Focusing of spherical nonlinear pulses in $\mathbb{R}^{1+3}$, II. Nonlinear caustic

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Abstract

We study spherical pulse like families of solutions to semilinear wave equations in space time of dimension 1+3 as the pulses focus at a point and emerge outgoing. We emphasize the scales for which the incoming and outgoing waves behave linearly but the nonlinearity has a strong effect at the focus. The focus crossing is described by a scattering operator for the semilinear equation, which broadens the pulses. The relative errors in our approximate solutions are small in the L^∞ norm.

1. Introduction

Consider the asymptotic behavior as $\varepsilon \to 0$ of solutions of the initial value problem

(1.1)
$$
\begin{cases}
\Box \mathbf{u}^\varepsilon + a|\partial_t \mathbf{u}^\varepsilon|^{p-1}\partial_t \mathbf{u}^\varepsilon = 0, \qquad (t,x) \in [0,T] \times \mathbb{R}^3, \\[2mm]
\mathbf{u}^\varepsilon\big|_{t=0} = \varepsilon^{J+1}U_0\left(r, \dfrac{r-r_0}{\varepsilon}\right), \\[2mm]
\partial_t \mathbf{u}^\varepsilon\big|_{t=0} = \varepsilon^J U_1\left(r, \dfrac{r-r_0}{\varepsilon}\right),
\end{cases}
$$

where $\Box := \partial_t^2 - \Delta_x$, a is a complex number, $r = |x|$, $r_0 > 0$, and, $1 < p < \infty$. The functions U_0 and U_1 are infinitely differentiable, bounded, and, there is a $z_0 > 0$ so that for all $r \geq 0$,

(1.2)
$$
\operatorname{supp} U_j(r,.) \subset [-z_0, z_0].
$$

The last assumption implies that at time $t = 0$ the solutions are families of spherical pulses supported in a $O(\varepsilon)$ neighborhood of $r = r_0$. Precisely the support is contained in the interval $|r - r_0| \leq \varepsilon z_0$, and only the values of the profiles $U_j(r, z)$ for these r play a role.

For a purely imaginary, the initial value problem is conservative in the sense that the wave equation energy is conserved, while for a with positive real part the problem is dissipative. The case of a positive real the nonlinear term is monotone and the evolution is contractive in the wave equation energy.

There are at least three reasons why the study of the nonlinear behavior of short pulses is of interest. First is the importance of short pulses in radar and in the technology of ultrashort lasers. Second is the fact that the Fourier Transform of pulses is very broad, $O(1/\varepsilon)$, in the codirections normal to the fronts and so the analysis forces one to deal with this bandwidth of scales which grows infinitely large in the limit of short wavelengths. In the case of wave trains, the band of frequencies that are important stays bounded but tends to infinity. Finally, when pulses focus they spend of time $O(\varepsilon)$ near the caustic, which makes the analysis of nonlinear caustics easier than the case of wave trains.

1.1. Linear spherical pulse families in $\mathbb{R}^{1+3}$

Consider first spherically symmetric solutions of the linear wave equation

$$\Box u = 0\,.$$

With the usual abuse of notation, $u = u(t, r)$ where u is a smooth even function of $r \in \mathbb{R}$. Then

$$w(t, r) \ := \ r\, u(t, r)$$

is a smooth odd function of r. In three space dimensions, the wave equation is equivalent to

$$w_{tt} - w_{rr} \ = \ 0\,.$$

The general odd solution w is

$$w \ = \ g(t + r) - g(t - r)\,.$$

This yields

$$u \ = \ \frac{g(t + r) - g(t - r)}{r} \quad \text{when} \quad r \neq 0\,, \qquad u(t, 0) = 2g'(t)\,.$$

For a smooth compactly supported f, the family with

$$g^{\varepsilon}(r) \ := \ f\Big(\frac{r - r_0}{\varepsilon}\Big)$$

yields

$$u^\varepsilon \; := \; \frac{1}{r}\Big[f\Big(\frac{t+r-r_0}{\varepsilon}\Big) - f\Big(\frac{t-r-r_0}{\varepsilon}\Big)\Big],$$

a family which near $t=0$ are incoming spherical pulses supported on a $O(\varepsilon)$ neighborhood of $|x| = r_0$.

To understand the heuristics in the next section it is important to note the following things. For r bounded away from the origin, $0 < \delta \le r$ one has $(\varepsilon\partial)^k u^\varepsilon = O(1)$. As the pulse approaches the origin it grows like $1/r$ until it gets to $r = O(\varepsilon)$. There the cancellation of the incoming and outgoing pulses regularizes the growth. One has $u^\varepsilon = O(\min\{1/\varepsilon, 1/r\})$. Thus the pulse is larger by a factor of order $1/\varepsilon$ in a small neighborhood of the focus. The $\varepsilon\partial$ derivatives are also amplified by $1/\varepsilon$ in this region.

One could think that since the singular behavior is restricted to a set which the wave passes in time $O(\varepsilon)$ that the singular behavior is negligible. However the small time is compensated by the amplification and the caustic crossing has the finite effect that the outgoing pulse family has profile equal to the negative of the incoming profile. The change in the solution is precisely 200%.

1.2. Two parameters

In the nonlinear case, there are two key parameters. The exponent J controls the amplitude of the solutions as a function of ε while p gives the rate of growth of the nonlinearity at infinity. We will see that to leading order the nonlinearity is negligible in $r \ge \delta > 0$ if and only if $J > 0$. Similarly it is negligible near $r = 0$ if and only if $J > (p-2)/(p-1)$. There are two critical values. The critical value $J = 0$ is the amplitude at which the nonlinear geometric optics is needed to describe the propagation away from the focus. For $J > 0$ linear geometric optics suffices.

A similar situation occurs at the focus. In this paper we treat the case of critically nonlinear focus, namely $J = (p-2)/(p-1)$. In that case, the leading order behavior of the pulse is affected by the nonlinear term as the pulse passes through the focus. We next present rough estimates which yield the critical values.

For short time, one finds two spherical pulses, one expanding outward and the other focusing inward. It is the inward propagating pulse and its behavior as it approaches and passes through the focal point at time $t \approx r_0$ that interests us.

When $J \ge 0$ and even for data which depend on the angular variables, the analysis of Alterman and Rauch [1] shows that before the focus the incoming pulse satisfies $\mathbf{u} \sim \varepsilon^{J+1} U(t, r, (t+r)/\varepsilon)$ so $\nabla_{t,x}\mathbf{u}^\varepsilon = O(\varepsilon^J)$, and $\Box \mathbf{u}^\varepsilon = O(\varepsilon^J)$. Note that the latter is one power of ε smaller than one might guess thanks to the fact that $t+r$ satisfies the eikonal equation.

The nonlinear term is $O(\varepsilon^{pJ})$ so is negligible compared to $\square\mathbf{u}$ precisely when $J > 0$. When $J = 0$, U is determined by a nonlinear transport equation, while when $J > 0$ the transport equation is linear. In the first case we say that there is *nonlinear propagation*. In the second case, there is *linear propagation*. In this classification, p plays no role.

Near $r = 0$ and in the linear case the explicit solutions have first (respectively second) derivative larger by a factor $1/\varepsilon$ (respectively $1/\varepsilon^2$) than they are for r bounded away from 0. Assuming that the "eikonal cancellation" continues to hold for our nonlinear problem one would have $|\nabla_{t,x}\mathbf{u}^\varepsilon|^p = O(\varepsilon^{p(J-1)})$, and $\square\mathbf{u}^\varepsilon = O(\varepsilon^{J-2})$. Comparing these two terms suggests that for $p(J-1) > J - 2$ the nonlinear term can be neglected at the focus. The value $J = (p-2)/(p-1)$ is the critical case for which the nonlinearity cannot be neglected. In the latter case, one expects that there is nonlinear behavior at the caustic. In the former there is a *linear caustic*. Powers J smaller than the critical power, $J < (p-2)/(p-1)$, are called *supercritical caustic*. These expectations are summarized in the following table.

	$J > \frac{p-2}{p-1}$	$J = \frac{p-2}{p-1}$	$J < \frac{p-2}{p-1}$
$J > 0$	linear caustic linear propagation	nonlinear caustic linear propagation	supercritical caustic linear propagation
$J = 0$	linear caustic nonlinear propagation	nonlinear caustic nonlinear propagation	supercritical caustic nonlinear propagation

In addition to this sixfold classification, there is an additional doubling because each class can be considered for pulses and for wave trains. The latter correspond to profiles U_j which are periodic in z. To avoid this compounding of cases, in this paper we will treat exclusively the case of pulses with critically nonlinear caustic and linear propagation.

Distinctions as in the table were computed formally in [10]. For nonlinear Schrödinger equations, they were justified in [4], [3], [5]. The present paper is an analogous treatment for the nonlinear wave equation (1.1) with a critically nonlinear focus and linear propagation. As far as we know, there is nothing known about the corresponding problem for the wave train case.

The paper [8] shows that the entries described as linear caustic are correct. That is, to leading order as $\varepsilon \to 0$, the nonlinear term can be neglected in the asymptotics at the caustic $\{r = 0\}$.

Other papers study the same circle of questions for wave trains as opposed to pulses. The emphasis has been on the case $J = 0$. When $1 < p < 2$, the nonlinearity is negligible at the caustic crossing. This has been proved in ([12], [15], [13], [2]). Though the nonlinear effects at the focus are negligible to leading order they alter the corrector terms.

The case of nonlinear caustic crossing by wave trains with $J = 0$, $p \geq 2$ which is supercritical is studied in ([11], [14]) in the very special cases $a > 0$ and $a < 0$ respectively. In either the accretive or dissipative cases, the effects on approaching the focus are so strong that the crossing is not important. In the accretive case, the solution blows up before reaching the caustic, and in the dissipative case the absorption is so effective that oscillations do not cross the caustic ([11]). In the dissipative case, not only are oscillations absorbed, but also singularities leading to a smoothing effect ([14]). We will treat the analogous cases for spherical pulses, announced in [6], in a subsequent paper.

1.3. Two parameters rescaled

Introduce $\varepsilon^{-J}\mathbf{u}^{\varepsilon} =: u^{\varepsilon}$ instead of $\mathbf{u}^{\varepsilon}$ so that the solutions have derivatives of order $O(1)$ away from the caustic. Define

$$\alpha := (p-1)J.$$

The initial value problem is transformed to

(1.3)
$$\begin{cases} \Box u^{\varepsilon} + a\,\varepsilon^{\alpha}|\partial_t u^{\varepsilon}|^{p-1}\,\partial_t u^{\varepsilon} = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ \\ u^{\varepsilon}\big|_{t=0} = \varepsilon U_0\left(r, \dfrac{r-r_0}{\varepsilon}\right), \qquad \partial_t u^{\varepsilon}\big|_{t=0} = U_1\left(r, \dfrac{r-r_0}{\varepsilon}\right). \end{cases}$$

Translating the previous table yields

	$\alpha + 2 > p$	$\alpha + 2 = p$	$\alpha + 2 < p$
$\alpha > 0$	linear caustic, linear propagation	nonlinear caustic, linear propagation	supercritical caustic, linear propagation
$\alpha = 0$	linear caustic, nonlinear propagation	nonlinear caustic, nonlinear propagation	supercritical caustic, nonlinear propagation

1.4. An inner problem predicts the answer

As the pulse width is $O(\varepsilon)$ and is carried by the incoming characteristic $t + r = r_0$ the interesting focusing is expected to occur in an $O(\varepsilon)$ neighborhood of the point $t = r_0$, $r = 0$. This suggests looking at the scaled functions $\underline{u}(t,x)$ defined by

$$u(t,r) = \underline{u}\left(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\right), \qquad \underline{u}(\tau,\rho) := u(\varepsilon\tau + r_0, \varepsilon\rho),$$

associated with the change of variables

$$\tau = \frac{t-r_0}{\varepsilon}, \quad \rho = \frac{r}{\varepsilon}, \qquad t = \varepsilon\tau + r_0, \quad r = \varepsilon\rho.$$

Equation (1.3) transforms to

$$\frac{1}{\varepsilon^2} \Box_{\tau,\rho}\, \underline{u} \; + \; a\,\varepsilon^\alpha \left| \frac{1}{\varepsilon}\, \underline{u}_\tau \right|^{p-1} \left(\frac{1}{\varepsilon}\, \underline{u}_\tau \right) \; = \; 0\,.$$

The ε dependence disappears exactly in the critical case $\alpha + 2 = p$ to yield

$$(1.4) \qquad\qquad \Box\, \underline{u} + a\, |\underline{u}_\tau|^{p-1} \underline{u}_\tau \; = \; 0\,.$$

This is identical to the starting equation. That invariance under scaling is a second notion of criticality. The fact that the descaled equation capturing the focusing behavior is nonlinear is another expression of the fact that the focus is nonlinear. For any higher power of α the descaled equation would have had a nonlinear term with a positive power of ε as prefactor.

The initial data for u are taken at time $t = 0$ and that translates to time $\tau = -r_0/\varepsilon$ for $\underline{u}$. In the limit $\varepsilon \to 0$ this yields for $\underline{u}$ an initial value problem with initial data taken at $\tau = -\infty$.

The values of u^ε observed $t = r_0 + O(1)$ are read from the values of $\underline{u}$ at time $\tau = O(1/\varepsilon) \to +\infty$. Thus the fact that the caustic crossing is described by a scattering operator for the nonlinear equation (1.4) is reasonable. The existence of such a scattering operator is one of the problems we address.

1.5. Characteristic coordinates

Since the initial data are spherical, so is the solution. With the usual abuse of notation,

$$u^\varepsilon(t, x) \; = \; u^\varepsilon(t, |x|), \qquad u^\varepsilon(t, |x|) \; \in \; C^\infty_{\text{even in } r}(\mathbb{R}_t \times \mathbb{R}_r)\,.$$

Introduce $v^\varepsilon := (v^\varepsilon_-, v^\varepsilon_+)$ where

$$(1.5) \quad \tilde{u}^\varepsilon(t, r) := r u^\varepsilon(t, r), \qquad v^\varepsilon_\mp := (\partial_t \pm \partial_r)\tilde{u}^\varepsilon\,, \qquad v^\varepsilon_\mp \in C^\infty(\mathbb{R}_t \times \mathbb{R}_r)\,.$$

Then (1.1) becomes

$$(1.6) \quad \begin{cases} (\partial_t \pm \partial_r)v^\varepsilon_\pm = \varepsilon^\alpha r^{1-p} g(v^\varepsilon_- + v^\varepsilon_+), \quad g(y) := b|y|^{p-1}y\,, \quad b := -a 2^{-p}\,, \\ (v^\varepsilon_- + v^\varepsilon_+)\big|_{r=0} = 0\,, \end{cases}$$

with initial data given by

$$(1.7) \qquad\qquad v^\varepsilon_\mp\big|_{t=0} = P_\mp\left(r, \frac{r - r_0}{\varepsilon} \right) \pm \varepsilon P_1\left(r, \frac{r - r_0}{\varepsilon} \right)\,,$$

where

$$(1.8) \qquad \begin{aligned} P_\mp(r, z) &:= r U_1(r, z) \pm r \partial_z U_0(r, z)\,, \\ P_1(r, z) &:= U_0(r, z) + r \partial_r U_0(r, z)\,. \end{aligned}$$

Our interest is in analyzing the case $0 < \alpha = p - 2$ which has linear propagation and nonlinear caustic.

The linear propagation is clear in both (1.3) and (1.6) since the nonlinear term has a prefactor $\varepsilon^{\alpha} \to 0$. Before the focus one sees an essentially linear incoming pulse family and after the focus essentially linear outgoing pulses. The subtlety is that the profile of the outgoing family is determined from the profile of the incoming family by solving a scattering problem for (1.4).

The first thing to do is to prove that the solutions $v^{\varepsilon} = (v_{+}^{\varepsilon}, v_{-}^{\varepsilon})$ exist for times $0 \le t \le \underline{t}$ with $\underline{t} > r_0$ so that one can talk about the caustic crossing. We prove this if the initial data are suitably small, or in the case of dissipative nonlinearity, $a > 0$ without smallness assumption.

Even local existence on an ε dependent time interval is not obvious since the nonlinear term is quite singular at $r = 0$ because of the r^{1-p} factor. This singularity is compensated by the fact that $v_{+}^{\varepsilon} + v_{-}^{\varepsilon}$ vanishes at $r = 0$. So, when u^{ε} is C^2 one has $v^{\varepsilon} \in C^1$ and $g(v_{+}^{\varepsilon} + v_{-}^{\varepsilon}) = O(r^p)$ which suffices to compensate for the singularity. In the differentiated equation for first derivatives $\partial_t v^{\varepsilon}$, the nonlinearity is precisely $O(1)$ near $r = 0$.

In all that follows, virtually all estimates are pointwise. This is one of the key benefits of the radial case. Reducing to essentially one space variable yields a variety L^{∞} estimates proved either by the method of characteristics or multipliers of Kruzkov type.

The proof of Theorem 1 is given in § 2.

