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Circle Actions and Higher Elliptic Genera

By

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Abstract

For a manifold with an S^1-action, we define generalized elliptic genera by using the orbit map and generalize Hirzebruch-Slodowy's formula in [9]. As a result, we have vanishing theorems of higher elliptic genera and higher twisted $\hat{A}$-genera. We also generalize elliptic genera of level N for stable almost complex manifolds and have a similar vanishing theorem.

§1. Introduction

Elliptic genera were introduced by S. Ochanine [13]. The $\hat{A}$-genus and the signature are special cases of elliptic genera. We know many results concerning the $\hat{A}$-genus and the signature which are related with group actions (cf. [10]). Some of these results were extended to the case of elliptic genera. For example, the vanishing theorem of the $\hat{A}$-genus [2] was extended to the rigidity theorem of elliptic genera by Bott-Taubes [4]. For a manifold with an involution, Hirzebruch and Slodowy [9] proved the relation between the elliptic genera of the manifold and the elliptic genera of the fixed point set, which is a generalization of an old formura for the signature. Moreover in [8], Hirzebruch defined elliptic genera of level N for almost complex manifolds and proved the rigidity of those genera.

On the other hand, the vanishing theorem of the $\hat{A}$-genus above was generalized to the vanishing theorem of the higher $\hat{A}$-genus by Browder-Hsiang [6]. In their proof, they first generalized the $\hat{A}$-genus by using an orbit map and proved the vanishing of the generalized $\hat{A}$-genus by using the equivariant surgery. H-T. Ku and M-C. Ku [11] generalized the signature in a similar way and proved the generalized G-signature theorem.

In this paper, we first define generalized elliptic genera in a similar way and generalize the Hirzebruch-Slodowy's theorem above. Consequently, we have some vanishing theorems of higher elliptic genera and higher twisted $\hat{A}$-genera. After that, we generalize the elliptic genera of level N for stable almost complex manifolds and have a similar vanishing theorem.

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§2. Elliptic Genera

Let Ω_*^{SO} be the oriented cobordism ring and Λ any commutative $\mathbf{Q}$-algebra with unit. A genus is a ring homomorphism

$$\varphi : \Omega_*^{SO} \to \Lambda$$

with $\varphi(1) = 1$. Since $\Omega_*^{SO} \otimes \mathbf{Q} = \mathbf{Q}[[\mathbf{C}P^2],[\mathbf{C}P^4],[\mathbf{C}P^6],\ldots], \varphi$ is determined by the logarithm

$$g(x) = \sum_{n \geq 0} \frac{\varphi(\mathbf{C}P^{2n})}{2n+1} x^{2n+1}.$$

Following Ochanine [14], we call φ an elliptic genus if $g(x)$ has the form

$$g(x) = \int_0^x \frac{dt}{\sqrt{1-2\delta t^2 + \varepsilon t^4}}$$

with $\delta, \varepsilon \in \Lambda$. We remark that for any elliptic genus φ one has $\delta = \varphi(\mathbf{C}P^2)$, $\varepsilon = \varphi(\mathbf{H}P^2)$.

Let E be a real vector bundle over X. We write $\Lambda^i(E)$ and $S^i(E)$ for the exterior and the symmetric powers of E respectively, and set

$$\Lambda_t(E) = \sum_{i \geq 0} \Lambda^i(E) t^i$$
$$S_t(E) = \sum_{i \geq 0} S^i(E) t^i.$$

Define

$$\mathscr{R}_q(E) = \bigotimes_{n \geq 0} (\Lambda_{q^n}(E) \otimes S_{q^n}(E))$$

and

$$\Theta_q(E) = \bigotimes_{n \geq 0} (\Lambda_{-q^{2n-1}}(E) \otimes S_{q^{2n}}(E)).$$

Then $\mathscr{R}_q(E)$ and $\Theta_q(E)$ are formal power series in q with coefficients in $KO(X)$. Moreover

$$\mathscr{R}_q(E \oplus F) = \mathscr{R}_q(E) \mathscr{R}_q(F)$$

and

$$\Theta_q(E \oplus F) = \Theta_q(E) \Theta_q(F),$$

hence $\mathscr{R}_q$ and Θ_q can be extended to $KO(X)$.

