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Existence of H-bubbles in a perturbative setting

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Abstract

Given a C^1 function $H\colon \mathbb{R}^3 \to \mathbb{R}$, we look for H-bubbles, i.e., surfaces in $\mathbb{R}^3$ parametrized by the sphere $\mathbb{S}^2$ with mean curvature H at every regular point. Here we study the case $H(u) = H_0(u)+\epsilon H_1(u)$ where H_0 is some "good" curvature (for which there exist H_0-bubbles with minimal energy, uniformly bounded in L^∞), ϵ is the smallness parameter, and H_1 is *any* C^1 function.

1. Introduction

In this work we study the existence of $\mathbb{S}^2$-type parametric surfaces in the Euclidean space $\mathbb{R}^3$, having prescribed mean curvature.

This geometrical problem can be stated in analytical form as follows: given a C^1 map $H\colon \mathbb{R}^3 \to \mathbb{R}$, find a smooth nonconstant function $\omega\colon \mathbb{R}^2 \to \mathbb{R}^3$ satisfying

$$(1.1) \qquad \begin{cases} \Delta\omega = 2H(\omega)\omega_x \wedge \omega_y \quad \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\omega|^2 < +\infty\ . \end{cases}$$

Indeed it is known that such a function ω turns out to be conformal and if $p = \omega(z)$ is a regular point, then the value $H(p)$ represents the mean curvature of the surface parametrized by ω at the point p. Moreover, denoting by $\sigma\colon \mathbb{S}^2 \to \mathbb{R}^2$ the stereographic projection, the mapping $\omega \circ \sigma\colon \mathbb{S}^2 \to \mathbb{R}^3$ defines an $\mathbb{S}^2$-type parametric surface in $\mathbb{R}^3$ having prescribed mean curvature H, briefly, an *H-bubble.*

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When the prescribed mean curvature is a nonzero constant $H(u) \equiv H_0$, Brezis and Coron [2] proved that the only nonconstant solutions to (1.1) are spheres of radius $|H_0|^{-1}$ anywhere placed in $\mathbb{R}^3$.

Only recently, the case in which H is nonconstant has been investigated. In particular, in [3] the authors studied the case of a bounded function $H \in C^1(\mathbb{R}^3)$ asymptotic to a constant at infinity. Under some global assumptions, the existence of an H-bubble having "minimal energy" is proved. This existence result constitutes the starting point of the present work and, for future convenience, let us recall its precise statement.

Firstly, we point out that problem (1.1) has a variational structure. More precisely, H-bubbles can be found as critical points of the energy functional

$$\mathcal{E}_H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y \ ,$$

where $Q\colon \mathbb{R}^3 \to \mathbb{R}^3$ is any vectorfield such that div $Q = H$.

When H is bounded, $\mathcal{E}_H$ turns out to be well defined (by continuous extension) and sufficiently regular on some Sobolev space.

Roughly, the integral

$$\int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$$

has the meaning of the algebraic H-weighted volume of the region enclosed by range u and it is essentially cubic in u. Therefore, the energy functional $\mathcal{E}_H$, which is unbounded from below and from above, actually admits a saddle type geometry.

In order to make precise the geometrical structure of $\mathcal{E}_H$, we consider the restriction of $\mathcal{E}_H$ to the space of smooth functions $C^1_c(\mathbb{R}^2, \mathbb{R}^3)$, and we introduce the value:

$$c_H = \inf_{\substack{u \in C^1_c(\mathbb{R}^2,\mathbb{R}^3) \\ u \neq 0}} \sup_{s>0} \ \mathcal{E}_H(su) \ , \tag{1.2}$$

which represents the mountain pass level along radial paths. In [3], we proved the following existence result.

Theorem 1.1 *Let $H \in C^1(\mathbb{R}^3)$ be such that*

(h_1) $H(u) \to H_\infty$ *as* $|u| \to \infty$, *for some* $H_\infty \in \mathbb{R}$,

(h_2) $\sup_{u \in \mathbb{R}^3} |\nabla H(u) \cdot u \ u| < 1$,

(h_3) $c_H < \dfrac{4\pi}{3H_\infty^2}$.

Then there exists an H-bubble ω with $\mathcal{E}_H(\omega) = c_H$. In addition $c_H = \inf_{\mathcal{B}_H} \mathcal{E}_H$ where $\mathcal{B}_H$ is the class of the H-bubbles.

We point out that the assumption (h_2) is important in order to get a positive lower bound for the minimal energy of H-bubbles. In addition, it is also used to get boundedness (with respect to the Dirichlet norm) of the Palais Smale sequences of $\mathcal{E}_H$ and to guarantee that the value c_H is an admissible minimax level. The assumption (h_3) is variational in nature and it is verified, for instance, whenever $H > H_\infty \geq 0$ on a suitably large set. Moreover, together with (h_1) and (h_2), the hypothesis (h_3) implies that some special Palais Smale sequences of $\mathcal{E}_H$ at level c_H are bounded even in the strong L^∞ topology.

The main difficulty is the lack of compactness, due to the fact that problem (1.1) is invariant with respect to the conformal group. This means that we deal with a problem on the image of ω, rather that on the mapping ω itself.

If H is constant there is also an invariance with respect to translations on the image. The assumption (h_3) forces H to be nonconstant and allows us to look for minimal H-bubbles in some bounded region, recovering some compactness, in a suitable way.

In this paper we investigate the following question: does the existence result stated above persist under perturbation of the prescribed curvature function? More precisely, we consider the case in which

$$H(u) = H_0(u) + \epsilon H_1(u) := H_\epsilon(u)$$

where $H_0 \in C^1(\mathbb{R}^3)$ satisfies (h_1)–(h_3), $|\epsilon|$ is small, and $H_1 \colon \mathbb{R}^3 \to \mathbb{R}$ is *any* C^1 function, not necessarily bounded.