Theorem 1 i. *If $a > 0$, $\varepsilon > 0$, and $v_{0,\pm}^{\varepsilon}(r)$ are uniformly Lipschitzean functions on $[0, \infty[$ in the following sense,*

$$\exists C > 0, \ \forall \varepsilon \in]0, 1], \ \|\varepsilon \partial_r v_{0,\pm}^{\varepsilon}\|_{L^{\infty}} \le C,$$

and satisfying the compatibility conditions

$$v_{0,+}^{\varepsilon}(0) + v_{0,-}^{\varepsilon}(0) = 0, \qquad \partial_r v_{0,+}^{\varepsilon}(0) - \partial_r v_{0,-}^{\varepsilon}(0) = 0,$$

then there is a unique uniformly Lipschitzean solution $v^{\varepsilon} \in C^1([0, \infty[\times[0, \infty[)$ to (1.6) satisfying the initial condition $v_{\pm}^{\varepsilon}(0, r) = v_{0,\pm}^{\varepsilon}(r)$. In addition,

$$\left\| v^{\varepsilon}, \ \varepsilon \partial_t v^{\varepsilon} \right\|_{L^{\infty}([0,\infty[\times[0,\infty[)} \le \left\| v^{\varepsilon}(0), \ \varepsilon \partial_t v^{\varepsilon}(0) \right\|_{L^{\infty}([0,\infty[)}.$$

ii. *For any $a \in \mathbb{C}$ there is are positive constants $K_1(a)$ and $K_2(a)$ so that if $0 < \varepsilon < 1$ and the initial data satisfy in addition*

$$\left\| v^{\varepsilon}(0), \ \varepsilon \partial_t v^{\varepsilon}(0) \right\|_{L^{\infty}([0,\infty[)} \le K_1(a)$$

then there is a unique uniformly Lipschitzean $C^1([0, \infty[\times[0, \infty[)$ solution to the initial value problem and

$$\left\| v^{\varepsilon}, \ \varepsilon \partial_t v^{\varepsilon} \right\|_{L^{\infty}([0,\infty[\times[0,\infty[)} \le K_2(a) \left\| v^{\varepsilon}(0), \ \varepsilon \partial_t v^{\varepsilon}(0) \right\|_{L^{\infty}([0,\infty[)}.$$

In both case, $v^{\varepsilon}(t, r)$ and $\partial_t v^{\varepsilon}(t, r)$ tend to zero as $r \to \infty$ provided that this is true at $t = 0$.

1.6. Linear geometric optics region

The description of the solution involves three overlapping regions which cover all of space time. The first is the region where linear propagation is all that needs to be considered.

Perform three approximations. First drop the ε^α nonlinear term. Next drop the $O(\varepsilon)$ terms in the initial conditions and replace $P_\pm(r, (r - r_0)/\varepsilon)$ by $P_\pm(r_0, (r - r_0)/\varepsilon)$. These modifications are expected to change the solution by $O(\varepsilon^\alpha)$ and $O(\varepsilon)$ respectively. Solve the resulting linear initial value problem explicitly.

One finds three waves; an outgoing pulse $(P_+(r_0, (t - r + r_0)/\varepsilon), 0)$, an incoming pulse $(0, P_-(r_0, (t + r - r_0)/\varepsilon))$, and an outgoing reflected pulse. Thanks to (1.2), for any $\delta > 0$ and for ε small there will be no reflected wave in the region (1.9) sketched in Figure 1. The proof of the next result which expresses these facts is given in § 3.
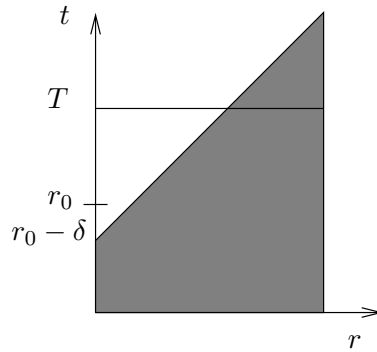

FIGURE 1: Region $\Omega_{\delta,T}^L$ for linear geometric optics without reflection

Theorem 2 *For any $\delta > 0$ and $T \in [r_0, \infty[$ define*

$$(1.9) \qquad \Omega_{\delta,T}^L \ := \ \left\{ (t, r) \ : \ 0 \le r < \infty \, , \ 0 \le t \le \min\{ t - r + r_0 - \delta \, , \ T \} \right\}.$$

Define v^ε from (1.5, 1.6, 1.7, 1.8) and an approximate solution

$$v_{\text{app}}^\varepsilon(t, r) \ := \ \left(P_+ \left(r_0, \frac{t - r + r_0}{\varepsilon} \right) \, , \ P_- \left(r_0, \frac{t + r - r_0}{\varepsilon} \right) \right).$$

*Then if $a > 0$ or the smallness hypothesis of part **ii.** of Theorem 1 are satisfied one has*

$$\left\| v^\varepsilon - v_{app}^\varepsilon \, , \ \varepsilon \partial_{t,r}(v^\varepsilon - v_{app}^\varepsilon) \right\|_{L^\infty(\Omega_{\delta,T}^L)} \ = \ O\!\left(\varepsilon^{\min\{1, p-2\}} \right).$$

1.7. The three key players

To describe the behavior near the focal point and after the focus involves three key functions ψ^L, ψ and Ψ^{out} that we now describe. Introducing the blown up coordinates in the characteristic form (1.6) yields

$$v_\pm^\varepsilon(t,r) \ := \ \psi_\pm^\varepsilon\left(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\right), \qquad \psi_\pm^\varepsilon(\tau,\rho) \ := \ v_\pm^\varepsilon(\varepsilon\tau + r_0, \varepsilon\rho),$$

$$(1.10) \quad (\partial_\tau \pm \partial_\rho)\psi_\pm^\varepsilon \ = \ \rho^{1-p}g(\psi_-^\varepsilon + \psi_+^\varepsilon) \ \text{ for } \rho > 0, \ \ \psi_-^\varepsilon(t,0) + \psi_+^\varepsilon(t,0) \ = \ 0,$$

$$\psi_\mp^\varepsilon\big|_{\tau=-\frac{r_0}{\varepsilon}} \ = \ P_\mp\left(\varepsilon\rho, \rho - \frac{r_0}{\varepsilon}\right) \pm \varepsilon P_1\left(\varepsilon\rho, \rho - \frac{r_0}{\varepsilon}\right).$$

The initial value of ψ_- differs by $O(\varepsilon)$ from $P_-(r_0, \rho - \frac{r_0}{\varepsilon})$. Define

$$\psi^L(\tau,\rho) \ := \ \big(-G(\tau-\rho), G(\tau+\rho)\big), \qquad G(\sigma) \ := \ P_-(r_0, \sigma).$$

Then ψ^L is a linear solution whose incoming component differs by $O(\varepsilon)$ at time $\tau = -r_0/\varepsilon$ from the incoming wave ψ_-^ε. Note that

$$\left\|\psi^L(\tau), \, \partial_\tau \psi^L(\tau)\right\|_{L^\infty([0,\infty[)}$$

is independent of $\tau \in \mathbb{R}$. Note also that hypothesis 1.2 guarantees that ψ^L has compact support in ρ for each τ.

Theorem 3 i. *There is a constant $K_3(a) > 0$ so that if*

$$a > 0 \qquad \text{or} \qquad \left\|\psi^L(\tau), \, \partial_\tau \psi^L(\tau)\right\|_{L^\infty([0,\infty[)} < K_3(a),$$

then there is one and only one uniformly Lipschitzean solution $\psi \in C^1(\mathbb{R} \times \mathbb{R}_+)$ to (1.10) which satisfies the Cauchy condition as $\tau \to -\infty$

$$\left\|\psi(\tau) - \psi^L(\tau), \, \partial_{\tau,\rho}\big(\psi(\tau) - \psi^L(\tau)\big)\right\|_{L^\infty([0,\infty[)} \ = \ o(1).$$

ii. *There is a uniformly Lipschitzean C^1 linear solution*

$$\Psi^{out}(\tau,\rho) \ := \ \big(F(\tau-\rho), \, -F(\tau+\rho)\big)$$

with $|F(\sigma)| = o(1)$ as $|\sigma| \to \infty$ so that as $\tau \to +\infty$,

$$\left\|\psi(\tau) - \Psi^{out}(\tau)\right\|_{L^\infty([0,\infty[)} \ = \ o(1).$$

For any $\gamma > 0$ one has

$$(1.11) \quad \left\|\psi(\tau) - \Psi^{out}(\tau), \, \partial_{\tau,\rho}\big(\psi(\tau) - \Psi^{out}(\tau)\big)\right\|_{L^\infty([\gamma\tau,\infty[)} \ = \ O(1/\tau^{p-2}).$$

In the case of small initial data one has

$$(1.12) \quad \left\|\psi(\tau) - \Psi^{out}(\tau), \, \partial_{\tau,\rho}\big(\psi(\tau) - \Psi^{out}(\tau)\big)\right\|_{L^\infty([0,\infty[)} \ = \ O(1/\tau^{p-2}).$$

Remarks.

1. The two key players introduced here are ψ and Ψ^{out}.

2. The mapping taking G to F is the scattering operator for (1.6).

3. The proofs of parts **i** and **ii** are in § 4.2 and § 4.1 respectively.

In a final section, the scattering operator is studied in the limit of small incoming states. This reveals a phenomenon of pulse broadening. Compactly supported incoming profiles G yield outgoing profiles F which are $O(\sigma^{1-p})$ as $\sigma \to \infty$ (see Corollary 6.2).

1.8. Description at and beyond the focus

The next theorem asserts that $v^\varepsilon(t,r) \approx \psi((t-r_0)/\varepsilon, r/\varepsilon)$ in the left hand region of Figure 2 and $v^\varepsilon(t,r) \approx \Psi^{\text{out}}((t-r_0)/\varepsilon, r/\varepsilon)$ in the right hand region. In the latter region the solution takes the form of an outgoing linearly propagating pulse whose profile is given by the scattering operator applied to the incoming profile. Note that the regions overlap and overlap with the region in Figure 1.
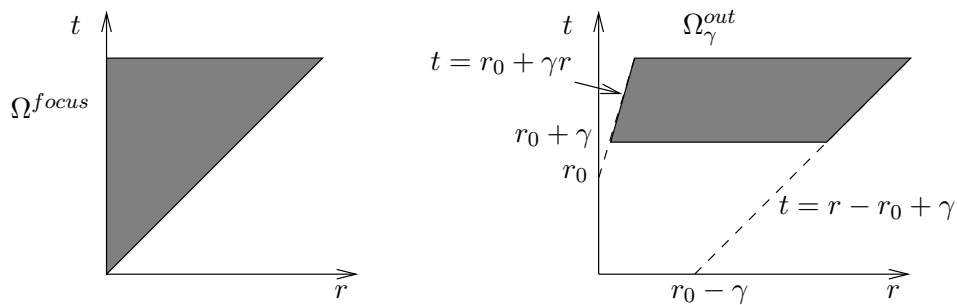

FIGURE 2: Regions Ω^{focus} and Ω^{out} where ψ and Ψ^{out} are good approximations.

Theorem 4 *For any small $\gamma > 0$ define*

$$\Omega^{focus} := \left\{(t,r) \ : \ t \geq r\right\},$$
$$\Omega^{out}_\gamma := \left\{(t,r) \ : \ \min\{r_0 + \gamma, r - r_0 + \gamma\} \leq t \leq r_0 + \gamma r\right\}.$$

i. *In Ω^{focus} one has $v^\varepsilon(t,r) \approx \psi((t-r_0)/\varepsilon, r/\varepsilon)$ in the sense that*

$$\left\| v^\varepsilon(t,r) - \psi\left(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\right) \right\|_{L^\infty(\Omega^{focus})} = O(\varepsilon^{\min(1,p-2)}).$$

In the case of small data one has in addition,

$$\left\| \varepsilon\partial_{t,r} \left(v^\varepsilon(t,r) - \psi\left(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\right) \right) \right\|_{L^\infty(\Omega^{focus})} = O(\varepsilon^{\min(1,p-2)}).$$

ii. *In Ω_γ^{out} one has $v^\varepsilon(t,r) \approx \Psi^{out}((t-r_0)/\varepsilon, r/\varepsilon)$ in the sense that*

$$\left\| v^\varepsilon(t,r) - \Psi^{out}\left(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\right) \, , \, \varepsilon\partial_{t,r}\left(v^\varepsilon(t,r) - \Psi^{out}\left(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\right)\right) \right\|_{L^\infty(\Omega_\gamma^{out})}$$

$$= O\!\left(\varepsilon^{\min(1,p-2)}\right).$$

Part **i** is proved in Proposition 5.1. Part **ii** then follows from **i** and Proposition 4.1.

Using (1.5), we can deduce from the above results the asymptotic behavior of $\partial_t u^\varepsilon$. Indeed,

$$\partial_t u^\varepsilon(t,r) = \frac{v_-^\varepsilon + v_+^\varepsilon}{2r},$$

when r is not so small, the asymptotic behavior of $\partial_t u^\varepsilon$ is given by that of v^ε, which is stated in Th. 2 and 4. When r is small, the boundary condition (1.6) actually prevents $\partial_t u^\varepsilon$ from being singular. In the region $r = O(\varepsilon)$, write

$$(v_-^\varepsilon + v_+^\varepsilon)(t,r) = r\partial_r(v_-^\varepsilon + v_+^\varepsilon)(t,r) + o(r).$$

From (1.6), we also have

$$\partial_r(v_-^\varepsilon + v_+^\varepsilon) = \partial_t(v_-^\varepsilon - v_+^\varepsilon).$$

Therefore, when we know the asymptotic behavior of $\partial_t v^\varepsilon$ near $r = 0$, we can describe $\partial_t u^\varepsilon$ outside the focus and near the focus. This is so when the initial data are small, as stated in Th. 4.

Corollary 1.1 *Assume that the equation is dissipative ($a > 0$) or that the initial data are small. Linear geometric optics is valid (in L^∞) before the focus, and after. Before the focus, $\partial_t u^\varepsilon$ is asymptotic to the superposition of an outgoing wave and an incoming wave, solving the linear wave equation; if $0 \le t < r_0$,*

$$\partial_t u^\varepsilon(t,r) = \frac{1}{2r}\left(\psi_-^L\left(\frac{r+t-r_0}{\varepsilon}\right) + P_+\left(r_0, \frac{r-t-r_0}{\varepsilon}\right) + O\left(\varepsilon^{\min(1,p-2)}\right)\right),$$

and beyond the caustic, $\partial_t u^\varepsilon$ is asymptotic to the superposition of a wave leaving the focus and of the same outgoing wave; if $t > r_0$,

$$\partial_t u^\varepsilon(t,r) = \frac{1}{2r}\left(\Psi_+^{out}\left(\frac{t-r-r_0}{\varepsilon}\right) + P_+\left(r_0, \frac{r-t-r_0}{\varepsilon}\right) + O\left(\varepsilon^{\min(1,p-2)}\right)\right).$$

The matching between these two régimes is described by the nonlinear scattering operator, mapping ψ^L to Ψ^{out}. Moreover, in the small data case, ψ is a "caustic profile": near the focus (for $r \le C\varepsilon$ for any $C > 0$),

$$\partial_t u^\varepsilon(t,r) = \frac{1}{2\varepsilon}\left((\partial_\tau\psi_- - \partial_\tau\psi_+)\left(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\right) + O\left(\varepsilon^{\min(1,p-2)}\right)\right).$$

Notice that since the scattering operator $G \mapsto F$ has some broadening properties (stated in Cor. 6.2), the wave $\partial_t u^\varepsilon$ is broadened when crossing the focal point $(t, r) = (r_0, 0)$.

In Section 2, we address the question of the existence of solutions, and deduce Theorem 2 in Section 3. Section 4 is devoted to scattering theory for (1.10), and we go back to pulses in Section 5, to prove Theorem 4. Finally, in Section 6, we analyze the broadening effect of the scattering operator constructed in Section 4.

A preliminary version of these results was announced in [7].

2. Existence results

In this section we prove Theorem 1. There are two difficulties. The first is that the equations are singular at $r = 0$. This singularity is exactly the result of the focusing effects. One needs to manage this singularity to prove even a local existence result. The second difficulty is to bound the total accumulated effect of the nonlinear terms over long time intervals in order to prove global existence and scattering results.

Consider the mixed problem

$$
\begin{cases}
(\partial_t \pm \partial_r)v_\pm^\varepsilon &= \varepsilon^\alpha r^{1-p} g(v_-^\varepsilon + v_+^\varepsilon), \\
v_-^\varepsilon + v_+^\varepsilon{}_{|r=0} &= 0, \\
v_\mp^\varepsilon{}_{|t=0} &= v_{0\mp}^\varepsilon(r).
\end{cases}
$$

In the critical case, $p - 2 = \alpha$, a scaling gets rid of ε. Introduce $\psi_\pm^\varepsilon$ defined by

$$
\psi_\pm^\varepsilon(t, r) := v_\pm^\varepsilon(\varepsilon t, \varepsilon r), \qquad v_\pm^\varepsilon(t, r) = \psi_\pm^\varepsilon\left(\frac{t}{\varepsilon}, \frac{r}{\varepsilon}\right).
$$

Then ψ^ε solves the mixed problem,

$$
\begin{cases}
(\partial_t \pm \partial_r)\psi_\pm^\varepsilon &= r^{1-p} g(\psi_-^\varepsilon + \psi_+^\varepsilon), \\
\psi_-^\varepsilon(t, 0) + \psi_+^\varepsilon(t, 0) &= 0, \\
\psi_\mp^\varepsilon{}_{|t=0} &= v_{0\mp}^\varepsilon(\varepsilon r).
\end{cases}
$$

Consider the general problem,

$$
(2.1) \quad
\begin{cases}
(\partial_t \pm \partial_r)\psi_\pm &= r^{1-p} g(\psi_- + \psi_+), \\
\psi_-(t, 0) + \psi_+(t, 0) &= 0, \\
\psi_\mp{}_{|t=0} &= \psi_{0\mp}.
\end{cases}
$$

The differential equation is quite singular at $r = 0$. Since $p > 2$, the factor r^{1-p} is not even locally integrable in r. This is handled by the fact that

$\psi_+ + \psi_-$, the argument of g, vanishes at $r = 0$ so one expects that it is $O(r)$. In that case, $g(\psi_+ + \psi_-) = O(r^p)$ which compensates the singular factor. In order to take advantage of this we construct solutions ψ with bounded first derivatives. Note that first derivatives of ψ are second derivatives of u.

2.1. Easy linear estimate

All of our estimates begin with the following linear result.