For a closed n-dimensional oriented smooth manifold M, we define

$$\Phi_1(M) = \left\langle \hat{L}(M) ch(\mathscr{R}_q(T(M) - [n]) \otimes \mathbf{C}), [M] \right\rangle$$

and

$$\Phi_2(M) = \langle \hat{A}(M) ch(\Theta_q(T(M) - [n]) \otimes \mathbf{C}), [M] \rangle,$$

where $T(M)$ is the tangent bundle of M, $\hat{L}$ and $\hat{A}$ are multiplicative sequences for characteristic power series $x/2\tanh(x/2)$ and $x/2\sinh(x/2)$ respectively. Note that Φ_1 and Φ_2 are genera with respect to the characteristic power series $Q_1(x)$ and $Q_2(x)$ respectively, where

$$Q_1(x) = \frac{x/2}{\tanh(x/2)} \prod_{n=1}^{\infty} \frac{(1+q^n e^{-x})(1+q^n e^x)/(1+q^n)^2}{(1-q^n e^{-x})(1-q^n e^x)/(1-q^n)^2}$$

$$Q_2(x) = \frac{x/2}{\sinh(x/2)} \prod_{n=1}^{\infty} \frac{(1-q^{2n-1}e^{-x})(1-q^{2n-1}e^x)/(1-q^{2n-1})^2}{(1-q^{2n}e^{-x})(1-q^{2n}e^x)/(1-q^{2n})^2}.$$

We now recall the following theorem due to D. Zagier.

Theorem 2.1 ([14], **cf.** [7]). (i) Φ_1 *is an elliptic genus with*

$$\delta = \frac{1}{4} + 6\sum_{n=1}^{\infty} \Big(\sum_{\substack{d|n \\ d \text{ odd}}} d\Big) q^n$$

$$\varepsilon = \frac{1}{16} \prod_{n=1}^{\infty} \left(\frac{1-q^n}{1+q^n}\right)^8.$$

(ii) Φ_2 *is an elliptic genus with*

$$\tilde{\delta} = -\frac{1}{8} - 3\sum_{n=1}^{\infty} \Big(\sum_{\substack{d|n \\ d \text{ odd}}} d\Big) q^n$$

$$\tilde{\varepsilon} = \sum_{n=1}^{\infty} \Big(\sum_{\substack{d|n \\ n/d \text{ odd}}} d^3\Big) q^n.$$

It is known that these genera have the modular properties. If we put $q = e^{2\pi i \tau}$ with $\tau \in \mathfrak{h}$ (upper half plane), then the values of these genera are modular forms on $\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \pmod 2 \right\}$. Let $M_k(2)$ denote the complex vector space of all modular forms of weight k on $\Gamma_0(2)$. Then $\delta, \tilde{\delta} \in M_2(2)$ and $\varepsilon, \tilde{\varepsilon} \in M_4(2)$. Moreover for the graded ring $M_*(2) = \oplus_{k \in \mathbf{Z}} M_k(2)$, we have

$$M_*(2) = \mathbf{C}[\delta, \varepsilon].$$

In particular, δ and ε are algebraically independent.

§3. Circle Actions and Higher Elliptic Genera

In this section, we consider manifolds with circle actions. We will assume from now on (unless stated explicitly otherwise) that all S^1-actions are smooth and effective. Hirzebruch and Slodowy proved the following theorem.

Theorem 3.1 ([9], cf. [3]). *Let M be a $4k$-dimensional closed spin manifold with an S^1-action. Let M^I be the fixed point set of the involution I which is induced by the S^1-action. Let F_λ be a connected component of M^I, and d_λ the codimension of F_λ in M. Then,*

$$\Phi_1(M) = \sum_{\lambda} \Phi_1(F_\lambda \circ F_\lambda)\varepsilon^{d_\lambda/4}$$

where $F_\lambda \circ F_\lambda$ is the self-intersection of F_λ and ε is as in Theorem 2.1, (i).

For an m-dimensional closed oriented smooth manifold M and $z \in H^{\cdot}(M;\mathbb{Q})$, we define

$$\Phi_1(M,z) = \left\langle \hat{L}(M)ch(\mathscr{R}(T(M)-[m])\otimes\mathbb{C})\cup z,[M]\right\rangle.$$

We can generalize the theorem above as follows.

Theorem 3.2. *Let M be an m-dimensional closed spin manifold with an S^1-action. Let I, F_λ and d_λ be as in Theorem* 3.1 *and $p: M \to M/S^1$ the orbit map. Then for $x \in H\ (M/S^1;\mathbb{Q})$*

$$\Phi_1(M,p\ x) = \sum_{\lambda} \Phi_1(F_\lambda \circ F_\lambda, i_\lambda p\ x)\varepsilon^{d_\lambda/4}$$

where $i_\lambda : F_\lambda \circ F_\lambda \to M$ is the inclusion.