Clearly, in general, none of the hypotheses (h_1)–(h_3) is fulfilled by H_ϵ. Furthermore, even the corresponding energy functional $\mathcal{E}_{H_\epsilon}$ is not well defined on the Sobolev spaces suited to study problem (1.1). Hence no "global" variational approach works, but a localizing argument has to be followed. To this aim, one takes advantage from the fact that the set of minimal H_0-bubbles is uniformly bounded in L^∞ (thanks to the condition (h_3)) and then a truncation on the perturbative term H_1 can be made. The main result of this paper is stated as follows.

Theorem 1.2 *Let $H_0 \in C^1(\mathbb{R}^3)$ be such that (h_1)–(h_3) hold, and let $H_1 \in C^1(\mathbb{R}^3)$. Then there is $\bar\epsilon > 0$ such that for every $\epsilon \in (-\bar\epsilon, \bar\epsilon)$ there exists an H_ϵ-bubble ω^ϵ. Furthermore, as $\epsilon \to 0$, ω^ϵ converges (geometrically) to some minimal H_0-bubble ω. More precisely*

$$\omega^\epsilon \circ \sigma \to \omega \circ \sigma \quad \text{in } C^1(\mathbb{S}^2, \mathbb{R}^3)\ .$$

We remark that the energy of ω^ϵ is close to the (unperturbed) minimal energy of H_0-bubbles. However in general we cannot say that ω^ϵ is a minimal H_ϵ-bubble.

Finally, we remark that Theorem 1.2 cannot be applied in case the unperturbed curvature H_0 is a constant, since assumption (h_3) is not satisfied. That case is studied in the work [5], with quite different techniques.

The paper is organized as follows. In Section 2 we introduce some notation and the functional setting. We also recall some useful results already discussed in other papers. In Section 3 we state some convergence properties for the energy functional, and finally in Section 4 we give the proof of Theorem 1.2.

2. Notation and preliminaries

Denoting by $\sigma\colon \mathbb{S}^2 \to \mathbb{R}^2$ the stereographic projection, to every map $u\colon \mathbb{S}^2 \to \mathbb{R}^3$ we associate the map $\bar u := u \circ \sigma^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^3$. Thus, for example, for $s \ge 1$, the norm of u in $L^s(\mathbb{S}^2, \mathbb{R}^3)$ is given by

$$\|u\|_s = \left(\int_{\mathbb{R}^2} |\bar u|^s \mu^2\right)^{1/s}$$

where

$$\mu(z) = \frac{2}{1+|z|^2}\ , \quad z = (x,y)\ .$$

Similarly, if $du(p)\colon T_p\mathbb{S}^2 \to \mathbb{R}^3$ denotes the gradient of u at $p \in \mathbb{S}^2$, and $z = \sigma(p)$, one has

$$|du(p)| = |\nabla \bar u(z)| \mu(z)^{-1} \tag{2.1}$$

where ∇ denotes the standard gradient in $\mathbb{R}^2$. Hence, the norm of $|du|$ in $L^s(\mathbb{S}^2)$ is given by

$$\|du\|_s = \left(\int_{\mathbb{R}^2} |\nabla \bar u|^s \mu^{2-s}\right)^{1/s}\ .$$

In the following, we simply write H^1 instead of $W^{1,2}(\mathbb{S}^2, \mathbb{R}^3)$ and we will often identify any map $u\colon \mathbb{S}^2 \to \mathbb{R}^3$ with $\bar u := u \circ \sigma^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^3$.

Now, let $H \in C^1(\mathbb{R}^3)$ be a given curvature, and let $Q\colon \mathbb{R}^3 \to \mathbb{R}^3$ be any smooth vectorfield such that div $Q = H$. For every $u \in H^1 \cap L^\infty$ let us set

$$\mathcal{V}_H(u) = \int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y\ .$$

If u is smooth enough, the integral defined by $\mathcal{V}_H(u)$ corresponds to the algebraic volume enclosed by the surface parametrized by u, with weight H, and it is independent of the choice of the vectorfield Q (see [9]). Here we choose

$$Q(u) = m_H(u)u \ , \quad m_H(u) = \int_0^1 H(su)s^2 \, ds \ .$$

Notice that the following identity holds on $\mathbb{R}^3$:

$$H(u) = 3m(u) + \nabla m(u) \cdot u. \tag{2.2}$$

Denoting by

$$\mathcal{D}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2$$

the Dirichlet integral of u, the energy functional $\mathcal{E}_H \colon H^1 \cap L^\infty \to \mathbb{R}$ can be written as:

$$\mathcal{E}_H(u) = \mathcal{D}(u) + 2\mathcal{V}_H(u) \ .$$

It is proved that if H is bounded, $\mathcal{E}_H$ admits a continuous extension on H^1 (see [9]) and for every $u \in H^1$ there exists the directional derivative of $\mathcal{E}_H$ at u along any $\varphi \in H^1 \cap L^\infty$ (see [7]), given by

$$\partial_\varphi \mathcal{E}_H(u) = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} H(u)\varphi \cdot u_x \wedge u_y \ . \tag{2.3}$$

Now, for any $H \in C^1(\mathbb{R}^3)$ define

$$M_H = \sup_{u \in \mathbb{R}^3} |\nabla H(u) \cdot u \, u| \ .$$

Using (2.3) with $\varphi = u$, and the identity (2.2), one easily obtains the following key estimate

$$3\mathcal{E}_H(u) \geq (1 - M_H)\mathcal{D}(u) + \partial_u \mathcal{E}_H(u) \quad \text{for } u \in H^1 \cap L^\infty \ . \tag{2.4}$$

Notice that, assuming $M_H < 1$ (namely, the hypothesis (h_2) of Theorem 1.1), (2.4) immediately implies that if $(u^n) \subset H^1 \cap L^\infty$ is a Palais-Smale sequence for $\mathcal{E}_H$ at some level $c \in \mathbb{R}$, then $\sup \|\nabla u^n\|_2 < +\infty$ and $c \geq 0$.

