Phi'^+$. Then (3.1) can be written as PQ where

$$P \equiv (\rho + \lambda, 2e_1)^s \prod_{j \in J_1 \backslash \{1\}} (\rho + \lambda, e_1 \pm e_j)^s \prod_{j \in J_2} (\rho + \lambda, e_1 \pm e_j)^{s-1}$$

and

$$Q \equiv \prod_{\alpha \in \Psi'^+} (\rho + \lambda, \alpha)^s \prod_{\alpha \in \Psi^+ \backslash \Psi'^+} (\rho + \lambda, \alpha)^{s-1}.$$

As Ψ^+ is a root system of type C_{n-1} containing Ψ'^+ a subroot system of type $C_{k-1} \times C_{n-k}$ (with $k \geq 2$, $n - k \geq 2$) it follows from Theorem 3.1 that Q is bounded if $s \leq 2/(2n-3)$ and $n - 1 \geq 4$, or if $s \leq 1/3$ and $n - 1 = 3$.

Since $(\rho + \lambda, e_1 + e_j) = O(m_M)$ for all j,

$$P \leq c m_M^{s(1+2(|J_1|-1))+|J_2|(s-1)},$$

and because $2 \leq |J_1|, |J_2| \leq n-2$, this expression is bounded for $s \leq 2/(2n-3)$.

Modifications of these arguments are needed for types A_n, B_n and D_n. ∎

We will refer to the maximal subroot systems of the proposition as the *better* maximal subroot systems and the others as the *worst* maximal subroot systems. To summarize, the worst maximal subroot systems are: type A_{n-1} in A_n; D_n in B_n; $C_1 \times C_{n-1}$ in C_n; D_{n-1} in D_n and A_3 in D_4. (See the list in section 2 of maximal subroot systems.)

Notice that any proper subroot system is either one of the worst types or is contained in one of the better types. For example, a proper subroot system of A_n which is not type A_{n-1} is either contained in one of the better subroot systems $A_k \times A_{n-k-1}$, or is contained in a proper subroot system of type A_{n-1}. We can show, in fact, that the second case falls within the first. The two maximal subroot systems of A_{n-1} are A_{n-2}, which is contained in the better subroot system $A_1 \times A_{n-2}$, and $A_k \times A_{n-1-k-1}$, which is contained in the better subroot system $A_k \times A_{n-k-1}$. The arguments for the other types are similar.

Clearly Proposition 3.3 continues to hold if $\Phi^+(g)$ is only assumed to be contained in one of the better maximal subroot systems. These observations imply that as long as $\Phi^+(g)$ is not one of the worst maximal subroot systems, then there is a constant $c(g)$ such that for all $\lambda \in \widehat{G}$,

$$\left| \frac{Tr\lambda(g)}{d_\lambda} \right| \leq c(g)(d_\lambda)^{-s},$$

provided s is as stated in the proposition.

As a consequence, for many orbital measures a stronger result than the main theorem can be proved.

Corollary 3.4 *Suppose G is a classical, compact, connected, simple Lie group, $g \notin Z(G)$ and $\Phi^+(g)$ is not one of the worst maximal subroot systems. Then $\widehat{\mu_g}^{k_0} \in l^2$.*

Proof. The previous corollary implies that

$$\left\| \widehat{\mu_g}^{k} \right\|_2^2 = \sum_{\lambda \in \widehat{G}} d_\lambda^2 \left| \frac{Tr\lambda(g)}{d_\lambda} \right|^{2k} \leq c(g) \sum_{\lambda \in \widehat{G}} d_\lambda^{2-2ks}$$

(for s as in the chart). It was shown in [2, Cor. 9] that

$$\sum_{\lambda \in \widehat{G}} d_\lambda^t < \infty$$

whenever $t < -\text{rank } G/|\Phi^+|$ and it is a straightforward calculation to check that $2 - 2k_0 s < -\text{rank } G/|\Phi^+|$. ∎

4. Completion of the Proof of the Main Theorem

In this section we will complete the proof of the main theorem.

Theorem 4.1 *Let G be a compact, connected, simple Lie group of type A_n, B_n, C_n or D_n. Then $\widehat{\mu_g}^k \in l^2$ for all continuous orbital measures on G if*

$$
k > k_0 \equiv \begin{cases}
n + 1/2 & \text{if } G \text{ is type } A_n \\
2n - 1/2 & \text{if } G \text{ is type } B_n \\
n - 1/4 & \text{if } G \text{ is type } C_n, \ n \neq 3 \\
13/4 & \text{if } G \text{ is type } C_3 \\
n - 3/4 & \text{if } G \text{ is type } D_n.
\end{cases}
$$

