

An analysis of quantum Fokker–Planck models: A Wigner function approach

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Abstract

The analysis of dissipative transport equations within the framework of open quantum systems with Fokker–Planck–type scattering is carried out from the perspective of a Wigner function approach. In particular, the well–posedness of the self–consistent whole–space problem in 3D is analyzed: existence of solutions, uniqueness and asymptotic behavior in time, where we adopt the viewpoint of mild solutions in this paper. Also, the admissibility of a density matrix formulation in Lindblad form with Fokker–Planck dissipation mechanisms is discussed. We remark that our solution concept allows to carry out the analysis directly on the level of the kinetic equation instead of on the level of the density operator.

1. Introduction

This paper is concerned with the analysis of a class of dissipative quantum models that arise when considering the motion of an ensemble of quantum particles (for example, electrons) interacting with a heat bath of oscillators in thermal equilibrium, and thus the effects of system–environment interactions have to be taken into account. Of particular practical interest are interaction mechanisms that can be described by Fokker–Planck scattering terms. On a kinetic level these models are typically represented by an initial value problem for the Wigner function:

$$(1.1) \quad \frac{\partial W}{\partial t} + (\xi \cdot \nabla_x)W + \Theta[V]W = L_{QFP}W, \quad x, \xi \in \mathbb{R}^3, \quad t > 0,$$

$$(1.2) \quad W(x, \xi, 0) = W^I(x, \xi).$$

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Here x and ξ denote respectively the position and velocity variables of the phase-space. We are interested in the case in which the potential V is the electrostatic Hartree (Coulomb) potential

$$(1.3) \quad V(x, t) = \frac{\alpha}{4\pi} \int_{\mathbb{R}_y^3} \frac{n(y, t)}{|x - y|} dy, \quad n(x, t) = \int_{\mathbb{R}_\xi^3} W(x, \xi, t) d\xi,$$

where $\alpha = +1$ or $\alpha = -1$ depending on the type of forces acting on the system (repulsive or attractive) and $n(x, t)$ is the particle position density. In this model, the nonlinear character of the Wigner–Fokker–Planck equation (1.1) stems from the self-consistent action of the potential (1.3) on the system through the pseudo-differential operator

$$(1.4) \quad \Theta[V]W(x, \xi, t) = \frac{i}{(2\pi)^3} \int_{\mathbb{R}_\eta^3} \int_{\mathbb{R}_{\xi'}^3} \frac{V(x + \frac{\hbar}{2m}\eta, t) - V(x - \frac{\hbar}{2m}\eta, t)}{\hbar} \\ \times W(x, \xi', t) e^{-i(\xi - \xi') \cdot \eta} d\xi' d\eta$$

with \hbar denoting the Planck constant and m the effective mass of the particles, while L_{QFP} represents the quantum Fokker–Planck operator

$$(1.5) \quad L_{QFP}W = \frac{D_{pp}}{m^2} \Delta_\xi W + 2\gamma \operatorname{div}_\xi(\xi W) + 2\frac{D_{pq}}{m} \operatorname{div}_x(\nabla_\xi W) + D_{qq} \Delta_x W.$$

Here $\gamma, D_{pp}, D_{pq}, D_{qq}$ are positive constants related to the interaction between the particles and the reservoir (see next section).

The above described quantum model with Fokker–Planck-type scattering governs the dynamical evolution of an electron ensemble in the single-particle Hartree approximation interacting dissipatively with an idealized heat bath consisting of an ensemble of harmonic oscillators [7]. This system generally fits the framework of open quantum-mechanical systems, present in a wide range of situations in quantum statistical mechanics [22], [16], where the particle-background interaction is important. Dissipative phenomena play a relevant role in microelectronics, essentially through the modeling of quantum transport of charge carriers in quantum semiconductor devices ([24], [36]). Some other significant fields of application are quantum optics, quantum Brownian motion ([7], [20], [21]) and damped quantum oscillators (A4 of [34]), as well as a variety of technological problems based on systems representing transport processes that operate far from equilibrium.

The rigorous well-posedness analysis of quantum Fokker–Planck models in the Wigner phase-space representation will be carried out in this paper for the simplest Markovian approximation of open quantum systems in the high-temperature limit (for which $D_{qq} = D_{pq} = 0$) and frictionless

case $\gamma = 0$. Denoting $\sigma = \frac{D_{pp}}{m^2}$, the system studied in this paper is

$$(1.6) \quad \frac{\partial W}{\partial t} + (\xi \cdot \nabla_x)W + \Theta[V]W - \sigma \Delta_\xi W = 0$$

with initial condition (1.2). Our main existence and uniqueness result for this *Wigner–Poisson–Fokker–Planck* (WFPF) *system* shall be given in Theorem 3.3, and the large–time behavior of the solution is characterized in Theorem 5.9.

In reference [19] it is shown that this model yields a mathematically “consistent” master equation (i.e. a positivity–preserving equation at the density matrix level) which does not take into account energy dissipation of the electron ensemble by the background. In spite of this, the frictionless models constitute the only physically relevant Fokker–Planck quantum models which make quantum entropy grow monotonously (see next section). Also, in [20] the simplest systematic Markovian approximation with dissipation is derived, consisting of equation (1.1) after disregarding the term $\frac{D_{pq}}{m} \operatorname{div}_x(\nabla_\xi W)$. This fact means that the friction model includes an elliptic term with respect to the position variable, namely $D_{qq} \Delta_x W$. Setting $D_{pq} = D_{qq} = 0$ gives the Caldeira–Leggett model [7] which does not belong to the Lindblad class, and thus positivity of the density matrix operator is not preserved under temporal evolution. In the next section we shall show in more detail how the Lindblad form of the evolution equation for the density matrix provides the necessary properties to ensure that the system is well–posed (in the sense that total charge and quantum density and entropy are well–defined, as well as mathematical “consistency” of the problem holds). The mathematical analysis of the frictionless WFPF problem becomes more complex due to the *a priori* lack of elliptic regularization in the x –variable.

The technique to be used to prove existence of mild solutions relies on the construction of a sequence of approximate problems whose solutions are shown to verify some appropriate bounds (independent of the regularization) in order to pass to the limit in the approximation parameter. In essence, these bounds are based on the regularization of the initial particle density, which also implies the corresponding elliptic regularization of the potential through the Poisson equation. For this purpose, we will take advantage of the Green function representation associated with the WFPF system, which provides for an equivalent fixed–point nonlinear integral equation useful for *a priori* estimates. The uniqueness (resp. stability) proof follows by estimating the difference of two different solutions in an adequate norm. On the other hand, the analysis of the asymptotic behavior of the solutions relies on a nonlinear technique consisting of the introduction of a rescaling group acting on the WFPF equation (thus of a sequence of rescaled problems),

the derivation of the necessary compactness properties for the rescaled solutions and potentials in order to pass to the limit in the scale parameter (somehow connected to the time variable, as we will see later) and the identification of the large-time limit.

In this paper we shall establish an $L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ -theory for the Wigner function $W(x, \xi, t)$. In contrast to collisionless Wigner models (cf. [30]) this is possible here due to the smoothing effects of the FP-operator. It allows a direct mathematical analysis of the kinetic equation without going back to the density operator formulation. Specifically, our main goals are, on the one hand, the L^1 -boundedness of the Wigner function locally uniformly in time, under very weak assumptions on the initial data; on the other hand, we also deduce strong regularity properties for the Wigner particle distribution, the density and the potential as well as optimal time decay estimates. However, these properties cannot be extended (at least with our techniques) for all times. Actually, global solutions (whose existence proof requires different techniques) shall be dealt with in a forthcoming paper. On the other hand, global existence and L^1 theory have been recently dealt with in [8] for the frictional problem (i.e. that containing the full FP-operator (1.5)). Finally, it is proved that the solution of the WFPF problem behaves like MG for large times, where M denotes the total charge of the system and G is the fundamental solution of the linear kinetic Fokker-Planck equation. Therefore, the quantum effects in the system appear to be lost in the long time limit.

Finally, a comparison of the results of this paper with those corresponding to the classical picture, i.e., for the Vlasov-Poisson-Fokker-Planck (VPFP) system, could help to completely clarify the context. The VPFP system has been extensively studied in last years, see for example [4, 5, 6, 9, 10, 11] and the references therein. Our existence result makes use of similar norms as those employed for the existence analysis of the VPFP system (see [5, 10]) which are the appropriate tools for time-dependent kinetic Fokker-Planck equations. Under some additional hypotheses involving the control of the second order velocity moment along with strong regularity assumptions on the initial density, in [31] the existence of global solutions and gain of regularity for the VPFP system is proved.

The paper is organized as follows: In Section 2 we present an overview of Wigner-Fokker-Planck problems in the context of open quantum systems and discuss its relation to completely positive and dissipative systems in Lindblad form. Section 3 is devoted to the existence of local-in-time solutions. In Section 4 we prove uniqueness and stability. Finally, in Section 5 the long time behavior of solutions of the Wigner-Poisson-Fokker-Planck system is studied.

2. Wigner–Fokker–Planck models

2.1. On the derivation of quantum Fokker–Planck models

We consider the motion of an electron in \mathbb{R}^d (or of an electron ensemble in the single–particle approximation) under the influence of an electric potential $V = V(x, t) : \mathbb{R}_x^d \times \mathbb{R}_t^+ \rightarrow \mathbb{R}$ and under interactions with a thermal bath of harmonic oscillators in equilibrium. In particular we shall be concerned with an extension of the so–called Caldeira–Leggett master equation [7] to the case of moderate/high temperatures as derived in [19]. For the following let $R(t) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be the density (matrix) operator of the electron (ensemble) at time $t \geq 0$, i.e. it is a linear, non negative, self–adjoint trace class operator, and let $\rho = \rho(x, y, t) \in L^2(\mathbb{R}_x^d \times \mathbb{R}_y^d)$ be its integral kernel, the “density matrix function”:

$$(2.1) \quad (R(t)f)(x) = \int_{\mathbb{R}_y^d} f(y)\rho(x, y, t) dy.$$

The model for the motion of the electron reads

$$(2.2) \quad \frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}(H_x - H_y)\rho - \gamma(x - y) \cdot (\nabla_x - \nabla_y)\rho + \left(D_{qq}|\nabla_x + \nabla_y|^2 - \frac{D_{pp}}{\hbar^2}|x - y|^2 + \frac{2i}{\hbar}D_{pq}(x - y) \cdot (\nabla_x + \nabla_y) \right)\rho,$$

where

$$(2.3) \quad H = -\frac{\hbar^2}{2m}\Delta_x + V(x, t)$$

is the electron Hamiltonian (H_x, H_y stand for copies of H acting on the x and, respectively, y –variable), m is the effective electron mass, \hbar the reduced Planck constant and, as stated above, V is the electric (or Hartree–) potential. The positive constants $\gamma, D_{pp}, D_{pq}, D_{qq}$ stem from the oscillator bath (cf. [19]):

$$(2.4) \quad \gamma = \frac{\eta}{2m}, \quad D_{pp} = \eta k_B T, \quad D_{qq} = \frac{\eta \hbar^2}{12m^2 k_B T}, \quad D_{pq} = \frac{\eta \Omega \hbar^2}{12\pi m k_B T}.$$

Here $\eta > 0$ is the coupling (damping) constant of the bath, k_B the Boltzmann constant, T the temperature of the bath and Ω the cut–off frequency of the reservoir oscillators. Note that (2.2)–(2.4) was derived in [19] as the Markovian approximation of the originally non–Markovian evolution of the electron in the oscillator bath. The latter is obtained from the full electron–oscillator model by tracing out the oscillator coordinates. For a somewhat more phenomenological derivation we refer to [17].

The assumptions on the parameters, which guarantee the validity of (2.2), can be found in [19]. Here we only remark that the main hypotheses are:

- (i) the reservoir memory time $\frac{1}{\Omega}$ is much smaller than the characteristic time scale of the electrons,
- (ii) weak coupling: $\gamma \ll \Omega$,
- (iii) medium/high temperature: $\Omega \lesssim k_B T / \hbar$.

We now introduce the Wigner function W of the electron ensemble, defined on the phase space $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ for $t \geq 0$:

$$(2.5) \quad W(x, \xi, t) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}_\eta^d} \rho\left(x + \frac{\hbar}{2m}\eta, x - \frac{\hbar}{2m}\eta, t\right) e^{-i\xi \cdot \eta} d\eta \in L^2(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$$

(cf. [38], [28], [29], [23], [27], [30]). Note that the self-adjointness of $R(t)$ implies $\rho(x, y, t) = \bar{\rho}(y, x, t)$ (“ $\bar{\cdot}$ ” stands for complex conjugation) which in turn implies that W is real-valued. It is a simple exercise to compute the evolution equation satisfied by (2.5). From (2.2) we obtain the kinetic equation (1.1) for W where the pseudo-differential operator is given by (1.4) and L_{QFP} is the quantum Fokker-Planck operator (1.5). In the classical limit $\hbar \rightarrow 0$ we have $D_{qq} \rightarrow 0, D_{pq} \rightarrow 0$ and, formally

$$(2.6) \quad L_{QFP}W \rightarrow \frac{D_{pp}}{m^2} \Delta_\xi W + 2\gamma \operatorname{div}_\xi(\xi W)$$

$$(2.7) \quad \Theta[V]W \rightarrow -\frac{1}{m} \nabla_x V \cdot \nabla_\xi W$$

such that the classical Vlasov-Fokker-Planck equation is recovered (see, for example, [10]):

$$(2.8) \quad \frac{\partial W}{\partial t} + (\xi \cdot \nabla_x)W - \frac{1}{m} \nabla_x V \cdot \nabla_\xi W = \frac{D_{pp}}{m^2} \Delta_\xi W + 2\gamma \operatorname{div}_\xi(\xi W).$$

We remark that [19] used an asymptotic expansion procedure in the parameter $\alpha = \gamma\hbar/k_B T$ to derive (2.2). The error of the asymptotics is $O(\alpha^3)$, thus (2.2) and, equivalently, (1.1)-(1.4) can be regarded as medium/high temperature model. When the terms in $L_{QFP}W$ with coefficients D_{qq} and D_{pq} (which both are $O(\frac{1}{T})$) are neglected, then the classical Fokker-Planck operator (Caldeira-Leggett master equation [7]) is obtained. In [20] it is stated that this approach, which gives a high temperature $O(\alpha^2)$ -accurate model, is sensible if the coherence length pertaining to the state of the electron is larger than the de Broglie wavelength $\lambda_{dB} = \hbar/\sqrt{4mk_B T}$. When the term with coefficient γ is also dropped, we obtain an $O(\alpha)$ -accurate model (very high temperature asymptotics) with $L_{QFP} = \frac{D_{pp}}{m^2} \Delta_\xi$. Then electron energy dissipation by the thermal bath is not described. The mathematical analysis presented in the following sections will be concerned with exactly this case. This simplified model requires the most complex mathematical analysis in the class of (Lindblad) equations (cf. below) due to its lack of x -ellipticity.

2.2. The density matrix equation in Lindblad form

Many important mathematical properties of open quantum systems can be shown to hold if the corresponding Markovian evolution equation for the density operator $R(t)$ is of so-called *Lindblad form* (cf. [26], [13], [21]).