The condition $M_H < 1$ enters in an important way also to infer some properties on the mountain pass value c_H defined by (1.2). This is stated by the next result, proved in [3].

Lemma 2.1 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $M_H < 1$. Then:*

(i) *If ω is an H-bubble, there holds $\mathcal{E}_H(\omega) \geq c_H$.*

(ii) *Given $u \in H^1 \cap L^\infty$, u nonconstant, one has $\sup_{s>0} \mathcal{E}_H(su) < +\infty$ if and only if there exists $s_1 > 0$ such that $\mathcal{E}_H(s_1 u) < 0$. If this is the case, then $\sup_{s>0} \mathcal{E}_H(su) = \max_{s \in [0,s_1]} \mathcal{E}_H(su)$.*

Remark 2.2 If $H(u) \equiv H_0 \in \mathbb{R} \setminus \{0\}$ for all $u \in \mathbb{R}^3$, then

$$c_{H_0} = \frac{4\pi}{3H_0^2}.$$

Indeed, given $u \in C_c^1(\mathbb{R}^2, \mathbb{R}^3)$, $u \neq 0$, the mapping $s \mapsto \mathcal{E}_{H_0}(su)$ admits a critical point $\bar{s} > 0$ if and only if $\mathcal{V}_{H_0}(u) < 0$. In this case

$$\sup_{s>0} \mathcal{E}_{H_0}(su) = \mathcal{E}_{H_0}(\bar{s}u) = \frac{1}{27\, H_0^2} \frac{\mathcal{D}(u)^3}{\mathcal{V}_1(u)^2} \ .$$

Then

$$c_{H_0} = \frac{S^3}{27\, H_0^2}$$

where

$$S = \inf_{\substack{u \in C_c^1(\mathbb{R}^2, \mathbb{R}^3) \\ u \neq 0}} \frac{\mathcal{D}(u)}{\mathcal{V}_1(u)^{2/3}}$$

is the classical isoperimetric constant (see [10]). Since $S = \sqrt[3]{36\pi}$ the conclusion follows.

3. Some convergence results

The first convergence result presented in this Section is given by an upper semicontinuity property for the mountain pass level defined by (1.2). This result has been proved in [3].

Lemma 3.1 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $M_H < 1$. Let $(H_n) \subset C^1(\mathbb{R}^3)$ be a sequence of functions satisfying $M_{H_n} < 1$, and such that $H_n \to H$ uniformly on compact sets of $\mathbb{R}^3$. Then $\limsup c_{H_n} \leq c_H$.*

The next result concerns the semicontinuity of the energy functional $\mathcal{E}_H$. In general $\mathcal{E}_H$ is not lower semicontinuous, because of possible concentration phenomena (see an example by Wente in [10]). However, as stated by the next Lemma, the lower semicontinuity holds true at least along a sequence of solutions.

First, let us introduce the following notation: for every $\rho > 0$ set

$$\bar{M}_{H,\rho} = 2 \sup_{|u|<\rho} |(H(u) - 3m_H(u))u| \ .$$

Notice that $\bar{M}_{H,\rho} \leq M_H$. In addition, if $H_n \to H$ uniformly on

$$\overline{B}_\rho = \{u \in \mathbb{R}^3 : |u| \leq \rho\},$$

then $\bar{M}_{H_n,\rho} \to \bar{M}_{H,\rho}$.

Lemma 3.2 *Let* $(H_n) \subset C^1(\mathbb{R}^3)$*,* $H \in C^1(\mathbb{R}^3)$ *and* $\rho > 0$ *be such that:*

(i) $H_n \to H$ *uniformly on* $\overline{B}_\rho$*,*

(ii) $\bar{M}_{H_n,\rho} \le 1$ *for every* $n \in \mathbb{N}$.

Let $(\omega^n) \subset H^1 \cap L^\infty$ *be such that for every* $n \in \mathbb{N}$*:*

(iii) ω^n *is an* H_n*-bubble,*

(iv) $\|\omega^n\|_\infty \le \rho$*,*

(v) $|\nabla\omega^n(0)| = \|\nabla\omega^n\|_\infty = 1$

(vi) $\|\nabla\omega^n\|_2 \le C$ *for some positive constant* C.

Then there exists an H*-bubble* ω *such that, for a subsequence,* $\omega^n \to \omega$ *weakly in* H^1 *and strongly in* $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$*. Moreover* $\mathcal{E}_H(\omega) \le \liminf \mathcal{E}_{H_n}(\omega^n)$*.*

To prove Lemma 3.2 we will use the following "ε-regularity" Lemma, inspired by a similar result due to Sacks and Uhlenbeck [8] (see also Lemma A.1 in [1]). For its proof, we refer to [3].

Lemma 3.3 *Let* $H \in C^1(\mathbb{R}^3) \cap L^\infty$*. Then there exist* $\varepsilon > 0$ *such that for every* $s > 1$ *there exists a constant* $C_s > 0$*, depending only on* $s, \|H\|_\infty$*, such that if* $u \in W^{2,s}_{loc}(D, \mathbb{R}^3)$ *solves* $\Delta u = 2H(u)u_x \wedge u_y$ *on an open domain* $\Omega \subseteq \mathbb{R}^2$*, then*

$$\|\nabla u\|_{L^2(D_R(z))} \le \varepsilon \ \Rightarrow \ \|\nabla u\|_{W^{1,s}(D_{R/2}(z))} \le C_s R^{\frac{2}{s}-s}\|\nabla u\|_{L^2(D_R(z))}$$