Lemma 2.1 *Suppose that w and $f = (f_+, f_-)$ are bounded continuous functions on $[0, T] \times [0, \infty[$ satisfying in the sense of distributions*

$$(\partial_t \pm \partial_r)w_\pm = f_\pm , \qquad w_+(t, 0) + w_-(t, 0) = 0 , \quad \text{for} \ \ 0 \le t \le T .$$

Denote by

$$M_\pm(t) \ := \ \|w_\pm(t)\|_{L^\infty([0, \infty[)} .$$

Then for $0 \le t \le T$ one has

$$M_\pm(t) \ \le \ \max\{M_+(0), M_-(0)\} \ + \ \max_{\text{characteristics } \Gamma} \ \int_\Gamma \max\{|f_+|, |f_-|\} \, dt ,$$

where the last maximum is over leftward, rightward, and reflected characteristics connecting points on the initial line $\{t = 0\}$ to points at time t.
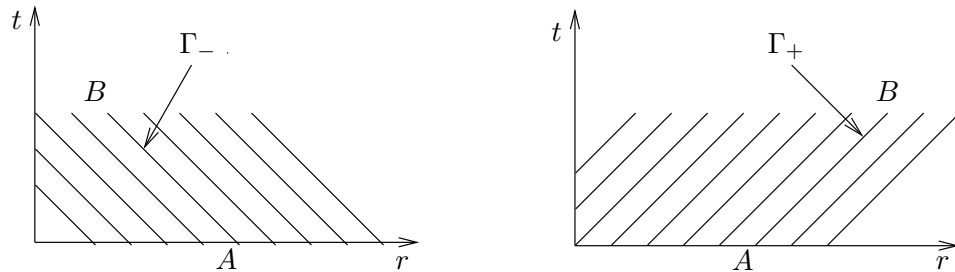

FIGURE 3: Leftward and rightward characteristics.

Proof. **Estimate for w_-**. Referring to the left part of Figure 3, the value of $w_-(B)$ is given by

$$w_-(B) = w_-(A) + \int_{\Gamma_-} f_-(t, r) \, dt ,$$

where Γ_- is the leftward characteristic from A to B.

Taking the supremum over $|w_-(B)|$ yields

$$M_-(t) \leq M_-(0) + \max_{\Gamma_-} \int_{\Gamma_-} |f_-(t,r)| \, dt \,,$$

where the maximum is over all leftward characteristics.

Estimate for w_+ in $t \leq r$. Referring to the right part of Figure 3, one has

$$w_+(B) = w_+(A) + \int_{\Gamma_+} f_+(t,r) \, dt \,,$$

where Γ_+ is the rightward characteristic connecting A to B. Estimating as above yields

$$\max_{r \geq t} |w_+(t,r)| \leq M_+(0) + \max_{\Gamma_+} \int_{\Gamma_+} |f_+(t,r)| \, dt \,,$$

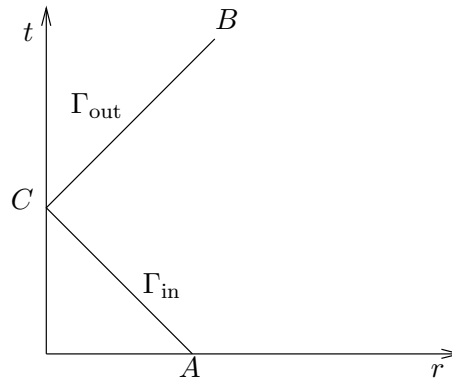where the maximum is over rightward characteristics.

FIGURE 4: Reflected characteristics Γ showing Γ_{in} and Γ_{out}.

Estimate for w_+ in $t \geq r$. Refer to Figure 4. Use three identities

$$w_+(B) = w_+(C) + \int_{\Gamma_{\text{out}}} f_+(t,r) \, dt \,,$$

$$w_+(C) = -w_-(C) \,, \qquad \text{from the boundary condition} \,,$$

and

$$w_-(C) = w_-(A) + \int_{\Gamma_{\text{in}}} f_-(t,r) \, dt \,.$$

Combining yields

$$\max_{r \leq t} |w_+(t,r)| \leq M_-(0) + \max_{\Gamma_{\text{reflected}}} \int \max\{|f_+(t,r)| \,, \, |f_-(t,r)|\} \, dt \,,$$

where the maximum is over reflected characteristics.

Combining the above three estimates proves the Lemma. ∎

2.2. Local estimates for the nonlinear term

To handle the r^{1-p} singularity in the nonlinear term we use the following estimates.

Proposition 2.2 *There is a constant C independent of $0 \leq t \leq T$ and T so that if $\psi \in Lip([0,T] \times [0,\infty[)$ satisfies (2.1), then*

$$(2.2) \qquad \|\psi_+(t,r) + \psi_-(t,r)\|_{L^\infty(\mathbb{R}_+)} \ \leq \ 2r \, \|\partial_t\psi(t)\|_{L^\infty(\mathbb{R}_+)}$$

$$(2.3) \quad \left\|r^{1-p}\,g(\psi_+(t) + \psi_-(t))\right\|_{L^\infty(\mathbb{R}_+)} \ \leq \ C \, \|\partial_t\psi(t)\|^{p-1}_{L^\infty(\mathbb{R}_+)} \, \|\psi(t)\|_{L^\infty(\mathbb{R}_+)} \, ,$$

$$(2.4) \qquad \left\|r^{1-p}\,g'(\psi_+(t) + \psi_-(t))\right\|_{L^\infty(\mathbb{R}_+)} \ \leq \ C \, \|\partial_t\psi(t)\|^{p-1}_{L^\infty(\mathbb{R}_+)} \, ,$$

$$(2.5) \quad \|\partial_r\psi(t)\|_{L^\infty(\mathbb{R}_+)} \leq C \left(\|\partial_t\psi(t)\|^{p-1}_{L^\infty(\mathbb{R}_+)} \, \|\psi(t)\|_{L^\infty(\mathbb{R}_+)} \ + \ \|\psi(t)\|_{L^\infty(\mathbb{R}_+)}\right).$$

Remark. The last estimate shows that in order to prove space time Lipschitz bounds it suffices to bound ψ and $\partial_t\psi$.

Proof. The Fundamental Theorem of Calculus implies that

$$|\psi_+(t,r) + \psi_-(t,r)| \leq r\|\partial_r\big(\psi_+(t) + \psi_-(t)\big)\|_{L^\infty} \, .$$

Using (2.1) one has

$$\partial_r(\psi_+(t) + \psi_-(t)) \ = \ \partial_t(\psi_+(t) - \psi_-(t)) \, .$$

Combining the preceding assertions yields (2.2).

In g, use (2.2) for $|\psi_+ + \psi_-|^{p-1}$ and leave the last factor alone to find

$$|g(\psi_+ + \psi_-)| \ \leq \ C \, r^{p-1} \, \|\partial_t(\psi_+ - \psi_-)\|^{p-1}_{L^\infty} \, |\psi_+ - \psi_-| \, .$$

Estimate (2.3) follows since the r^{p-1} cancels the r^{1-p} on the left hand side of (2.3).

For the function g' which is homogeneous of degree $p-1$, (2.2) yields (2.4) directly. Expressing $\pm\partial_r\psi_\pm = r^{1-p}g - \partial_t\psi_\pm$ and using (2.3) yields (2.5). ∎

2.3. A priori estimate

To prove existence, the solution is constructed as a limit of solutions of approximate equations. The passage to the limit is based on *a priori* estimates. The estimate which yields local existence is contained in the following Proposition.

Proposition 2.3 (*A priori* estimate) *There is a constant $C > 0$ so that if ψ is a uniformly Lipschitzean solution of (2.1) on $[0,T] \times [0,\infty[$ and*

$$M_\pm^0(t) := \|\psi_\pm(t)\|_{L^\infty([0,\infty[)}, \qquad M_\pm^1(t) := \|\partial_t\psi_\pm(t)\|_{L^\infty([0,\infty[)},$$

$$m(t) := \sum_\pm \Big(M_\pm^0(t) + M_\pm^1(t) \Big),$$

then for t satisfying $C t m(0)^{p-1} < 1$ one has,

(2.6)
$$m(t) \le \frac{2\,m(0)}{\big(1 - C\,t\,m(0)^{p-1}\big)^{1/(p-1)}}\,.$$

Proof of Proposition 2.3. The linear estimate from Lemma 2.1 yields

(2.7) $\quad M_\pm^0(t) \le \max\{M_+^0(0), M_-^0(0)\} + \displaystyle\int_0^t \|r^{1-p}\,g(\psi_+(s)+\psi_-(s))\|_{L^\infty([0,\infty[)}\,ds.$

Using (2.3) in (2.7) yields

(2.8) $\qquad M_\pm^0(t) \;\le\; \max\{M_+^0(0), M_-^0(0)\} \;+\; C\displaystyle\int_0^t m(s)^p\,ds\,.$

The time derivatives of ψ satisfy

(2.9) $\qquad (\partial_t \pm \partial_r)\partial_t\psi_\pm = r^{1-p}g'(\psi_- + \psi_+)(\partial_t\psi_+ + \partial_t\psi_-)\,.$

Differentiating the boundary condition at $\{r = 0\}$ with respect to time yields

$$\partial_t\psi_+(t,0) + \partial_t\psi_-(t,0) = 0\,.$$

Lemma 2.1 implies the estimate

(2.10) $\quad M_\pm^1(t) \le \max\{M_+^1(0), M_-^1(0)\} +$

$$+ \int_0^t \big\|r^{1-p}g'(\psi_-(s) + \psi_+(s))(|\partial_t\psi_+(s)| + |\partial_t\psi_-(s)|)\big\|_{L^\infty}\,ds\,.$$

Using (2.4) yields

(2.11) $\qquad M_\pm^1(t) \;\le\; \max\{M_+^1(0), M_-^1(0)\} + C\displaystyle\int_0^t m(s)^p\,ds\,.$

Adding (2.8) and (2.11) yields the main result,

$$m(t) \le 2m(0) + C\int_0^t m(s)^p\,ds\,.$$

This implies that m is no larger than the solution of

$$y(t) = 2m(0) + C\int_0^t y(s)^p ds\,.$$

The function y is the solution of $y' = Cy^p$ with $y(0) = 2m(0)$. The formula for y is the right hand side of (2.6). This completes the proof of (2.6). $\blacksquare$

2.4. Local existence

The basic local existence theorem is the following. Note that in this result, $\partial_t \psi(0)$ is computed from the initial data $\psi_0 = \psi(0)$ using the differential equation (2.1).

Proposition 2.4 (Local existence) *There is a constant $\underline{C} > 0$ so that for all T and initial data $\psi_0 \in C^1([0, \infty))$ satisfying*

(2.12)
$$T \, \|\psi_0, \partial_r \psi_0\|^{p-1}_{L^\infty([0,\infty[)} \; \leq \; \underline{C} \,,$$

and the compatibility conditions

$$\psi_{0+}(0) + \psi_{0-}(0) \; = \; 0 \,, \qquad \text{and} \qquad \partial_r \psi_{0+}(0) - \partial_r \psi_{0-}(0) \; = \; 0 \,,$$

there is a unique solution $\psi \in C^1([0, T] \times [0, \infty[)$ of (2.1). In addition there is a constant K so that for these ψ,

$$\|\psi, \partial_t \psi\|_{L^\infty([0,T]\times[0,\infty[)} \; \leq \; K \, \|\psi_0, \partial_r \psi_0\|_{L^\infty([0,\infty[)} \,.$$

Before embarking on the proof, we recall the origin of the compatibility conditions. For a continuous function satisfying the boundary condition at $r = 0$ one has

$$\psi_{0+}(0) + \psi_{0-}(0) \; = \; \lim_{t \to 0} \left(\psi_+(t, 0) + \psi_-(t, 0) \right) \; = \; 0 \,.$$

For a C^1 function satisfying the boundary condition,

$$r^{1-p} g(\psi_+ + \psi_-) = r^{1-p} O(r)^p = O(r)$$

as $r \to 0$. Thus for such solutions the differential equation (2.1) yields

$$\partial_t \psi_\pm(t, 0) \pm \partial_r \psi_\pm(t, 0) = 0 \,.$$

Using this one derives the second compatibility condition as follows

$$\begin{aligned}
0 \; &= \; \lim_{t \to 0} \; \partial_t \psi_+(t, 0) + \partial_t \psi_-(t, 0) \\
&= \; \lim_{t \to 0} \; -\partial_r \psi_+(t, 0) + \partial_r \psi_-(t, 0) = -\partial_r \psi_{0+}(0) + \partial_r \psi_{0-}(0) \,.
\end{aligned}$$

Proof of uniqueness. Suppose that ψ and $\tilde{\psi}$ are two solutions and denote by $\delta := \psi - \tilde{\psi}$ the difference. Then δ satisfies

$$\begin{aligned}
(\partial_t \pm \partial_r)\delta_\pm &= r^{1-p}\big(g(\psi_+ + \psi_-) - g(\tilde{\psi}_+ + \tilde{\psi}_-)\big) \,, \\
\big(\delta_+ + \delta_-\big)(t, 0) &= 0 \,, \\
\delta(0, r) &= 0 \,.
\end{aligned}$$

Taylor's Theorem with remainder expresses

(2.13) $g(\psi_+ + \psi_-) - g(\tilde{\psi}_+ + \tilde{\psi}_-) = h_{p-1}(\psi_+ + \psi_-, \tilde{\psi}_+ + \tilde{\psi}_-)(\delta_+ + \delta_-),$

where $h_{p-1} \in C^\infty(\mathbb{C}^2 \setminus \{(0,0)\})$ is homogeneous of degree $p - 1 > 0$.

Using (2.2) yields

$$|h_{p-1}(\psi_+ + \psi_-, \tilde{\psi}_+ + \tilde{\psi}_-)| \leq C r^{p-1}.$$

Inserting this in the differential equation for $\delta_\pm$ yields

$$|(\partial_t \pm \partial_r)\delta_\pm| \leq C |\delta_+ + \delta_-|.$$

In addition one has the boundary and initial conditions for δ so Lemma 2.1 yields the integral identity

$$\|\delta(t)\|_{L^\infty([0,\infty[)} \leq C \int_0^t \|\delta(s)\|_{L^\infty([0,\infty[)} \, ds.$$

Gronwall's inequality implies that $\delta = 0$.   ∎

Proof of existence. Define a sequence of functions $k_n(s)$ converging to the function $k_\infty(s) := s|s|^{p-1}$ as follows. First $k_n(s) = k_\infty(s)$ for $-n \leq s \leq n$. Second, k_n is an odd function of s. And finally, for $s \geq n$, the graph of k_n is equal to the graph of the tangent line to k_∞ at $s = n$. Then k_n is C^1 with bounded derivative. k_n converges uniformly on compacts to k_∞. And, for all s

$$|k_n(s)| \leq |k_\infty(s)| \qquad \text{and} \qquad |k_n'(s)| \leq |k_\infty'(s)|.$$

Define

$$g_n(s) = a k_n(s).$$

Define approximate solutions ψ^n as solutions of

$$(\partial_t \pm \partial_r)\psi_\pm^n = (r + 1/n)^{1-p} g_n(\psi_+^n + \psi_-^n).$$
$$\psi_+^n(t, 0) = -\psi_-^n(t, 0),$$
$$\psi^n|_{t=0} = \psi_0.$$

Recall that the initial data belong to $C^1([0, \infty[)$ with $\psi_0, \partial_r \psi_0 \in L^\infty([0, \infty[)$ and satisfy a pair of compatibility conditions. Since $h_n' \in L^\infty(\mathbb{R})$, it is classical that there is a unique global solution $\psi^n \in C^1([0, \infty[\times [0, \infty[)$. For each fixed T and n $\partial_{t,r}\psi^n \in L^\infty([0, T] \times [0, \infty[)$.

Repeat the derivation of the estimates in Proposition 2.3 for the problem defining ψ^n. At each point that one encounters an integral involving $(r + 1/n)^{p-1} g_n$ or $(r + 1/n)^{p-1} g_n'$ the integral can be estimated from above by replacing g_n by g and $r + 1/n$ by r.

In this way one shows that the ψ^n satisfy the estimates of Proposition 2.3 with the same constant C. Precisely, define

$$M^0_\pm(n,t) := \|\psi^n_\pm(t)\|_{L^\infty([0,\infty[)} , \qquad M^1_\pm(n,t) := \|\partial_t\psi^n_\pm(t)\|_{L^\infty([0,\infty[)} ,$$

and

$$m(n,t) := \sum_\pm \Big(M^0_\pm(n,t) + M^1_\pm(n,t) \Big)$$

Then with the same constant C as in Proposition 2.3, one has

$$m(n,t) \ \leq \ \frac{2\,m(n,0)}{\big(1 - C\,t\,m(n,0)^{p-1}\big)^{1/(p-1)}} \,,$$

$$\|\partial_r\psi^n(t)\|_{L^\infty([0,\infty[)} \ \leq \ m(n,t) + C\,m(n,t)^p\,.$$

Note that $M^0_\pm(n,0) = M^0_\pm(0) = \|\psi_{0\pm}\|_{L^\infty}$ is independent of n. For the time derivatives in M^1 one has

$$\partial_t\psi^n_\pm(0) = \mp\partial_r\psi^n_{0\pm} + (r+1/n)^{1-p}\,g_n(\psi^n_{0+} + \psi^n_{0-})\,.$$

The first summand on the right is independent of n but the second depends on n. One gets an n independent bound as follows.

$$|\partial_t\psi^n_\pm(0)| \leq |\mp\partial_r\psi^n_{0\pm}| + |(r+1/n)^{1-p}\,g_n(\psi^n_{0+} + \psi^n_{0-})|$$

$$\leq |\partial_r\psi^n_{0\pm}| + |r^{1-p}g(\psi^n_{0+} + \psi^n_{0-})|$$

$$\leq |\partial_r\psi_{0\pm}| + r^{1-p}\Big(\frac{2r}{r+1}\Big)^p\big(\|\psi_0\|_{L^\infty} + \|\partial_r\psi_0\|_{L^\infty}\big)^p\,,$$

where the last estimate uses Lemma 2.2. Therefore

$$m(n,0) \leq \|\psi_0\,,\,\partial_r\psi_{0\pm}\|_{L^\infty} + C\,\|\psi_0\,,\,\partial_r\psi_0\|^p_{L^\infty}\,.$$

The *a priori* estimate gives us bounds provided

$$C\,T\,m(n,0)^{p-1} < 1\,.$$

This is guaranteed as soon as

$$C\,T\,\big(\|\psi_0\,,\,\partial_r\psi_{0\pm}\|_{L^\infty} + C\,\|\psi_0\,,\,\partial_r\psi_0\|^p_{L^\infty}\big)^{p-1} < 1\,.$$

Choose $\underline{C}$ so that when (2.12) is satisfied it follows that

$$C\,T\,\big(\|\psi_0\,,\,\partial_r\psi_{0\pm}\|_{L^\infty} + C\,\|\psi_0\,,\,\partial_r\psi_0\|^p_{L^\infty}\big)^{p-1} < 1/2\,.$$

If ψ_0 and T satisfy the condition of Proposition 2.4 it follows that

$$m(n,t) \le 2^{p/(p-1)} m(n,0) \,,$$

for $0 \le t \le T$. Together with the companion estimate for $\partial_r \psi^n$, this shows that the family ψ^n is uniformly bounded in $\mathrm{Lip}([0,T] \times [0,\infty[)$.