The evolution equation

$$(2.9) \quad \frac{d}{dt}R = -\frac{i}{\hbar}[H_E, R] + A(R),$$

where H_E is self-adjoint, is called of Lindblad form, if linear operators L_j , $j = 1, 2, 3, \dots$, exist, such that

$$(2.10) \quad A(R) = \sum_{j=1}^{\infty} L_j R L_j^* - \frac{1}{2}(LR + RL)$$

with

$$(2.11) \quad L := \sum_{j=1}^{\infty} L_j^* L_j$$

holds. Here $[A, B] := AB - BA$ denotes the commutator of the operators A, B . Also $H_E, L_j, j = 1, 2, \dots$, have to satisfy additional assumptions such that $-\frac{i}{\hbar}[H_E, \cdot] + A(\cdot)$ generates a C^0 -semigroup on the space of trace class operators on $L^2(\mathbb{R}^d)$. For details we refer to [13]. The main properties of evolution equations in Lindblad form are:

- (a) *conservation of positivity* (cf. [15], [26], [13]):

$$(2.12) \quad R(0) \geq 0 \implies R(t) \geq 0, \quad \forall t > 0$$

(in the sense of positive definite operators). In fact, the Lindblad form even gives complete positivity of the evolution semigroup (cf. [26]). The positivity of $R(t)$ implies that the Husimi transform $W^h = W^h(x, \xi, t)$ of $R(t)$ is pointwise nonnegative on $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ (cf. [27], [23]):

$$(2.13) \quad W^h(x, \xi, t) := W(x, \xi, t) \star_x \Gamma_{\frac{\hbar}{m}}(x) \star_\xi \Gamma_{\frac{\hbar}{m}}(\xi) \geq 0, \quad \text{on } \mathbb{R}_x^d \times \mathbb{R}_\xi^d,$$

where W is the Wigner function (2.5) and Γ_σ the Gaussian

$$(2.14) \quad \Gamma_\sigma(u) = \frac{1}{(\pi\sigma)^{d/2}} \exp\left(-\frac{|u|^2}{\sigma}\right).$$

As a consequence, a simple calculation leads to

$$\int_{\mathbb{R}_x^d} \int_{\mathbb{R}_\xi^d} W^h(x, \xi, t) d\xi dx = \int_{\mathbb{R}_x^d} \int_{\mathbb{R}_\xi^d} W(x, \xi, t) d\xi dx.$$

Note that $\text{Tr}R(t)$, the total charge of the electron ensemble, is left invariant by the evolution: $\text{Tr}R(t) = \text{Tr}R(0)$, $\forall t > 0$, so that

$$(2.15) \quad M = \text{Tr}R(0) = \int_{\mathbb{R}^d} n(x, t) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W^h(x, \xi, t) d\xi dx \geq 0.$$

We remark that the Husimi transform shall not be used for proving existence of local-in-time solutions in §3. However, it shall be needed in §5 for establishing the large-time behavior of the WFPF solution. The positivity of W^h is crucial for deriving estimates on the kinetic energy and the inertial momentum (cf. (2.29), (2.30)).

- (b) *dissipativity* (in the space of trace class operators, cf. § X.8 of [33], [32]): the inequality $\langle A(R), \text{sgn}(R) \rangle_{HS} \leq 0$ holds, where $\langle A, B \rangle_{HS} = \text{Tr}(AB^*)$ is the usual scalar product on the Hilbert space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$. $A(R)$ is then a dissipative operator in the sense of the semigroup generator.
- (c) *entropy growth* (cf. [3]): If

$$(2.16) \quad \sum_{j=1}^{\infty} L_j L_j^* \leq \sum_{j=1}^{\infty} L_j^* L_j,$$

then the quantum entropy (cf. [37]) $S(R) := -\text{Tr}(R \ln R)$ satisfies

$$(2.17) \quad \frac{d}{dt} S(R(t)) \geq 0$$

for all operators $R(t)$.

Obviously, the properties (b) and (c) refer to the irreversibility of the evolution equation (2.9) (in nontrivial cases).

We shall now try to write the equation (2.2) in Lindblad form. Therefore we set

$$(2.18) \quad L_j = r x_j + \delta \partial_{x_j}; \quad r, \delta \in \mathbb{C}, \quad j = 1, \dots, d,$$

where x_j stands here for the operator representing multiplication by the j -th position coordinate. Clearly, $L_j^* = \bar{r} x_j - \bar{\delta} \partial_{x_j}$. Also we set

$$(2.19) \quad L_{d+j} = w x_j; \quad L_{2d+j} = \varphi \partial_{x_j}; \quad w, \varphi \in \mathbb{C}, \quad j = 1, \dots, d.$$

Moreover, we define the “adjusted” Hamiltonian (see [21])

$$(2.20) \quad H_E = H - i \frac{\hbar \mu}{2} \sum_{j=1}^d \{x_j, \partial_{x_j}\}, \quad \mu \in \mathbb{R},$$

where $\{A, B\} = AB + BA$ denotes the anti-commutator of the operators A and B . Obviously, H_E is (formally) self-adjoint. A lengthy but simple exercise gives the integral kernel $a = a(x, y)$ of

$$-\frac{i}{\hbar}[H_E, R] + \sum_{j=1}^{3d} L_j R L_j^* - \frac{1}{2} \left(\sum_{j=1}^{3d} L_j^* L_j R + R \sum_{j=1}^{3d} L_j^* L_j \right).$$

It has the form

$$(2.21) \quad a(x, y) = -\frac{i}{\hbar}(H_x - H_y)\rho + d(\operatorname{Re}(\delta\bar{r}) - \mu)\rho \\ - \frac{1}{2}(|r|^2 + |w|^2)|x - y|^2\rho + \frac{1}{2}(|\delta|^2 + |\varphi|^2)|\nabla_x + \nabla_y|^2\rho \\ - \left((\mu - i\operatorname{Im}(\delta\bar{r}))x \cdot \nabla_x - \delta\bar{r}y \cdot \nabla_x - \delta\bar{r}x \cdot \nabla_y + (\mu + i\operatorname{Im}(\delta\bar{r}))y \cdot \nabla_y \right)\rho.$$

A comparison shows that we can choose the parameters $r, w, \delta, \varphi \in \mathbb{C}$ and $\mu \in \mathbb{R}$ such that the right-hand side of (2.2) comes out, iff

$$(2.22) \quad |(r, w)|^2 = \frac{2D_{pp}}{\hbar^2}, \quad |(\delta, \varphi)|^2 = 2D_{qq}, \quad \mu = \operatorname{Re}(\delta\bar{r}) = \gamma, \quad \operatorname{Im}(\delta\bar{r}) = \frac{2}{\hbar}D_{pq}.$$

We easily can conclude that we can find parameters $r, w, \delta, \varphi, \mu$ satisfying these equations iff the reservoir parameters are such that the following matrix is positive definite:

$$(2.23) \quad \begin{pmatrix} D_{qq} & D_{pq} + \frac{i}{2}\hbar\gamma \\ D_{pq} - \frac{i}{2}\hbar\gamma & D_{pp} \end{pmatrix} \geq 0.$$

In terms of the original thermal bath constants, this condition reads (see (2.4))

$$(2.24) \quad \frac{\hbar\Omega}{k_B T} \leq \sqrt{3}\pi$$

or $\eta = 0$ (no coupling to the thermal bath, trivial case). (2.24) is satisfied for medium-high temperatures. We remark that (2.23), (2.24) can be found in [19], [18].

Under condition (2.23) one possible choice of the Lindblad operators is given by:

$$r = \frac{\sqrt{2D_{pp}}}{\hbar}, \quad \delta = \frac{\hbar\gamma - 2iD_{pq}}{\sqrt{2D_{pp}}}, \quad w = 0, \quad \varphi^2 = 2D_{qq} - |\delta|^2.$$

This implies $\sum_{j=1}^{3d}(L_j^* L_j - L_j L_j^*) = -2d\gamma$, and entropy growth for all initial density matrices can only be concluded in the frictionless case $\gamma = 0$ (see (2.16), (2.17)).

We shall now argue that the condition (2.23) is also necessary for (2.2) to be of Lindblad form. The above representation in Lindblad form is (as usual) not unique. However, only the mixed x/y -terms in (2.2) are relevant for the validity of the Lindblad form. They arise from the operators $L_j R L_j^*$, $j = 1, \dots, 3d$. Since these operators are positive, the cancellation of different Lindblad operators is not possible. From the structure of (2.21) and (2.2) we readily see that only Lindblad operators of the form (2.18)–(2.19) can be used to represent the right-hand side of (2.2). Assume now that (2.2) can be represented by two operators of the form (2.18) (the other cases are trivial). Using (the obvious generalization of) the relations (2.22) we estimate:

$$|\gamma + \frac{2i}{\hbar} D_{pq}|^2 = |\bar{\delta}_1 r_1 + \bar{\delta}_2 r_2|^2 \leq (|\delta_1|^2 + |\delta_2|^2)(|r_1|^2 + |r_2|^2) \leq \frac{4}{\hbar^2} D_{qq} D_{pp},$$

and (2.23) follows.

Note that the Caldeira–Leggett master equation (with $D_{pq} = D_{qq} = 0$) is not of Lindblad form and hence the conservation of positivity of $R(t)$ is not guaranteed. In fact one can easily construct an initial density matrix $R(0)$, such that the positivity will be lost under temporal evolution with the Caldeira–Leggett equation. The very high temperature model (γ, D_{pq}, D_{qq} set to zero), however, is of Lindblad form.

2.3. Equilibrium states

The dissipativity of the quantum Fokker–Planck operator (we assume now that the Lindblad condition (2.23) holds) immediately raises the question of possible equilibrium states of L_{QFP} . Therefore we rewrite $L_{QFP}W$ as follows

$$(2.25) \quad \frac{D_{pp}}{m^2} \operatorname{div}_\xi \left(e^{-\frac{\gamma m^2}{D_{pp}}|\xi|^2} \nabla_\xi (e^{\frac{\gamma m^2}{D_{pp}}|\xi|^2} W) \right) + 2 \frac{D_{pq}}{m} \operatorname{div}_x (\nabla_\xi W) + D_{qq} \Delta_x W,$$

multiply by $z := W \exp(\frac{\gamma m^2}{D_{pp}}|\xi|^2)$ and integrate by parts. We obtain

$$(2.26) \quad \int_{\mathbb{R}_x^d} \int_{\mathbb{R}_\xi^d} e^{\frac{\gamma m^2}{D_{pp}}|\xi|^2} W (L_{QFP}W) d\xi dx = - \int_{\mathbb{R}_x^d} \int_{\mathbb{R}_\xi^d} e^{-\frac{\gamma m^2}{D_{pp}}|\xi|^2} \times \left(\frac{D_{pp}}{m^2} |\nabla_\xi z|^2 - \frac{2D_{pq}}{m} \nabla_x z \cdot \nabla_\xi z + D_{qq} |\nabla_x z|^2 \right) d\xi dx.$$

We estimate, using (2.23),

$$\det \begin{pmatrix} \frac{D_{pp}}{m^2} & \frac{D_{pq}}{m} \\ \frac{D_{pq}}{m} & D_{qq} \end{pmatrix} \geq \frac{\hbar^2 \gamma^2}{4m^2}.$$

For $\gamma > 0$ we conclude that (2.26) is strictly positive unless $z = \text{const.}$, which gives $W_e \equiv 0$ as the unique state with $L_{QFP}W_e = 0$. For $\gamma = 0$ we easily obtain the necessary and sufficient condition

$$\nabla_\xi W_e + m\sqrt{\frac{D_{qq}}{D_{pp}}}\nabla_x W_e = 0$$

for $L_{QFP}W_e = 0$ to hold. This implies

$$W_e(x, \xi) = h\left(-m\sqrt{\frac{D_{qq}}{D_{pp}}}\xi + x\right)$$

for some scalar function h , which again implies $W_e = 0$ for $W_e \in L^2(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$. Thus, the Lindblad condition excludes nontrivial equilibrium states of the quantum Fokker–Planck operator. This seems quite natural, since relaxation towards a nontrivial steady electron state should arise as consequence of the presence of an external potential or through the evolution of the particle quasi–probability function in a bounded domain with appropriate boundary conditions (see, for example, [4]).

Also we remark that the Lindblad condition (2.23) and $\gamma > 0$ (non–vanishing friction) imply that L_{QFP} is uniformly elliptic in $(x, \xi) \in \mathbb{R}^{2d}$.

2.4. Propagation of moments

Another important question concerns the behavior of the zeroth, first and second order velocity moments of the solution W of (1.1)–(1.4). We define the electron position density

$$(2.27) \quad n(x, t) = \int_{\mathbb{R}_\xi^d} W(x, \xi, t) d\xi,$$

the electron current density

$$(2.28) \quad J(x, t) = \int_{\mathbb{R}_\xi^d} \xi W(x, \xi, t) d\xi,$$

the electron kinetic energy density

$$(2.29) \quad e(x, t) = \int_{\mathbb{R}_\xi^d} \frac{|\xi|^2}{2} W(x, \xi, t) d\xi$$

and the electron inertial momentum density

$$(2.30) \quad \mathcal{I}(\xi, t) = \int_{\mathbb{R}_x^d} |x|^2 W(x, \xi, t) dx.$$

Instead of the ‘usual’ continuity equation we obtain here

$$(2.31) \quad \frac{\partial n}{\partial t} + \operatorname{div}_x J = D_{qq} \Delta_x n,$$

and hence we have conservation of the total charge:

$$(2.32) \quad \int_{\mathbb{R}_x^d} n(x, t) \, dx \equiv \int_{\mathbb{R}_x^d} n(x, t = 0) \, dx = M, \quad \forall t > 0.$$

The current density satisfies

$$(2.33) \quad \frac{d}{dt} \int_{\mathbb{R}_x^d} J(x, t) \, dx + \frac{1}{m} \int_{\mathbb{R}_x^d} \nabla_x V(x, t) n(x, t) \, dx = -2\gamma \int_{\mathbb{R}_x^d} J(x, t) \, dx.$$

Also, the energy density solves

$$(2.34) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_x^d} e(x, t) \, dx + \frac{1}{m} \int_{\mathbb{R}_x^d} \nabla_x V(x, t) \cdot J(x, t) \, dx = \\ = d \frac{D_{pp}}{m^2} M - 4\gamma \int_{\mathbb{R}_x^d} e(x, t) \, dx, \end{aligned}$$

and the inertial momentum satisfies

$$(2.35) \quad \frac{d}{dt} \int_{\mathbb{R}_\xi^d} \mathcal{I}(\xi, t) \, d\xi = 2 \int_{\mathbb{R}_x^d} x \cdot J(x, t) \, dx + 2D_{qq} M d.$$

For future reference we state the following well-known (formal) identities for positive density matrices $R(t) \geq 0$:

$$(2.36) \quad n(x, t) = \rho \left(x + \frac{\hbar\eta}{2m}, x - \frac{\hbar\eta}{2m}, t \right) \Big|_{\eta=0} \geq 0$$

and

$$(2.37) \quad \int_{\mathbb{R}_x^d} e(x, t) \, dx = \frac{1}{2} \operatorname{Tr}(-\Delta_x R(t)) \geq 0.$$

Also, by duality, $\int_{\mathbb{R}_\xi^d} \mathcal{I}(\xi, t) \, d\xi = \operatorname{Tr}(|x|^2 R(t)) \geq 0$. Clearly, (2.36) implies

$$(2.38) \quad \int_{\mathbb{R}_x^d} n(x, t) \, dx = \operatorname{Tr} R(t).$$

In the case of Poisson coupling $-\Delta_x V = \alpha n$ ($\alpha = \pm 1$) the equation (2.33) becomes

$$(2.39) \quad \frac{d}{dt} \int_{\mathbb{R}_x^d} J(x, t) \, dx = -2\gamma \int_{\mathbb{R}_x^d} J(x, t) \, dx,$$

which gives exponential relaxation of the total current:

$$(2.40) \quad \int_{\mathbb{R}_x^d} J(x, t) dx = e^{-2\gamma t} \int_{\mathbb{R}_x^d} J(x, t = 0) dx.$$

Similarly, (2.34) becomes

$$(2.41) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}_x^d} \left(e(x, t) + \frac{\alpha}{2m} |\nabla_x V(x, t)|^2 \right) dx = \\ & = d \frac{D_{pp}}{m^2} M - 4\gamma \int_{\mathbb{R}_x^d} e(x, t) dx - \frac{\alpha D_{qq}}{m} \int_{\mathbb{R}_x^d} n(x, t)^2 dx. \end{aligned}$$

Also we shall use the Husimi energy E^h and inertial momentum I^h , which are related to $e(x, t)$ and, respectively, $\mathcal{I}(\xi, t)$ by means of

$$(2.42) \quad E^h(t) = \int_{\mathbb{R}_x^d} \int_{\mathbb{R}_\xi^d} \frac{|\xi|^2}{2} W^h d\xi dx = \int_{\mathbb{R}_x^d} \int_{\mathbb{R}_\xi^d} \frac{|\xi|^2}{2} W d\xi dx + \frac{dM\sigma}{4},$$

$$(2.43) \quad I^h(t) = \int_{\mathbb{R}_\xi^d} \int_{\mathbb{R}_x^d} |x|^2 W^h dx d\xi = \int_{\mathbb{R}_\xi^d} \int_{\mathbb{R}_x^d} |x|^2 W dx d\xi + \frac{dM\sigma}{2}.$$

We remark that the formal equations (2.29) and (2.36)–(2.38) only hold under restrictive regularity assumptions on the density matrix $R(t)$. On the contrary, the Husimi energy is well–defined for all positive density matrices $R(t)$ with finite trace and finite kinetic energy $\frac{1}{2}\text{Tr}(-\Delta_x R(t))$ (see [27], [1]).