for every disc $\overline{D_R(z)} \subset \Omega$ *with* $R \in (0,1)$*.*

Proof of Lemma 3.2. From the assumption (iv) and (vi), there exists $\omega \in H^1 \cap L^\infty$ such that, for a subsequence, $\omega^n \to \omega$ weakly in H^1. Now fix an arbitrary $r > 0$ and let us prove that $\omega^n \to \omega$ strongly in $C^1(\overline{D}_r, \mathbb{R}^3)$. By Lemma 3.3, for every $n \in \mathbb{N}$ and for $s > 2$ fixed, there exists $\varepsilon_n > 0$ (which in fact depends only on $\|H_n\|_{L^\infty(B_\rho)}$) and $C_{s,n} > 0$ for which

$$\|\nabla\omega^n\|_{L^2(D_R(z))} \le \varepsilon_n \ \Rightarrow \ \|\nabla\omega^n\|_{H^{1,s}(D_{R/2}(z))} \le C_{s,n} R^{\frac{2}{s}-2}\|\nabla\omega^n\|_{L^2(D_R(z))}$$

for every $z \in \mathbb{R}^2$ and for every $R \in (0,1)$. By (i), one has $\varepsilon_n \ge \varepsilon > 0$ and $C_{s,n} \le C_s$ for every $n \in \mathbb{N}$. Since $\|\nabla\omega^n\|_\infty = 1$, there exists $R \in (0,1)$ and a finite covering $\{D_{R/2}(z_i)\}_{i\in I}$ of $\overline{D}_r$ such that $\|\nabla\omega^n\|_{L^2(D_R(z_i))} \le \varepsilon$ for every $n \in \mathbb{N}$ and $i \in I$. Since $\|\omega^n\|_\infty \le \rho$, we have that

$$\|\omega^n\|_{H^{2,s}(D_{R/2}(z_i))} \le \bar{C}_{s,\rho}$$

for some constant $\bar{C}_{s,\rho} > 0$ independent of $i \in I$ and $n \in \mathbb{N}$.

Then the sequence (ω^n) is bounded in $H^{2,s}(D_r,\mathbb{R}^3)$. For $s>2$ the space $H^{2,s}(D_r,\mathbb{R}^3)$ is compactly embedded into $C^1(\overline{D}_r,\mathbb{R}^3)$. Hence $\omega^n\to\omega$ strongly in $C^1(\overline{D}_r,\mathbb{R}^3)$. By a standard diagonal argument, one concludes that $\omega^n\to\omega$ strongly in $C^1_{loc}(\mathbb{R}^2,\mathbb{R}^3)$.

Now we prove that ω is an H-bubble. Indeed, for every $n\in\mathbb{N}$, if $h\in C^\infty_c(\mathbb{R}^2,\mathbb{R}^3)$ then

$$\int_{\mathbb{R}^2}\nabla\omega^n\cdot\nabla h+2\int_{\mathbb{R}^2}H_n(\omega^n)h\cdot\omega^n_x\wedge\omega^n_y=0\,.$$

Since $\omega^n\to\omega$ strongly in $C^1_{loc}(\mathbb{R}^2,\mathbb{R}^3)$, passing to the limit, one immediately infers that ω is a weak solution to (1.1). By regularity theory of H-systems (see, e.g., [6]), ω is a classical, conformal solution to (1.1). In addition ω is nonconstant, since

$$|\nabla\omega(0)|=\lim|\nabla\omega^n(0)|=1.$$

Hence ω is an H-bubble.

Finally, let us show that

$$\mathcal{E}_H(\omega)\le\liminf\mathcal{E}_{H_n}(\omega^n).$$

Again by the strong convergence in $C^1_{loc}(\mathbb{R}^2,\mathbb{R}^3)$ and by (i) and (iv), for every $R>0$, one has

$$\mathcal{E}_{H_n}(\omega^n,D_R)\to\mathcal{E}_H(\omega,D_R) \tag{3.1}$$

where we denote

$$\mathcal{E}_{H_n}(\omega^n,\Omega)=\frac{1}{2}\int_\Omega|\nabla\omega^n|^2+2\int_\Omega m_{H_n}(\omega^n)\omega^n\cdot\omega^n_x\wedge\omega^n_y$$

(and similarly for $\mathcal{E}_H(\omega,\Omega)$). Now, fixing $\epsilon>0$, let $R>0$ be such that

$$\mathcal{E}_H(\omega,\mathbb{R}^2\setminus D_R)\ \le\ \epsilon \tag{3.2}$$

$$\int_{\mathbb{R}^2\setminus D_R}|\nabla\omega|^2\ \le\ \epsilon\,. \tag{3.3}$$

By (3.2) and (3.1) we have

$$\begin{aligned}\mathcal{E}_H(\omega)\ &\le\ \mathcal{E}_H(\omega,D_R)+\epsilon\\ &=\ \mathcal{E}_{H_n}(\omega^n,D_R)+\epsilon+o(1)\\ &=\ \mathcal{E}_{H_n}(\omega^n)-\mathcal{E}_{H_n}(\omega^n,\mathbb{R}^2\setminus D_R)+\epsilon+o(1)\end{aligned} \tag{3.4}$$

with $o(1)\to 0$ as $n\to+\infty$.