The Arzela-Ascoli Theorem implies that a subsequence converges uniformly on compacts in $[0,T] \times [0,\infty[$. The limit $\psi \in \mathrm{Lip}([0,T] \times [0,\infty[)$ is a solution of the the initial value problem (2.1).

Since the data are continuously differentiable it is classical that the second compatibility condition guarantees that the solution $\psi \in C^1([0,T] \times [0,\infty[)$. For completeness we recall the principal ideas.

That ψ_- is continuously differentiable does not require compatibility. The right hand side of $(\partial_t - \partial_r)\psi_- = r^{1-p} g(\psi_+ + \psi_-)$ is continuous in $r > 0$ and converges to zero uniformly on $[0,T]$ as $r \to 0$ so is continuous in $r \ge 0$. It suffices to show that $(\partial_t + \partial_r)\psi_-$ is also continuous. For that differentiate the differential equation to obtain

$$
\begin{aligned}
(\partial_t - \partial_r)(\partial_t + \partial_r)\psi_- &= (1-p)r^{-p} g(\psi_+ + \psi_-) \\
&\quad + r^{1-p} g'(\psi_+ + \psi_-)\left((\partial_t + \partial_r)\psi_- + r^{1-p} g(\psi_+ + \psi_-)\right).
\end{aligned}
$$

Denote $B_- := (\partial_t + \partial_r)\psi_-$ and by a_j functions in $L^\infty([0,T] \times [0,\infty[)$ which are continuous on $[0,T] \times]0,\infty[$, that is in $\{r > 0\}$. The above equation is of the form

$$(\partial_t - \partial_r)B_- = a_1 B_- + a_2 \,.$$

Since $B(0,r)$ is a continuous function of r it follows that B is continuous in $[0,T] \times [0,\infty[$.

For ψ_+ it suffices to show that $B_+ := (\partial_t - \partial_r)\psi_+$ is continuous. Using the fact that one already knows that ψ_- is continuously differentiable one has

(2.14) $$(\partial_t + \partial_r)B_+ = a_1 B_+ + a_2 \,.$$

It is known that $B_+(0,r)$ is $C([0,\infty[)$ and it follows that $B_+ \in C^1(\{r \ge t\})$. Similarly, the boundary condition at $r = 0$ implies that $B_+(t,0) = -B_-(t,0) \in C([0,T])$ and it follows that $B_+ \in C(\{r \le t\})$ so $\psi_+ \in C^1(\{r \le t\})$.

To complete the proof one needs to verify that the limits of $\partial_t \psi_+$ from above and below the line $\{r = t\}$ agree. Each of these limits satisfies the same equation (2.14), so the conclusion follows if they have the same initial values at $t = r = 0$. The limit from below has initial value equal to

$$\lim_{r \to 0} \partial_t \psi_+(0,r) = -\partial_r \psi_{0+}(0) \,,$$

where the differential equation is used to compute the limit as in the derivation of the compatibility condition after the statement of (2.4). On the other hand the limit from above has initial value equal to

$$\lim_{t\to 0+} \partial_t \psi_+(t,0) = -\lim_{t\to 0+} \partial_t \psi_-(t,0) = -\partial_t \psi_-(0,0) = -\partial_r \psi_-(0,0) = -\partial_r \psi_{0-}(0).$$

The second compatibility yields the equality and the proof that $\psi \in C^1$ is complete.

Since C^1 solutions are unique, it follows that all subsequences of the ψ^n have subsequences converging to ψ and it follows that the whole sequence ψ^n converges to ψ.

Passing to the limit in our estimates for the derivatives of $\|\psi^n, \partial_{t,r}\psi^n\|_{L^\infty}$ proves the estimate of Proposition 2.4. ∎

2.5. Global existence for small data

To prove global existence for small data one must estimate the accumulated effect of the nonlinear term over long time intervals. This is controlled using the integrability at $r \to \infty$ of the factor r^{1-p}.

Proposition 2.5 (Small data global existence) *There are constants K_1 and $K_1' > 0$ so that for all initial data $\psi_0 \in C^1([0,\infty))$ satisfying*

$$\|\psi_0, \partial_r \psi_0\|_{L^\infty([0,\infty[)} \leq K_1,$$

and the compatibility conditions

$$\psi_{0+}(0) + \psi_{0-}(0) = 0, \qquad \text{and} \qquad \partial_r \psi_{0+}(0) - \partial_r \psi_{0-}(0) = 0,$$

there is a unique solution $\psi \in C^1([-\infty,\infty] \times [0,\infty[)$ of (2.1). In addition,

$$\|\psi, \partial_t \psi\|_{L^\infty([-\infty,\infty]\times[0,\infty[)} \leq K_1' \|\psi_0, \partial_r \psi_0\|_{L^\infty([0,\infty[)}.$$

To prove this one needs *a priori* estimates which show that the Lipschitz norm does not grow without bound. The proof that the solution is C^1 uses the compatibility conditions as in the local existence proof.

The estimate is like that of the previous section in its handling of the singularity at $r = 0$. The difference is for large times. The key is to use the fact that $r^{1-p}g(\psi_+ + \psi_-)$ decays like $1/r^{p-1}$ for large r so is absolutely integrable along infinitely long characteristics.

According to the linear estimate Lemma 2.1, the quantity

$$(2.15) \qquad \max_{\text{characteristics } \Gamma} \int_\Gamma r^{1-p} |g(\psi_- + \psi_+)| \, dt$$

estimates from above the accumulated effect of the nonlinear term on the size of the solution.

Similarly

$$(2.16) \qquad \max_{\text{characteristics } \Gamma} \int_\Gamma r^{1-p} |g'(\psi_- + \psi_+)| \max\{|\partial_t \psi_+|, |\partial_t \psi_-|\} \, dt$$

bounds the accumulated effect on the time derivatives.

Lemma 2.6 (Total interaction estimate) *Suppose that* $-\infty < T_1 < T_2 < \infty$ *and* ψ *is a uniformly Lipschitzean,* C^1 *solution of (2.1) on* $[T_1, T_2] \times [0, \infty[$. *With* $M_\pm^j(t)$ *and* $m(t)$ *as in Proposition 2.3 there is a constant* C *independent of* ψ, T_1, T_2, *so that both (2.15) and (2.16) with characteristics* Γ *in* $T_1 \leq t \leq T_2$ *are bounded above by*

$$(2.17) \qquad C \sup_{T_1 \leq t \leq T_2} m(t)^p.$$

Proof. The bound (2.2) is too rough for large r where the bound

$$|\psi_+(t, r) + \psi_-(t, r)| \leq 2\|\psi(t)\|_{L^\infty([0,\infty[)}$$

is preferable. Combining the two yields

$$(2.18) \qquad |\psi_+(t, r) + \psi_-(t, r)| \leq \frac{4r}{1+r} \|\psi(t), \partial_t \psi(t)\|_{L^\infty([0,\infty[)}.$$

Inserting this in (2.15) yields the upper bound

$$C \sup_t m(t)^p \max_{\text{characteristics } \Gamma} \int_\Gamma r^{1-p} \left(\frac{r}{r+1}\right)^p dt.$$

The integrals are no larger than

$$2 \int_0^\infty \frac{r}{(1+r)^p} \, dr$$

which is a finite constant. This yields the bound (2.17) for (2.15).

In the same way, using (2.18), (2.16) is bounded by

$$C \sup_t m(t)^p \int_\Gamma r^{1-p} \left(\frac{r}{1+r}\right)^{p-1} dt.$$

The integral is no larger than

$$2 \int_0^\infty \frac{1}{(1+r)^{p-1}} \, dr$$

which is a finite constant since $p > 2$. $\blacksquare$

Proposition 2.7 (Small data *a priori* estimate) *Suppose that* $-\infty < T_1 \leq 0 < T_2 < \infty$ *and* ψ *is a uniformly Lipschitzean, C^1 solution of (2.1) on $[T_1, T_2] \times [0, \infty[$. With $m(t)$ as in Proposition 2.3 there is a constant C independent of ψ, T_1, T_2, so that*

$$(2.19) \qquad \max_{T_1 \leq t \leq T_2} m(t) \ \leq \ m(0) + C \max_{T_1 \leq t \leq T_2} m(t)^p \,.$$

Proof. The linear estimate applied to ψ shows that

$$M_\pm^0(t) \leq \max\{M_+^0(0)\,,\ M_-^0(0)\} + \max_{\text{characteristics } \Gamma} \int_\Gamma r^{1-p}|g(\psi_- + \psi_+)| \ dt \,.$$

The linear estimate applied to $\partial_t \psi$ yields

$$M_\pm^1(t) \leq \max\{M_+^0(0)\,,\ M_-^0(0)\}+$$
$$\max_{\text{characteristics } \Gamma} \int_\Gamma r^{1-p}|g'(\psi_- + \psi_+)| \max\{|\partial_t\psi_+|\,,\ |\partial_t\psi_-|\} \ dt \,.$$

Adding these two estimates and bounding the integrals using Lemma 2.6 yields the desired estimate. ∎

Proof of Proposition 2.5. With C as in estimate (2.19), choose $\delta > 0$ so that $C\,(2\mu)^p < \mu$ for all $\mu \leq \delta$. Consider initial data which satisfies $m(0) < \delta$.

We show that such solutions are global and that $m(t) \leq 2m(0)$ for all $-\infty < t < \infty$. The proof is presented for positive t. The case of $t < 0$ is essentially the same.

The local existence theorem implies that there is a unique maximal solution $\psi \in C^1([0, T^*[\times[0, \infty[)$ and if $T^* < \infty$, then

$$\liminf_{t \to T^*} m(t) \ = \ \infty \,.$$

We show that $m(t) < 2m(0)$ for all $0 \leq t < T^*$. In particular this implies that $T^* = \infty$.

If it were not true that $\max m(t) \ < 2m(0)$, there would exist

$$0 < T = \inf\{t \in [0, T^*[:\ m(t) = 2m(0)\} \,.$$

Proposition 2.7 implies that

$$2m(0) = m(T) \leq \max_{0 \leq t \leq T} m(t) \ \leq \ m(0) + C \max_{0 \leq t \leq T} m(t)^p$$
$$\leq m(0) + C\,(2m(0))^p \ < \ m(0) + m(0) = 2m(0) \,.$$

This contradiction proves that

$$T^* = \infty \qquad \text{and that} \qquad m(t) < 2m(0) \quad \text{for all } t > 0. \qquad ∎$$

2.6. Global existence in the monotone dissipative case

When $a > 0$, the solution is global in time even for large initial data. We restrict our calculations to the case of real solutions. Complex solutions require only standard modifications.

The nonlinearity is given by $g(y) = -a2^{-p}|y|^{p-1}y$. If $a > 0$, the classic energy estimate asserts that $\sum_\pm \|\psi_\pm(t)\|_{L^2}^2$ is a non-increasing function of time. The key is to derive analogous L^∞ bounds.

Proposition 2.8 (Forward solvability in the dissipative case) *If $a > 0$, real solution ψ to problem (2.1) exists for all $t > 0$. In addition for any $1 \le q < \infty$,*

$$(2.20) \qquad \Big(\sum_\pm \|\psi_\pm(t)\|_{L^q([0,\infty[)}^q\Big)^{1/q} \qquad \text{and} \qquad \Big(\sum_\pm \|\partial_t\psi_\pm(t)\|_{L^q([0,\infty[)}^q\Big)^{1/q}$$

are nonincreasing functions of $t \ge 0$. If $q = \infty$, then for $t \ge 0$,

$$\max_\pm \|\psi_\pm(t)\|_{L^\infty([0,\infty[)} \le \max_\pm \|\psi_\pm(0)\|_{L^\infty([0,\infty[)}$$

$$\max_\pm \|\partial_t\psi_\pm(t)\|_{L^\infty([0,\infty[)} \le \max_\pm \|\partial_t\psi_\pm(0)\|_{L^\infty([0,\infty[)}.$$

If ψ and $\tilde\psi$ are two real solutions then for $1 \le q < \infty$,

$$(2.21) \qquad \Big(\sum_\pm \|\psi_\pm(t) - \tilde\psi_\pm(t)\|_{L^q([0,\infty[)}^q\Big)^{1/q}$$

is a nonincreasing function of $t \ge 0$. Finally, for $t \ge 0$,

$$\max_\pm \|\psi_\pm(t) - \tilde\psi_\pm(t)\|_{L^\infty([0,\infty[)} \le \max_\pm \|\psi_\pm(0) - \tilde\psi_\pm(0)\|_{L^\infty([0,\infty[)}.$$

Proof. Given our local existence theorem it suffices to prove (2.20) and (2.21).

The equations are

$$(2.22) \qquad (\partial_t - \partial_r)\psi_- = br^{1-p}|\psi_- + \psi_+|^{p-1}(\psi_- + \psi_+),$$
$$(2.23) \qquad (\partial_t + \partial_r)\psi_+ = br^{1-p}|\psi_- + \psi_+|^{p-1}(\psi_- + \psi_+),$$

with $b = -a2^{-p} < 0$.

For $q \ge 1$ define

$$g_{q-1}(s) := \frac{d}{ds}|s|^q = q|s|^{q-1}\operatorname{sgn} s.$$

Then g_{q-1} is a nonincreasing odd function of s which is homogeneous of degree $q - 1$.

Multiply (2.22) by $g_{q-1}(\psi_-)$, and (2.23) by $g_{q-1}(\psi_+)$. Summing yields

$$\partial_t(|\psi_-|^q + |\psi_+|^q) + \partial_r(|\psi_+|^q - |\psi_-|^q) =$$
$$= b r^{1-p}\left(g_{q-1}(\psi_-) + g_{q-1}(\psi_+)\right) g_p(\psi_- + \psi_+)/(p+1)\,.$$

The signs of both $g_{q-1}(\psi_-) + g_{q-1}(\psi_+)$ and $g_p(\psi_- + \psi_+)$ are equal to the sign of the larger of $\psi_\pm$. Therefore

$$\partial_t(|\psi_-|^q + |\psi_+|^q) \; - \; \partial_r(|\psi_-|^q - |\psi_+|^q) \; \le \; 0.$$

For any fixed $T > 0$ and $R > T$, integrate over the truncated light cone $\{0 \le t \le T\} \cap \{0 \le r \le R-t\}$. Since $|\psi_+|^q = |\psi_-|^q$ when $r = 0$, integration by parts yields

$$\int_0^{R-T} |\psi_-(T,r)|^q + |\psi_+(T,r)|^q \; dr \le$$

$$\le \int_0^R |\psi_-(0,r)|^q + |\psi_+(0,r)|^q \; dr - \int_{r=R-t} 2|\psi_+|^q dt\,.$$

Note that the boundary contribution at $r = 0$ vanishes thanks to the boundary condition. Note also that the boundary term at $r = R - t$ is nonnegative so,

$$\int_0^{R-T} |\psi_-(T,r)|^q + |\psi_+(T,r)|^q \; dr \le \int_0^R |\psi_-(0,r)|^q + |\psi_+(0,r)|^q \; dr\,.$$

Letting $R \to \infty$ proves the $\|\psi\|_{L^q}$ estimate of the Proposition for $q < \infty$.

For the sup norm estimate, first fix R and use

$$\|f\|_{L^\infty} = \lim_{q \to \infty} \|f\|_{L^q},$$

to find

$$\max_\pm \|\psi_\pm(T)\|_{L^\infty([0,R-T])} \; \le \; \max_\pm \|\psi_\pm(0)\|_{L^\infty([0,R])}\,.$$

Letting $R \to \infty$ proves the $\|\psi\|_{L^\infty}$ estimate.

The proof of (2.21) is similar. Multiply

$$\left(\partial_t \pm \partial_r\right)\left(\psi_\pm - \tilde\psi_\pm\right) \; = \; b\, r^{1-p}\left(g_p(\psi_+ - \psi_-) - g_p(\tilde\psi_+ - \tilde\psi_-)\right)/(p+1)$$

by $g_{q-1}(\psi_\pm - \tilde\psi_\pm)$ to find

$$\partial_t(|\psi_+ - \tilde\psi_+|^q + |\psi_- - \tilde\psi_-|^q) + \partial_r(|\psi_+ - \tilde\psi_+|^q - |\psi_- - \tilde\psi_-|^q) =$$
$$= \frac{b r^{1-p}}{p+1}\left(g_{q-1}(\psi_+ - \tilde\psi_+) + g_{q-1}(\psi_- - \tilde\psi_-)\right)\left(g_p(\psi_- + \psi_+) - g_p(\tilde\psi_- + \tilde\psi_+)\right)\,.$$

The signs of each of the last two factors on the right are both equal to the sign of the larger of $\psi_\pm - \tilde\psi_\pm$. Therefore the right hand side is nonpositive. The proof of (2.21) is then completed as above.