3. Existence of local–in–time solutions

In this section, local–in–time existence of mild solutions of the WFPF system in 3D is discussed. Here, “local in time” means that for a fixed $T > 0$ there exists a ball of “admissible” initial data for which the corresponding solutions are defined on $(0, T]$. Moreover, T becomes arbitrarily large for “sufficiently small” initial data. We shall focus our attention on the study of the frictionless WFPF equation.

Physically, the relevant situation is to consider initial data W^I that correspond to a positive density matrix operator $R^I = R[W^I]$ (see (2.1), (2.5)). Since our model is in Lindblad form this would then imply $R[W(t)] \geq 0$, $t \geq 0$, and hence the (formal) identities (2.32), (2.36) yield the following *a priori* estimate on the position density $n(x, t) = \int_{\mathbb{R}_\xi^3} W(x, \xi, t) d\xi$:

$$(3.1) \quad \|n(t)\|_{L^1(\mathbb{R}_x^3)} = \|n^I\|_{L^1(\mathbb{R}_x^3)} = M.$$

Indeed, in Proposition 3.10 below we shall prove $W(t) \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ for $t \in [0, T]$, and this then gives sense to $\|n(t)\|_{L^1(\mathbb{R}_x^3)}$.

In this section, however, we will not require $R^I \geq 0$ as it is not needed from a mathematical viewpoint. (3.1) is then replaced by

$$(3.2) \quad \|n(t)\|_{L^1(\mathbb{R}_x^3)} \leq \|n[R_+^I]\|_{L^1(\mathbb{R}_x^3)} + \|n[R_-^I]\|_{L^1(\mathbb{R}_x^3)},$$

where $R^I = R_+^I - R_-^I$ is the spectral decomposition of R^I into its positive and negative part.

The mathematical analysis of the frictionless WFPF model is in some sense more complicated than for the model with friction since the collision term only acts in the velocity variable ξ . Thus, it is necessary to prove the regularizing effect of the equation with respect to the position coordinate by a more detailed analysis than for the friction model. In the sequel we assume $d = 3$ and the normalization condition $\hbar = m = 1$ for simplicity of the calculations. Then, an important part of our efforts will be devoted to prove that the density $n(x, t)$ is regularized as far as $L^\infty(\mathbb{R}^3 \times (0, T))$, provided that the initial averaged density $\int_{\mathbb{R}_\xi^3} W^I(x - t\xi, \xi) d\xi$, involving the displaced variable $x - t\xi$, belongs to $L^{p_0}(\mathbb{R}^3)$ for an appropriate p_0 . Also, if $p_0 > \frac{9}{8}$ and $W^I \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ we shall see that the potential $V(\cdot, t) \in W^{1,p}(\mathbb{R}^3)$ with $p > p(p_0)$ and, in particular, $V(\cdot, t) \in L^\infty(\mathbb{R}^3)$. In addition, if $p_0 > \frac{3}{2}$ we shall prove that $\nabla_x V(\cdot, t) \in L^\infty(\mathbb{R}^3)^3$. Then, the equivalent integral WFPF equation will be shown to admit a unique solution $W(x, \xi, t) \in C([0, T]; L^1 \cap L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$, where T depends on the initial data and q depends on p_0 and on the regularity of W^I .

The main point in deriving these estimates lies in proving the required regularity for the potential and the Wigner function in order to reformulate the nonlinear term of the WFPF equation as a convolution with respect to the momentum variable:

$$(3.3) \quad -\Theta[V]W(x, \xi, t) = \Phi(x, \xi, t) \star_\xi W(x, \xi, t),$$

where

$$(3.4) \quad \Phi(x, \xi, t) = -\frac{i}{(2\pi)^3} \int_{\mathbb{R}_\eta^3} \left(V(x + \frac{\eta}{2}, t) - V(x - \frac{\eta}{2}, t) \right) e^{-i\xi \cdot \eta} d\eta.$$

In fact, we easily deduce that

$$(3.5) \quad \begin{aligned} \Phi &= -i\mathcal{F}_{\eta \rightarrow \xi}^{-1} \left(V(x + \frac{\eta}{2}, t) - V(x - \frac{\eta}{2}, t) \right) \\ &= -16\text{Re}[ie^{2i\xi \cdot x} \mathcal{F}_{x \rightarrow \xi}^{-1} V(2\xi, t)], \end{aligned}$$

where we denoted the inverse Fourier transform

$$\mathcal{F}_{x \rightarrow y}^{-1} f = (2\pi)^{-3} \int_{\mathbb{R}_x^3} f(x) e^{-ix \cdot y} dx.$$

Thus, it suffices to control $\|\mathcal{F}_{x \rightarrow \xi}^{-1} V(\cdot, t)\|_{L^1(\mathbb{R}_\xi^3)}$ and $\|W(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}$ to give sense to the convolution. In order to give a rigorous sense to $\Theta[V]W$ we shall later show (under some mild assumptions on the initial data) that $V(\cdot, t) \in L^q(\mathbb{R}_x^3)$ for some $q > 3$ and $(\mathcal{F}_{\xi \rightarrow \eta} W)(x, \cdot, t) \in L^r(\mathbb{R}_\eta^3)$ for some $r < 6$ which implies that the inverse Fourier transform of (1.4) is well defined. Therefore the equality (3.3) holds. In this case, the WFPF equation can be equivalently written as

$$(3.6) \quad \frac{\partial W}{\partial t} + (\xi \cdot \nabla_x)W - \sigma \Delta_\xi W = \Phi \star_\xi W,$$

with the right–hand side being local in position and non–local in velocity, see also [2].

We start our study with some notations and definitions.

3.1. The fundamental solution and the concept of mild solution

We are now concerned with a description of the fundamental solution of the FP equation, as well as with the statement of some of its properties. The concept of mild solution of the WFPF problem is then introduced as a solution of an equivalent integral equation involving the fundamental solution of the linear FP operator.

The Green’s function G associated with the linear kinetic FP problem is the fundamental solution of

$$(3.7) \quad L[W] \stackrel{\text{def}}{=} \frac{\partial W}{\partial t} + (\xi \cdot \nabla_x)W - \sigma \Delta_\xi W = 0,$$

satisfying $W(t = 0) = \delta(x, \xi)$. This fundamental solution can be written as follows (see, for instance, [10] or [5] for details)

$$(3.8) \quad G(x, \xi, z, v, t) = G_0(x - z - tv, \xi - v, t), \quad x, \xi, z, v \in \mathbb{R}^3, t \geq 0,$$

where

$$(3.9) \quad G_0(x, \xi, t) = \frac{(3/4)^{\frac{3}{2}}}{(\pi\sigma)^3 t^6} \exp \left\{ -\frac{3|x|^2 + 3|x - t\xi|^2 - t^2|\xi|^2}{2\sigma t^3} \right\}.$$

As usual, formulae (3.8) and (3.9) are found by Fourier transforming the linear equation (3.7) in the (x, ξ) –variables and then integrating the resulting linear first–order hyperbolic equation along the characteristics, cf. [10], [11], [5]. In the following lemma we list some of the properties of G that will be useful in the sequel to obtain regularity and compactness for the mild solutions of the WFPF system (see [10], [5], [9]).

Lemma 3.1 *The fundamental solution G associated with the kinetic FP equation, given by relations (3.8) and (3.9), satisfies the following properties:*

(i) *For any $t \geq 0$ and $x, \xi, z, v \in \mathbb{R}^3$ we have*

$$\begin{aligned} \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, t) dz dv &= \int_{\mathbb{R}_\xi^3} \int_{\mathbb{R}_x^3} G(x, \xi, z, v, t) dx d\xi = 1, \\ \int_{\mathbb{R}_x^3} G(x, \xi, z, v, t) dx &= \frac{1}{(4\pi\sigma t)^{\frac{3}{2}}} \exp\left\{-\frac{|\xi - v|^2}{4\sigma t}\right\}, \\ \int_{\mathbb{R}_\xi^3} G(x, \xi, z, v, t) d\xi &= \frac{1}{((4/3)\pi\sigma)^{\frac{3}{2}} t^{\frac{9}{2}}} \exp\left\{-\frac{|x - z - tv|^2}{(4/3)\sigma t^3}\right\}. \end{aligned}$$

(ii) *For any $h \geq 0$ and for any $x, \xi, z, v \in \mathbb{R}^3, t \geq 0$, the following equality is satisfied:*

$$\begin{aligned} \int_{\mathbb{R}_\xi^3} G(x - h\xi, \xi, z, v, t) d\xi &= \\ &= \frac{1}{(4\pi\sigma(t^3/3 + ht^2 + h^2t))^{\frac{3}{2}}} \exp\left\{-\frac{|x - z - (t + h)v|^2}{4\sigma(t^3/3 + ht^2 + h^2t)}\right\}. \end{aligned}$$

(iii) *For any $\epsilon > 0$ and for any $x, \xi \in \mathbb{R}^3, t \geq 0$, the fundamental solution G is scale-invariant (or, equivalently, self-similar) in the following sense:*

$$G_0(x, \xi, t) = \epsilon^{-12} G_0(\epsilon^{-3}x, \epsilon^{-1}\xi, \epsilon^{-2}t).$$

In the sequel we shall consider the WFPF equation as a nonlinear perturbation of equation (3.7). In this context, let us consider the pseudo-differential term $\Theta[V]W$ as a force term in the right-hand side of (3.7). Then, if we assume that (W, V) is a regular solution of the WFPF equation, this problem may be reformulated by using the relation (3.3) in terms of the integral equation

$$\begin{aligned} W(x, \xi, t) &= \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, t) W^I(z, v) dz dv \\ (3.10) \quad &+ \int_0^t \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, s) (\Phi \star_\xi W)(z, v, t - s) dz dv ds. \end{aligned}$$

We remark that, under some regularity conditions on the Wigner function and the potential that will be proved in the sequel, the problems WFPF and (3.10) will be shown to be equivalent. We may consider the solution W to be split into two parts $W = W^1 + W^2$, the first of which W^1 , the linear

part, just depending on the initial data W^I and the second one W^2 basically depending upon the potential V through the pseudo–differential operator $\Theta[V]$. This decomposition is particularly suitable for a convolution, in order to get estimates and stability results for the Wigner function W . A similar method was first proposed by G.H. Cottet and J. Soler in [14] for the same purpose in the context of Navier–Stokes equations. In fact, we adopt formula (3.10) as our definition of mild solution (see also [11], [32]). Here we state the precise concept:

Definition 3.2 *We call the pair (W, V) , belonging to $C([0, T]; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ and to $C((0, T]; L^\infty(\mathbb{R}^3))$, respectively, a mild solution of the WFPF equation (3.6) with initial data $W^I \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ if $\Phi \star_\xi W$ is locally integrable with respect to the Lebesgue measure $d(x, \xi, t)$, and if (W, V) solves the integral equation (3.10) for $0 \leq t \leq T$, and $V = \frac{\alpha}{4\pi} \frac{1}{|x|} \star n$ with $n(\cdot, t) \in L^1(\mathbb{R}^3)$ for $0 \leq t \leq T$.*

Also, we shall say that $f = f(x, \xi, t) \in S_p$ if

$$(3.11) \quad \max_{h>0} \left\{ \left\| \int_{\mathbb{R}_\xi^3} f(x - h\xi, \xi, t) d\xi \right\|_{L^p(\mathbb{R}^3)} \right\} < \infty.$$

Our main result in this section is the following:

Theorem 3.3 *Let $W^I \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap S_{p_0}$ be the initial condition of the WFPF problem such that $p_0 > 9/8$. Then, the WFPF problem admits a mild solution $W \in C([0, T]; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap S_r)$, with $r > \frac{3}{2}$ and the decay bounds established in (3.34), defined on a maximal time interval $[0, T)$. If in addition $W^I \in L^p(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$, then $W \in C([0, T); L^1 \cap L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$, where p_0, p and q are related as in (3.45). Also, the density and the potential satisfy the additional regularity properties:*

$$V(\cdot, t) \in L^\infty(\mathbb{R}^3), \quad \nabla_x V(\cdot, t) \in L^q(\mathbb{R}^3)^3, \quad n(\cdot, t) \in L^p(\mathbb{R}^3),$$

such that $q = q(p_0)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. Furthermore, if $p_0 > \frac{3}{2}$ then $\nabla_x V \in L^\infty(\mathbb{R}^3)^3$ and if $p_0 > \frac{9}{2}$ then $n \in L^\infty(\mathbb{R}^3)$, with the bounds given in Proposition 3.9. Here, T depends on σ, p_0 and W^I . Also, T is arbitrarily large for initial data W^I with $S_{p_0}^I$ and $\|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}$ sufficiently small.

The rest of the section is devoted to the proof of this theorem. At this point we should distinguish between the mild solution to Eq. (3.6) (with convoluted nonlinearity) for $p_0 > 9/8$, in terms of which Theorem 3.3 is stated, and the mild solution to Eq. (1.6) (with the original form of the pseudo–differential operator) for $p_0 > 7/6$. The equivalence between both of them is sketched in the last remark of §3.2.