Since every ω^n is an H_n-bubble, using the divergence theorem, for any $R>0$ one has

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus D_R}|\nabla\omega^n|^2 = 3\mathcal{E}_{H_n}(\omega^n,\mathbb{R}^2\setminus D_R)-\int_{\partial D_R}\omega^n\cdot\frac{\partial\omega^n}{\partial\nu} +2\int_{\mathbb{R}^2\setminus D_R}\left(H_n(\omega^n)-3m_{H_n}(\omega^n)\right)\omega^n\cdot\omega^n_x\wedge\omega^n_y\ .$$

Moreover, by definition of $\bar{M}_{H_n,\rho}$, we can estimate

$$2\int_{\mathbb{R}^2\setminus D_R}\left(H_n(\omega^n)-3m_{H_n}(\omega^n)\right)\omega^n\cdot\omega^n_x\wedge\omega^n_y\le\frac{\bar{M}_{H_n,\rho}}{2}\int_{\mathbb{R}^2\setminus D_R}|\nabla\omega^n|^2.$$

Thus, we obtain

$$-\mathcal{E}_{H_n}(\omega^n,\mathbb{R}^2\setminus D_R) \le -\frac{1}{3}\int_{\partial D_R}\omega^n\cdot\frac{\partial\omega^n}{\partial\nu}-\frac{1-\bar{M}_{H_n,\rho}}{6}\int_{\mathbb{R}^2\setminus D_R}|\nabla\omega^n|^2 \tag{3.5}$$

$$\le -\frac{1}{3}\int_{\partial D_R}\omega^n\cdot\frac{\partial\omega^n}{\partial\nu}\ , \tag{3.6}$$

because of the assumption (ii). Using again the C^1_{loc} convergence of ω^n to ω, as well as the fact that ω is an H-bubble, we obtain that

$$\begin{aligned}\lim_{n\to+\infty}\left|\int_{\partial D_R}\omega^n\cdot\frac{\partial\omega^n}{\partial\nu}\right| &= \left|\int_{\partial D_R}\omega\cdot\frac{\partial\omega}{\partial\nu}\right|\\ &= \left|\int_{\mathbb{R}^2\setminus D_R}\left(\omega\cdot\Delta\omega+|\nabla\omega|^2\right)\right|\\ &= \left|\int_{\mathbb{R}^2\setminus D_R}\left(2H(\omega)\omega\cdot\omega_x\wedge\omega_y+|\nabla\omega|^2\right)\right|\\ &\le (\|\omega\|_\infty\|H\|_\infty+1)\int_{\mathbb{R}^2\setminus D_R}|\nabla\omega|^2\\ &\le (\|\omega\|_\infty\|H\|_\infty+1)\,\epsilon \qquad (3.7)\end{aligned}$$

thanks to (3.3). Finally, (3.4), (3.6) and (3.7) imply

$$\mathcal{E}_H(\omega)\le\mathcal{E}_{H_n}(\omega^n)+C\epsilon+o(1)$$

for some positive constant C independent of ϵ and n. Hence, the conclusion follows. ∎

As a consequence of Lemma 3.2, one obtains the following result.

Lemma 3.4 *Let $H \in C^1(\mathbb{R}^3)$ satisfy (h_1)–(h_3). Then there exists $\rho > 0$ (depending on $\|H\|_\infty$, on $\mu_H := 1 - M_H > 0$ and on $\delta_H := \frac{4\pi}{3H_\infty^2} - c_H > 0$) such that*

$$\|\omega\|_\infty \le \rho$$

for every H-bubble ω with $\mathcal{E}_H(\omega) = c_H$.

Proof. First, notice that ω is smooth by the regularity theory for H-systems (see for example [6]). In addition, if $\mathcal{E}_H(\omega) = c_H$, then by (2.4), we get

$$\int_{\mathbb{R}^2} |\nabla\omega|^2 \le \frac{6c_H}{1 - M_H} \ . \tag{3.8}$$

Moreover, by a corollary to a Grüter's estimate [6] (see [3, proof of Theorem 1.1]), one has

$$\text{diam}\ \omega \le C\left(1 + \int_{\mathbb{R}^2} |\nabla\omega|^2\right)$$

where

$$\text{diam}\ \omega = \sup_{z,z' \in \mathbb{R}^2} |\omega(z) - \omega(z')|,$$

and C is a positive constant depending only on $\|H\|_\infty$. Hence

$$\text{diam}\ \omega \le \frac{C(\|H\|_\infty)}{1 - M_H} =: \rho \ . \tag{3.9}$$

To conclude, it is enough to show that there exists $\rho_0 > 0$ such that $|\omega(0)| \le \rho_0$, for every H-bubble ω with $\mathcal{E}_H(\omega) = c_H$. Arguing by contradiction, suppose that $|\omega^n(0)| \to +\infty$ for a sequence (ω^n) of H-bubbles with $\mathcal{E}_H(\omega^n) = c_H$. Set

$$\bar{\omega}^n = \omega^n - \omega^n(0) \qquad \text{and} \qquad H_n(u) = H(u + \omega^n(0)).$$

Then $H_n \to H_\infty$ uniformly on compact sets. This also implies that $\bar{M}_{H_n,\rho} \to 0$ as $n \to +\infty$. Moreover, thanks to the conformal invariance, we may suppose $|\nabla\bar{\omega}^n(0)| = \|\nabla\bar{\omega}^n\|_\infty = 1$. Clearly, for every $n \in \mathbb{N}$, $\bar{\omega}^n$ is an H_n-bubble. In addition, (3.8) and (3.9) imply that

$$\sup(\|\bar{\omega}^n\|_\infty + \|\nabla\bar{\omega}^n\|_2) < +\infty.$$

Hence we are in position to apply Lemma 3.2, obtaining that

$$\liminf \mathcal{E}_{H_n}(\bar{\omega}^n) \ge \mathcal{E}_{H_\infty}(\bar{\omega})$$

where $\bar{\omega}$ is some H_∞-bubble. But $\mathcal{E}_{H_n}(\bar{\omega}^n) = \mathcal{E}_H(\omega^n) = c_H$, and, by Lemma 2.1, $\mathcal{E}_{H_\infty}(\bar{\omega}) \ge c_{H_\infty}$. Hence, thanks to Remark 2.2, we obtain $c_H \ge \frac{4\pi}{3H_\infty^2}$, in contradiction with (h_3). This concludes the proof. ■

Finally we state a sufficient condition in order to have strong convergence in H^1 along a sequence of H_n-bubbles, when H_n converges to some limit curvature H.