Proof. Following the proof of [6, Theorem 1.8] (cf. [5]), for any $x \in H^I(M/S^1;\mathbb{Q})$ there exist a transverse framed S^1-submanifold N of $M \times \mathbb{R}^k$ and $c \in \mathbb{Q}-\{0\}$ such that

$$j\ [N] = cp\ x \cap [M]$$

where S^1 acts trivially on the $\mathbb{R}^k$ and $j: N \to M \times \mathbb{R}^k$ is the inclusion. Then

$$\begin{aligned}
\Phi_1(N) &= \left\langle j\ (\hat{L}(M)ch(\mathscr{R}(T(M)-[m])\otimes\mathbb{C})),[N]\right\rangle \\
&= \left\langle \hat{L}(M)ch(\mathscr{R}(T(M)-[m])\otimes\mathbb{C}), j\ [N]\right\rangle \\
&= \left\langle \hat{L}(M)ch(\mathscr{R}(T(M)-[m])\otimes\mathbb{C}), cp\ x\cap[M]\right\rangle \\
&= c\left\langle \hat{L}(M)ch(\mathscr{R}(T(M)-[m])\otimes\mathbb{C})\cup p^{'}x,[M]\right\rangle \\
&= c\Phi_1(M,p\ x).
\end{aligned}$$

We put $\tilde{F}_\lambda = (F_\lambda \times \mathbb{R}^k) \cap N$. We denote the normal bundle of $\tilde{F}_\lambda$ in N and the normal bundle of F_λ in M by $\tilde{\nu}_\lambda$ and ν_λ respectively.

Let $j_\lambda : \tilde{F}_\lambda \to F_\lambda \times \mathbb{R}^k$ and $h_\lambda : F_\lambda \to M$ be the inclusions. We denote the Euler classes of $\tilde{\nu}_\lambda$ and ν_λ by $e(\tilde{\nu}_\lambda)$ and $e(\nu_\lambda)$ respectively. Then

$$\begin{aligned}
\Phi_1(\tilde{F}_\lambda \circ \tilde{F}_\lambda) &= \left\langle \hat{L}(\tilde{F}_\lambda)\hat{L}(\tilde{\nu}_\lambda)^{-1} ch(\mathscr{R}(T(\tilde{F}_\lambda) - \tilde{\nu}_\lambda - [m-l]) \otimes \mathbb{C}) \cup e(\tilde{\nu}_\lambda), [\tilde{F}_\lambda] \right\rangle \\
&= \left\langle j_\lambda(\hat{L}(F_\lambda)\hat{L}(\nu_\lambda)^{-1} ch(\mathscr{R}(T(F_\lambda) - \nu_\lambda - [m]) \otimes \mathbb{C}) \cup e(\nu_\lambda)), [\tilde{F}_\lambda] \right\rangle \\
&= \left\langle \hat{L}(F_\lambda)\hat{L}(\nu_\lambda)^{-1} ch(\mathscr{R}(T(F_\lambda) - \nu_\lambda - [m]) \otimes \mathbb{C}) \cup e(\nu_\lambda), j_{\lambda !}[\tilde{F}_\lambda] \right\rangle \\
&= \left\langle \hat{L}(F_\lambda)\hat{L}(\nu_\lambda)^{-1} ch(\mathscr{R}(T(F_\lambda) - \nu_\lambda - [m]) \otimes \mathbb{C}) \cup e(\nu_\lambda), ch_\lambda^{!} p^{'}(x) \cap [F_\lambda] \right\rangle \\
&= c\left\langle \hat{L}(F_\lambda)\hat{L}(\nu_\lambda)^{-1} ch(\mathscr{R}(T(F_\lambda) - \nu_\lambda - [m]) \otimes \mathbb{C}) \cup e(\nu_\lambda) \cup h_\lambda^{'} p^{'} x, [F_\lambda] \right\rangle \\
&= c\Phi_1(F_\lambda \circ F_\lambda, i_\lambda^{'} p\ x).
\end{aligned}$$

If $m - l \not\equiv 0$ (mod 4), then $\Phi_1(M, p^{'} x) = 0$ and $\Phi_1(F_\lambda \circ F_\lambda, i_\lambda^{'} p^{'} x) = 0$ for any λ. If $m - l \equiv 0 \pmod 4$, it follows from Hirzebruch-Slodowy's theorem above that

$$\Phi_1(N) = \sum_\lambda \Phi_1(\tilde{F}_\lambda \circ \tilde{F}_\lambda)\varepsilon^{d_\lambda/4}.$$