We notice that $W^1(x, \xi, t)$ in the above decomposition actually solves the linear WFP problem $L[W] = 0$ ($V \equiv 0$) with initial datum W^I . In addition, we have (see [9], [5], [11])

Lemma 3.4 *If we define*

$$(3.12) \quad f(x, \xi, t) = \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, t) g(z, v) dz dv,$$

then the following decay estimate

$$\|f(\cdot, \cdot, t)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \leq C t^{-6(\frac{1}{p} - \frac{1}{q})} \|g\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}, \quad 1 \leq p \leq q \leq \infty, \quad t > 0,$$

holds, where C is a positive constant. In particular, for $p = q$ we have

$$\|f(\cdot, \cdot, t)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \leq \|g\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}, \quad t \geq 0.$$

The same result holds true when

$$(3.13) \quad f(x, \xi, t) = \int_0^t \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, t - s) g(z, v, s) dz dv ds.$$

In this case, for every $1 \leq p \leq q \leq \infty$, we have

$$\|f(\cdot, \cdot, t)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \leq C \int_0^t (t - s)^{-6(\frac{1}{p} - \frac{1}{q})} \|g(\cdot, \cdot, s)\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} ds.$$

In order to give simpler expressions for the corresponding position densities $n^k = n(W^k)$, with $k = 1, 2$, we now introduce the following notation:

$$(3.14) \quad \mathcal{N}(x) = \frac{1}{((4/3)\pi\sigma)^{\frac{3}{2}}} e^{-\frac{|x|^2}{2}}, \quad d_h(t) = \frac{t^3}{3} + ht^2 + h^2t.$$

Then, we find the following integral representation for the “two-parts” density (with the notation $n^{1,2} = \int W^{1,2} d\xi$):

$$(3.15) \quad n^1(x, t) = \frac{1}{((4/3)\pi\sigma)^{\frac{3}{2}} t^{\frac{9}{2}}} \int_{\mathbb{R}_z^3} \exp\left\{-\frac{|x - z|^2}{(4/3)\sigma t^3}\right\} \times \int_{\mathbb{R}_v^3} W^I(z - tv, v) dv dz,$$

$$(3.16) \quad n^2(x, t) = \frac{1}{((4/3)\pi\sigma)^{\frac{3}{2}}} \int_0^t s^{-\frac{9}{2}} \int_{\mathbb{R}_z^3} \exp\left\{-\frac{|x - z|^2}{(4/3)\sigma s^3}\right\} \times \int_{\mathbb{R}_v^3} (\Phi \star_\xi W)(z - sv, v, t - s) dv dz ds,$$

where we have used Lemma 3.1 (i) and the change of variables $z - tv \mapsto z$.

Formulae (3.15) and (3.16) illustrate the fact that the solution operator acts on the density just as a convolution in the position variable by a Gaussian which spreads with time, producing a regularizing effect on the system (see [5], [10]). Indeed, formulae (3.15)–(3.16) can be rewritten as

$$(3.17) \quad n^1(x, t) = t^{-\frac{9}{2}} \mathcal{N}\left(\frac{x}{\sqrt{(2/3)\sigma t^3}}\right) \star_x \int_{\mathbb{R}^3} W^I(x - tv, v) dv,$$

$$(3.18) \quad n^2(x, t) = \int_0^t s^{-\frac{9}{2}} \mathcal{N}\left(\frac{x}{\sqrt{(2/3)\sigma s^3}}\right) \star_x \int_{\mathbb{R}^3} (\Phi \star_\xi W)(x - sv, v, t - s) dv ds.$$

The next crucial ingredient lies in deriving *a priori* estimates. For this purpose it is not sufficient to assume L^p -bounds of the initial density $n^I = n(W^I)$. As one can see from (3.17) and (3.18) we shall also need to control some appropriate L^p -norms of

$$(3.19) \quad n_h^I(x) = n_h(W^I)(x) = \int_{\mathbb{R}^3} W^I(x - h\xi, \xi) d\xi,$$

uniformly with respect to h , for all $h \geq 0$.

Thus, according to Lemma 3.1 (ii) we can express the density averages $n_h^k = n_h(W^k)$, following the notation introduced in (3.19), as follows:

$$(3.20) \quad n_h^1(x, t) = \int_{\mathbb{R}^3} W^1(x - h\xi, \xi, t) d\xi = \frac{1}{(3d_h(t))^{\frac{3}{2}}} \mathcal{N}\left(\frac{x}{\sqrt{2\sigma d_h(t)}}\right) \star_x \int_{\mathbb{R}^3} W^I(x - (t+h)v, v) dv,$$

$$(3.21) \quad n_h^2(x, t) = \int_{\mathbb{R}^3} W^2(x - h\xi, \xi, t) d\xi = \int_0^t \frac{1}{(3d_h(s))^{\frac{3}{2}}} \mathcal{N}\left(\frac{x}{\sqrt{2\sigma d_h(s)}}\right) \star_x \int_{\mathbb{R}^3} (\Phi \star_\xi W)(x - (s+h)v, v, t - s) dv ds.$$

3.2. *A priori* estimates

In this section we shall derive *a priori* estimates for mild solutions of (3.10), i.e. for $W \in C([0, T]; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$. This regularity assumption implies the following estimate for the particle density $n = \int W d\xi$:

$$(3.22) \quad \|n(t)\|_{L^1(\mathbb{R}_x^3)} \leq \|W(t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}, \quad 0 \leq t \leq T.$$

For a positive initial density matrix R^I and a smooth enough solution, we would of course have the charge conservation (3.1). The main goal of this section is to develop the main tools to derive the uniform boundedness (with respect to $h \geq 0$) of $n_h(\cdot, t)$ in $L^1 \cap L^q$ for some $q > \frac{3}{2}$, which implies the following regularity result on the potential: $V(\cdot, t) \in W^{1,q}(\mathbb{R}^3)$ for some $q > 3$ and $V(\cdot, t) \in L^\infty(\mathbb{R}^3)$ by a Sobolev imbedding.

In Proposition 3.9 we shall prove that $W(\cdot, \cdot, t) \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$, $t > 0$, and hence the convolution in (3.6) is well-defined and the weak formulation (3.6) of the WFPF problem makes sense. Note that for the usual Wigner–Poisson problem without Fokker–Planck scattering kernel the property $W(\cdot, \cdot, t) \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ is generally not satisfied. Actually, this can be considered as a specific property due to the regularizing effect of the Fokker–Planck operator.

Our first task is to prove the L^p -boundedness of the averaged densities n_h^k . For that we define for an arbitrary $f = f(x, \xi, t)$

$$S_p(t, f) = \max\{\|n_h(f)(\cdot, t)\|_{L^p(\mathbb{R}^3)}, h \geq 0\}$$

for any $1 \leq p \leq \infty$, and we denote by

$$S_p^I(f) = \max\{\|n_h^I(f)\|_{L^p(\mathbb{R}^3)}, h \geq 0\} = S_p(0, f).$$

In the case $f = |W|$ the functional dependence of S_p will be omitted in the sequel, i.e. $S_p(t, |W|) = S_p(t)$, and $S_p^j(t)$, $j = 1, 2$, is defined as $S_p^j(t) = S_p(t, |W^j|)$ (compare (3.20), (3.21)). Notice that the S_p norms are natural in this context, since they allow in a natural way to estimate the particle density: $\|n(\cdot, t)\|_{L^p(\mathbb{R}^3)} \leq S_p(t)$.

From now on, unless otherwise specified, C will denote various positive constants depending on generic parameters of the problem. Besides, we shall (by a slight abuse of notation) also denote by S_p the space consisting of all functions f with bounded $S_p(t, f)$ norm (cf. (3.11)).

Lemma 3.5 *Consider f and g related as in (3.12). Then, the following decay estimates hold for any $1 \leq p \leq q \leq \infty$:*

$$S_q(t, f) \leq Ct^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_p(g),$$

and hence from (3.10)

$$(3.23) \quad S_q^1(t) \leq Ct^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_p^I.$$

An analogous result holds true when f and g are related as in (3.13). In this case

$$(3.24) \quad S_q(t, f) \leq C \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_p(s, g) ds.$$

Proof. We focus here on the proof of (3.24), since the first part of the lemma is proved similarly. From the properties of the fundamental solution G we have

$$\begin{aligned} \|n_h(f)(\cdot, t)\|_{L^q(\mathbb{R}_x^3)} &= \left\| \int_0^t \frac{ds}{(4\pi\sigma d_h(t-s))^{\frac{3}{2}}} \exp\left\{-\frac{|x|^2}{4\sigma d_h(t-s)}\right\} \right. \\ &\quad \left. \star_x \int_{\mathbb{R}_v^3} g(x - (t-s+h)v, v, s) dv \right\|_{L^q(\mathbb{R}_x^3)} \\ (3.25) \quad &\leq C \int_0^t ds (t-s)^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} \times \left\| \int_{\mathbb{R}_v^3} g(x - (t-s+h)v, v, s) dv \right\|_{L^p(\mathbb{R}_v^3)} \end{aligned}$$

with $p \leq q$, where we have applied the Young inequality to estimate the convolution in the x variable. Now, taking maximum with respect to the parameter $h \geq 0$ on both sides of this inequality yields the assertion (3.24). ■

Note that this result directly provides for a bound of $n^1(x, t)$ in $L^q(\mathbb{R}^3)$:

$$(3.26) \quad \|n^1(\cdot, t)\|_{L^q(\mathbb{R}^3)} \leq S_q^1(t) \leq Ct^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_p^I, \quad 1 \leq p \leq q \leq \infty.$$

We now proceed to estimate S_q^2 and, hence, $\|n^2(\cdot, t)\|_{L^q(\mathbb{R}^3)}$.

Lemma 3.6 *Let $1 \leq p \leq q \leq \infty$ and $\frac{3}{2} < r \leq 2$ be fixed, and assume that $W \in C([0, T]; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ is a mild solution of (3.10), satisfying (3.22). Then, the following estimates*

$$(3.27) \quad \|\mathcal{F}_{x \rightarrow y}^{-1} V(\cdot, t)\|_{L^1(\mathbb{R}^3)} \leq C \left(S_r(t) + \sqrt{MS_r(t)} \right),$$

$$(3.28) \quad S_q^2(t) \leq C \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_p(s) \left(S_r(s) + \sqrt{MS_r(s)} \right) ds$$

hold, where $M = \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} W^I d\xi dx$ denotes the total charge of the system.

Proof. Set $f = |W^2|$ and $g = \Phi \star_\xi W$ in (3.24) (cf. (3.10)). Then we have

$$(3.29) \quad S_q^2(t) \leq C \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_p(s, |\Phi \star_\xi W|) ds$$

for every $1 \leq p \leq q \leq \infty$. From (3.5), an easy computation leads to

$$(3.30) \quad S_p(t, |\Phi \star_\xi W|) \leq CS_p(t) \|\mathcal{F}_{x \rightarrow y}^{-1} V(\cdot, t)\|_{L^1(\mathbb{R}^3)},$$

since

$$\begin{aligned} &\int_{\mathbb{R}_\xi^3} |\Phi \star_\xi W|(x - h\xi, \xi, t) d\xi \leq \\ &\leq C \int_{\mathbb{R}_\xi^3} \int_{\mathbb{R}_\eta^3} |\mathcal{F}_{x \rightarrow \eta}^{-1} V(2\eta, t)| |W(x - h\xi, \xi - \eta, t)| d\eta d\xi \\ &= C \int_{\mathbb{R}_\eta^3} |\mathcal{F}_{x \rightarrow \eta}^{-1} V(2\eta, t)| |n_h(|W|)(x - h\eta, t)| d\eta, \end{aligned}$$

and hence

$$\begin{aligned} \left\| \int_{\mathbb{R}^3_\xi} |\Phi \star_\xi W|(x - h\xi, \xi, t) d\xi \right\|_{L^p(\mathbb{R}^3_x)} &\leq \\ &\leq C \int_{\mathbb{R}^3_\eta} |\mathcal{F}_{x \rightarrow \eta}^{-1} V(2\eta, t)| \|n_h(|W|)(x - h\eta, t)\|_{L^p} d\eta \\ &\leq C \|\mathcal{F}_{x \rightarrow y}^{-1} V(\cdot, t)\|_{L^1(\mathbb{R}^3)} \|n_h(|W|)(\cdot, t)\|_{L^p(\mathbb{R}^3_x)}. \end{aligned}$$

Finally, we also have

$$\|\mathcal{F}_{x \rightarrow y}^{-1} V(\cdot, t)\|_{L^1} = \frac{1}{4\pi} \left\| \mathcal{F}_{x \rightarrow y}^{-1} \left(\frac{1}{|x|} \star n(x, t) \right) (\cdot, t) \right\|_{L^1} = \left\| \frac{1}{|\cdot|^2} (\mathcal{F}_{x \rightarrow y}^{-1} n)(\cdot, t) \right\|_{L^1}.$$

We now estimate the L^1 norm of $|\cdot|^{-2} (\mathcal{F}_{x \rightarrow y}^{-1} n)(\cdot, t)$ in B_R and B_R^c separately, with B_R denoting the ball of radius R in \mathbb{R}^3 centered at the origin. On the one hand, we find for B_R^c

$$\left\| \frac{1}{|\cdot|^2} (\mathcal{F}_{x \rightarrow y}^{-1} n)(\cdot, t) \right\|_{L^1(B_R^c)} \leq C \|\mathcal{F}_{x \rightarrow y}^{-1} n(\cdot, t)\|_{L^{r'}(\mathbb{R}^3)}.$$

Hence,

$$(3.31) \quad \|\mathcal{F}_{x \rightarrow y}^{-1} V(\cdot, t)\|_{L^1(B_R^c)} \leq C \|n(\cdot, t)\|_{L^r(\mathbb{R}^3)} \leq CS_r(t).$$

On the other hand, using analogous arguments we obtain for B_R :

$$\left\| \frac{1}{|\cdot|^2} (\mathcal{F}_{x \rightarrow y}^{-1} n)(\cdot, t) \right\|_{L^1(B_R)} \leq C \|\mathcal{F}_{x \rightarrow y}^{-1} n(\cdot, t)\|_{L^{\tilde{q}}(\mathbb{R}^3)} \leq C \|n(\cdot, t)\|_{L^{\tilde{q}'}(\mathbb{R}^3)}$$

with $3 < \tilde{q} \leq \infty$. Therefore, the interpolation inequality for L^p spaces yields

$$(3.32) \quad \begin{aligned} \|\mathcal{F}_{x \rightarrow y}^{-1} V(\cdot, t)\|_{L^1(B_R)} &\leq C \|n(\cdot, t)\|_{L^1(\mathbb{R}^3)}^\theta \|n(\cdot, t)\|_{L^r(\mathbb{R}^3)}^{1-\theta} \\ &\leq CM^\theta S_r(t)^{1-\theta}, \end{aligned}$$

where $1/\tilde{q}' = \theta + (1 - \theta)/r$ for the fixed $r \in (\frac{3}{2}, 2]$. In (3.32) we used the bound (3.1) for n . Combining (3.31) and (3.32) yields (3.27), where we have fixed \tilde{q} such that $\theta = \frac{1}{2}$. Now, from (3.30) and (3.27) we obtain

$$(3.33) \quad S_p(t, |\Phi \star_\xi W|) \leq CS_p(t) \left(S_r(t) + \sqrt{MS_r(t)} \right)$$

and the proof concludes by inserting (3.33) into (3.29). ■

In the following result we derive the dominant time decay rates for $S_q(t)$ on bounded time intervals. For that, as easily deduced from Lemma 3.6, it is enough to control the $S_r(t)$ norm for some $r > \frac{3}{2}$.