To prove the estimates for $\partial_t \psi$, differentiate (2.22) and (2.23) with respect to time to find

$$(2.24) \qquad (\partial_t - \partial_r)\partial_t \psi_- = bpr^{1-p}|\psi_- + \psi_+|^{p-1}\partial_t(\psi_- + \psi_+),$$

$$(2.25) \qquad (\partial_t + \partial_r)\partial_t \psi_+ = bpr^{1-p}|\psi_- + \psi_+|^{p-1}\partial_t(\psi_- + \psi_+).$$

Take $g_{q-1}(\partial_t \psi_-)$ and $g_{q-1}(\partial_t \psi_+)$ as multipliers and add to find that

$$\partial_t(\,|\partial_t\psi_+|^q + |\partial_t\psi_-|^q) + \partial_r\,(|\partial_t\psi_+|^q - |\partial_t\psi_-|^q) =$$
$$= bpq\,r^{1-p}\,|\psi_- + \psi_+|^{p-1}\left(|\partial_t\psi_+|^{q-1} + |\partial_t\psi_-|^{q-1}\right)(\partial_t\psi_+ + \partial_t\psi_-)\ \leq\ 0\,.$$

Integrating by parts as above the boundary contributions at $r = 0$ and $r = R - t$ are respectively zero and nonnegative and one finds

$$\int_0^{R-T} |\partial_t\psi_+(T,r)|^q + |\partial_t\psi_-(T,r)|^q\ dr \leq \int_0^R |\partial_t\psi_+(0,r)|^q + |\partial_t\psi_-(0,r)|^q\ dr.$$

The $\|\partial_t\psi\|_{L^q}$ estimates follow, and the $\|\partial_t\psi\|_{L^\infty}$ as well. ∎

Remark. For later use note that the case $q = 2$ yields the identity

$$\int_0^\infty |\psi_+(T,r)|^2 + |\psi_-(T,r)|^2\ dr + \frac{a}{2^{p-1}}\int_0^T\int_0^\infty \frac{|\psi_+(t,r) + \psi_-(t,r)|^{p+1}}{r^{p-1}}dt\,dr$$
$$= \int_0^\infty |\psi_+(0,r)|^2 + |\psi_-(0,r)|^2\ dr\,.$$

3. Before the focus

This short section proves Theorem 2. The most important observation is that hypothesis (1.2) implies that the initial data for both v^ε and for $v^\varepsilon_{\text{app}}$ are supported in the set $\{r \geq r_0 - C\varepsilon\}$. Then the finite speed of propagation implies that for ε small

$$\left(\text{supp}\, v^\varepsilon \,\cup\, \text{supp}\, v^\varepsilon_{\text{app}}\right) \cap \Omega^L_{\delta,T}\ \subset\ \{r \geq \delta/2\}\,.$$

Thus the singular factor r^{1-p} in the nonlinear term is uniformly bounded on the support of $g(v^\varepsilon_+ + v^\varepsilon_-)$. Since the family v^ε as well as its $\varepsilon\partial_t$-derivatives are uniformly bounded it follows that

$$\left|\varepsilon^{p-2}g(v^\varepsilon_+ + v^\varepsilon_-)\right| + \left|\varepsilon\partial_t\,\varepsilon^{p-2}g(v^\varepsilon_+ + v^\varepsilon_-)\right|\ \leq\ \frac{C(\delta)\varepsilon^{p-2}}{1 + r^{p-1}}\,.$$

Therefore if w^ε denotes either $v^\varepsilon - v^\varepsilon_{\text{app}}$ or $\varepsilon\partial_t(v^\varepsilon - v^\varepsilon_{\text{app}})$ one has

$$\left|(\partial_t \pm \partial_r)w^\varepsilon_\pm\right|\ \leq\ \frac{C(\delta)\varepsilon^{p-2}}{1 + r^{p-1}}\,.$$

The initial data vanish identically near the origin and satisfy

$$\left| w_{\pm}^{\varepsilon}(0, r) \right| = O(\varepsilon) \,.$$

Lemma 2.1 implies the estimates for $v^{\varepsilon} - v_{\mathrm{app}}^{\varepsilon}$ and $\varepsilon \partial_t (v^{\varepsilon} - v_{\mathrm{app}}^{\varepsilon})$ in Theorem 2. The estimates for $\varepsilon \partial_r (v^{\varepsilon} - v_{\mathrm{app}}^{\varepsilon})$ follow from the equations satisfied by v^{ε} and $v_{\mathrm{app}}^{\varepsilon}$. ∎

4. Large time asymptotics and scattering

The passage of pulses through the focal point is described by a scattering operator for (2.1). In this section, we establish the needed scattering theory by studying the asymptotic behavior of solutions as $t \to +\infty$, and solving an initial value problem with initial data at $t = -\infty$. The parameter ε does not appear.

4.1. Asymptotics as $t \to +\infty$

First consider the linear problem. For initial data which tend to zero as $r \to \infty$ the exact solution has the form

$$\psi = \left(\psi_+, \psi_- \right) = \left(F(t - r), -F(t + r) \right) .$$

with $F(\sigma)$ tending to zero as $|\sigma| \to \infty$. On the outgoing characteristic $\Gamma_+(\sigma) := \{t - r = \sigma\}$, one has $\psi_+ = F(\sigma)$, and $\psi_- \to 0$ as $t \to \infty$.

Suppose next that ψ is a uniformly Lipschitzean, C^1 solution of (2.1). We show that for fixed σ, the values of ψ_+ change along the $\Gamma_+(\sigma)$ to tend to a limit as $t \to +\infty$ and that limit defines a linear solution to which ψ tends as t and r tend to infinity.

To find the limiting behavior of ψ_+ along $\Gamma_+(\sigma)$, write

$$\psi_+(t, r) = \psi_+(t - r, 0) + \int_{\Gamma_+} r^{1-p} g(\psi_+ + \psi_-) \, dt \,,$$

where Γ_+ is the forward characteristic connecting $(t - r, 0)$ to (t, r). Since $p > 2$ and ψ is bounded, the integral is absolutely convergent so passing to the limit $t \to +\infty$ yields

$$F(\sigma) = \lim_{t \to +\infty} \psi_+(\sigma + t, t) = \psi_+(\sigma, 0) + \int_{\Gamma_+(\sigma)} r^{1-p} g(\psi_+ + \psi_-) \, dt \,.$$

Differentiating shows that F defined by this formula is a continuously differentiable uniformly Lipschitzean function on $]-\infty, \infty[$.

One can define F just as easily when ψ is only defined in $r \geq R$ in which case

$$(4.1) \quad F(\sigma) = \lim_{t \to \infty} \psi_+(\sigma+t, t) = \psi_+(\sigma+R, R) + \int_{\Gamma_+(\sigma) \cap \{r \geq R\}} r^{1-p} g(\psi_+ + \psi_-) \, dt.$$

Define Ψ^{out} by

$$(4.2) \quad \Psi^{\text{out}}(t, r) = \big(F(t-r), -F(t+r)\big).$$

Then Ψ^{out} is a linear solution whose positive component has the same asymptotics along $\Gamma(\sigma)$ as does ψ_+, that is

$$\lim_{t \to \infty} \left.\big(\psi_+ - \Psi_+^{\text{out}}\big)\right|_{\Gamma_+(\sigma)} = 0.$$

Proposition 4.1 ($r \geq h(t) \to \infty$ asymptotics) *Suppose that for some $R > 0$, $\psi \in C^1$ is a uniformly Lipschitzean solution on $\mathbb{R}_t \times [R, \infty[$ and satisfies*

$$\lim_{r \to \infty} |\psi(0, r)| + |\partial_r \psi(0, r)| = 0.$$

i. *Define F and Ψ^{out} by (4.1) and (4.2) respectively. Then F is uniformly Lipschitzean on $]-\infty, \infty[$ with*

$$\lim_{\sigma \to -\infty} \big(|F(\sigma)| + |F'(\sigma)|\big) = 0.$$

ii. *For any increasing function function $h(t)$ satisfying $\lim_{t \to +\infty} h(t) = +\infty$ one has*

$$\lim_{T \to +\infty} \left\| \psi_-, \partial_{t,r} \psi_- \right\|_{L^\infty(\{t \geq T\} \cap \{r \geq h(t)\})} = 0,$$

$$\lim_{T \to +\infty} \left\| (\psi_+ - \Psi_+^{\text{out}}), \partial_{t,r}(\psi_+ - \Psi_+^{\text{out}}) \right\|_{L^\infty(\{t \geq T\} \cap \{r \geq h(t)\})} = 0.$$

iii. *If $h(t) = \gamma t$ with $\gamma > 0$ and the initial data $\psi|_{t=0}$ vanish for large r, then as $T \to \infty$*

$$\left\| (\psi - \Psi^{\text{out}}), \partial_{t,r}(\psi - \Psi^{\text{out}}) \right\|_{L^\infty(\{t \geq T\} \cap \{r \geq h(t)\})} = O\left(\frac{1}{T^{p-2}}\right).$$

Remarks.

1. This result is weak because convergence is only proved in $\{r \geq h(t)\}$. On the other hand, the proof only uses the differential equation in $\{r \geq h(t)\}$ and there is no smallness assumption.

2. Part **iii** proves (1.11) of Theorem 3 and part **ii** of Theorem 4.

Proof of Proposition 4.1. For $\sigma < -R$ one has

$$F(\sigma) \;=\; \psi_+(0, -\sigma) \;+\; \int_{\Gamma_+(\sigma) \cap \{t \geq 0\}} r^{1-p} \, g(\psi_+ + \psi_-) \, dt \,.$$

Since the integration lies in $r \geq |\sigma|$ both of the terms on the right are $o(1)$. A similar argument applies to F'.

To prove **ii**, let Γ_+ denote the infinite forward $\partial_t + \partial_r$ characteristic with foot at (t, r). Integrating the differential equation satisfied by ψ_+ along $\Gamma_+(\sigma)$ starting at t, r yields the integral identity

$$(4.3) \qquad \psi_+(t, r) \;=\; \Psi_+^{\mathrm{out}}(t, r) - \int_{\Gamma_+} r^{1-p} \, g(\psi_+ + \psi_-) \, dt \,.$$

Similarly denote by Γ_- the backward $\partial_t - \partial_r$ characteristic connecting (t, r) to $(0, t+r)$. Integrating the equation for ψ_- on Γ_- yields

$$\psi_-(t, r) \;=\; \psi_-(0, t + r) + \int_{\Gamma_-} r^{1-p} \, g(\psi_+ + \psi_-) \, dt \,.$$

Differentiating with respect to time yields

$$(4.4) \qquad \partial_t \psi_+(t, r) \;=\; \partial_t \Psi_+^{\mathrm{out}}(t, r) - \int_{\Gamma_+} r^{1-p} \, g'(\psi_+ + \psi_-) \partial_t(\psi_+ + \psi_-) \, dt$$
$$+ \, r^{1-p} g(\psi_+ + \psi_-)(t, r),$$

$$\partial_t \psi_- \;=\; \partial_r \psi_-(0, t + r) + \int_{\Gamma_-} r^{1-p} \, g'(\psi_+ + \psi_-) \partial_t(\psi_+ + \psi_-) \, dt$$
$$+ \, r^{1-p} g(\psi_+ + \psi_-)(t, r) - (r + t)^{1-p} g(\psi_+ + \psi_-)(0, r + t) \,.$$

For t, r in the region $t \geq T$ and $r \geq h(t)$ one has $r \geq h(T)$ on the entire characteristic Γ_-. This together with the fact that $\psi(0, t+r)$ and $\partial_r \psi(0, t+r)$ tend to zero suffice to show that

$$\lim_{T \to \infty} \big\| \psi_- \, , \, \partial_t \psi_- \big\|_{L^\infty(\{t \geq T\} \cap \{r \geq h(t)\})} \;=\; 0 \,,$$

which proves the incoming component of the desired conclusion.

Similarly, one has $r \geq h(t)$ in the integrals in (4.3) and (4.4) so these integrals tend to zero which suffices to prove that for $j = 0, 1$,

$$\lim_{T \to \infty} \big\| \partial_t^j (\psi_+ - \Psi_+^{\mathrm{out}}) \big\|_{L^\infty(\{t \geq T\} \cap \{r \geq h(t)\})} \;=\; 0 \,.$$

That the r derivative tends to zero follows using the differential equation.

To prove **iii** note that in the proof of **ii** the contributions from $t = 0$ vanish for T large because of the compact support and that the integrals are bounded above by

$$C \int_{\gamma t}^{\infty} \frac{1}{r^{p-1}} \, dr = \frac{C'}{t^{p-2}} \, .$$

Finally, the terms $r^{1-p} g(\psi_+ + \psi_-)(t, r)$ are decaying at least like T^{1-p}, which is even better. This completes the proof of Proposition 4.1   ∎

This proposition tells us nothing about the behavior of solutions for fixed r as t tends to infinity, nor of the behavior of $F(\sigma)$ as $\sigma \to +\infty$. The values of F for large σ correspond to waves which appear at large times. If F did not tend to zero, then such waves would appear for infinitely large times. The outgoing wave would correspond to a solution of the linear wave equation which did not tend to zero as $t \to +\infty$ with r fixed. We are able to rule this behavior out in two distinct ways. When the real part of a is nonpositive we can use the fact that the total energy is bounded above. For general a, we are able to prove a sharp quantitative decay estimate in time under a smallness hypothesis on the initial data.

Proposition 4.2 (Decay of Ψ^{out}) *Suppose that $\psi \in \text{Lip}(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+})$ is a uniformly Lipschitzean solution of (2.1).*
i. *If the real part of a is nonnegative and the initial data have finite energy in the sense that $\int_0^{\infty} |\psi(0, r)|^2 \, dr < \infty$ then*

$$\int_{-\infty}^{\infty} |F(\sigma)|^2 \, d\sigma \; < \; \infty \, .$$

Since F' is bounded, this implies $\lim_{|\sigma| \to \infty} F(\sigma) = 0$.
ii. *If a is a nonnegative real, $a \geq 0$, and the initial data satisfy*

$$\int_0^{\infty} |\psi_t(0, r)|^2 \, dr < \infty,$$

then

$$\int_{-\infty}^{\infty} |F'(\sigma)|^2 \, d\sigma \; < \; \infty \, .$$

Proof. To prove the first assertion, use the energy law

$$\partial_t(|\psi_+|^2 + |\psi_-|^2) + \partial_r(-|\psi_+|^2 + |\psi_-|^2) \; = \; -2(\text{Re}\, a)|\psi_+ + \psi_-|^{p+1} \, .$$

Since the real part of a is nonnegative, the right hand side is nonpositive. Integrating over the set $\{0 \leq t \leq T\} \cap \{0 \leq r \leq R - t\}$ shows that for any $R > t > 0$,

$$\int_0^{R-t} |\psi_+(t, r)|^2 + |\psi_-(t, r)|^2 \, dr \; \leq \; \int_0^{R} |\psi_+(0, r)|^2 + |\psi_-(0, r)|^2 \, dr \, .$$

Passing to the limit $R \to \infty$ using the Monotone Convergence Theorem yields

$$\int_0^\infty |\psi_+(t,r)|^2 + |\psi_-(t,r)|^2 \ dr \ \leq \ \int_0^\infty |\psi_+(0,r)|^2 + |\psi_-(0,r)|^2 \ dr \,.$$

On the other hand, $\psi_+(t, t - \sigma) \to F(\sigma)$ as $t \to \infty$ so the Dominated Convergence Theorem implies that as $t \to \infty$

$$\int_{t-\beta}^{t-\alpha} |\psi_+(t,r)|^2 \ dr \to \int_\alpha^\beta |F(\sigma)|^2 \ d\sigma \,.$$

Combining the last two estimates and letting $\alpha \to -\infty$ and $\beta \to +\infty$ implies that

$$\int_{-\infty}^\infty |F(\sigma)|^2 \ d\sigma \ \leq \ \int_0^\infty |\psi_+(0,r)|^2 + |\psi_-(0,r)|^2 \ dr \,,$$

proving **i**.

To prove **ii**, use the fact that

$$\mathrm{Re}\left(a \left(\partial_t \overline{\psi}_+ + \partial_t \overline{\psi}_- \right) \partial_t \big(g(\psi_+ + \psi_-) \big) \right) \ \geq \ 0$$

to find the derived dissipation inequality

$$\partial_t(|\partial_t \psi_+|^2 + |\partial_t \psi_-|^2) + \partial_r(-|\partial_t \psi_+|^2 + |\partial_t \psi_-|^2) \ \leq \ 0 \,.$$

Integrating as above this implies that

$$\int_0^\infty |\partial_t \psi_+(t,r)|^2 + |\partial_t \psi_-(t,r)|^2 \ dr \ \leq \ \int_0^\infty |\partial_t \psi_+(0,r)|^2 + |\partial_t \psi_-(0,r)|^2 \ dr \,.$$

The definition of F implies that as $t \to \infty$

$$\partial_t \psi_+(t + \sigma, t) \to F'(\sigma) \,,$$

so

$$\int_{t-\beta}^{t-\alpha} |\partial_t \psi_+(t,r)|^2 \ dr \to \int_\alpha^\beta |F'(\sigma)|^2 \ d\sigma \,.$$

Combined with the preceding estimate, this yields

$$\int_{-\infty}^\infty |F'(\sigma)|^2 \ d\sigma \ \leq \ \int_0^\infty |\partial_t \psi_+(0,r)|^2 + |\partial_t \psi_-(0,r)|^2 \ dr \ < \ \infty \,,$$

proving **ii**. ∎

The previous result describes the behavior of the outgoing wave, equivalently the behavior of the linear solution to which the solution is asymptotic as $t \to +\infty$. The next result describes the behavior of the solution ψ itself.