Lemma 3.5 *Let $(H_n) \subset C^1(\mathbb{R}^3)$, $H \in C^1(\mathbb{R}^3)$ and $\rho > 0$ be such that:*

(i) *$H_n \to H$ uniformly on $\overline{B}_\rho$,*

(ii) *$\bar{M}_{H,\rho} < 1$.*

Let $(\omega^n) \subset H^1 \cap L^\infty$ and $\omega \in H^1 \cap L^\infty$ be such that:

(iii) *ω is an H-bubble and ω^n is an H_n-bubble for every $n \in \mathbb{N}$,*

(iv) *$\|\omega^n\|_\infty \le \rho$ for every $n \in \mathbb{N}$,*

(v) *$\omega^n \to \omega$ weakly in H^1 and strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$,*

(vi) *$\mathcal{E}_{H_n}(\omega^n) \to \mathcal{E}_H(\omega)$.*

Then $\omega^n \to \omega$ strongly in H^1.

Proof. First, notice that assumption (v) guarantees that

$$\nabla\omega^n \to \nabla\omega \qquad \text{in } L^2_{loc}.$$

Therefore, in order to have strong convergence of the gradients in L^2 it suffices to prove that

$$\forall \epsilon > 0\ ,\ \exists R > 0 \ \text{ such that } \quad \limsup \int_{\mathbb{R}^2 \setminus D_R} |\nabla\omega^n|^2 \le \epsilon\ . \tag{3.10}$$

Furthermore, assumption (iv), together with Rellich theorem, will readily lead to the conclusion. In order to prove (3.10), we will use some estimates already seen in the proof of Lemma 3.2.

In particular, fixing an arbitrary $\varepsilon > 0$ and $R > 0$ according to (3.2) and (3.3), by (3.4), (3.5), and (3.7), we have

$$\frac{1 - \bar{M}_{H_n,\rho}}{6} \int_{\mathbb{R}^2 \setminus D_R} |\nabla\omega^n|^2 \le \mathcal{E}_{H_n}(\omega^n) - \mathcal{E}_H(\omega) + C\varepsilon + o(1)\ .$$

Thanks to (i), for $n \in \mathbb{N}$ large one has $\bar{M}_{H_n,\rho} \le \bar{M}$ for some $\bar{M} < 1$. Moreover $\mathcal{E}_{H_n}(\omega^n) \to \mathcal{E}_H(\omega)$, by hypothesis, and then we obtain

$$\frac{1 - \bar{M}}{6} \int_{\mathbb{R}^2 \setminus D_R} |\nabla\omega^n|^2 \le C\varepsilon + o(1)\ .$$

that, up to an insignificant multiplicative constant (independent of ϵ), is the desired estimate (3.10), since $\bar{M} < 1$. ■

Finally, we show that strong convergence in H^1 of a sequence of H_n-bubbles implies convergence in $C^1(\mathbb{S}^2,\mathbb{R}^3)$.

Lemma 3.6 *Let $(H_n)\subset C^1(\mathbb{R}^3)$, $H\in C^1(\mathbb{R}^3)$ and $\rho>0$ be such that $H_n\to H$ uniformly on $\overline{B}_\rho$. Let $(\omega^n)\subset H^1\cap L^\infty$ and $\omega\in H^1\cap L^\infty$ be such that:*

(i) *ω is an H-bubble and ω^n is an H_n-bubble for every $n\in\mathbb{N}$,*

(ii) *$\|\omega^n\|_\infty\le\rho$ for every $n\in\mathbb{N}$,*

(iii) *$\omega^n\to\omega$ strongly in H^1.*

Then $\omega^n\circ\sigma\to\omega\circ\sigma$ in $C^1(\mathbb{S}^2,\mathbb{R}^3)$.

Proof. Setting $\hat\omega^n(z)=\omega^n\left(z^{-1}\right)$ (in complex notation), one easily checks that $\hat\omega^n$ is an H_n-bubble. Similarly, $\hat\omega(z)=\omega\left(z^{-1}\right)$ is an H-bubble. Let $\varepsilon>0$ be given according to Lemma 3.3 and let $R>2$ be such that

$$\int_{\mathbb{R}^2\setminus D_{1/R}}|\nabla\omega|^2<\varepsilon\ .$$

Since $\omega^n\to\omega$ strongly in H^1, for $n\in\mathbb{N}$ large enough one has

$$\int_{D_R}|\nabla\hat\omega^n|^2=\int_{\mathbb{R}^2\setminus D_{1/R}}|\nabla\omega^n|^2<\varepsilon\ .$$

Hence, by Lemma 3.3, one obtains

$$\|\nabla\hat\omega^n\|_{W^{1,s}(D_{R/2})}\le C_sR^{\frac{2}{s}-s}\|\nabla\hat\omega^n\|_{L^2(D_R)}\le C$$

for every $n\in\mathbb{N}$ large enough. Since for $s>2$ $W^{1,s}(D_{R/2})$ is compactly embedded into $C^0(D_{R/2})$, we obtain

$$\hat\omega^n\to\hat\omega\text{ in }C^1(D_{R/2})\ . \tag{3.11}$$

Now, let $z_1,\dots,z_k$ and $r\in(0,1)$ be such that $\overline{D_{2/R}}\subset D_r(z_1)\cup\ldots\cup D_r(z_k)$ and $\int_{D_{2r}(z_j)}|\nabla\omega|^2<\varepsilon$ for every $j=1,\dots,k$. Applying again Lemma 3.3, we infer that for every $j=1,\dots,k$ one has $\|\nabla\omega^n\|_{W^{1,s}(D_r(z_j))}\le C$ and then $\omega^n\to\omega$ in $C^1(D_r(z_j))$. Hence

$$\omega^n\to\omega\text{ in }C^1(D_{2/R})\ . \tag{3.12}$$

Since $|\nabla\hat\omega^n(z)|=|z|^{-2}|\nabla\omega^n(z^{-1})|$, (3.11) and (3.12) imply

$$\sup_{z\in\mathbb{R}^2}|\omega^n(z)-\omega(z)|\to0$$
$$\sup_{z\in\mathbb{R}^2}|\nabla(\omega^n-\omega)(z)\mu(z)^{-1}|\to0.$$