Proposition 3.7 *Let W be a mild solution of (3.10) on $[0, T]$.*

(a) *Let p_0 with $\frac{9}{8} < p_0 < 2$ be fixed, and W^I be the initial datum for the WFPF system such that $S_{p_0}^I$ is bounded. Then, there exists a $T > 0$ depending on $S_{p_0}^I$ and M such that, for every $0 < t \leq T$, the estimate*

$$(3.34) \quad S_q(t) \leq C(T)t^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{q})}$$

is satisfied, where $q = q(p_0) > \frac{3}{2}$ is arbitrarily close to $\frac{3}{2}$ for p_0 sufficiently close to $\frac{9}{8}$. Estimate (3.34) is also valid for $q = p_0$ and hence, by interpolation, for any q such that $p_0 \leq q \leq q(p_0)$. If $p_0 > \frac{3}{2}$ one can choose $q(p_0) > 3$ in (3.34).

(b) *If $p_0 > \frac{9}{2}$, then (3.34) holds true for $q = \infty$.*

Proof. Part (a): (3.23) and Lemma 3.6 yield the following estimate for $S_q(t)$:

$$(3.35) \quad S_q(t) \leq C_1 t^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{q})} S_{p_0}^I + C_2 \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p} - \frac{1}{q})} S_p(s) \left(S_r(s) + \sqrt{MS_r(s)} \right) ds,$$

for any $1 \leq p \leq q \leq \infty$ with $q \geq p_0$ and C_1 and C_2 represent different positive constants. The constant $r = r(p_0)$ with $\max(\frac{3}{2}, p_0) < r \leq 2$ will be chosen later in the proof.

First step. Estimating $S_{p_0}(t)$ and $S_r(t)$. (3.35) provides the following bound for the choice $q = p = p_0$:

$$(3.36) \quad S_{p_0}(t) \leq C_1 S_{p_0}^I + C_2 \int_0^t S_{p_0}(s) \left(S_r(s) + \sqrt{MS_r(s)} \right) ds,$$

and the following one for the choice $q = r$ and $p = p_0$:

$$(3.37) \quad S_r(t) \leq C_1 t^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} S_{p_0}^I + C_2 \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} S_{p_0}(s) \left(S_r(s) + \sqrt{MS_r(s)} \right) ds.$$

Define

$$K_p(t) = \begin{cases} t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{p})} S_p(t) & \text{for } p > p_0, \\ S_p(t) & \text{for } p \leq p_0 \end{cases}$$

and, for continuous-in- t $K_p(t)$, denote by $K_p := \max\{K_p(t), 0 \leq t \leq T\}$. Then, from (3.36) and (3.37) it is easily deduced that

$$(3.38) \quad K_{p_0}(t) \leq C_1 S_{p_0}^I + C_2 B \left(1, 1 - \frac{9}{2} \left(\frac{1}{p_0} - \frac{1}{r} \right) \right) K_{p_0} K_r + C_2 B \left(1, 1 - \frac{9}{4} \left(\frac{1}{p_0} - \frac{1}{r} \right) \right) K_{p_0} \sqrt{MK_r}$$

and

$$(3.39) \quad \begin{aligned} K_r(t) &\leq C_1 S_{p_0}^I + C_2 t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} B\left(1 - \frac{9}{2}\left(\frac{1}{p_0} - \frac{1}{r}\right), 1 - \frac{9}{2}\left(\frac{1}{p_0} - \frac{1}{r}\right)\right) K_{p_0} K_r \\ &+ C_2 t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} B\left(1 - \frac{9}{2}\left(\frac{1}{p_0} - \frac{1}{r}\right), 1 - \frac{9}{4}\left(\frac{1}{p_0} - \frac{1}{r}\right)\right) K_{p_0} \sqrt{M K_r}, \end{aligned}$$

with $B(a, b)$ denoting the Beta function defined by

$$B(a, b) = \int_0^t (t - s)^{a-1} s^{b-1} ds.$$

We recall that, when the arguments a, b are positive, then this integral is convergent and equals $B(a, b) = C(a, b)t^{a+b-1}$, where $C(a, b)$ is a constant depending on a and b . For our fixed $p_0 > \frac{9}{8}$ one can now easily choose a $r = r(p_0) > \max(\frac{3}{2}, p_0)$, such that the Beta functions appearing in (3.38) and (3.39) are convergent. Finally, if we define $K := \max\{K_{p_0}, K_r\}$ we obtain from (3.38), (3.39):

$$(3.40) \quad K \leq C_1 S_{p_0}^I + \tilde{C}_2 T^{1 - \frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} K^2 + \tilde{C}_2 T^{1 - \frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} \sqrt{M} K^{\frac{3}{2}}.$$

For small values of $T \geq 0$, $K = K(T)$ is continuous in T and $K(0) \leq C_1 S_{p_0}^I$ (from (3.40)). Using $K^{\frac{3}{2}} \leq K^2 + 1$ in (3.40) hence yields the *a priori* estimate:

$$\begin{aligned} K &\leq \frac{1 - \sqrt{1 - 4\tilde{C}_2\alpha(T, M)T^{1 - \frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})}(C_1 S_{p_0}^I + \tilde{C}_2\sqrt{M}T^{1 - \frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})})}}{2\tilde{C}_2\alpha(T, M)T^{1 - \frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})}} \\ &< \frac{1}{2\tilde{C}_2 T^{1 - \frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})}}, \end{aligned}$$

as long as $T = T(S_{p_0}^I, M)$ is “small enough” such that the radicand of the square root is positive. Here we used the definition

$$\alpha(T, M) = 1 + \sqrt{M}T^{\frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} > 1.$$

This concludes the first step of the proof.

Second step. Estimating $S_q(t)$. We first prove an estimate for q “close enough” and greater than 3. Putting $p = r$ in formula (3.35) we have

$$(3.41) \quad \begin{aligned} S_q(t) &\leq C_1 t^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{q})} S_{p_0}^I \\ &+ C_2 \int_0^t (t - s)^{-\frac{9}{2}(\frac{1}{r} - \frac{1}{q})} (S_r(s)^2 + \sqrt{M}S_r(s)^{\frac{3}{2}}) ds. \end{aligned}$$

The uniform bound on $K_r(t)$ yields

$$(3.42) \quad S_r(s) \leq \gamma(M, T) s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})},$$

where the function γ is of the form $\gamma(M, T) = C'_1(M, T)S_{p_0}^I + C'_2(M, T)$.

Hence, the integrability conditions of the Beta functions in (3.41) give $\frac{1}{r} - \frac{1}{q} < \frac{2}{9}$ and $\frac{1}{p_0} - \frac{1}{r} < \frac{1}{9}$. These inequalities and the range of $r \in (\frac{3}{2}, 2]$ imply

$$\frac{3}{2} < r(p_0) \leq q < \min\left(\frac{3p_0}{3-p_0}, \frac{18}{5}\right)$$

as possible choices for $q = q(p_0)$. For future reference we remark that $p_0 > \frac{3}{2}$ allows to choose $q(p_0) > 3$. We insert (3.42) in (3.41) and observe that the first term of the right-hand side of (3.41) dominates (due to the inequality $\frac{1}{p_0} - \frac{1}{r} < \frac{1}{9}$). This finishes Part (a) of the proof.

Part (b): To obtain a bound for $S_\infty(t)$ we now consider $p = q \geq p_0 > \frac{9}{2}$ in formula (3.35), which becomes

$$S_q(t) \leq C_1 t^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{q})} S_{p_0}^I + C_2 \int_0^t S_q(s) \left(S_r(s) + \sqrt{MS_r(s)} \right) ds,$$

with $r = r(\tilde{p}_0)$ to be fixed later on. In terms of K_q , as defined in the first part of this proof, it reads

$$\begin{aligned} K_q(t) &\leq C_1 S_{p_0}^I + C_2(T, M, S_{\tilde{p}_0}^I) T^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{q})} \int_0^t K_q(s) s^{-\frac{9}{2}(\frac{1}{p_0} + \frac{1}{\tilde{p}_0} - \frac{1}{q} - \frac{1}{r(\tilde{p}_0)})} ds \\ (3.43) \quad &+ C_3(T, M, S_{\tilde{p}_0}^I) T^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{q})} \int_0^t K_q(s) s^{-\frac{9}{4}(\frac{2}{p_0} + \frac{1}{\tilde{p}_0} - \frac{2}{q} - \frac{1}{r(\tilde{p}_0)})} ds, \end{aligned}$$

where we have used that

$$S_{r(\tilde{p}_0)}(t) \leq C(T, M, S_{\tilde{p}_0}^I) t^{-\frac{9}{2}(\frac{1}{\tilde{p}_0} - \frac{1}{r(\tilde{p}_0)})},$$

for some $\tilde{p}_0 \in (\frac{9}{8}, 2)$ large enough and

$$\max\left(\frac{3}{2}, \tilde{p}_0\right) < r(\tilde{p}_0) \leq 2 \quad \text{with} \quad \frac{1}{\tilde{p}_0} - \frac{1}{r(\tilde{p}_0)} < \frac{2}{9}$$

(first step of the proof). We remark that $S_{\tilde{p}_0}^I$ is bounded due to the interpolation $S_{\tilde{p}_0}^I \leq S_{p_0}^I + S_1^I$, and $S_1^I = \|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}$. Considering the relations

$$(3.44) \quad \frac{1}{p_0} + \frac{1}{\tilde{p}_0} < \frac{2}{9} + \frac{1}{q} + \frac{1}{r(\tilde{p}_0)}, \quad \frac{1}{p_0} - \frac{1}{q} < \frac{2}{9},$$

for $\frac{9}{8} < \tilde{p}_0 < p_0 \leq q$, we see that the exponents in the power functions in the integrals of (3.43) exceed -1 .

Hence we can apply the Gronwall lemma to (3.43) and obtain $K_q(t) \leq C$, with $p_0 \leq q$ such that (3.44) is fulfilled, where C is a positive constant depending on $T, S_{p_0}^I$ and $\|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}$. Thus, for $p_0 > \frac{9}{2}$ we may choose $q = q(p_0) = \infty$ and this concludes the proof. ■

This result will be useful when discussing the regularizing effect of the system on the density and the potential as a function of the choice of p_0 , starting from the hypothesis $W^I \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap S_{p_0}$.

Let us now recall a general result on singular integrals of convolution type whose proof can be found in [35]. We shall adapt the statement of the lemma to the particular case in which $V = \frac{\alpha}{4\pi} \frac{1}{|x|} \star n$ is the self-consistent Hartree potential.

Lemma 3.8 *The following assertions hold true for V :*

(i) *If $1 < p < q < \infty$ satisfy $\frac{1}{q} = \frac{1}{p} - \frac{2}{3}$ and $n(\cdot, t) \in L^p(\mathbb{R}^3)$, then $V(\cdot, t) \in L^q(\mathbb{R}^3)^3$ and*

$$\|V(\cdot, t)\|_{L^q(\mathbb{R}^3)^3} \leq C(p, q) \|n(\cdot, t)\|_{L^p(\mathbb{R}^3)},$$

where $C(p, q)$ is a positive constant which depends on p and q . Furthermore, if $n(\cdot, t) \in L^p(\mathbb{R}^3)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$, then $\nabla_x V(\cdot, t) \in L^q(\mathbb{R}^3)$ and

$$\|\nabla_x V(\cdot, t)\|_{L^q(\mathbb{R}^3)} \leq C(p, q) \|n(\cdot, t)\|_{L^p(\mathbb{R}^3)}.$$

(ii) *If*

$$1 \leq p < \frac{3}{2} < q \leq \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad \theta = \left(\frac{1}{q'} - \frac{1}{3}\right) / \left(\frac{1}{p} - \frac{1}{q}\right)$$

and $n(\cdot, t) \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, we have $V \in L^\infty(\mathbb{R}^3)$ and

$$\|V(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C(p, q) \|n(\cdot, t)\|_{L^p(\mathbb{R}^3)}^\theta \|n(\cdot, t)\|_{L^q(\mathbb{R}^3)}^{1-\theta}.$$

Also, if $n(\cdot, t) \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ with

$$1 \leq p < 3 < q \leq \infty \quad \text{and} \quad \theta = \left(\frac{1}{q'} - \frac{2}{3}\right) / \left(\frac{1}{p} - \frac{1}{q}\right),$$

then $\nabla_x V \in L^\infty(\mathbb{R}^3)^3$ and

$$\|\nabla_x V(\cdot, t)\|_{L^\infty(\mathbb{R}^3)^3} \leq C(p, q) \|n(\cdot, t)\|_{L^p(\mathbb{R}^3)}^\theta \|n(\cdot, t)\|_{L^q(\mathbb{R}^3)}^{1-\theta}.$$

(iii) *If $n(\cdot, t) \in L^p(\mathbb{R}^3)$ with $1 < p < \infty$, then $\partial_{x_i} \partial_{x_j} V(\cdot, t) \in L^p(\mathbb{R}^3)$ and the following estimates hold:*

$$\|\partial_{x_i} \partial_{x_j} V(\cdot, t)\|_{L^p(\mathbb{R}^3)} \leq C(p) \|n(\cdot, t)\|_{L^p(\mathbb{R}^3)}, \quad 1 \leq i, j \leq 3.$$

Some *a priori* regularities for the particle density and the potential are stated in the following proposition, which is a direct consequence of the estimates given in Proposition 3.7 and Lemma 3.8.

Proposition 3.9 *Assume W to be a mild solution of (3.10) on $[0, T]$. Let $p_0 > \frac{9}{8}$ be fixed such that $W^I \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap S_{p_0}$. Then, for arbitrary small $\epsilon > 0$, there exists $T = T(S_{p_0}^I, M) > 0$ such that the following regularity properties are verified for $0 < t \leq T$:*

(i) $\|V(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq Ct^{3-\frac{9}{2p_0}-\epsilon}$.

(ii) *If $\frac{3}{2} < p_0 < 2$, then $\nabla_x V \in L^\infty(\mathbb{R}^3)^3$ and*

$$\|\nabla_x V(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq Ct^{\frac{3}{2}-\frac{9}{2p_0}-\epsilon}.$$

(iii) *If $p_0 > \frac{9}{2}$, then the density $n \in L^\infty(\mathbb{R}^3)$ and*

$$\|n(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq Ct^{-\frac{9}{2p_0}}.$$

The constants C in (i) and (ii) may depend on ϵ .

Finally, as consequence of the regularization effect of the Fokker–Planck kernel, we also find the following bound for the L^q norm of the Wigner function.

Proposition 3.10 *Assume W to be a mild solution of (3.10) on $[0, T]$. Let $\frac{9}{8} < p_0 < 2$ be fixed such that $S_{p_0}^I$ is bounded. Then, the following estimate holds in $(0, T]$ for every $1 \leq p \leq q \leq \infty$ satisfying*

(3.45)
$$\frac{6}{p} < \frac{6}{q} + 4 - \frac{9}{2p_0},$$

$$\|W(\cdot, \cdot, t)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \leq Ct^{-6(\frac{1}{p}-\frac{1}{q})} \|W^I\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)},$$

where C is a positive constant depending on T and M .