Therefore

$$\Phi_1(M, p\ x) = \sum_\lambda \Phi_1(F_\lambda \circ F_\lambda, i_\lambda^{'} p\ x)\varepsilon^{d_\lambda/4}. \qquad \square$$

Let M be a closed oriented smooth manifold and $K(\pi,1)$ an Eilenberg-MacLane space. For a map $f : M \to K(\pi,1)$ and $x \in H^{'}(K(\pi,1);\mathbb{Q})$, we call $\Phi_1(M, f^{'} x)$ a higher elliptic genus (cf. [12]).

From Theorem 3.2 and [6, Theorem 1.1], we have the following corollary.

Corollary 3.3. *Let M, F_λ and d_λ be the same as in Theorem 3.2. Suppose that $f : M \to K(\pi,1)$ is a map with $f_! : \pi_1(M) \to \pi$ surjective and that $\alpha : \pi \to \pi' = \pi / f_! i_! (\pi_1(S^1))$ is the quotient map where $i : S^1 \to M$ is the inclusion induced by the S^1-action. Then for $x \in H\ (K(\pi',1);\mathbb{Q})$*

$$\Phi_1(M, f\ \alpha\ x) = \sum_\lambda \Phi_1(F_\lambda \circ F_\lambda, i_\lambda f\ \alpha\ x)\ \varepsilon^{d_\lambda/4}.$$

§4. Vanishing Theorems

Let M be a closed connected spin manifold with an S^1-action, and P a Spin-structure for M. The S^1-action is said to be of even type if it lifts to an action on

P. Otherwise is said to be of odd type. Let I be the element of order 2 in S^1. If the fixed point set M^I of I is not empty, then

$$codim(M^I) = \begin{cases} 0 \pmod 4 & \text{if the action is even,} \\ 2 \pmod 4 & \text{if the action is odd} \end{cases}$$

(see [1]). We get the following theorem by Theorem 3.2.

Theorem 4.1. *Let M be a closed connected spin manifold with an odd type S^1-action. Let $p: M \to M/S^1$ be the orbit map. Then for any $x \in H\ (M/S^1;\mathbf{Q})$,*

$$\Phi_1(M, p' x) = 0.$$

Proof. Let I be the element of order 2. If $M^I = \phi$, the theorem is clear by Theorem 3.2.

In case $M^I \neq \phi$, let F_λ be a connected component of M^I. Since the S^1-action is odd, $d_\lambda = codim\ F_\lambda \equiv 2 \pmod 4$. By Theorem 3.2,

$$\Phi_1(M, p\ x) = \sum_\lambda \Phi_1(F_\lambda \circ F_\lambda, i_\lambda p\ x)\varepsilon^{d_\lambda/4}.$$

As we see in the proof of Theorem 3.2, $\Phi_1(M, p\ x)$ equals the elliptic genus of some manifold up to a constant multiplication. So is $\Phi_1(F_\lambda \circ F_\lambda, i_\lambda p\ x)$. Hence they are polynomials in δ and ε with coefficients in $\mathbf{Q}$. Since $d_\lambda - 2 \equiv 0 \pmod 4$, $\sum_\lambda \Phi_1(F_\lambda \circ F_\lambda, i_\lambda p\ x)\varepsilon^{(d_\lambda-2)/4} \in \mathbf{Q}[\delta,\varepsilon]$. However $(\sum_\lambda \Phi_1(F_\lambda \circ F_\lambda, i_\lambda p\ x)\varepsilon^{(d_\lambda-2)/4})\varepsilon^{1/2} = \Phi_1(M, p\ x) \in \mathbf{Q}[\delta,\varepsilon]$. Hence $\sum_\lambda \Phi_1(F_\lambda \circ F_\lambda, i_\lambda p\ x)\varepsilon^{(d_\lambda-2)/4} = 0$. As a result, $\Phi_1(M, p' x) = 0$. □

For an m-dimensional closed oriented smooth manifold M and $z \in H'(M;\mathbf{Q})$, we define

$$\Phi_2(M,z) = \left\langle \hat{A}(M) ch(\Theta_q(T(M) - [m]) \otimes \mathbf{C}) \cup z, [M] \right\rangle.$$