Proposition 4.3 (Decay of ψ)

i. *For any a there is a constant $K(a) > 0$ so that if*

$$\|\psi_t\|_{L^\infty([0,\infty[\times[0,\infty[)} \leq K(a), \quad \text{and} \quad \sup_{r \geq 0} r^{p-1}\big(|\psi(0,r)| + |\psi_t(0,r)|\big) < \infty,$$

then

$$\sup_{t \geq 0, \ r \geq 0} (1 + |t \mp r|^{p-1})\big(|\psi_\pm(t,r)| + |\partial_t \psi_\pm(t,r)|\big) < \infty,$$

and therefore

$$\sup_{-\infty < \sigma < \infty} \sigma^{p-1}\big(|F(\sigma)| + |F'(\sigma)|\big) < \infty.$$

ii. *Suppose that $a \geq 0$ is a nonnegative real number and*

$$\psi \in \mathrm{Lip}([0,\infty[\times[0,\infty[)$$

is a uniformly Lipschitzean solution of (2.1) with $\lim_{r \to \infty} |\psi(0,r)| = 0$ and $\psi(0,\cdot) \in L^2([0,\infty[)$. Then for any $R > 0$

$$\lim_{t \to \infty} \sup_{0 \leq r \leq R} |\psi(t,r)| = 0.$$

Proof. To prove **i**, introduce $\langle s \rangle := (1 + s^2)^{1/2}$. Then $(\partial_t + \partial_r)\langle t - r \rangle = 0$ so

$$(\partial_t + \partial_r)\big(\langle t - r \rangle^{p-1}\psi_+\big) = \langle t - r \rangle^{p-1} r^{1-p} g(\psi_+ + \psi_-).$$

Use (2.18) to find

$$\big|(\partial_t + \partial_r)\langle t - r \rangle^{p-1}\psi_+\big| \leq \frac{C \, \|\psi_t(t)\|_{L^\infty}^{p-1} \, \langle t - r \rangle^{p-1}}{\langle r \rangle^{p-1}} \left(|\psi_+| + |\psi_-|\right).$$

Similarly,

$$\big|(\partial_t - \partial_r)\langle t + r \rangle^{p-1}\psi_-\big| \leq \frac{C \, \|\psi_t(t)\|_{L^\infty}^{p-1} \, \langle t + r \rangle^{p-1}}{\langle r \rangle^{p-1}} \left(|\psi_+| + |\psi_-|\right).$$

Introduce

$$w_+ := \langle t - r \rangle^{p-1}\psi_+, \qquad w_- := \langle t + r \rangle^{p-1}\psi_-.$$

Then

$$\big|(\partial_t + \partial_r)w_+\big| \leq C \, \|\psi_t(t)\|_{L^\infty}^{p-1} \left\{ \frac{1}{\langle r \rangle^{p-1}} |w_+| + \frac{\langle t - r \rangle^{p-1}}{\langle t + r \rangle^{p-1}\langle r \rangle^{p-1}} |w_-| \right\},$$

$$\big|(\partial_t - \partial_r)w_-\big| \leq C \, \|\psi_t(t)\|_{L^\infty}^{p-1} \left\{ \frac{\langle t + r \rangle^{p-1}}{\langle t - r \rangle^{p-1}\langle r \rangle^{p-1}} |w_+| + \frac{1}{\langle r \rangle^{p-1}} |w_-| \right\}.$$

Assume next that $\psi(0,r)$ vanishes for r large. Then the basic linear estimate shows that

$$\|w(t)\|_{L^\infty} \leq \|w(0)\|_{L^\infty} +$$

$$+ C \|\psi_t\|_{L^\infty(\mathbb{R}^2_+)}^{p-1} \max_{s\in[0,t]} \|w(s)\|_{L^\infty} \max_\Gamma \int_\Gamma \sum_\pm \frac{\langle t\pm r\rangle^{p-1}}{\langle t\mp r\rangle^{p-1}\langle r\rangle^{p-1}} + \frac{1}{\langle r\rangle^{p-1}} \, dt \,,$$

where the second max is over characteristic leading backward to $t = 0$ perhaps with a reflection at $r = 0$.

In the integral on the right use the triangle inequality $|t\pm r| \leq |t\mp r|+2r$ to bound the integrand from above by

$$\frac{C}{\langle r\rangle^{p-1}} + \sum_\pm \frac{C}{\langle t\mp r\rangle^{p-1}} \,.$$

It follows that the integrals are bounded independent of Γ and t so

$$\|w\|_{L^\infty([0,T]\times[0,\infty[)} \leq \|w(0)\|_{L^\infty} + C \|\psi_t\|_{L^\infty(\mathbb{R}^2_+)}^{p-1} \|w\|_{L^\infty([0,T]\times[0,\infty[)} \,.$$

A similar argument works for the time derivatives. One has the equations

$$(\partial_t \pm \partial_r)\big(\langle t\mp r\rangle^{p-1}\partial_t\psi_\pm\big) = \langle t-r\rangle^{p-1} r^{1-p} g'(\psi_+ + \psi_-)(\partial_t\psi_+ + \partial_t\psi_-) \,.$$

Introduce

$$z_\pm := \langle t\mp r\rangle^{p-1}\partial_t\psi_\pm \,,$$

and use (2.18) to show that

$$\big|(\partial_t + \partial_r)z_+\big| \leq C \|\psi_t\|_{L^\infty}^{p-1} \left\{ \frac{1}{\langle r\rangle^{p-1}} |z_+| + \frac{\langle t-r\rangle^{p-1}}{\langle t+r\rangle^{p-1}\langle r\rangle^{p-1}} |z_-| \right\},$$

$$\big|(\partial_t - \partial_r)z_-\big| \leq C \|\psi_t\|_{L^\infty}^{p-1} \left\{ \frac{\langle t+r\rangle^{p-1}}{\langle t-r\rangle^{p-1}\langle r\rangle^{p-1}} |z_+| + \frac{1}{\langle r\rangle^{p-1}} |z_-| \right\}.$$

Reasoning as above produces a constant C independent of T so that

$$\|w,z\|_{L^\infty([0,T]\times[0,\infty[)} \leq \|w(0),z(0)\|_{L^\infty} + C \|\psi_t\|_{L^\infty(\mathbb{R}^2_+)}^{p-1} \|w,z\|_{L^\infty([0,T]\times[0,\infty[)} \,.$$

When $C \|\psi_t\|_{L^\infty([0,\infty[\times[0,\infty[)}^{p-1} < 1$, this yields the bound

$$(4.5) \qquad \|w,z\|_{L^\infty([0,T]\times[0,\infty[)} \leq \frac{\|w(0),z(0)\|_{L^\infty([0,\infty[)}}{1 - C \|\psi_t\|_{L^\infty([0,\infty[\times[0,\infty[)}^{p-1}} \,.$$

For initial data which do not vanish for large r but satisfy the hypotheses of **i**, apply this inequality to the solution w^n, z^n with initial data $\chi(r/n)\psi(0,r)$ cut off at $r \sim n$. Passing to the limit $n \to \infty$ proves that estimate (4.5) holds for w, z. Since the right hand side of the estimate is independent of T it proves **i** of the Proposition.

To prove **ii**, let $T \to \infty$ in the remark after Proposition 2.8 to find

$$\int_0^\infty \int_0^\infty \frac{|\psi_+(t,r) + \psi_-(t,r)|^{p+1}}{r^{p-1}} dt\, dr \; < \; \infty.$$

Since ψ is uniformly Lipschitzean, this implies that for any $0 < \mu < R$

$$\lim_{t\to\infty} \sup_{\mu \le r \le R} |\psi_+(t,r) + \psi_-(t,r)| \; = \; 0.$$

On the other hand since $\psi_+(t,r) + \psi_-(t,r)$ vanishes when $r = 0$ and is uniformly Lipschitzean one has for $0 \le r \le \mu$,

$$\left| \psi_+(t,r) + \psi_-(t,r) \right| \; \le \; C\mu.$$

Combining these shows that

$$(4.6) \qquad \lim_{t\to\infty} \sup_{0 \le r \le R} |\psi_+(t,r) + \psi_-(t,r)| \; = \; 0.$$

Express $\psi_-(t,r)$ as the sum of $\psi_-(0, t+r)$ and the integral over the backward characteristic from (t,r) to $(0, t+r)$. When $t + r \to +\infty$, the initial value tends to zero by hypothesis and the integral tends to zero thanks to (4.6). This yields

$$\lim_{t\to\infty} \sup_{0 \le r \le R} |\psi_-(t,r)| \; = \; 0.$$

Combined with (4.6) this completes the proof of **ii**. ∎

4.2. Cauchy problem with data at $t = -\infty$

The problem addressed in this section is equivalent to the existence of Moeller's wave operators. Suppose

$$\psi^L \; \in \; \mathrm{Lip}\big(\,]-\infty, \infty[\times [0, \infty[\,\big)$$

is a uniformly Lipschitzean C^1 linear solution with

$$\lim_{r\to\infty} \left| \psi^L(t,r), \partial_{t,r}\psi^L(t,r) \right| \; = \; 0.$$

The Cauchy problem with initial data at $t = -\infty$ is to find a uniformly Lipschitzean $\psi \in C^1\big(\,]-\infty, \infty[\times [0, \infty[\,\big)$ satisfying

$$(\partial_t \pm \partial_r)\psi_\pm = r^{1-p} g(\psi_- + \psi_+),$$
$$\psi_-(t,0) + \psi_+(t,0) = 0 \quad \text{for} \quad -\infty < t < \infty,$$

and the initial condition at $t = -\infty$,

$$(4.7) \qquad \lim_{t\to-\infty} \left\| \psi(t) - \psi^L(t), \, \partial_{t,r}\big(\psi(t) - \psi^L(t)\big) \right\|_{L^\infty([0,\infty[)} \; = \; 0.$$

A similar definition holds for Cauchy data at $t = +\infty$. A first example is given by the next result.

Proposition 4.4 (Corollary to Propositions 4.1 and 4.3) *Suppose that ψ is as in part **i** of Proposition 4.3 and that Ψ^{out} is the linear solution describing its behavior as in Proposition 4.1. Then ψ has Cauchy data equal to Ψ^{out} at $t = +\infty$ and one has the stronger result*

$$(4.8) \qquad \left\| \psi(t) - \Psi^{\mathrm{out}}(t) , \, \partial_{t,r}\big(\psi(t) - \Psi^{\mathrm{out}}(t)\big) \right\|_{L^\infty([0,\infty[)} \;=\; O(1/t^{p-2}) .$$

Remarks.
1. We do not prove the analogous result for part **ii** of Proposition 4.3 which is an interesting open question.
2. This proves (1.12) of Theorem 3.

Proof. In Proposition 4.1 take $h(t) = \gamma t$ with $0 < \gamma < 1$, and break the right hand side of (4.8) into $r \leq \gamma t$ and $r \geq \gamma t$. The proposition shows that the contribution from $r \geq \gamma t$ is $O(1/t^{p-2})$. On the other hand, the contribution from $r \leq \gamma t$ is $O(1/t^{p-1})$ thanks to the decay estimate from Proposition 4.3. ∎

Proposition 4.5 (Cauchy problem with data at $t = -\infty$)
i. Small data. *Suppose that K_1 and K_1' are as in Proposition 2.5 and that*

$$\|\psi^L, \partial_r \psi^L\|_{L^\infty([0,\infty[)} \;\leq\; K_1 ,$$

then there is one and only one solution of the Cauchy problem with initial data at $t = -\infty$ as in (4.7). The solution satisfies

$$\|\psi, \partial_{t,r}\psi\|_{L^\infty([0,\infty[\times[0,\infty[)} \leq K_1' \|\psi^L(t), \partial_t \psi^L(t)\|_{L^\infty([0,\infty[)} ,$$

the right hand side being independent of t.
ii. Dissipative case. *In the case $a > 0$, one can take arbitrarily large data and the unique solution satisfies the stronger estimate*

$$\sum_{\pm, j \leq 1} \|\partial_{t,r}^j \psi_\pm(t)\|_{L^\infty([0,\infty[)} \;\leq\; \sum_{\pm, j \leq 1} \|\partial_t^j \psi_\pm^L(0)\|_{L^\infty([0,\infty[)} .$$

Remark. This proves part **i** of Theorem 3 and completes the proof of that Theorem.

Proof of uniqueness. Suppose that ψ and $\tilde{\psi}$ are two solutions and define $\delta := \psi - \tilde{\psi}$. Use (2.13) to find

$$(\partial_t \pm \partial_r)\delta_\pm \;=\; C \, r^{1-p} \, h_{p-1}(\psi_+ + \psi_- , \, \tilde{\psi}_+ + \tilde{\psi}_-) \, (\delta_+ + \delta_-) .$$

Therefore,

$$\left| (\partial_t \pm \partial_r)\delta_\pm \right| \;\leq\; C \, r^{1-p} \left(\frac{r}{1+r} \right)^{p-1} \|\psi(t), \tilde{\psi}(t)\|_{\mathrm{Lip}}^{p-1} \left(|\delta_+| + |\delta_-| \right) .$$

In addition one has the homogeneous boundary condition $\delta_+ = -\delta_-$ at $r = 0$.

Using Lemma 2.1 to estimate $\delta(T)$ in terms of $\delta(T - N)$ and letting $N \to \infty$ yields the following bound with $T < 0$,

$$\|\delta(T)\|_{L^\infty} \leq C \max_{\text{characteristics } \Gamma} \int_\Gamma r^{1-p}\big(|\psi^{p-1}| + |\tilde{\psi}^{p-1}|\big)\big(|\delta_+| + |\delta_-|\big)\,dt\,,$$

where the maximum is over leftward, rightward, and reflected characteristics lying in $\{t \leq T\}$. Therefore

$$\sup_{t \leq T}\|\delta(t)\|_{L^\infty} \leq C\left[\max_\Gamma \int_\Gamma r^{1-p}\big(|\psi^{p-1}| + |\tilde{\psi}^{p-1}|\big)\,dt\right]\sup_{t \leq T}\|\delta(t)\|_{L^\infty}\,.$$

To prove uniqueness we show that the max in brackets on the right is $o(1)$ as $T \to -\infty$. The integrals over Γ are split into two parts.

For the part of the integral in $r \geq |t|/2$, the functions ψ and $\tilde{\psi}$ are estimated by a fixed constant and the integral is no larger than

$$C\int_{|t|/2}^\infty r^{1-p}\,dr$$

which is $o(1)$ as $t \to -\infty$.

For the part in $r \leq |t|/2$, ψ^L and $\partial_t\psi^L$, tend uniformly to zero as $t \to -\infty$. The initial condition at $t = -\infty$ implies that over this region $\|\psi, \tilde{\psi}, \partial_t\psi, \partial_t\tilde{\psi}\|_{L^\infty} = o(1)$ as $t \to -\infty$. Proposition 2.2 shows that the integrand is estimated by

$$C\,r^{1-p}\frac{r^{p-1}}{(1+r)^{p-1}}\|\psi(t), \tilde{\psi}(t), \partial_t\psi, \partial_t\tilde{\psi}\|_{L^\infty([0,|T|/2])}$$

with a constant independent of Γ and $T \leq -1$. Since the sup norm tends to zero this is

$$= o(1)\,\frac{1}{(1+r)^{p-1}}\,.$$

Since $p > 2$ the integrals over Γ are $o(1)$ uniformly in Γ. This completes the proof of uniqueness.

Proof of existence. For $\{t \geq -n\}$ define ψ^n to be the solution of the initial value problem

$$(\partial_t \pm \partial_r)\psi_\pm^n = r^{1-p}g(\psi_-^n + \psi_+^n),$$
$$\psi_-^n(t,0) + \psi_+^n(t,0) = 0 \quad \text{for} \quad 0 \leq t \leq T\,,$$
$$\psi^n|_{t \leq -n} = \psi^L|_{t \leq -n}\,.$$

The initial data are C^1 and uniformly Lipschitzean. The global existence theorem implies that in both cases of the proposition there is a unique global solution with

$$(4.9) \qquad \|\psi^n\,,\,\partial_t\psi^n\|_{L^\infty(]-n,\infty[\times[0,\infty[)} \;\le\; K_1'\,\|\psi^L(t),\partial_t\psi^L(t)\|_{L^\infty([0,\infty[)}\,.$$

The Arzela-Ascoli Theorem implies that there is a subsequence converging uniformly on compact subsets to a function ψ with

$$\|\psi\,,\,\partial_t\psi\|_{L^\infty(]-\infty,\infty[\times[0,\infty[)} \;\le\; K_1'\,\|\psi^L(t),\partial_t\psi^L(t)\|_{L^\infty([0,\infty[)}\,.$$

That the limit is a solution of the differential equation and the boundary condition at $\{r=0\}$ follows upon passing to the limit in the equation for ψ^n. That the limit is C^1 follows as in the local existence proof, without need of compatibility conditions.

It remains to prove that the initial condition at $t=-\infty$ is satisfied.

Lemma 4.6 (Key estimate) *There is a constant C and a function $f(T)\to 0$ as $T\to-\infty$ depending on ψ^L but independent of n and $\sigma>0$ so that for $-n\le T$,*

$$\|(\psi^n-\psi^L)\,,\,\partial_t(\psi^n-\psi^L)\|_{L^\infty(\{-n\le t\le\min\{-n+\sigma,T\})} \;\le$$
$$\le C\,\|(\psi^n-\psi^L),\partial_t(\psi^n-\psi^L)\|^p_{L^\infty(\{-n\le t\le\min\{-n+\sigma,T\})} \;+\; f(T)\,.$$

Proof that Lemma 4.6 implies Proposition 4.5. With C as in the key estimate, choose $0<\varepsilon_0<1$ so that for all $0<\varepsilon<\varepsilon_0$,

$$C\,\varepsilon^p < \varepsilon/3\,.$$

For $0<\varepsilon<\varepsilon_0$, choose $\underline{T}$ so that $f(\underline{T})<\varepsilon/3$. We claim that for all n

$$\|(\psi^n-\psi^L),\partial_t(\psi^n-\psi^L)\|_{L^\infty(\{-\infty<t\le\underline{T}\})} \;\le\; \varepsilon\,.$$

If the claim were true, passing to the limit $n\to\infty$ yields

$$\|(\psi-\psi^L),\partial_t(\psi-\psi^L)\|_{L^\infty(\{-\infty<t\le\underline{T}\})} \;\le\; \varepsilon\,.$$

This proves that

$$\lim_{T\to-\infty}\|(\psi-\psi^L),\partial_t(\psi-\psi^L)\|_{L^\infty(\{-\infty<t\le\underline{T}\})} \;=\; 0\,.$$

The convergence of the r derivatives follows from the differential equation.