Proof. First we choose a constant $r \in (\frac{3}{2}, 2]$ sufficiently close to $\frac{3}{2}$ such that

(3.45a)
$$\frac{6}{p} < \frac{6}{q} + 1 - \frac{9}{2} \left(\frac{1}{p_0} - \frac{1}{r} \right)$$

holds, and such that r lies in the interval of admissible $q(p_0)$ for our fixed p_0 (see the proof of Proposition 3.7 (a)). Applying Lemma 3.4 to (3.10) yields

(3.46)
$$\begin{aligned} \|W(\cdot, \cdot, t)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} &\leq C_1 t^{-6(\frac{1}{p}-\frac{1}{q})} \|W^I\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \\ &+ C_2 \int_0^t \left(S_r(s) + \sqrt{MS_r(s)} \right) \|W(\cdot, \cdot, s)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} ds, \end{aligned}$$

where we estimated

$$\|(\Phi \star_{\xi} W)(\cdot, \cdot, s)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)} \leq C \left(S_r(s) + \sqrt{MS_r(s)} \right) \|W(\cdot, \cdot, s)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)}$$

via the Young inequality and we used (3.27). Then, from Proposition 3.7, (3.46) becomes

$$(3.47) \quad \|W(\cdot, \cdot, t)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)} \leq C_1 t^{-6(\frac{1}{p}-\frac{1}{q})} \|W^I\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)} + C_2 \left(S_{p_0}^I \right) \int_0^t \left(s^{-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})} + \sqrt{M} s^{-\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} \right) \|W(\cdot, \cdot, s)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)} ds.$$

Define $N_{p,q}(t) := t^{6(\frac{1}{p}-\frac{1}{q})} \|W(\cdot, \cdot, t)\|_{L^q(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)}$. Then, from (3.47) it is a simple matter to obtain the estimate

$$N_{p,q}(t) \leq C_1 \|W^I\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)} + C_2 \left(S_{p_0}^I \right) T^{6(\frac{1}{p}-\frac{1}{q})} \int_0^t s^{-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})-6(\frac{1}{p}-\frac{1}{q})} (1 + \sqrt{M} s^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})}) N_{p,q}(s) ds.$$

Now the proof ends as a straightforward consequence of the Gronwall inequality, since (3.45a) guarantees the integrability of the coefficient in the last integral. ■

Remark. As a consequence of Proposition 3.10, choosing $p_0 > \frac{7}{6}$ implies that $W(\cdot, \cdot, t) \in L^q(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)$ with some $q > \frac{6}{5}$, which implies

$$(\mathcal{F}_{\xi \mapsto \eta} W)(x, \cdot, t) \in L^{q'}(\mathbb{R}_{\eta}^3)$$

a.e. in $x \in \mathbb{R}^3$ with some $q' < 6$. This completes the equivalence of the pseudo-differential operator (1.4) with its convolution form (cf. (3.3)).

3.3. Sequence of approximate solutions and passage to the limit

We shall now define a sequence of linearized problems formally approximating the WFPF equation. For every $n \in \mathbb{N}_0$ we consider mild solutions of the equation

$$\frac{\partial}{\partial t} W^{n+1} + (\xi \cdot \nabla_x) W^{n+1} - \sigma \Delta_{\xi} W^{n+1} = -\theta[V^n]W^n,$$

with

$$V^n(x, t) = \frac{\alpha}{4\pi} \frac{1}{|x|} \star \int_{\mathbb{R}_{\xi}^3} W^n(x, \xi, t) d\xi$$

and $W^n(x, \xi, 0) = W^I(x, \xi)$ for every $n \geq 1$.

We consider $W^0(x, \xi, t) = 0$ to avoid regularity problems with the definition of $\theta[V^0]W^0$. Using the integral formulation, the sequence $\{W^n\}$ is defined iteratively by

$$(3.48) \quad \begin{aligned} W^{n+1}(x, \xi, t) &= \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, t) W^n(z, v) dz dv \\ &+ \int_0^t \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, s) (\Phi^n \star_\xi W^n)(z, v, t-s) dz dv ds, \end{aligned}$$

where we have used the convoluted form of the nonlinear term, with $\Phi^n = \Phi(V^n)$. If $W^n \in C([0, T]; L^1(\mathbb{R}^6))$ and $t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} W^n \in L^\infty(0, T; S_r)$ for some $\frac{3}{2} < r \leq 2$, then estimates like in the proof of Proposition 3.10 show that again $W^{n+1} \in C([0, T]; L^1(\mathbb{R}^6))$. Estimate (3.34) shows that also $t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} W^{n+1} \in L^\infty(0, T; S_r)$, and hence the sequence $\{W^n\}$ is well-defined.

We are now concerned with the passage to the limit as $n \rightarrow \infty$. For this purpose, we shall firstly prove that $\{W^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in a suitable space in order to give sense to the limit solution. Note that the estimates derived in Lemmata 3.5 and 3.6 and Propositions 3.7 and 3.9 are still valid independently of n .

Lemma 3.11 *Assume that W^I belongs to $L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap S_{p_0}$ with $p_0 > \frac{9}{8}$. Then, there exists a (small enough) $T > 0$ such that $\{W^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(0, T; S_{p_0})$ and $\{t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} W^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(0, T; S_r)$, with $0 < r - \frac{3}{2}$ sufficiently small.*

Proof. Consider the difference $W^{n+1} - W^n$ in the norm of the spaces S_r , for $\frac{3}{2} < r \leq 2$. We can estimate

$$S_r(t, W^{n+1} - W^n) \leq C \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p} - \frac{1}{r})} S_p(s, \Phi^n \star_\xi W^n - \Phi^{n-1} \star_\xi W^{n-1}) ds$$

according to Lemma 3.5. Now, introducing the term $\Phi^n \star_\xi W^{n-1}$ in the above expression we deduce

$$\begin{aligned} S_q(t, W^{n+1} - W^n) &\leq \\ &\leq C \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p} - \frac{1}{q})} \left\{ S_p(s, |W^n - W^{n-1}|) \left(S_r(s, |W^n|) + \sqrt{M S_r(s, |W^n|)} \right) \right. \\ &\quad \left. + S_p(s, |W^{n-1}|) \left(S_r(s, |W^n - W^{n-1}|) + \sqrt{2M S_r(s, |W^n - W^{n-1}|)} \right) \right\} ds \end{aligned}$$

for $1 \leq p \leq q \leq \infty$, as in Lemma 3.6, with $M = \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} W^I d\xi dx$. Following a similar reasoning as in Proposition 3.7, we set

$$K_p^n(t) = \begin{cases} t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{p})} S_p(t, |W^n|) & \text{for } p > p_0 \\ S_p(t, |W^n|) & \text{for } p \leq p_0 \end{cases}$$

and denote by $K_p^n := \max\{K_p^n(t), 0 \leq t \leq T\}$. Then, applying exactly the same ideas as in the proof of Proposition 3.7, we obtain

$$K^{n+1} \leq CT^{1-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})} \left[2K + T^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} \sqrt{MK} \right] K^n + CT^{1-\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} K \sqrt{MK^n},$$

where we denoted

$$K := \max\{K_r^n, K_{p_0}^{n-1}\} \text{ and } K^n := \max\{K_r|W^n - W^{n-1}|, K_{p_0}|W^n - W^{n-1}|\}.$$

Now, if we set $\bar{K}^n := \max\{K^n, \sqrt{K^n}\}$ then we have $K^{n+1} \leq \lambda(K, T, M)\bar{K}^n$, with

$$\lambda(K, T, M) = CT^{1-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})} \left[\left(2 + \sqrt{MT}^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} \right) K + \sqrt{MT}^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} \sqrt{K} \right].$$

We already know from (3.40) that

$$K \leq C_1 S_{p_0}^I + CT^{1-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})} \left(\sqrt{MT}^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} + (1 + \sqrt{MT}^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})}) K \right) K,$$

which implies

$$\sqrt{K} \leq \left(\frac{1 - C\sqrt{MT}^{1-\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})}}{2C(1 + \sqrt{MT}^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})})T^{1-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})}} \right)^{1/2}$$

for small values of T . Then it is a simple matter to observe that a sufficient condition to guarantee that $\lambda < 1$ is given by “small enough” times T such that

$$T^{1-\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} < \frac{1}{C\sqrt{M}}, \quad S_{p_0}^I \leq \frac{\left(CT^{1-\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})} - 1 \right)^2}{4CC_1(1 + \sqrt{MT}^{\frac{9}{4}(\frac{1}{p_0}-\frac{1}{r})})T^{1-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})}}.$$

Iterating this bound we have $K^{n+1} \leq \lambda^n \bar{K}^1$, which concludes the proof. ■

Then, the sequence of approximate solutions W^n converges to a certain function W in $L^\infty(0, T; S_{p_0})$, as well as

$$t^{\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})} W^n \text{ is convergent in } L^\infty(0, T; S_r).$$

This implies that

$$n(W^n) \longrightarrow n(W) \quad \text{in } L^\infty(0, T; L^{p_0}(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)),$$

and

$$t^{\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})} n(W^n) \longrightarrow t^{\frac{9}{2}(\frac{1}{p_0}-\frac{1}{r})} n(W) \quad \text{in } L^\infty(0, T; L^r(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)),$$

with $r > \frac{3}{2}$.

As a consequence, using Lemma 3.8 (ii) we deduce that the sequence $t^{\frac{9}{2p_0}-3}V^n$ is a Cauchy sequence in $L^\infty(0, T; L^\infty(\mathbb{R}_x^3)^3)$.

Since W^{n+1} is a solution of the integral equation (3.48), the Cauchy property proved in Lemma 3.11 implies that the integral formula (3.10) is verified by W by passing to the limit in (3.48), for which we use Lemma 3.4 and the arguments of Proposition 3.7. Similar to the above L^1 -argument for (3.48) we verify $W \in C([0, T]; L^1(\mathbb{R}^6))$, and hence W is a mild solution of the WFPF problem.

Now the passage to the limit as n goes to infinity is justified and this proves Theorem 3.3.

Notice that the 'smallness' assumption on the size of the initial data for existence of solutions to be proved in $[0, T)$, with T large, stems from the T -dependence on $S_{p_0}^I$ and M in the proof of Proposition 3.7.

The conditions in Theorem 3.3 on the initial data are satisfied e.g. for an initial density matrix function $\rho^I \in \mathcal{S}$, the Schwartz space, which also implies $W^I \in \mathcal{S}$.

4. Uniqueness and stability

We now turn to show the uniqueness result. Assume that there exist two different solutions W_1 and W_2 of the WFPF problem satisfying the bounds proved in the previous section. We set $w(x, \xi, t) = W_1(x, \xi, t) - W_2(x, \xi, t)$ and $n(w)(x, t) = \int_{\mathbb{R}_\xi^3} w(x, \xi, t) d\xi$. It is clear that w solves the following problem:

$$(4.1) \quad \frac{\partial w}{\partial t} + (\xi \cdot \nabla_x)w + \theta[V_1 - V_2]W_1 + \theta[V_2]w = \sigma \Delta_\xi w, \quad w(x, \xi, 0) = 0,$$

with $V_1 = V(W_1)$ and $V_2 = V(W_2)$.

If we now consider the problem (4.1) as a nonlinear perturbation of a heat equation, we find the following integral formulation for the Wigner function w :

$$\begin{aligned} w(x, \xi, t) = & - \int_0^t \int_{\mathbb{R}_\xi^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, s) \theta[V_1 - V_2]W_1(z, v, t - s) dz dv ds \\ & - \int_0^t \int_{\mathbb{R}_\xi^3} \int_{\mathbb{R}_z^3} G(x, \xi, z, v, s) \theta[V_2]w(z, v, t - s) dz dv ds. \end{aligned}$$

Now, application of Lemma 3.6 with $p = q = r$ easily gives

$$(4.2) \quad \begin{aligned} S_r(t, w) \leq & C(T) \int_0^t \left\{ s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \left(2 + \sqrt{M} s^{\frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} \right) S_r(s, w) \right. \\ & \left. + s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \sqrt{\|w(\cdot, \cdot, s)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} S_r(s, w)} \right\} ds, \end{aligned}$$

where we have estimated

$$M(w)(t) = \|n(W_1 - W_2)(\cdot, t)\|_{L^1(\mathbb{R}_x^3)} \leq \|w(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}.$$

On the other hand, repeating the same type of arguments leading to Proposition 3.10 yields

$$\begin{aligned} & \|w(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \leq \\ & \leq C(T) \int_0^t \left\{ s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \left(1 + \sqrt{M} s^{\frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} \right) \|w(\cdot, \cdot, s)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \right. \\ (4.3) \quad & \left. + S_r(w, s) + \sqrt{M(w)(s)S_r(s, w)} \right\} ds. \end{aligned}$$

Denoting $X(t) = \max\{S_r(t, w), \|w(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}\}$, the estimate (4.2) becomes

$$(4.4) \quad S_r(t, w) \leq C(T) \int_0^t s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \left(3 + \sqrt{M} s^{\frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} \right) X(s) ds,$$

while (4.3) becomes

$$(4.5) \quad \|w(\cdot, \cdot, t)\|_{L^1} \leq C(T) \int_0^t \left\{ 2 + s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \left(1 + \sqrt{M} s^{\frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} \right) \right\} X(s) ds.$$

Combining (4.4) and (4.5) it is a simple matter to observe that

$$(4.6) \quad X(t) \leq C(T) \int_0^t \left\{ 2 + s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \left(3 + \sqrt{M} s^{\frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} \right) \right\} X(s) ds,$$

which implies that $S_r(t, w) = 0$ and $\|w(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} = 0$ for every $t > 0$ via the Gronwall lemma. Then $W_1 = W_2$, and thus the existence of a unique mild solution W of the WFPF system satisfying the estimates proved in Section 3 is guaranteed.

Note that the same arguments yield the stability of mild solutions with respect to small perturbations of the initial data in $L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap S_r$, but then on the right-hand side of (4.6) there also appears a constant-in-time term depending on $S_r(W_1^I - W_2^I)$ and $\|W_1^I - W_2^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}$:

$$X(t) \leq C_1 X(0) + C_2(T) \int_0^t \left\{ 2 + s^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \left(3 + \sqrt{M} s^{\frac{9}{4}(\frac{1}{p_0} - \frac{1}{r})} \right) \right\} X(s) ds,$$

which allows to conclude the stability via the Gronwall inequality.

5. Large–time behavior

The aim of this section is to give a description of the asymptotic behavior of global solutions $W(x, \xi, t)$ of the three–dimensional frictionless WFPF system. In particular, we shall prove that global mild solutions W (which satisfy certain t –uniform a–priori bounds that are motivated by the local–in–time analysis of Section 3) converge as $t \rightarrow \infty$ towards the total charge of the system times the Green’s function G , defined by (3.8) and (3.9). This means that the quantum effect due to the Coulomb potential term vanishes as $t \rightarrow \infty$.

In this section we shall need the positivity of the Husimi function, and hence $R(t) \geq 0$ and $W(t) \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ as announced in §2. We shall also consider a solution $W(t)$ defined for $t \in [0, \infty)$, which satisfies

$$(5.1) \quad \|W\|_{L^\infty(0, \infty; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))} \leq C \|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)},$$

$$(5.2) \quad M_{p_0} = \sup\{S_{p_0}(t), 0 \leq t < \infty\} < \infty$$

for $p_0 > \frac{9}{8}$ and

$$(5.3) \quad M_r = \sup\{t^{\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} S_r(t), 0 \leq t < \infty\} < \infty$$

for $\frac{3}{2} < r \leq 2$. These bounds, which were shown in Section 3 to hold locally in time, also appear to be inherent to the properties of global solutions of Vlasov–Fokker–Planck equations, as can be seen in [11].