Since δ and ε are algebraically independent, we may replace Φ_1 and ε in Theorem 3.2 with Φ_2 and $\tilde{\varepsilon}$ (in Theorem 2.1 (ii)). We denote the coefficient of q^i in $\Theta_q(T(M))$ by $\Theta^i(T(M))$. Then we have

Theorem 4.2. *Let M be an m-dimensional closed spin manifold with an S^1-action, and let $p: M \to M/S^1$ be the orbit map. Suppose that I is the element of order 2 in S^1. Then, for a non-negative integer i with $i < \dfrac{codim\ M^I}{4}$ and for $x \in H\ (M/S^1;\mathbf{Q})$, we have*

$$\left\langle \hat{A}(M) ch(\Theta^i(T(M)) \otimes \mathbf{C}) \cup p\ x, [M] \right\rangle = 0.$$

Proof. Let F_λ and d_λ be those in Theorem 3.2. Then

$$\Phi_2(M, p^! x) = \sum_\lambda \Phi_2(F_\lambda \circ F_\lambda, i_\lambda^! p^! x)\ \tilde{\varepsilon}^{d_\lambda/4}.$$

Since the constant term of $\tilde{\varepsilon} \in \mathbf{Q}[[q]]$ is zero, the coefficient of q^i in $\tilde{\varepsilon}^{d_\lambda/4}$ is zero for $i < \frac{d_\lambda}{4}$. Therefore we have

$$\left\langle \hat{A}(M) ch(\Theta^i(T(M)-[m]) \otimes \mathbf{C}) \cup p^! x, [M] \right\rangle = 0$$

for $i < \frac{codim\ M^I}{4}$. Since $\Theta^i(T(M)) = \sum_{l=0}^{i} \Theta^l(T(M)-[m])\Theta^{i-l}([m])$,

$$\left\langle \hat{A}(M) ch(\Theta^i(T(M)) \otimes \mathbf{C}) \cup p^! x, [M] \right\rangle = 0$$

for $i < \frac{codim\ M^I}{4}$. □

We get the following corollaries from Theorems 4.1, 4.2 and [6, Theorem 1.1].

Corollary 4.3. *Let M be a closed connected spin manifold with an odd type S^1-action. Let $f: M \to K(\pi, 1)$ and $\alpha: \pi \to \pi' = \pi / f_* i_*(\pi_1(S^1))$ be as in Corollary 3.3. Then for $x \in H\ (K(\pi', 1); \mathbf{Q})$*

$$\Phi_i(M, f\ \alpha\ ) = 0 \qquad (i = 1, 2).$$

Corollary 4.4. *Let M be a closed spin manifold with an S^1-action and I the element of order 2 in S^1. Let $f: M \to K(\pi, 1)$ and $\alpha: \pi \to \pi' = \pi / f_* i_*(\pi_1(S^1))$ be as in Corollary 3.3. Then for a non-negative integer k with $k < \frac{codim\ M^I}{4}$ and for $x \in H^*(K(\pi', 1); \mathbf{Q})$*

$$\left\langle \hat{A}(M) ch(\Theta^k(T(M)) \otimes \mathbf{C}) \cup f\ \alpha^* x, [M] \right\rangle = 0.$$

§5. Higher Elliptic Genera of Level N

In the following, N is a fixed integer greater than 1 and the "variable" x runs through the complex numbers. $\mathfrak{h}$ is the upper half-plane of the complex numbers, $\tau \in \mathfrak{h}$ and $q = e^{2\pi i\tau}$. Let $L = 2\pi i(\mathbf{Z}\tau + \mathbf{Z})$ be a lattice and $\alpha = 2\pi i(\frac{k}{N}\tau + \frac{l}{N})$ with $0 \leq k < N$, $0 \leq l < N$ and $\alpha \neq 0$. In order to define the genus for stable almost complex manifolds, we introduce the function

$$\Phi(x) = (1 - e^{-x}) \prod_{n=1}^{\infty} (1 - q^n e^{-x})(1 - q^n e^{x}) / (1 - q^n)^2$$

and we put

$$f(x)=e^{(k/N)x}\Phi(x)\Phi(-\alpha)/\Phi(x-\alpha).$$

The function $f(x)$ is elliptic with respect to a sublattice $\tilde{L}$ of index N in L (see [7], [8]).