Proof. It remains only to prove this theorem for orbital measures μ_g with $\Phi^+(g)$ one of the worst maximal subroot systems: type A_{n-1} in A_n, type D_n in B_n, type $C_1 \times C_{n-1}$ in C_n, type D_{n-1} in D_n and type A_3 in D_4. The previous method will not work for these orbital measures as the (optimal) pointwise estimates on the trace function are not adequate. Instead we will find an upper bound on the l^2 norm of $\widehat{\mu_g}^k$ by appealing to (2.3) and using the fact that

$$
\left\| \widehat{\mu_g}^k \right\|_2^2 = \sum_{\lambda \in \widehat{G}} d_\lambda^2 \left| \frac{Tr\lambda(g)}{d_\lambda} \right|^{2k} \leq c \sum_{\Phi' = w(\Phi(g))} \sum_{\lambda \in \widehat{G}} \prod_{\alpha \in \Phi'^+} (\rho + \lambda, \alpha)^2 \prod_{\alpha \in \Phi^+ \backslash \Phi'^+} (\rho + \lambda, \alpha)^{2-2k}
$$

where the outer sum is taken over the finitely many subroot systems, Φ', Weyl conjugate to $\Phi(g)$. Thus it suffices to show that for $k > k_0$,

$$
(4.1) \qquad \sum_{\lambda \in \widehat{G}} \left(\prod_{a \in \Phi'^+} (\rho + \lambda, \alpha)^2 \prod_{\alpha \in \Phi^+ \backslash \Phi'^+} (\rho + \lambda, \alpha)^{2-2k} \right) < \infty
$$

for each such positive subroot system Φ'^+. This alternate approach will allow us to take advantage of the fact that while $|Tr\lambda(g)|/d_\lambda = O(d_\lambda^{-s_0})$ for certain λ, it is much smaller for others.

We will continue to adhere to the convention that $m_M = \max_i m_i$, where $\rho + \lambda = \sum m_i \lambda_i$ and c will denote a constant which may vary. Our strategy requires us to consider each type separately. Again, we will give the details for type C_n and sketch the main ideas for the other types.

Type $C_1 \times C_{n-1}$ in C_n: We will leave C_3 for the reader and proceed inductively. So assume $n \geq 4$, $k > n - 1/4$, and Φ'^+ is a type $C_1 \times C_{n-1}$ subroot system in type C_n. It will have positive roots

$$
\{2e_{n_0}\} \bigcup \{e_i \pm e_j, 2e_l : 1 \leq i < j \leq n, 1 \leq l \leq n : i, j, l \neq n_0\}
$$

for some $n_0 \in \{1, \ldots, n\}$.

Case $M < n_0$: Let Ψ^+ be the set of all positive roots on the letters $\{2, \ldots, n\}$, i.e.,

$$\Psi^+ = \{e_i \pm e_j, 2e_l : 2 \le i < j \le n, 2 \le l \le n\}$$

and let $\Psi'^+ = \Phi'^+ \cap \Psi^+$. With this notation we may factor

$$\prod_{\alpha \in \Phi^+ \backslash \Phi'^+} (\rho + \lambda, \alpha)^{2-2k} \prod_{\alpha \in \Phi'^+} (\rho + \lambda, \alpha)^2$$

as PQ, where

$$P \equiv (\rho + \lambda, e_1 \pm e_{n_0})^{2-2k}(\rho + \lambda, 2e_1)^2 \prod_{j \ne 1, n_0} (\rho + \lambda, e_1 \pm e_j)^2$$

and

$$Q \equiv \prod_{\alpha \in \Psi^+ \backslash \Psi'^+} (\rho + \lambda, \alpha)^{2-2k} \prod_{\alpha \in \Psi'^+} (\rho + \lambda, \alpha)^2.$$

Since $M < n_0$, we have $(\rho + \lambda, e_1 \pm e_{n_0}) = O(m_M)$. This ensures that $P \le cm_M^{4n-2-4k} \le cm_1^{4n-2-4k}$.

Now Ψ'^+ is a type $C_1 \times C_{n-2}$ subroot system in a type C_{n-1} root system Ψ^+ (on the letters $\{2, \ldots, n\}$). For $\alpha \in \Psi^+$, $(\rho + \lambda, \alpha) = (\sum_{i=2}^n m_i \lambda_i, \alpha)$, thus the induction assumption may be applied to Q to give the conclusion

$$\sum_{m_2, \ldots, m_n} \prod_{\alpha \in \Psi^+ \backslash \Psi'^+} (\rho + \lambda, \alpha)^{2-2k} \prod_{\alpha \in \Psi'^+} (\rho + \lambda, \alpha)^2 < \infty.$$