5.1. Rescaling and a priori estimates

For any $\epsilon > 0, t \geq 0$ and $x, \xi \in \mathbb{R}^3$ we define the following sequence of rescaled solutions

$$(5.4) \quad W_\epsilon(x, \xi, t) = \epsilon^{-12} W(\epsilon^{-3}x, \epsilon^{-1}\xi, \epsilon^{-2}t),$$

keeping the same self–similarity factors as the fundamental solution G , as stated in Lemma 3.1 (iii). Also, denote by $W_\epsilon^I(x, \xi) = W_\epsilon(x, \xi, 0)$. Then, after a simple change of variables we deduce the following expression for the rescaled density

$$(5.5) \quad n_\epsilon(x, t) = n(W_\epsilon)(x, t) = \epsilon^{-9} n(\epsilon^{-3}x, \epsilon^{-2}t).$$

Moreover, we set $V_\epsilon = \frac{\alpha}{4\pi|x|} \star n_\epsilon$, which yields

$$V_\epsilon(x, t) = \epsilon^{-3} V(\epsilon^{-3}x, \epsilon^{-2}t).$$

We also rescale the Husimi transform W^h (cf. (2.13)) according to our group of scale transformations. Let W_ϵ^h denote the rescaled Husimi function defined by $W_\epsilon^h = W_\epsilon \star_{x,\xi} \Gamma_\epsilon(x, \xi)$, where Γ_ϵ stands for the Gaussian function

$$\Gamma_\epsilon(x, \xi) = \epsilon^{-12} \pi^{-3} e^{-(|\epsilon^{-3}x|^2 + |\epsilon^{-1}\xi|^2)},$$

so that

$$W_\epsilon^h(x, \xi, t) = \epsilon^{-12} W^h(\epsilon^{-3}x, \epsilon^{-1}\xi, \epsilon^{-2}t).$$

Straightforward computations lead to the following

Lemma 5.1 *For any $\epsilon > 0, t > 0$ and $1 \leq p \leq \infty$, the following equalities hold:*

- (i) $\|W_\epsilon(\cdot, \cdot, t)\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} = \epsilon^{-\frac{12}{p'}} \|W(\cdot, \cdot, \epsilon^{-2}t)\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}.$
- (ii) $\|n_\epsilon(\cdot, t)\|_{L^p(\mathbb{R}^3)} = \epsilon^{-\frac{9}{p'}} \|n(\cdot, \epsilon^{-2}t)\|_{L^p(\mathbb{R}^3)}.$
- (iii) $\|V_\epsilon(\cdot, t)\|_{L^p(\mathbb{R}^3)} = \epsilon^{\frac{9}{p}-3} \|V(\cdot, \epsilon^{-2}t)\|_{L^p(\mathbb{R}^3)}.$

We shall now derive the equation satisfied by the rescaled pair (W_ϵ, V_ϵ) .

Lemma 5.2 *Let the pair (W, V) be a mild solution of the WFPF equation. Then, for any $\epsilon > 0$ fixed, (W_ϵ, V_ϵ) is a mild solution of the problem*

$$(5.6a) \quad \frac{\partial W_\epsilon}{\partial t} + (\xi \cdot \nabla_x)W_\epsilon + \epsilon \theta_\epsilon[V_\epsilon]W_\epsilon = \sigma \Delta_\xi W_\epsilon$$

$$(5.6b) \quad W_\epsilon(t = 0) = \epsilon^{-12} W_I(\epsilon^{-3}x, \epsilon^{-1}\xi)$$

in $\mathbb{R}_x^3 \times \mathbb{R}_\xi^3 \times (0, T)$, where the rescaled pseudo-differential operator $\theta_\epsilon[V_\epsilon]$ is given by

$$(5.7) \quad \begin{aligned} \theta_\epsilon[V_\epsilon]W_\epsilon(x, \xi, t) &= \frac{i}{(2\pi)^3} \int_{\mathbb{R}_\eta^3} \int_{\mathbb{R}_y^3} \left(V_\epsilon(x + \epsilon^4 \frac{y}{2}, t) - V_\epsilon(x - \epsilon^4 \frac{y}{2}, t) \right) \\ &\quad \times W_\epsilon(x, \eta, t) e^{-i(\xi-\eta) \cdot y} dy d\eta. \end{aligned}$$

In the sequel we shall denote by $WPFPE_\epsilon$ the rescaled problem (5.6)–(5.7) with initial data $W_\epsilon(x, \xi, 0) = W_\epsilon^I(x, \xi)$. As for the non-rescaled system, we observe that W_ϵ admits a 'two-parts' integral representation $W_\epsilon = W_\epsilon^1 + W_\epsilon^2$ with $W_\epsilon^k, k = 1, 2$, defined as in formula (3.10) up to the natural action of the scaling group, where the self-similarity of G stated in Lemma 3.1 (iii) and the expression (5.7) defining the rescaled pseudo-differential operator have been taken into account.

Our sequence of rescaled solutions W_ϵ contains information about the long time behavior of solutions of the WFPF problem in the following way: We first prove that $\{W_\epsilon\}_\epsilon$ converges towards a function g in an appropriate sense as $\epsilon \rightarrow 0$. Indeed, after passing to the limit with the scale parameter $\epsilon \rightarrow 0$ in the rescaled problem (5.6), we show that the nonlinear term is asymptotically “killed” and identify g as the unique distributional solution of the linear kinetic FP equation subject to the initial condition $g(x, \xi, 0) = M\delta_0$, i.e. $g = MG$. Now, it is enough to observe that the convergence $W_\epsilon \rightarrow g$ is equivalent to $\tau^6 W(\tau^{\frac{3}{2}}x, \tau^{\frac{1}{2}}\xi, \tau) \rightarrow g(x, \xi, 1)$, as $\tau \rightarrow \infty$ by setting $t = 1$ and $\epsilon = \tau^{-\frac{1}{2}}$. In order to develop this process we shall need uniform estimates with respect to ϵ which allow for the passage to the limit. Notice that, with the same notation (affected by the scaling) as in Section 3, the rescaled densities $n_\epsilon^k = n(W_\epsilon^k)$, $k = 1, 2$ are derived in an analogous way as those for the non-rescaled system.

In the following lemma we collect some extensions of the properties proved for the original WFPF problem to the rescaled WFPF $_\epsilon$ problem. These properties are easily transferred to the rescaled equation up to the obvious changes in the proofs already given. Let

$$S_{p,\epsilon}^k(t) = \max\{\|n_{\epsilon,h}^k(|W|)(\cdot, t)\|_{L^p(\mathbb{R}^3)}, h \geq 0\}$$

for any $1 \leq p \leq \infty$, and denote by

$$S_{p,\epsilon}^I = \max\{\|n_{\epsilon,h}^I(|W|)\|_{L^p(\mathbb{R}^3)}, h \geq 0\}.$$

Lemma 5.3 *The following assertions hold true for mild solutions of the WFPF equation:*

- (i) Let $1 \leq q \leq \infty$. Then, $S_{q,\epsilon}(t) = \epsilon^{-\frac{9}{q}} S_q(\epsilon^{-2}t)$.
- (ii) Let $1 \leq p \leq q \leq \infty$. Then, $S_{q,\epsilon}^1(t) \leq Ct^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_{p,\epsilon}^I$.
- (iii) Let $1 \leq p \leq q \leq \infty$, $\frac{3}{2} < r \leq 2$. Then,

$$S_{q,\epsilon}^2(t) \leq C\epsilon \int_0^t (t-s)^{-\frac{9}{2}(\frac{1}{p}-\frac{1}{q})} S_{p,\epsilon}(s) \left(S_{r,\epsilon}(s) + \sqrt{MS_{r,\epsilon}(s)} \right) ds.$$

- (iv) If $\epsilon S_{p_0,\epsilon}^I$ is small enough for $p_0 > \frac{9}{8}$ fixed, then there exists a positive $T = T(W^I)$ such that

$$(5.8) \quad S_{q,\epsilon}(t) \leq \left(C_1(T) S_{p_0,\epsilon}^I + C_2(T) \right) t^{-\frac{9}{2}(\frac{1}{p_0}-\frac{1}{q})}$$

for every $\frac{9}{8} < q \leq r$ and $0 < t \leq T$. Here, C_1 and C_2 are uniformly bounded in $\epsilon < 1$. In addition, if $p_0 > \frac{3}{2}$ then $S_{q,\epsilon}(t)$ satisfies (5.8) for $q > 3$. Also, if $p_0 > \frac{9}{2}$ then (5.8) is still valid for $q = \infty$.

Proof. The estimate (iii) follows from re-establishing Lemma 3.6 for the mild solution of the scaled WFPF equation (5.6a). Similarly, the estimate (iv) can be established for the scaled WFPF equation (5.6a) like Proposition 3.7. ■

Before proving the convergence properties of W_ϵ which rigorously allow for passing to the limit $\epsilon \rightarrow 0$, we shall clarify how the nonlinear term of the rescaled problem WFPF $_\epsilon$ will be asymptotically simplified. To this aim, we will use essentially the scaling properties of the system and the bounds established in Lemma 5.3. However, we firstly need the following result based on the regularizing nature of the FP equation.

Lemma 5.4 *Let $p_0 > \frac{9}{8}$ fixed and $W^I \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ be such that the hypotheses (5.1), (5.2), and (5.3) are satisfied. Then, the following estimate*

$$S_{p_0,\epsilon}(t) \leq C(W^I, t)\epsilon^{-\frac{9}{p'}}$$

is fulfilled for $0 < \frac{9}{8} - p$ sufficiently small, $t > 0$ and ϵ small enough, where $C(W^I, t)$ is a positive constant depending on the initial data W^I and on time, but not on ϵ . Furthermore, for $0 < \delta < t$ we have

$$\|\mathcal{F}_{x \rightarrow y}^{-1} V_\epsilon(\cdot, t)\|_{L^1(\mathbb{R}^3)} \leq C(W^I, \delta, T)(t - \delta)^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \epsilon^{-\frac{9}{p'}}.$$

Proof. We first estimate $S_{p_0,\epsilon}(t)$. For that, we choose $1 < \tilde{p} < p < p_0$ and use Lemma 5.3 (ii) and (iii) to obtain the following bound:

$$\begin{aligned} S_{p_0,\epsilon}(t) &\leq C_1 t^{-\frac{9}{2}(\frac{1}{p} - \frac{1}{p_0})} S_{p,\epsilon}^I \\ &\quad + C_2 \epsilon \int_0^t (t - s)^{-\frac{9}{2}(\frac{1}{\tilde{p}} - \frac{1}{p_0})} S_{\tilde{p},\epsilon}(s) \left(S_{r,\epsilon}(s) + \sqrt{M S_{r,\epsilon}(s)} \right) ds \\ &\leq C_1 t^{-\frac{9}{2}(\frac{1}{p} - \frac{1}{p_0})} S_{p,\epsilon}^I + C_2 \epsilon \int_0^t (t - s)^{-\frac{9}{2}(\frac{1}{\tilde{p}} - \frac{1}{p_0})} s^{-\frac{9}{2}(\frac{1}{\tilde{p}} - \frac{1}{r})} \\ &\quad \times \max\{S_{\tilde{p},\epsilon}(s), 0 \leq s \leq t\} \max\left\{s^{\frac{9}{2}(\frac{1}{\tilde{p}} - \frac{1}{r})} S_{r,\epsilon}(s), 0 \leq s \leq t\right\} ds \\ &\quad + C_2 \sqrt{M} \epsilon \int_0^t (t - s)^{-\frac{9}{2}(\frac{1}{\tilde{p}} - \frac{1}{p_0})} s^{-\frac{9}{4}(\frac{1}{\tilde{p}} - \frac{1}{r})} \\ (5.9) \quad &\quad \times \max\{S_{\tilde{p},\epsilon}(s), 0 \leq s \leq t\} \max\left\{s^{\frac{9}{4}(\frac{1}{\tilde{p}} - \frac{1}{r})} \sqrt{S_{r,\epsilon}(s)}, 0 \leq s \leq t\right\} ds. \end{aligned}$$

It is easy to observe that

$$\begin{aligned} \max\{S_{\tilde{p},\epsilon}(s), 0 \leq s \leq t\} &\leq \epsilon^{-\frac{9}{\tilde{p}'}} \max\{S_{\tilde{p}}(\tau), 0 \leq \tau \leq \epsilon^{-2}t\} =: M_{\tilde{p},\epsilon}, \\ \max\left\{s^{\frac{9}{2}(\frac{1}{\tilde{p}} - \frac{1}{r})} S_{r,\epsilon}(s), 0 \leq s \leq t\right\} &\leq \\ &\leq \epsilon^{-\frac{9}{p_0'}} \max\left\{\tau^{\frac{9}{2}(\frac{1}{\tilde{p}} - \frac{1}{r})} S_r(\tau), 0 \leq \tau \leq \frac{t}{\epsilon^2}\right\} =: M_{r,\epsilon}, \end{aligned}$$

and

$$\begin{aligned} \max\left\{s^{\frac{9}{4}\left(\frac{1}{p}-\frac{1}{r}\right)}\sqrt{S_{r,\epsilon}(s)}, 0 \leq s \leq t\right\} &\leq \\ &\leq \epsilon^{-\frac{9}{2p_0}} \max\left\{\tau^{\frac{9}{4}\left(\frac{1}{p}-\frac{1}{r}\right)}\sqrt{S_r(\tau)}, 0 \leq \tau \leq \frac{t}{\epsilon^2}\right\} \leq \sqrt{M_{r,\epsilon}}. \end{aligned}$$

Then, a simple computation shows that the r.h.s. of the estimate (5.9) for $S_{p_0,\epsilon}(t)$ can be bounded by

$$C_1 t^{-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{p_0}\right)} \epsilon^{-\frac{9}{p'}} S_p^I + C_2 \epsilon T^{1-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{p_0}+\frac{1}{p}-\frac{1}{r}\right)} M_{\tilde{p},\epsilon} \left(M_{r,\epsilon} + \sqrt{MT}^{\frac{9}{4}\left(\frac{1}{p}-\frac{1}{r}\right)} \sqrt{M_{r,\epsilon}}\right),$$

where we have used Lemma 5.3 (i). We now estimate $M_{\tilde{p},\epsilon}$ and $M_{r,\epsilon}$. An interpolation argument for the norm $S_{\tilde{p}}$ between S_1 and S_{p_0} gives

$$M_{\tilde{p},\epsilon} \leq C \|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}^{\frac{1}{2}} M_{p_0}^{\frac{1}{2}} \epsilon^{-\frac{9}{p'}},$$

where we used hypothesis (5.1) and $M_{p_0} = \max\{S_{p_0}(t), 0 \leq t < \infty\}$. By hypothesis (5.3) we have $M_{r,\epsilon} \leq M_r \epsilon^{-\frac{9}{p_0'}} < \infty$. Therefore, we have

$$\begin{aligned} S_{p_0,\epsilon}(t) &\leq C_1 t^{-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{p_0}\right)} \epsilon^{-\frac{9}{p'}} S_p^I + C_2 T^{1-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{p_0}+\frac{1}{p}-\frac{1}{r}\right)} \|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}^{\frac{1}{2}} M_{p_0}^{\frac{1}{2}} \\ &\quad \times \max\{M_r, \sqrt{M_r}\} \epsilon^{1-9\left(\frac{1}{p'}+\frac{1}{p_0'}\right)} \left(1 + \sqrt{MT}^{\frac{9}{4}\left(\frac{1}{p}-\frac{1}{r}\right)} \epsilon^{\frac{9}{2p'}}\right) \end{aligned}$$

and the proof of the first assertion concludes by identifying

$$\begin{aligned} C(W^I, t) &= C_1 t^{-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{p_0}\right)} S_p^I + C_2 T^{1-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{p_0}+\frac{1}{p}-\frac{1}{r}\right)} \|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)}^{\frac{1}{2}} M_{p_0}^{\frac{1}{2}} \\ &\quad \times \max\{M_r, \sqrt{M_r}\} \left(1 + \sqrt{MT}^{\frac{9}{4}\left(\frac{1}{p}-\frac{1}{r}\right)}\right) \end{aligned}$$

when choosing p close enough to $\frac{9}{8}$ and $\frac{18}{17} < \tilde{p} < p$ close enough to $\frac{18}{17}$.