We next prove the claim. Since $\psi^n - \psi^L$ is $C^1 \cap \mathrm{Lip}$, tends to zero with its first derivatives as $r \to \infty$ and is equal to 0 at $t = -n$ it follows that

$$\|(\psi^n - \psi^L), \partial_t(\psi^n - \psi^L)\|_{L^\infty(\{-n < t \le -n+\sigma\})}$$

is a continuous function of σ which vanishes for $\sigma = 0$. In particular for small positive σ

$$\|(\psi^n - \psi^L), \partial_t(\psi^n - \psi^L)\|_{L^\infty(\{-n < t \le -n+\sigma\})} \le \varepsilon .$$

The proof proceeds by showing that it must remain less than ε for all $\sigma \le T + n$. If the claim were false for ψ^n there would be a first σ with $-n + \sigma \le \underline{T}$ for which it was false. For that value of σ, the key estimate yields the contradiction

$$\varepsilon = \|(\psi^n - \psi^L), \partial_t(\psi^n - \psi^L)\|_{L^\infty(\{-n \le t \le -n+\sigma\})} \le C\,\varepsilon^p + f(\underline{T}) \le \varepsilon/3 + \varepsilon/3.$$

Proof of Lemma 4.6. Let t be such that $-n \le t \le \min(-n + \sigma, T)$. Integrating the differential equation from time t backward to time $-n$ yields

$$(4.10) \qquad \psi_\pm^n(t,r) - \psi_\pm^L(t,r) \;=\; \int_{\Gamma_\pm} r^{1-p}\, g(\psi_+^n + \psi_-^n)\, dt .$$

where Γ denotes a backward characteristic possibly reflected at $r = 0$ connecting (t,r) to $\{t = -n\}$. Similarly

$$(4.11) \quad \partial_t \psi_\pm^n(t,r) - \partial_t \psi_\pm^L(t,r) \;=\; \int_{\Gamma_\pm} r^{1-p}\, g'(\psi_+^n + \psi_-^n)\,(\partial_t \psi_+^n + \partial_t \psi_-^n)\, dt .$$

Since when $r \ge |t|/2$, we have $r \ge |T|/2$, (4.9) yields

$$\left| \int_{\Gamma \cap \{r \ge |t/2|\}} \right| \;=\; O(|T|^{2-p}) ,$$

as $T \to -\infty$. These terms are absorbed in the $f(T)$ term of the key estimate.
In the region $r \le |t/2|$ the key fact is that

$$\|\psi^L\|_{\mathrm{Lip}(\{t \le T\} \cap \{r \le |t/2|\})} = o(1)$$

as $T \to -\infty$. Thus replacing ψ^n by $\psi^n - \psi^L$ in (4.10) yields, thanks to the middle two estimates of Proposition 2.2,

$$\left| \int_{\Gamma \cap \{r \le |t/2|\}} \right| \;\le\; C\, \|(\psi^n - \psi^L), \partial_t(\psi^n - \psi^L)\|_{L^\infty([-n,t] \times [0,\infty[)}^p + o(1) .$$

Similarly for (4.11),

$$\left| \int_{\Gamma \cap \{r \le |t/2|\}} r^{1-p} g'(\psi_+^n + \psi_-^n)\, \partial_t(\psi_+^n + \psi_-^n)\, dt \right| \le$$
$$\le C\, \|(\psi^n - \psi^L), \partial_t(\psi^n - \psi^L)\|_{L^\infty([-n,t] \times [0,\infty[)}^p + o(1) .$$

The proofs of Lemma 4.6 and Proposition 4.5 are complete. ∎

Corollary 4.7 *The solution in Proposition 4.5 satisfies*

$$\lim_{r \to \infty} \Big(|\psi(t,r)| + |\partial_t \psi(t,r)| + |\partial_r \psi(t,r)| \Big) = 0 \,.$$

Proof. For a challenge number $\varepsilon > 0$ choose $T << 0$ so that

$$\|(\psi - \psi^L)(T), \partial_t(\psi - \psi^L)(T)\|_{L^\infty([0,\infty[} \leq \frac{\varepsilon}{2K_1'} \,.$$

Choose $R > 0$ so that

$$\|\psi^L(T), \partial_t \psi^L(T)\|_{L^\infty([R,\infty[)} \leq \frac{\varepsilon}{2K_1'} \,.$$

Consider t, r in the domain of determinacy of the set $\{t = T\} \times \{r \geq R\}$. This is the set $r \geq R + |t - T|$. The Cauchy data in $\{t = T\} \times \{r \geq R\}$ has size at most ε/K_1' and a simple consequence of the fundamental estimate is that

$$\|\psi, \partial_t \psi\|_{L^\infty(r \geq R + |t-T|)} \leq \varepsilon \,.$$

This shows that for any $t > 0$ and $j = 0, 1$

$$\limsup_{r \to \infty} |\partial_t^j \psi(t,r)| \leq \varepsilon \,.$$

This completes the proof of the corollary. ∎

4.3. Definition of the scattering operator

With the notation of the preceding section one has

$$\psi^L(t,r) = \big(- G(t - r), G(t + r) \big)$$

for a unique $G \in \mathrm{Lip}(]-\infty,\infty[)$ with $\lim_{|\sigma|\to\infty} |G(\sigma)| = 0$. The function G gives the profile of the incoming wave near $t = -\infty$.

On the other hand, Proposition 4.1 shows that for $r \geq h(t)$ the solution of the initial value problem with data ψ^L at $t = -\infty$ satisfies for $t \to +\infty$

$$\psi = \Psi^{\mathrm{out}} + o(1) := \big(F(t - r), -F(t + r) \big) + o(1) \,.$$

The function F gives the profile of the outgoing spherical wave at $t = +\infty$.

Definition. *The scattering operator S is the mapping sending G to F.*

In the linear case, $a = 0$, the scattering operator defined here is multiplication by -1. This reflects the fact that the profile of the outgoing spherical wave is equal to minus the profile of the incoming wave as described in subsection 1.1.

The trio of functions $\psi^L, \psi, \Psi^{\mathrm{out}}$ are the main players in the description of the caustic crossing. We have chosen to use a different notation in the incoming linear solution ψ^L and the outgoing solution Ψ^{out} to emphasize that the sense in which ψ approaches the two limits is quite different. It is possible that in fact ψ has Cauchy data equal to Ψ^{out} at $t = +\infty$ but we do not prove that for large solutions.

Note that the domain of S always contains the functions G which are small in Lipschitz norm and tend to zero with their derivatives at $\pm\infty$. In case $a \geq 0$ the smallness is not required. The image of the scattering operator is contained in the uniformly Lipschitzean functions on $\mathbb{R}$ which tend to zero at $-\infty$. Additional properties of the image function F are given in Propositions 4.2 and 4.3.

5. Analysis of the focus crossing

Rescale as in § 1.4. There are two ways. One is to introduce characteristic variables in (1.4) and the other is to rescale the characteristic equations satisfied by v^ε introducing

$$v_\pm^\varepsilon(t, r) \ := \ \psi_\pm^\varepsilon\left(\frac{t - r_0}{\varepsilon}, \frac{r}{\varepsilon}\right), \qquad \psi_\pm^\varepsilon(\tau, \rho) \ := \ v_\pm^\varepsilon(\varepsilon\tau + r_0, \varepsilon\rho).$$

Both ways yield the equations

$$(5.1) \quad (\partial_\tau \pm \partial_\rho)\psi_\pm^\varepsilon \ = \ \rho^{1-p} g(\psi_-^\varepsilon + \psi_+^\varepsilon) \quad \text{for } \rho > 0, \qquad \left(\psi_-^\varepsilon + \psi_+^\varepsilon\right)_{\rho=0} \ = \ 0,$$

and the initial conditions

$$\psi_\mp^\varepsilon\big|_{\tau=-\frac{r_0}{\varepsilon}} \ = \ P_\mp\left(\varepsilon\rho, \rho - \frac{r_0}{\varepsilon}\right) \pm \varepsilon P_1\left(\varepsilon\rho, \rho - \frac{r_0}{\varepsilon}\right).$$

The key to analyzing this initial value problem is to note three things. First, the initial condition for $\psi_\pm$ is within $O(\varepsilon)$ of $P_\pm\left(r_0, \rho - \frac{r_0}{\varepsilon}\right)$. Second, the minus part of this is within $O(\varepsilon)$ of the value at time $\tau = -r_0/\varepsilon$ of the linear solution

$$(5.2) \qquad\qquad \psi^L(\tau, \rho) = \left(F(\tau + \rho), -F(\tau - \rho)\right),$$

$$(5.3) \qquad\qquad F(\sigma) \ := \ P_-(r_0, \sigma).$$

In that sense the problem resembles the Cauchy problem with data at $t = -\infty$. Finally, the P_+ part tends to launch an outgoing wave which lives in the region $\rho \sim r_0/\varepsilon >> 1$ where the nonlinearity is negligible thanks to the ρ^{1-p} factor. The outgoing wave moves away from the origin and does not affect the solution near the focal point at $\tau = \rho = 0$. The next Lemma verifies these expectations. The quantitative versions indicated with "resp." apply to our pulse families, since the functions P have compact support in z thanks to the hypothesis 1.2.

Proposition 5.1 *Suppose that ψ^L as in (5.2) is a uniformly Lipschitzean linear solution and suppose that ψ is a uniformly Lipschitzean solution of the nonlinear problem (5.1) with Cauchy data at $t = -\infty$ equal to those of ψ^L. Suppose that for $0 < \varepsilon < \varepsilon_0$, ψ^ε is a uniformly Lipschitzean family of solutions of (5.1) in $\tau \geq -r_0/\varepsilon$ satisfying at $\tau = -r_0/\varepsilon$*

$$\left\| \left[\psi^\varepsilon_+ - \psi^L_+ \, , \, \partial_t(\psi^\varepsilon_+ - \psi^L_+) \right]_{\tau=-r_0/\varepsilon} \right\|_{L^\infty([\mathbf{0},\mathbf{r_0}/\mathbf{2}\varepsilon])} = o(1), \qquad resp. \ 0,$$

and

$$\left\| \left[\psi^\varepsilon_- - \psi^L_- \, , \, \partial_t(\psi^\varepsilon_- - \psi^L_-) \right]_{\tau=-r_0/\varepsilon} \right\|_{L^\infty([0,\infty])} = o(1), \qquad resp. \ O(\varepsilon),$$

as $\varepsilon \to 0$.
i. *Then*

$$(5.4) \quad \left\| \psi^\varepsilon - \psi \, , \, \partial_t(\psi^\varepsilon - \psi) \right\|_{L^\infty(\{\tau-\rho=-r_0/\varepsilon\})} = o(1), \quad resp. \ O(\varepsilon^{\min\{1,p-2\}}).$$

ii. *There is a $\mu_0 > 0$ so that if $\|\psi\|_{\mathrm{Lip}([0,\infty[\times[0\infty[)} < \mu_0$ then*

$$(5.5) \quad \left\| \psi^\varepsilon - \psi \, , \, \partial_t(\psi^\varepsilon - \psi) \right\|_{L^\infty(\{\tau-\rho\geq-r_0/\varepsilon\})} = o(1), \quad resp. \ O(\varepsilon^{\min\{1,p-2\}}).$$

iii. *If $a > 0$ is nonnegative and ψ^L and ψ^ε are real valued, then without this smallness assumption one has convergence but not necessarily convergence of derivatives,*

$$(5.6) \qquad \left\| \psi^\varepsilon - \psi \right\|_{L^\infty(\{\tau-\rho\geq-r_0/\varepsilon\})} = o(1), \quad resp. \ O(\varepsilon^{\min\{1,p-2\}}).$$

Remarks.
1. The subtlety in this Lemma is that the initial data for the outgoing component ψ^ε_+ need not be close to those of ψ^L_+, the latter being $o(1)$. Nevertheless, one still has good approximation on $\tau - \rho \geq -r_0/\varepsilon$. This region where the estimates in parts **ii, iii** take place is sketched together with some characteristics in Figure 5.
2. Parts **ii** and **iii** of this Proposition yield part **i** of Theorem 4 completing the proof of that result.

Proof. First we prove **i**. The values on the line $\tau - \rho = -r_0/\varepsilon$ with $\rho \leq r_0/4\varepsilon$ are determined by the data at $\{\tau = -r_0/\varepsilon \, , \, \rho \leq r_0/2\varepsilon\}$. By hypothesis this data is $o(1)$ (resp. 0). The small data estimates show that the solution is $o(1)$ (resp. 0) in Lipschitz norm on the domain of determinacy of this data segment so

$$\left\| \psi^\varepsilon - \psi \, , \, \partial_t(\psi^\varepsilon - \psi) \right\|_{L^\infty(\{\tau-\rho=-r_0/\varepsilon\}\cap\{\rho\leq r_0/4\varepsilon\})} = o(1) \quad (resp. \ 0).$$
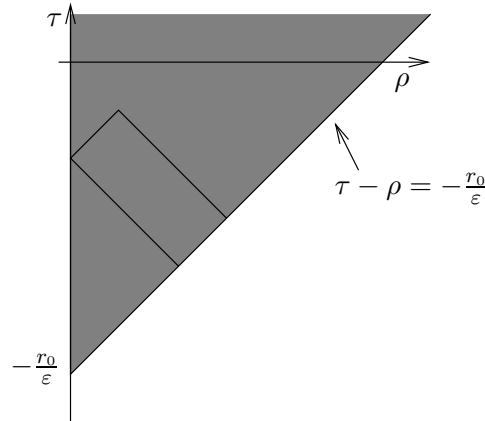
FIGURE 5: The region $\{\tau - \rho \geq -r_0/\varepsilon\}$ plus two characteristic paths.

For the minus component in $\{\tau - \rho = -r_0/\varepsilon\} \cap \{\rho \geq r_0/\varepsilon\}$, write for $j \leq 1$,

$$(5.7) \qquad \partial_\tau^j (\psi_-^\varepsilon - \psi_-) = \partial_\tau^j (\psi_-^\varepsilon - \psi_-)(-r_0/\varepsilon, \tau + \rho + r_0/\varepsilon)$$
$$+ \int_{\Gamma_-} \rho^{1-p} \, \partial_\tau^j \big[g(\psi_+^\varepsilon + \psi_-^\varepsilon) - g(\psi_+ + \psi_-) \big] \, d\tau \,,$$

where Γ_- is the minus characteristic from (τ, ρ) meeting the initial line $\{\tau = -r_0/\varepsilon\}$ at $(-r_0/\varepsilon, \tau + \rho + r_0/\varepsilon)$. Since this characteristic lies in $\rho \geq r_0/4\varepsilon$, and ψ^L and the family ψ^ε are uniformly Lipschitzean, the integrals are bounded above by

$$C \int_{r_0/\varepsilon}^\infty \frac{1}{r^{p-1}} \; = \; O(\varepsilon^{p-2}) \,.$$

The initial value for $\partial_\tau^j (\psi_-^\varepsilon - \psi_-)$ is $o(1)$ (resp. $O(\varepsilon)$) by hypothesis. This proves the estimate in part **i** for the minus component.

For the plus component on $\tau - \rho = -r_0/\varepsilon$ write

$$\partial_\tau^j (\psi_+^\varepsilon - \psi_+) = \partial_\tau^j (\psi_+^\varepsilon - \psi_+)\left(\frac{-3r_0}{4\varepsilon}, \frac{r_0}{4\varepsilon} \right)$$
$$+ \int_{\Gamma_+} \rho^{1-p} \, \partial_\tau^j \big[g(\psi_+^\varepsilon + \psi_-^\varepsilon) - g(\psi_+ + \psi_-) \big] \, d\tau,$$

where Γ_+ is

the segment on $\tau - \rho = -r_0/\varepsilon$ connecting (τ, ρ) to $(-3r_0/4\varepsilon, r_0/4\varepsilon)$.

As above, these quantities are $o(1)$ (resp. $O(\varepsilon^{p-2})$). This completes the proof of **i.**

The strategy for proving **ii** is to use **i** as initial data in the region

$$\left\{\tau - \rho \geq -\frac{r_0}{\varepsilon}\right\}.$$

Toward that end introduce

(5.8) $$\delta_{\pm}^{\varepsilon} := \psi_{\pm}^{\varepsilon} - \psi_{\pm}.$$

Subtracting the equations satisfied by ψ from that for ψ^{ε} yields

(5.9) $$\left(\partial_{\tau} \pm \partial_{\rho}\right)\delta_{\pm}^{\varepsilon} = b\rho^{1-p}\Big(g(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon}) - g(\psi_{+} + \psi_{-})\Big) =: f^{\varepsilon}.$$

Define for $\tau \geq -r_0/\varepsilon$

(5.10) $$m^{\varepsilon}(\tau) := \left\|\delta^{\varepsilon}(\tau, \rho), \, \partial_{\tau}\delta^{\varepsilon}(\tau, \rho)\right\|_{L^{\infty}(\{0 \leq \rho \leq \tau + r_0/\varepsilon\})}$$

Using **i** and integrating (5.9) backward along possibly broken characteristics Γ as in Figure 5, shows that

(5.11) $$m^{\varepsilon}(\tau) \leq o(1) \left(\text{resp. } O(\varepsilon^{\min\{1, p-2\}})\right) + \max_{\Gamma} \int_{\Gamma} |f^{\varepsilon}| + |\partial_{\tau}f^{\varepsilon}| \, d\tau.$$

To estimate f^{ε} write

$$g(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon}) - g(\psi_{+} + \psi_{-}) = h_{p-1}(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon}, \, \psi_{+} + \psi_{-})(\delta_{+}^{\varepsilon} + \delta_{-}^{\varepsilon})$$

where h_{p-1} is homogeneous of degree $p - 1$. Using (2.18) one has

(5.12) $$|f^{\varepsilon}(\tau, \rho)| \leq C \left(\|\psi\|_{\text{Lip}} + m^{\varepsilon}(\tau)\right)^{p-1} \frac{1}{1 + \rho^{p-1}} \, m^{\varepsilon}(\tau).$$

To estimate $\partial_{\tau}f$ use

(5.13) $$\partial_{\tau}f^{\varepsilon} = b\rho^{1-p}\Big(g'(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon})\partial_{\tau}(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon}) - g'(\psi_{+} + \psi_{-})\partial_{\tau}(\psi_{+} + \psi_{-})\Big),$$

and

$$g'(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon}) - g'(\psi_{+} + \psi_{-}) = h_{p-2}(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon}, \, \psi_{+} + \psi_{-})(\delta_{+}^{\varepsilon} + \delta_{-}^{\varepsilon}),$$

with h_{p-2} continuous and homogeneous of degree $p - 2$.