Since $4n - 2 - 4k < -1$, we can combine these facts to obtain

$$\sum_{\lambda : M < n_0} \prod_{\alpha \in \Phi^+ \backslash \Phi'^+} (\rho + \lambda, \alpha)^{2-2k} \prod_{\alpha \in \Phi'^+} (\rho + \lambda, \alpha)^2$$

$$\le \; c \sum_{m_1} m_1^{4n-2-4k} \sum_{m_2, \ldots, m_n} \prod_{\alpha \in \Psi^+ \backslash \Psi'^+} (\rho + \lambda, \alpha)^{2-2k} \prod_{\alpha \in \Psi'^+} (\rho + \lambda, \alpha)^2 < \infty.$$

Type A_{n-1} in A_n or Type D_{n-1} in D_n: Before explaining the case $M \ge n_0$ we remark that a similar induction argument works for these two types, as well. We again factor the summands in (4.1) as PQ, where bounds on Q can be obtained by the induction assumption and P can be bounded by finding a suitable number of roots α with $(\rho + \lambda, \alpha) \ge m_M$. For type A_n we take Q to be the product of the factors involving positive roots built on either the letters $\{2, \ldots, n+1\}$ or $\{1, \ldots, n\}$, depending on whether $M < n_0$ or $M \ge n_0$ (giving the problem of type A_{n-2} in A_{n-1}). In the first case we can use the fact that $(\rho + \lambda, \alpha) \ge m_M$ for $\alpha = e_1 - e_{n_0}$ to bound P. In the second case, the root $\alpha = e_{n_0} - e_{n+1}$ has the desired property.

For type D_n one can observe that it suffices to assume $n_0 > 1$ and thus we may apply an induction argument by taking Q to be the product of the factors involving the roots built on the letters $\{2, \ldots, n\}$. Finding a suitable number of roots α with $(\rho + \lambda, \alpha) \geq m_M$ is somewhat more delicate in this case. For example, if $M \geq n_0$, then $(\rho + \lambda, \alpha) \geq m_M$ for $\alpha = e_i + e_{n_0}$ for all $i \neq n_0$ (or $i \neq n_0, n$) provided $M \neq n - 1$ ($M = n - 1$) and for $\alpha = e_{n_0} - e_n$ if $M = n - 1$.

Type $C_1 \times C_{n-1}$ in C_n (ctd):

Case $M \geq n_0$: Here we take a different approach and instead factor

$$\prod_{\alpha \in \Phi^+ \setminus \Phi'^+} (\rho + \lambda, \alpha)^{2-2k} \prod_{\alpha \in \Phi'^+} (\rho + \lambda, \alpha)^2$$

as $P'Q'$ where

$$P' = \prod_{i \neq n_0} (\rho + \lambda, e_i + e_{n_0})^{2-2k} \prod_i (\rho + \lambda, 2e_i)^2 \prod_{i < j; i, j \neq n_0} (\rho + \lambda, e_i + e_j)^2$$

and

$$Q' = \prod_{i \neq n_0} (\rho + \lambda, e_i - e_{n_0})^{2-2k} \prod_{i < j; i, j \neq n_0} (\rho + \lambda, e_i - e_j)^2.$$

Note that $(\rho + \lambda, e_i - e_j) = (\sum_{k=1}^{n-1} m_k \lambda_k, e_i - e_j)$ for any $j > i$. Thus Q' is the summand in (4.1) which arises from a subroot system of type A_{n-2} in A_{n-1}, on the letters $\{1, \ldots, n\}$. As $k > n - 1/2$ it follows from the analysis for this case that

$$\sum_{m_1, \ldots, m_{n-1}} \prod_{i \neq n_0} (\rho + \lambda, e_i - e_{n_0})^{2-2k} \prod_{i < j; i, j \neq n_0} (\rho + \lambda, e_i - e_j)^2 < \infty.$$

Thus it suffices to prove $P' \leq c m_n^{-t}$ for some $t > 1$, and this is what we will verify. It is convenient to let $b_i = \max_{l \geq i} m_l$. Then $(\rho + \lambda, 2e_i)$ and $(\rho + \lambda, e_i + e_j)$ for $j > i$ are $O(b_i)$, so

$$
\begin{aligned}
P' &\leq c \prod_{j < n_0} b_j^{2-2k} \prod_{j > n_0} b_{n_0}^{2-2k} \prod_i b_i^2 \prod_{i < j; i, j \neq n_0} b_i^2 \\
&= c \prod_{i < n_0} b_i^{2(n-i-1)+4-2k} \prod_{i > n_0} b_i^{2+2(n-i)} \left(b_{n_0}^{2+(2-2k)(n-n_0)} \right).
\end{aligned}
$$

Now $b_i \leq m_M$ and $2 + 2(n - i) > 0$, thus

$$\prod_{i > n_0} b_i^{2+2(n-i)} \leq \prod_{i > n_0} m_M^{2+2(n-i)} = m_M^{(n-n_0+1)(n-n_0)}.$$

Since $M \geq n_0$, $b_i = m_M$ for $i \leq n_0$, hence

$$\prod_{i < n_0} b_i^{2(n-i-1)+4-2k} = m_M^{(2n+2-2k-n_0)(n_0-1)}.$$

Combining these observations, and using the fact that $n \geq 4$, gives the bound

$$P' \leq cm_M^{2k+n^2+n-2kn} \leq cm_n^{-5/2},$$

completing the proof of this case.