Let M be a compact stable almost complex maifold and c the total Chern class of M. If we write formally

$$c=\prod_{i=1}^{d}(1+x_i),$$

then the elliptic genus of level N is defined as

$$\varphi_N(M)=\langle \prod_{i=1}^{d}\frac{x_i}{f(x_i)},[M]\rangle.$$

It is known that if M has complex dimension m, $\varphi_N(M)$ is a modular form of weight m on $\Gamma_1(N)=\left\{A\in SL_2(\mathbb{Z})A\equiv\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}(\mathrm{mod}\,N)\right\}$.

We consider the case where M has an S^1-action which preserves the stable almost complex structure. For each fixed point p, the circle acts in the stable tangent space $\tilde{T}_p$, hence integers $m_1, m_2, \ldots, m_d$ are defined such that $g\in S^1$ acts by the diagonal matrix $(g^{m_1}, g^{m_2}, \ldots, g^{m_d})$. Let ν be an index for the connected components $(M^{S^1})_\nu$ of the fixed point set M^{S^1}. The numbers $m_1, m_2, \ldots, m_d$ depend on ν and $m_1+m_2+\ldots+m_d$ also depends on ν.

Definition. The S^1-action on M is called N-balanced if for the components $(M^{S^1})_\nu$ of the fixed point set the residue class of $m_1+m_2+\ldots+m_d$ modulo N does not depend on ν. If the action is N-balanced, the common residue class of $m_1+m_2+\ldots+m_d$ is called the type of the action and denoted by t.

In [8], Hirzebruch proved the following theorem.

Theorem 5.1 ([8]). *Let M be a compact stable almost complex manifold with the first Chern class $c_1\equiv 0\,(\mathrm{mod}\,N)$. If M has an S^1-action which preserves the stable almost complex structure and the type t of the action is $\not\equiv 0(\mathrm{mod}\,N)$, then $\varphi_N(M)=0$.*

We can consider generalized elliptic genera of level N for a stable almost complex manifold in a similar way of previous sections. For a stable almost complex manifold M with the total Chern class

$$c(M)=\prod_{i=1}^{d}(1+x_i)$$

and for $z \in H^{'}(M;\mathbb{Q})$, we define

$$\varphi_N(M,z) = \left\langle \prod_{i=1}^{d} \frac{x_i}{f(x_i)} \cup z, [M] \right\rangle .$$

We can generalize Hirzebruch's theorem above as follows.

Theorem 5.2. *Let M be a compact stable almost complex manifold with the first Chern class $c_1 \equiv 0 \pmod N$. If M has an S^1-action which preserves the stable almost complex structure and the type t of the action is $\not\equiv 0 \pmod N$, then for $x \in H^{'}(M/S^1;\mathbb{Q})$*

$$\varphi_N(M, p\ x) = 0$$

where $p: M \to M/S^1$ is the orbit map of the S^1-action.

Proof. As we saw in the proof of Theorem 3.2, for any $x \in H^l(M/S^1;\mathbb{Q})$, there exist a closed framed transverse S^1-submanifold X *of* $M \times \mathbb{R}^k$ and $c \in \mathbb{Q} - \{0\}$ such that

$$cp\ (x) \cap [M] = j\ [X]$$

where $j: X \to M \times \mathbb{R}^k$ is the inclusion. Then, $c\varphi_N(M, p^{'} x) = \varphi_N(X)$.

Since X is a framed submanifold of a stable almost complex manifold $M \times \mathbb{R}^k$, X is also a stable almost complex manifold. If $c_1(M) \equiv 0 \pmod N$, $c_1(X) = j\ c_1(M \times \mathbb{R}^k) \equiv 0 \pmod N$. If the type of the action on M is $\not\equiv 0 \pmod N$, the type of the action on $M \times \mathbb{R}^k$ is $\not\equiv 0 \pmod N$ and the type of the action on X is also $\not\equiv 0 \pmod N$. Hence $\varphi_N(X) = 0$ from Theorem 5.1. As a result, $\varphi_N(M, p\ x) = 0$. □

From this theorem and [6, Theorem 1.1], we have

Corollary 5.3. *Let M be a compact stable almost complex manifold with the first Chern class $c_1 \equiv 0 \pmod N$. Suppose that M has an S^1-action which preserves the stable almost complex structure and that the type t of the action is $\not\equiv 0 \pmod N$. Let $f: M \to K(\pi, 1)$ and $\alpha: \pi \to \pi' = \pi / f\ i\ (\pi_1(S^1))$ be as in Corollary* 3.3. *Then for $x \in H\ (K(\pi', 1);\mathbb{Q})$*

$$\varphi_N(M, f\ \alpha\ x) = 0 .$$

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