The proof of the second assertion stems from an application of Lemma 5.3 (iv) with initial condition given by $W_\epsilon(x, \xi, \delta)$ instead of $W_\epsilon^I(x, \xi)$. First we must check that the quantity $\epsilon S_{p_0,\epsilon}(\delta)$ is small enough, which is guaranteed since $\epsilon S_{p_0,\epsilon}(\delta) \leq C(W^I, \delta) \epsilon^{1-\frac{9}{p'}}$ for $\frac{18}{17} < p < \frac{9}{8}$, thus $1 - \frac{9}{p'} > 0$. As consequence, Lemma 5.3 (iv) yields

$$\begin{aligned} \|\mathcal{F}_{x \rightarrow y}^{-1} V_\epsilon(\cdot, t)\|_{L^1(\mathbb{R}^3)} &\leq C \left(S_{r,\epsilon}(t) + \sqrt{MS_{r,\epsilon}(t)}\right) \\ &\leq (t - \delta)^{-\frac{9}{2}\left(\frac{1}{p_0}-\frac{1}{r}\right)} (C_1(T) S_{p_0,\epsilon}(\delta) + C_2(T)) \\ &\quad + \sqrt{M} (t - \delta)^{-\frac{9}{4}\left(\frac{1}{p_0}-\frac{1}{r}\right)} \sqrt{C_1(T) S_{p_0,\epsilon}(\delta) + C_2(T)}. \end{aligned}$$

Finally, the result follows from the first part of the Lemma. ■

These bounds give the maximum rate of decay (with respect to the scaling group) of the pseudo-differential term in the rescaled equation (5.6). If we now reformulate this term as a convolution, i.e. $-\theta_\epsilon[V_\epsilon]W_\epsilon = \Phi_\epsilon \star_\xi W_\epsilon$ with

$$\begin{aligned} \Phi_\epsilon &= -i\mathcal{F}_{y \rightarrow \xi}^{-1} \left(V_\epsilon(x + \frac{\epsilon^4 y}{2}, t) - V_\epsilon(x - \frac{\epsilon^4 y}{2}, t) \right) \\ &= -16 \epsilon^{-12} \operatorname{Re} \left(i e^{2i\epsilon^{-4}\xi \cdot x} \left(\mathcal{F}_{x \rightarrow \xi}^{-1} V_\epsilon \right) \left(2 \frac{\xi}{\epsilon^4}, t \right) \right) \end{aligned}$$

and $\Phi_\epsilon(x, \xi, t) = \epsilon^{-6}\Phi(\epsilon^{-3}x, \epsilon^{-1}\xi, \epsilon^{-2}t)$, we find that, using the same kind of ideas developed in the previous sections in order to estimate

$$\|\mathcal{F}_{x \rightarrow y}^{-1} V(\cdot, t)\|_{L^1(\mathbb{R}^3)},$$

the following estimate holds:

Proposition 5.5 *Under the hypotheses of Lemma 5.4, we have*

$$\|(\Phi_\epsilon \star_\xi W_\epsilon)(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} \leq C(W^I, \delta) \|W^I\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} (t - \delta)^{-\frac{9}{2}(\frac{1}{p_0} - \frac{1}{r})} \epsilon^{-\frac{9}{p'}}$$

for $0 < \frac{9}{8} - p$ sufficiently small and ϵ small enough, where $C(W^I, \delta)$ is a positive constant depending on W^I and δ but not on ϵ . As a consequence, the nonlinear term of the WFPF $_\epsilon$ problem decays like $\epsilon^{1-\frac{9}{p'}}$ in $L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$.

The proof is an easy consequence of the Young inequality and the second assertion in Lemma 5.4.

5.2. Compactness in L^1

We recall the following result due to Bouchut and Dolbeault ([6, p. 510]):

Lemma 5.6 *Let $\sigma > 0$, $T > 0$ and $1 \leq p < \infty$ and consider the solution $f \in C([0, T]; L^p(\mathbb{R}^{2N}))$ of*

$$L_0 f \equiv \partial_t f + v \cdot \nabla_x f - \sigma \Delta_v f = h \text{ in } (0, T) \times \mathbb{R}^{2N}, \quad f(\cdot, 0) = f_0.$$

Assume that $f_0 \in F$ a bounded set of $L^p(\mathbb{R}^{2N})$ and $h \in H$ a bounded subset of $L^q((0, T); L^p(\mathbb{R}^{2N}))$, with $1 < q \leq \infty$. Then, for any $\eta > 0$ and ω bounded open subset of \mathbb{R}^{2N} , f is compact in $C([\eta, T]; L^p(\omega))$.

Our proof of compactness in L^1 is based on a straightforward application of this result. We have

Corollary 5.7 Denote by

$$h_\epsilon(x, \xi, t) = -\epsilon\theta[V_\epsilon]W_\epsilon(x, \xi, t).$$

Then, for ϵ sufficiently small there is $\delta > 0$ such that, for every $0 < \eta < T$, we have

$$\|h_\epsilon\|_{L^\infty([\eta, T]; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))} \leq C(W^I, \delta)\epsilon^\delta.$$

The proof is a direct consequence of Proposition 5.5. We shall also need the following bounds for the kinetic energy and the inertial momentum

$$E(t) = \int_{\mathbb{R}_x^d} e(x, t) dx, \quad I(t) = \int_{\mathbb{R}_\xi^d} \mathcal{I}(\xi, t) d\xi.$$

Proposition 5.8 Assume that $S_{6/5}^I$ is bounded and that the initial kinetic energy $E(0)$ and the initial inertial momentum $I(0)$ are finite. Then, there exist positive constants $C = C(T)$ such that, for every $t \geq 0$:

(i) $E(t) \leq C(1 + t)$.

(ii) $I(t) \leq C(1 + t)^3$.

Proof. (i) From Lemma 3.8 (i) and (3.34) with $p_0 = q = \frac{6}{5}$ we first deduce

$$\|\nabla_x V(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq C\|n(\cdot, t)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \leq C(T).$$

Then, the result can be deduced from the energy equation (see (2.41))

$$(5.10) \quad \frac{d}{dt} \int_{\mathbb{R}_x^d} \left(e(x, t) + \frac{\alpha}{2} |\nabla_x V(x, t)|^2 \right) dx = 3\sigma M.$$

Integrating (5.10) over $(0, t)$ yields

$$E(t) = E(0) + 3\sigma Mt + \frac{\alpha}{2} \left\{ \|\nabla_x V(\cdot, 0)\|_{L^2(\mathbb{R}^3)}^2 - \|\nabla_x V(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 \right\}.$$

Now, using the bound for the potential energy we find $E(t) \leq C_1(1 + t)$, which concludes the proof of (i).

(ii) follows from the inertial momentum equation (2.35)

$$\frac{d}{dt} I = 2 \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} (x \cdot \xi) W(x, \xi, t) d\xi dx.$$

Integrating this equation with respect to time and taking into account the identity

$$\int_0^t \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} (x \cdot \xi) W^h(x, \xi, t) d\xi dx = \int_0^t \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} (x \cdot \xi) W(x, \xi, t) d\xi dx,$$

we obtain

$$I(t) = I(0) + 2 \int_0^t \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} (x \cdot \xi) W^h(x, \xi, s) \, d\xi \, dx \, ds.$$

Then, an application of the Hölder inequality leads to

$$I(t) \leq I(0) + C \int_0^t \sqrt{I^h(s)} \sqrt{E^h(s)} \, ds.$$

The proof concludes by using the relations (2.42) and (2.43). ■

Using Proposition 5.8 and the relation between the Husimi kinetic energy and the kinetic energy given in (2.42), the following estimate for the growth in time of the kinetic energy associated with the rescaled Husimi function holds:

$$\begin{aligned} \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} \frac{|\xi|^2}{2} W_\epsilon^h(x, \xi, t) \, d\xi \, dx &= \epsilon^2 \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} \frac{|\xi|^2}{2} W^h(x, \xi, \epsilon^{-2}t) \, d\xi \, dx \\ &\leq C \epsilon^2 (1 + \epsilon^{-2}t) \leq C(1 + t). \end{aligned}$$

Also, for the inertial momentum associated with W_ϵ^h we find

$$\begin{aligned} \int_{\mathbb{R}_\xi^3} \int_{\mathbb{R}_x^3} |x|^2 W_\epsilon^h(x, \xi, t) \, dx \, d\xi &= \epsilon^6 \int_{\mathbb{R}_\xi^3} \int_{\mathbb{R}_x^3} |x|^2 W^h(x, \xi, \epsilon^{-2}t) \, dx \, d\xi \\ &\leq C \epsilon^6 (1 + \epsilon^{-2}t)^3 \leq C(1 + t)^3. \end{aligned}$$

Now, for $\eta > 0$ fixed we can apply Lemma 5.6 to the family

$$\mathcal{F} = \{W_\epsilon(x, \xi, t), \epsilon \leq \epsilon_0\},$$

with $\epsilon_0 = \epsilon_0(\eta)$ sufficiently small, and deduce the following result:

Theorem 5.9 *Let (W, V) be a mild solution of the WPFEP equation (1.6) with initial condition $W^I \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$, such that the initial kinetic energy $E(0)$ and the initial inertial momentum $I(0)$ are finite. Let also $W(t) \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$, $R(t) \geq 0$ and $S_{p_0}^I$ be bounded with $p_0 \geq \frac{6}{5}$ fixed, such that the hypotheses (5.1), (5.2) and (5.3) are fulfilled. Then,*

$$\lim_{t \rightarrow \infty} \|W(\cdot, \cdot, t) - MG_0(\cdot, \cdot, t)\|_{L^1(\omega_t)} = 0$$

for any ω bounded open subset of \mathbb{R}^6 , where $\omega_t = \{(t^{\frac{3}{2}}x, t^{\frac{1}{2}}\xi) \text{ s.t. } (x, \xi) \in \omega\}$.

Also,

$$\lim_{t \rightarrow \infty} \|W^h(\cdot, \cdot, t) - MG_0(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)} = 0.$$

Proof. Fix $0 < \eta < T$ such that $\eta \leq t \leq T$ and consider the families

$$\mathcal{F} = \{W_\epsilon(\cdot, \cdot, \cdot), 0 < \epsilon \leq \epsilon_0(\eta)\}, \quad H = \{h_\epsilon(\cdot, \cdot, t), 0 < \epsilon \leq \epsilon_0(\eta)\},$$

with $h_\epsilon = -\epsilon\theta[V_\epsilon]W_\epsilon$. Then, by Lemma 5.1 (i) and hypothesis (5.1) \mathcal{F} is a bounded subset of $L^\infty(0, \infty; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$. Also, H is a bounded subset of $L^\infty([\eta, T]; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ by Corollary 5.7. Lemma 5.6 now implies that \mathcal{F} is compact in $C([\eta, T]; L^1(\omega))$ for every $0 < \eta < T$, ω being an arbitrary bounded subset of $\mathbb{R}_x^3 \times \mathbb{R}_\xi^3$. Let g denote one of its accumulation points. Then, according to Corollary 5.7 we can pass to the limit $\epsilon \rightarrow 0$ in the rescaled problem (5.6) and obtain that the nonlinear term vanishes at the limit, so that g is the unique weak solution of the linear equation (3.7) with initial condition $g(\cdot, 0) = M\delta_0$ (cf. (5.6b)), δ_0 representing the Dirac mass centered at 0 and M being the total charge of the system. Thus, $g(x, \xi, t)$ coincides with $MG_0(x, \xi, t)$ in the sense of distributions because G_0 is the fundamental solution of the linear operator L defined in (3.7). The first assertion of the Theorem follows straightforwardly from the self-similarity of G_0 (cf. Lemma 3.1 (iii)) by setting $t = 1$ and $\tau = \epsilon^{-2}$, then performing the change of variables $x \mapsto \tau^{\frac{3}{2}}x$, $\xi \mapsto \tau^{\frac{1}{2}}\xi$.

Now it is a simple matter to see that the sequence of rescaled Husimi functions W_ϵ^h is also compact in $C([\eta, T]; L^1(\omega))$. To this end, choose $R > 0$ such that $\omega \subset B_R$ and let χ_R be the characteristic function associated with B_R , where B_R denotes the ball in \mathbb{R}^6 of radius R centered at the origin. Then, we have

$$\begin{aligned} \|W_\epsilon^h - g\|_{L^1(\omega)} &\leq \|(W_\epsilon - g) *_{x,\xi} \Gamma_\epsilon\|_{L^1(\omega)} + \|g *_{x,\xi} \Gamma_\epsilon - g\|_{L^1(\omega)} \\ &\leq \|(W_\epsilon - g) *_{x,\xi} \Gamma_\epsilon \chi_R\|_{L^1(\omega)} + \|(W_\epsilon - g) *_{x,\xi} \Gamma_\epsilon (1 - \chi_R)\|_{L^1(\omega)} + \|g *_{x,\xi} \Gamma_\epsilon - g\|_{L^1(\omega)} \\ &\leq \|W_\epsilon - g\|_{L^1(B_{2R})} + C\|\Gamma_\epsilon\|_{L^1(B_R^c)} + \|g *_{x,\xi} \Gamma_\epsilon - g\|_{L^1(\mathbb{R}^6)}. \end{aligned}$$

Here, $C = C(\|W^I\|_{L^1(\mathbb{R}^6)}, \|g\|_{L^1(\mathbb{R}^6)})$ and B_R^c denotes the complementary set of B_R . Passing to the limit as $\epsilon \rightarrow 0$ proves the compactness of W_ϵ^h in $L^1(\omega)$.

Now, it suffices to observe that the kinetic energy and the inertial momentum associated with W_ϵ^h satisfy

$$E(W_\epsilon^h)(t) \leq C(T), \quad I(W_\epsilon^h)(t) \leq C(T), \quad 0 \leq t \leq T, \quad \epsilon \leq 1.$$

As consequence, we claim that the sequence W_ϵ^h is compact in the whole space $C([\eta, T]; L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ for every $0 < \eta < T$, as follows from the estimate

$$\int_{\mathbb{R}_x^3} \int_{\mathbb{R}_\xi^3} |W_\epsilon^h - g| d\xi dx \leq \int_{B_R} |W_\epsilon^h - g| d\xi dx + \frac{C(T)}{R^2},$$

where we have used that $(|\xi|^2 + |x|^2)W_\epsilon^h, (|\xi|^2 + |x|^2)g \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ with bounds independent of ϵ . Note that we used the positivity of W_ϵ^h for this L^1 -bound. The proof concludes by identifying $t = 1$ and $\epsilon = \tau^{-\frac{1}{2}}$. ■

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