Type D_n in B_n: In this case $\Phi^+ \setminus \Phi'^+$ consists of all the positive roots of length one. If $\alpha = e_i$ is such a root, then

$$(\rho + \lambda, \alpha) = O(m_i + \cdots + m_n) \geq (\rho + \lambda, \beta)$$

for $\beta = e_i \pm e_j$ and any $i < j$. From this it follows that (4.1)

$$\sum_\lambda \prod_{\alpha \in \Phi^+ \setminus \Phi'^+} (\rho + \lambda, \alpha)^{2-2k} \prod_{\alpha \in \Phi'^+} (\rho + \lambda, \alpha)^2$$

$$\leq c \sum_{m_1,\ldots,m_n} \left(\prod_{i=1}^n (m_i + \cdots + m_n) \right)^{4n-4i-2k+2} \leq c \sum_{m_1,\ldots,m_n} \left(\prod_{i=1}^n m_i \right)^{4n-2k-2},$$

and this is finite as $4n - 2k - 2 < -1$.

Type A_3 in D_4: This type can be done by explicit calculation. Alternatively, one can effectively reduce the problem to that of type D_3 in D_4 by considering a suitable automorphism of the root system. ∎

Remark 4.1 *The main theorem stated in the introduction results from Corollary 3.2 and Theorem 4.1. It would be interesting to know if there was a less computational proof of this result, or a proof which did not involve consideration of each Lie type separately. Also, it remains to carry out the analysis for the exceptional groups. The maximal subroot systems of E_6, E_7 and E_8 seem to be too cumbersome for our approach.*

5. Applications

5.1. Convolutions

Since the convolution of two L^2 functions is continuous we immediately obtain the following corollary of the theorem.

Corollary 5.1 *If μ is a continuous orbital measure, then μ^{2n-1} is a continuous function if G is type A_{n-1} or D_n, μ^{2n} is continuous if G is type C_n, $n \geq 4$, and μ^{4n-1} is a continuous function if G is type B_n.*

A similar result will hold for any central, continuous measure compactly supported on the conjugates of a set of the form $\{x \in T : \Phi^+(x) = \Phi^+\}$ for some fixed set Φ^+ (c.f., [2, Cor. 7]).

5.2. L^p-Improving Measures

A measure μ is called L^p-improving if there is some $p < 2$ such that

$$\mu * L^p \subseteq L^2.$$

Young's inequality implies that all functions in L^q, for some $q > 1$, are examples of L^p-improving measures. A question of current interest is to understand which singular measures on compact groups are L^p-improving.

In [6] surface measures on compact, connected analytic submanifolds which generate the group were shown to be L^p-improving. Continuous orbital measures are an example of this phenomena. The measure μ_g is supported on the conjugacy class containing g, and this manifold generates G since a suitable k-fold product supports the non-zero, absolutely continuous measure μ_g^k and thus must have positive Haar measure.

A more refined problem is to determine the minimal p such that $\mu * L^p \subseteq L^2$. This was done for for orbital measures corresponding to regular elements (those with $\Phi^+(g)$ empty) in [7] where it was shown that

$$\mu_g * L^p \subseteq L^2 \qquad \text{if and only if} \qquad p \geq 1 + r/(2\dim G - r).$$

Our theorem gives results for all continuous orbital measures.

Proposition 5.2 *If $g \notin Z(G)$ then $\mu_g * L^p \subseteq L^2$ for $p > p_0$ where:*

(i) $p_0 = 2 - 4/(2n + 3)$ *when G is type A_n ;*

(ii) $p_0 = 2 - 4/(4n + 1)$ *when G is type B_n;*

(iii) $p_0 = 2 - 8/(4n + 3)$ *when G is type C_n, $n \neq 3$; $p_0 = 26/17$ if $n = 3$;*

(iv) $p_0 = 2 - 8/(4n + 1)$ *when G is type D_n.*

Proof. The main theorem implies that for $k > k_0$ the operator

$$T_k(f) = \mu_g^k * f$$

maps $L^1(G)$ into $L^2(G)$ whenever $g \notin Z(G)$. Since the identity map obviously maps $L^2(G)$ into $L^2(G)$, an application of Stein's interpolation theorem [8] gives that $\mu_g * L^p \subseteq L^2$ for the choices of p_0 listed. ∎

Remark 5.1 *It would be interesting to know if these results are optimal for the orbital measures corresponding to the worst subroot systems. Clearly they are not optimal for other continuous, orbital measures as better results can be obtained by invoking Proposition 3.3.*

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