The right hand side of (5.13) is equal to

$$b\rho^{1-p}\Big(g'(\psi_{+} + \psi_{-})\partial_{\tau}(\delta_{+}^{\varepsilon} + \delta_{-}^{\varepsilon}) +$$
$$+ h_{p-2}(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon}, \psi_{+} + \psi_{-})(\delta_{+}^{\varepsilon} + \delta_{-}^{\varepsilon})\partial_{\tau}(\psi_{+}^{\varepsilon} + \psi_{-}^{\varepsilon})\Big).$$

Using (2.18) again, estimate

$$\left|g'(\psi_+ + \psi_-)\right| \leq \left(\frac{C\rho}{1+\rho}\right)^{p-1} \|\psi\|_{\mathrm{Lip}}^{p-1},$$

$$\left|h_{p-2}(\psi_+^\varepsilon + \psi_-^\varepsilon, \psi_+ + \psi_-)\right| \leq \left(\frac{C\rho}{1+\rho}\right)^{p-2} (\|\psi\|_{\mathrm{Lip}} + m^\varepsilon(\tau))^{p-2},$$

$$\left|\delta_+^\varepsilon(\tau, \rho) + \delta_-^\varepsilon(\tau, \rho)\right| \leq \frac{C\rho}{1+\rho} m^\varepsilon(\tau).$$

Inserting these in (5.13) yields

$$(5.14) \qquad |\partial_\tau f^\varepsilon(\tau, \rho)| \leq C \left(\|\psi\|_{\mathrm{Lip}} + m^\varepsilon(\tau)\right)^{p-1} \frac{1}{1+\rho^{p-1}} m^\varepsilon(\tau),$$

the same form that we found for $|f^\varepsilon(\tau, \rho)|$.

Use (5.12) and (5.14) in (5.11). The key is that the integral of $1/(1+\rho^{p-1})$ over any possibly reflected characteristic, is bounded independent of the characteristic so for any $T > -r_0/\varepsilon$ and $-r_0/\varepsilon \leq \tau \leq T$

$$m^\varepsilon(\tau) \leq o(1) \ (\text{resp. } O(\varepsilon^{\min\{1, p-2\}}))$$

$$+ C \left(\|\psi\|_{\mathrm{Lip}} + \sup_{-r_0/\varepsilon \leq \sigma \leq T} m^\varepsilon(\sigma)\right)^{p-1} \sup_{-r_0/\varepsilon \leq \sigma \leq T} m^\varepsilon(\sigma).$$

Choose μ_0 so that $C(2\mu_0)^{p-1} = 1/2$. Denote the $o(1)$ term by $\zeta(\varepsilon)$. Choose $\varepsilon_0 > 0$ so that $\zeta(\varepsilon) < \mu_0/4$ for $0 < \varepsilon < \varepsilon_0$. We will show that for $\varepsilon < \varepsilon_0$ and $\|\psi\|_{\mathrm{Lip}} < \mu_0$ one has $m^\varepsilon(\tau) < 2\zeta(\varepsilon)$ for all $\tau \geq -r_0/\varepsilon$, which is the desired conclusion.

To prove that $m^\varepsilon(\tau) < 2\zeta(\varepsilon)$, define

$$\mathbf{m}^\varepsilon(\tau) := \sup_{0 \leq s \leq \tau} m^\varepsilon(s).$$

The preceding estimates show that when $\|\psi\|_{\mathrm{Lip}} < \mu_0$ one has

$$\mathbf{m}^\varepsilon(\tau) \leq \zeta(\varepsilon) + C\left(\mu_0 + \mathbf{m}^\varepsilon(\tau)\right)^{p-1} \mathbf{m}^\varepsilon(\tau).$$

If $\mathbf{m}^\varepsilon(\tau) \leq \mu_0$ it follows that

$$\mathbf{m}^\varepsilon(\tau) < \zeta(\varepsilon) + C(2\mu_0)^{p-1}\mathbf{m}^\varepsilon(\tau) = \zeta(\varepsilon) + \mathbf{m}^\varepsilon(\tau)/2,$$

and therefore for $\varepsilon < \varepsilon_0$,

$$(5.15) \qquad\qquad \mathbf{m}^\varepsilon(\tau) < 2\zeta(\varepsilon) < \mu_0/2.$$

This proves that in fact $\mathbf{m}^\varepsilon < \mu_0$ since if that were not true there would be a first τ_* where $\mathbf{m}^\varepsilon(\tau_*) = \mu_0$, and at that value of τ the above estimate leads to the contradiction $\mu_0 < \mu_0/2$.

It then follows that (5.15) holds for all τ which is the conclusion of **ii**.

To prove **iii**, the argument follows the proof of Proposition 2.8. Multiplying (5.9) by $g_{q-1}(\delta_\pm)$ and adding shows that

$$\partial_\tau\big(|\delta_+|^q + |\delta_-|^q\big) \;+\; \partial_\rho\big(|\delta_+|^q - |\delta_-|^q\big) \;\leq\; 0\,.$$

For $\tau > -r_0/\varepsilon$, integrating over $\big\{(\underline{\tau}, \rho)/\ \underline{\tau} - \rho \geq -r_0/\varepsilon,\ \underline{\tau} \leq \tau,\ \text{and } \rho \geq 0\big\}$ yields

$$\int_0^{\tau + r_0/\varepsilon} |\delta_+(\tau, \rho)|^q + |\delta_-(\tau, \rho)|^q\, d\rho \;\leq\; 2\int_0^{\tau + r_0/\varepsilon} |\delta_-(\rho - r_0/\varepsilon, \rho)|^q\, d\rho.$$

Passing to the limit $q \to \infty$ shows that in the shaded region in Fig. 5 one has

$$\|\delta_\pm\|_{L^\infty(\{\tau - \rho \geq -r_0/\varepsilon\})} \leq \|\delta_-\|_{L^\infty(\{\tau - \rho = -r_0/\varepsilon\})} \;=\; o(1)\ (\text{resp. } O(\varepsilon^{\min\{1, p-2\}}))\,,$$

and the proof of the proposition is complete. ∎

6. Pulse broadening

The passage of a pulse through the caustic is described by a scattering operator. If the incoming pulse has profile with compact support as in (1.2) then the scattering solution has Cauchy data at time $t = -\infty$ given by a linear solution

(6.1) $$\psi^L \;=\; (-G(t - r)\,,\, G(t + r))$$

with G compactly supported. If $G(\sigma)$ is supported in $|\sigma| \leq R$, then the incoming linear spherical wave solution is supported in a spherical shell $|r + t| \leq R$. The outgoing wave is supported in $|r - t| \leq R$.

Part **i** of Proposition 4.3 shows that for G small, the outgoing linear solution

$$\Psi^{\mathrm{out}} \;=\; (F(t - r)\,,\, -F(t + r))$$

has $F(\sigma) = O(|\sigma|^{1-p})$. In this subsection we show that this estimate is sharp thereby demonstrating that the passage through the nonlinear caustic transforms a compactly supported profile to a pulse with algebraically decaying tail.

We can show that the estimate is sharp in two contexts. The first is by studying the simply coupled system

(6.2) $$\Box u \;=\; 0\,, \qquad \Box v + a\, u_t |u_t|^{p-1} \;=\; 0\,.$$

The function u satisfies

$$(\partial_t \mp \partial_r)(ru) \;=\; \psi_\pm^L.$$

and it is assumed that this linear solution ψ^L is compactly supported in r for each t and is given by (6.1). The solution v is then expressed in terms of quadratures.

Exactly the same computation arises in computing scattering solutions of (1.4) by Picard iteration. If the first iterate is the linear solution, ψ^L, then the second iterate is exactly v. Even more interesting, the same function v is the term of leading order in the perturbation series of the scattering operator for small data. In this section, we show that the solution v leads exactly to algebraic decay in the outgoing profile and also derive the above small data behavior of the scattering operator. The small data strategy resembles the strategy in §6 of [9].

6.1. The simply coupled system

In the simply coupled system (6.2), introduce the characteristic variables,

$$\chi_\pm := (\partial_t \mp \partial_r)(rv),$$

to find the initial boundary value problem

(6.3)
$$\begin{cases} (\partial_t \pm \partial_r)\chi_\pm = r^{1-p}\, g(\psi_+^L + \psi_-^L), \\ \chi\big|_{t=-\infty} = 0, \\ \chi_+(t,0) + \chi_-(t,0) = 0. \end{cases}$$

Define the complex number

$$A = A(\psi^L) := \frac{a}{2^p} \int G(\sigma)\, |G(\sigma)|^{p-1}\, d\sigma.$$

To compute $\chi(\underline{t},\underline{r})$ in $\underline{r} - \underline{t} > R$, refer to the right hand graph in Figure 6.
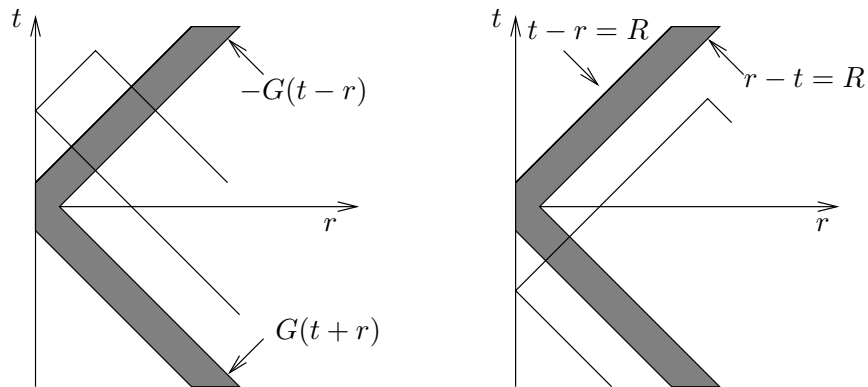

FIGURE 6: Computing $\chi(t,r)$ in $t - r > R$ and $r - t > R$ respectively.

The characteristic for χ_- does not intersect the support of the source term so $\chi_- = 0$.

For $\underline{r} - \underline{t} \to +\infty$, the quantity $\underline{r} - \underline{t}$ measures both the distance of the $\partial_t + \partial_r$ characteristic to the outgoing source wave and also the distance r from the points of intersection of the characteristic with the incoming source wave. Precisely, in that support one has $r = \underline{r} - \underline{t} + O(1)$ and therefore

$$r^{1-p} = \frac{1}{(\underline{r} - \underline{t})^{1-p}} + O((\underline{r} - \underline{t})^{-p}).$$

It follows that

$$\chi_-(\underline{t}, \underline{r}) = \frac{-A}{(\underline{r} - \underline{t})^{p-1}} + O((\underline{r} - \underline{t})^{-p}).$$

In an analogous way, when $\underline{r} - \underline{t} \to -\infty$, the left hand graph in Figure 6 yields

$$\chi_\mp(\underline{t}, \underline{r}) = \frac{\pm A}{(\underline{t} - \underline{r})^{p-1}} + O((\underline{t} - \underline{r})^{-p}).$$

Recall that the function $F(\sigma)$ defining the outgoing linear wave Ψ^{out} is defined as the limiting value of $\chi_+(t, r)$ on the characteristic $\Gamma_+(\sigma) := \{t - r = \sigma\}$. Note that for $|t - r| > R$, the values of χ_+ are constant outside a compact subset of $\Gamma_+(\sigma)$ and the estimates above prove that

$$|F(\sigma)| = |A|/|\sigma|^{p-1} + O(|\sigma|^{-p}).$$

This shows that our decay estimates are sharp and there are algebraic tails. Note also that up to a constant factor, the decaying tails are all of the form $c(1 + O(1/|\sigma|))/|\sigma|^{p-1}$.

6.2. Scattering of small solutions

In this subsection we study the scattering operator applied to incoming linear solutions $\delta\psi^L(t, r)$ where ψ^L is the fixed linear solution from the previous subsection. The scattering solution is the unique solution (for δ small) of

(6.4)
$$\begin{cases} (\partial_t \pm \partial_r)\psi = r^{1-p}\, g(\psi_+ + \psi_-), \\ (\psi - \delta\psi^L)_{t=-\infty} = 0, \\ \psi_+(t, 0) + \psi_-(t, 0) = 0. \end{cases}$$

Make a change of dependent variable,

(6.5)
$$\phi := \psi/\delta, \qquad \psi := \delta\phi,$$

to find with

(6.6)
$$\mu := |\delta|^{p-1} > 0,$$

(6.7)
$$\begin{cases} (\partial_t \pm \partial_r)\phi = \mu\, r^{1-p}\, g(\phi_+ + \phi_-), \\ (\phi - \psi^L)_{t=-\infty} = 0, \\ \phi_+(t, 0) + \phi_-(t, 0) = 0. \end{cases}$$

The solution, $\phi(t, r, \mu)$, depends on $\mu = |\delta|^{p-1} \to 0$. Formally expand

$$\phi(t, r, \mu) \sim \sum_{j=0}^{\infty} \mu^j \, \eta_j(t, r),$$

and insert into the initial boundary value problem. Equating like powers of μ first implies that each coefficient satisfies the boundary condition

$$\eta_{j,+}(t, 0) + \eta_{j,-}(t, 0) = 0.$$

One also finds initial boundary value problems which for the two leading coefficients η_0 and η_1 are,

$$(\partial_t \pm \partial_r)\eta_{0,\pm} = 0, \qquad \left(\eta_0 - \psi^L\right)_{t=-\infty} = 0,$$
$$(\partial_t \pm \partial_r)\eta_{1,\pm} = r^{1-p} \, g(\eta_{0,+} + \eta_{0,-}), \qquad \eta_1\big|_{t=-\infty} = 0.$$

The first initial boundary value problems has the unique solution $\eta_0 = \psi^L$. The problem determining η_1 is then identical to the problem determining χ in the previous subsection so we know that $\eta_1 = \chi$ has algebraically decaying tails in $|t - r| \to \infty$ when ψ^L is compactly supported with $A \neq 0$.

In order to draw a rigorous conclusion from this computation, we prove the following error estimate.

Proposition 6.1 *Suppose that ψ^L is a C^1 linear solution compactly supported in r for each time t. Suppose that $\eta_0 = \psi^L$, $\eta_1 = \chi$, and $\phi(t, r, \mu)$ is the solution of (6.7) as above. Then as $\mu \to 0$,*

$$\left\| \phi - (\eta_0 + \mu\eta_1), \, \partial_t(\phi - (\eta_0 + \mu\eta_1)) \right\|_{L^\infty([0,\infty[\times]-\infty,\infty[)} = O(\mu^2).$$

Proof. The first step is to estimate $\phi - \phi^L = \phi - \eta_0$ using the differential equation

$$(\partial_t \pm \partial_r)(\phi_\pm - \phi^L_\pm) = -\mu \, r^{1-p} \, g(\phi_+ + \phi_-),$$

supplemented by homogeneous initial and boundary conditions,

$$(\phi - \phi^L)_+(t, 0) + (\phi - \phi^L)_-(t, 0) = 0, \qquad (\phi - \phi^L)_{t=-\infty} = 0.$$

Noting that for $j = 0, 1$,

$$\sup_{\text{characteristics}} \int \left| \partial_t^j \mu \, r^{1-p} \, g(\phi_+ + \phi_-) \right| dt = O(\mu),$$

the basic linear estimate implies that

$$\left\| \phi - \eta_0, \, \partial_t(\phi - \eta_0)) \right\|_{L^\infty([0,\infty[\times]-\infty,\infty[)} = O(\mu).$$

Next estimate $E := \phi - (\eta_0 + \mu\eta_1)$ using the differential equation

$$(\partial_t \pm \partial_r)E_\pm = -\mu \, r^{1-p} \left[g(\phi_+ + \phi_-) - g(\phi^L_+ + \phi^L_-) \right] := f,$$

supplemented by the following homogeneous initial and boundary conditions: $E\big|_{t=-\infty} = 0$ and $E_+(t, 0) + E_-(t, 0) = 0$.

Using our estimate for $\phi - \phi^L$ it is not hard to show that

$$\sup_{\text{characteristics}} \int |f| + |\partial_t f| \, dt \;=\; O(\mu^2) \, ,$$

and the basic linear estimate completes the proof of the proposition.   ∎

Rewriting in terms of the problems at the beginning of the subsection yields the following corollary.

Corollary 6.2 *Let $\psi(t, r, \delta)$ be the scattering solution defined by* (6.4) *and χ be the solution of* (6.3). *Then for small δ, one has*

$$\left\| \psi - \delta\big(\psi^L + |\delta|^{p-1}\chi\big), \partial_t\big(\psi - \delta\big(\psi^L + |\delta|^{p-1}\chi\big)\big) \right\|_{L^\infty([0,\infty[\times]-\infty,\infty[)} = O\big(|\delta|^{1+2(p-1)}\big).$$

In particular, if

$$\Psi^{\text{out}}(t, r, \delta) \;=\; \delta\left(F(t-r,\delta), -F(t+r,\delta)\right)$$

is the outgoing linear solution corresponding to $\psi(t, r, \delta)$, then F has algebraic tails for $|\sigma| \to \infty$, in the sense that when $|\delta| \ll 1$,

$$|F(\sigma, \delta) + G(\sigma)| = |\delta|^{p-1}\Big(\frac{|A|}{|\sigma|^{p-1}} \;+\; O\big(|\delta|^{p-1}\big)\Big).$$

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