Then, by (2.1), the thesis follows. ■

4. Proof of Theorem 1.2

Let $H_0 \in C^1(\mathbb{R}^3)$ satisfy (h_1)–(h_3) and let $R_0 > 0$ be such that $\|\omega\|_\infty \le R_0$ for every minimal H_0-bubble ω, according to Lemma 3.4.

Fixing $H_1 \in C^1(\mathbb{R}^3)$, for every $\epsilon \in (-1,1)$ let

$$H_\epsilon = H_0 + \epsilon H_1 \ .$$

Now, let $\tilde H_1 \colon \mathbb{R}^3 \to \mathbb{R}$ of class C^1 with $\|\tilde H_1\|_\infty < +\infty$, $\|\nabla \tilde H_1\|_\infty < +\infty$, and $\tilde H_1(u) = H_1(u)$ as $|u| \le \tilde R$, for some $\tilde R > R_0$.

Moreover, let $0 < r_\epsilon < R_\epsilon$ and let $\chi_\epsilon \colon \mathbb{R}^3 \to \mathbb{R}$ be a C^1, radial, cut off function such that $\chi_\epsilon(u) = 1$ as $|u| \le r_\epsilon$, $\chi_\epsilon(u) = 0$ as $|u| \ge R_\epsilon$, and $|\nabla \chi_\epsilon(u)| \le \frac{2}{R_\epsilon - r_\epsilon}$ for all u. Here r_ϵ and R_ϵ are asked to satisfy the following conditions:

$$\tilde R < r_\epsilon \quad \text{for every } |\epsilon| < 1 \ , \tag{4.1}$$

$$R_\epsilon - r_\epsilon \ge 1 \quad \text{for every } |\epsilon| < 1 \ , \tag{4.2}$$

$$|\epsilon| R_\epsilon^2 \to 0 \quad \text{as } \epsilon \to 0 \ , \tag{4.3}$$

$$r_\epsilon \to +\infty \quad \text{as } \epsilon \to 0 \ . \tag{4.4}$$

Finally, for $|\epsilon| < 1$ let

$$\tilde H_\epsilon = H_0 + \epsilon \chi_\epsilon \tilde H_1 \ .$$

Our strategy is to show firstly that for $|\epsilon|$ small there exists a minimal $\tilde H_\epsilon$-bubble. Secondly, we will prove that if ω^ϵ is a minimal $\tilde H_\epsilon$-bubble with $|\nabla \omega^\epsilon(0)| = \|\nabla \omega^\epsilon\|_\infty = 1$, then for every sequence $\epsilon_n \to 0$ there exists a minimal H_0-bubble ω such that, for a subsequence, $\omega^{\epsilon_n} \circ \sigma \to \omega \circ \sigma$ in $C^1(\mathbb{S}^2, \mathbb{R}^3)$. In particular, this will imply that any minimal $\tilde H_\epsilon$-bubble stays in the region $|u| < \tilde R$, and thus it is an H_ϵ-bubble.

Lemma 4.1 *There exists $\epsilon_0 > 0$ (depending on $\|\tilde H_1\|_\infty$ and on $\|\nabla \tilde H_1\|_\infty$) such that for every $|\epsilon| < \epsilon_0$:*

(i) *$\tilde H_\epsilon(u) \to H_\infty$ as $|u| \to +\infty$,*

(ii) *$\sup_{u \in \mathbb{R}^3} |\nabla \tilde H_\epsilon(u) \cdot u\, u| := M_{\tilde H_\epsilon} \le \bar M < 1$,*

(iii) *$c_{\tilde H_\epsilon} \le \dfrac{4\pi}{3H_\infty^2} - \delta_0$, for some $\delta_0 > 0$ independent of ϵ.*

Proof. Part (i) follows by the fact that $\tilde H_\epsilon(u) = H_0(u)$ as $|u| \ge R_\epsilon$. For the same reason, $|\nabla \tilde H_\epsilon(u) \cdot u\, u| = |\nabla H_0(u) \cdot u\, u| \le M_{H_0} < 1$ as $|u| \ge R_\epsilon$. For $|u| < R_\epsilon$ direct computations easily give

$$|\nabla \tilde H_\epsilon(u) \cdot u\, u| \le M_{H_0} + |\epsilon|\, R_\epsilon^2\, \|\nabla \tilde H_1\|_\infty + \frac{2|\epsilon| R_\epsilon^2}{R_\epsilon - r_\epsilon} \|\tilde H_1\|_\infty \ .$$

Hence (ii) follows, thanks to (4.2)–(4.3). Finally, let us check (iii). By hypothesis, there exists $u \in C^1_c(D, \mathbb{R}^3)$, $u \neq 0$, such that

$$\sup_{s>0} \mathcal{E}_{H_0}(su) \leq \frac{4\pi}{3H_\infty^2} - 2\delta_0$$

for some $\delta_0 > 0$. By Lemma 2.1, part (i),

$$\sup_{s>0} \mathcal{E}_{H_0}(su) = \max_{s\in[0,s_1]} \mathcal{E}_{H_0}(su) \qquad \text{for some } s_1 > 0.$$

Setting $\rho_1 = s_1\|u\|_\infty$, by (4.4), $r_\epsilon \geq \rho_1$ for $|\epsilon|$ small, and then,

$$\mathcal{E}_{\tilde{H}_\epsilon}(su) = \mathcal{E}_{H_0}(su) + \epsilon\mathcal{V}_{\tilde{H}_1}(su) \text{ for } s \in [0, s_1]\ . \tag{4.5}$$

In particular, $\mathcal{E}_{\tilde{H}_\epsilon}(s_1 u) = \mathcal{E}_{H_0}(s_1 u) + o(1)$ with $o(1) \to 0$ as $\epsilon \to 0$. Hence, for $|\epsilon|$ small, $\mathcal{E}_{\tilde{H}_\epsilon}(s_1 u) < 0$ and, using again Lemma 2.1, part (i),

$$\sup_{s>0} \mathcal{E}_{\tilde{H}_\epsilon}(su) = \max_{s\in[0,s_1]} \mathcal{E}_{\tilde{H}_\epsilon}(su).$$

Finally, (4.5) yields

$$\max_{s\in[0,s_1]} \mathcal{E}_{\tilde{H}_\epsilon}(su) = \max_{s\in[0,s_1]} \mathcal{E}_{H_0}(su) + o(1)\ .$$

In conclusion, for $|\epsilon|$ small enough, one obtains

$$\sup_{s>0} \mathcal{E}_{\tilde{H}_\epsilon}(su) \leq \frac{4\pi}{3H_\infty^2} - \delta_0\ ,$$

namely, (iii). ■

In the next step we state the existence of minimal $\tilde{H}_\epsilon$-bubbles satisfying some uniform estimates.

Lemma 4.2 *There exists $\epsilon_0 > 0$ such that for every $|\epsilon| < \epsilon_0$ there is a $\tilde{H}_\epsilon$-bubble ω^ϵ satisfying:*

(i) $\mathcal{E}_{\tilde{H}_\epsilon}(\omega^\epsilon) = c_{\tilde{H}_\epsilon}$,

(ii) $\|\nabla\omega^\epsilon\|_2 \leq C$,

(iii) $\|\omega^\epsilon\|_\infty \leq C$,

where C is a positive constant independent of ϵ.

Proof. The existence of $\tilde{H}_\epsilon$-bubbles ω^ϵ with $\mathcal{E}_{\tilde{H}_\epsilon}(\omega^\epsilon) = c_{\tilde{H}_\epsilon}$, for $|\epsilon|$ small, is guaranteed by Lemma 4.1 and by Theorem 1.1. The estimate (ii) follows by (2.4), by (i) and by Lemma 4.1, parts (ii) and (iii). Finally (iii) can be deduced by Lemma 3.4, noticing that $M_{\tilde{H}_\epsilon} \leq \bar{M} < 1$ and $c_{\tilde{H}_\epsilon} \leq \frac{4\pi}{3H_\infty^2} - \delta_0$ with $\delta_0 > 0$. ■

For $|\epsilon|$ small, let ω^ϵ be the $\tilde{H}_\epsilon$-bubble given by Lemma 4.2. Note that, since $r_\epsilon \to +\infty$, whereas ω^ϵ are uniformly bounded in L^∞, for $|\epsilon|$ small ω^ϵ is in fact an $(H_0 + \epsilon \tilde{H}_1)$-bubble. Actually, we need a sharper estimate on the L^∞ norms of ω^ϵ, and precisely $\|\omega^\epsilon\|_\infty \le \tilde{R}$, so that we can conclude that ω^ϵ are H_ϵ-bubbles. This is the last step, and it will be accomplished in the sequel.

Notice that, by the invariance of problem (1.1) with respect to dilation, translation and inversion, we may suppose that

$$\|\nabla \omega^\epsilon\|_\infty = |\nabla \omega^\epsilon(0)| = 1 \ . \tag{4.6}$$

By the uniform bounds (ii) and (iii) stated in Lemma 4.2, using Lemma 3.2, and reminding that $\tilde{H}_\epsilon \to H_0$ uniformly on compact sets, we may also suppose that there exists an H_0-bubble ω such that (for a subsequence) $\omega^\epsilon \to \omega$ weakly in H^1 and in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$. Moreover

$$\mathcal{E}_{H_0}(\omega) \le \liminf \mathcal{E}_{\tilde{H}_\epsilon}(\omega^\epsilon) \ . \tag{4.7}$$

Lemma 4.3 *$\omega^\epsilon \circ \sigma \to \omega \circ \sigma$ in $C^1(\mathbb{S}^2, \mathbb{R}^3)$.*

Proof. Notice that $\mathcal{E}_{\tilde{H}_\epsilon}(\omega^\epsilon) = c_{\tilde{H}_\epsilon}$ and, since $M_{\tilde{H}_\epsilon} < 1$, by Lemma 3.1, $\limsup c_{\tilde{H}_\epsilon} \le c_{H_0}$. Hence, by (4.7) and by Lemma 2.1 we infer that ω is a minimal H_0-bubble, namely $\mathcal{E}_{H_0}(\omega) = c_{H_0}$, and

$$\mathcal{E}_{H_0}(\omega) = \lim \mathcal{E}_{\tilde{H}_\epsilon}(\omega^\epsilon) \ . \tag{4.8}$$

Hence all the hypotheses stated in Lemma 3.5 are fulfilled and thus we conclude that $\omega^\epsilon \to \omega$ strongly in H^1. Then we apply Lemma 3.6 to obtain the thesis. ∎

Thus, in particular, $\omega^\epsilon \to \omega$ uniformly on $\mathbb{R}^2$, and, since ω is a minimal H_0-bubble, $\|\omega\|_\infty \le R_0$ and $\|\omega^\epsilon\|_\infty < \tilde{R}$ for $|\epsilon|$ small enough. Hence ω^ϵ is an H_ϵ-bubble and Lemma 4.3 completes the proof of Theorem 1.2.

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